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# ON THE DYNAMICAL BOUNDARY EFFECT IN AN ELASTOVISCOPLASTIC HALF-SPACE SUBJECTED TO TRANSVERSE-LONGITUDINAL IMPACT

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We consider a half-space subjected to a skew impact loading with the tangential component of the applied stress being sufficiently large and exceeding the material yield stress. In this case, the classical elastoplastic model is inconsistent and admits no solutions. This gives rise to the question of the model to be used for solving this problem. The elastoviscoplastic model is the simplest generalization of the elastoplastic model for which a solution of the problem exists. This model may be regarded as a regularization of the classical elastoplastic model which is invariant to the change of time scale. As shown in [1, 2], for small relaxation times (compared with the characteristic time of the problem), the solutions obtained on the basis of the elastoviscoplastic model converge to solutions of elastoplastic equations in the region of slow variation of the solutions. In the region of their fast variation and on the jumps of the elastoviscoplastic solution inside the body, there appear boundary effects (boundary layers). Near the boundary of the body, boundary effects appear if there is no agreement between the static condition of plasticity and the boundary conditions [3]. A simple problem of this type is that of tangential impact along the boundary of a half-space. If the impact velocity exceeds some critical value (at which the elastoplastic solution ceases to exist), then a boundary layer appears near the surface in which plastic strains are localized up to complete material fracture [4].

In this paper, a general problem of longitudinal-transverse impact is considered. We study the effect of normal compressive stresses acting on the surface of the body on the process of localization of shear strains. The question is whether there exists a boundary layer in the general case of skew loading, and if it exists, what is the relation between the tangential and the normal forces ensuring that this boundary layer is nontrivial. In the case of normal impact, there is no strain localization, since the constrained plane deformation of the material leads to hardening and prevents stress localization. As shown in the present study, in the case of a longitudinal-transverse impact, strain localization is always observed, provided that the tangential component of the external load exceeds the yield stress of the material, $\sigma_{12}^0 > \tau_s$, and there is no strain localization if $\sigma_{12}^0 \le \tau_s$. However, for $\sigma_{11}^0 > \sigma_s$, the existence of the boundary layer depends on the relation between physical softening of the material under shear strain and its geometrical hardening. Moreover, in these two cases, the boundary layers are of different types. In this paper, we mainly examine the first case, $\sigma_{12}^0 > \tau_s$.

We construct an analytical solution of the problem by an asymptotic method and examine the structure of shear plastic strain localization strips. By a numerical method we obtain a solution of the problem in the general case of a material subject to hardening and investigate propagation of combined plastic loading and load-relief waves.

It is shown that for $t \gg \tau$ ($\tau$ is the relaxation time) and $\sigma_{12}^0 > \tau_s$, the localization does not depend on the magnitude of the normal component of the applied load. In the case of a softening elastoplastic medium, the stress-strain state in the boundary layer tends to a purely shear state on which hydrostatic pressure is imposed. If a plasticity condition of von Mises type holds, this hydrostatic pressure does not affect the material plasticity, and therefore, has no influence on plastic strain localization.

It should be emphasized that the effect described above is of a general character, since waves of arbitrary shape may be locally regarded as plane waves. This effect takes place not only in the case of plane waves in a half-space, but also for an arbitrary body whose surface is subjected to a tangential load the intensity of which exceeds the material yield stress (ultimate strength, for materials with hardening). Moreover, the same effect is observed in the case of quasi-static loading, since for an external dynamical load that tends to a static value, the solution coincides with the static solution of the problem at large distances from the wave front.

## 1. INTRODUCTION

Longitudinal-transverse waves in inelastic materials were considered in many publications. An interest in this problem is due to the fact that it allows one to study interaction of nonlinear combined waves in a complex stress state. In the 1960's, some problems of this kind were solved for elastoplastic materials [6, 7], and in the 1970–80's, for elastoviscoplastic materials the mechanical properties of which depend on strain rates [6, 7]. However, all these works deal exceptionally with either perfectly plastic media or media subject to hardening with a constantly growing diagram of the material subjected to an external loading in which the tangential stress (deviatoric component) does not exceed the shear yield stress of the material. This is due to the fact that for an external loading with $\sigma_{12} > \tau_s$, there is no solution for an elastoplastic material. Obviously, an external loading — by forces or a kinematic loading — has no relation to the material mechanical properties and may be of an arbitrary intensity. If this intensity is high, the material can transfer to a state of fracture or a close state which cannot be described by the model adopted for the description of the state of hardening. This is the case for the elastoplastic model of materials, for which a solution exists only if the external loading is such that the deviatoric components of the stress tensor at the points of the surface do not exceed the shear yield stress. This situation may arise also in the internal layers of a body, for instance, in the case of colliding or interacting waves. In that case, the zones in which the stress-strain state is incompatible with the constitutive equations are usually small as compared with the dimensions of the body or the region in which a solution of the problem is sought. Therefore, it is natural to use a regularization method for solving this problem and introduce into the model additional small terms that determine effects characterized by either a small internal scale of length or high rate of additional internal processes neglected in the elastoplastic model.

In recent years, many investigations have been carried out in that direction and certain results have been obtained for some more complex models. However, all these works are based on numerical analyses, and the solutions obtained frequently do not have due mathematical justification [8–11].

The elastoviscoplastic model is one of the simplest and well-studied models allowing for a natural regularization of the elastoplastic model to solve the corresponding ill-posed problems. Moreover, this model provides a physically more adequate description of the material behavior for large loading rates and is quite reasonable to be used as a basis for the investigation of dynamical processes [1, 6].

Solutions of ill-posed dynamical problems were previously studied only for the case of wave propagation in uniaxial stress states in beams or in a half-space [3, 4].

At the same time, in order to study and understand processes occurring in the general case of contacting bodies, it is necessary, primarily, to examine processes of interaction between combined longitudinal-transverse waves, since the contact gives rise to both normal and tangential forces.

In the last 10–15 years, several experimental works have been done regarding the longitudinal-transverse impact of plates with the aim of studying plastic flow and plastic strain localization for shear rates of $10^5$ to $10^6$ s$^{-1}$ [12], as well as some works on the numerical simulation of this process [13]. These investigations show that the elastoviscoplastic model adequately describes the properties of various materials in complex stress-strain states with high-rate loading.

## 2. STATEMENT OF THE PROBLEM AND BASIC EQUATIONS

Consider the elastoviscoplastic half-space $x_1 \geq 0$. Its surface is subjected to tangential and normal stresses of arbitrary intensity. These stresses are independent of the coordinates $x_2$ and $x_3$, and their time-dependence is expressed by an arbitrary law

$$\sigma_{11} = \sigma^0_{11}(t), \quad \sigma_{12} = \sigma^0_{12}(t) \qquad \text{at} \quad x = 0. \tag{2.1}$$

We will also consider the kinematic boundary conditions

$$v_1 = v^0_1(t), \quad v_2 = v^0_2(t) \qquad \text{at} \quad x = 0, \tag{2.2}$$

or the boundary conditions of mixed type

$$\sigma_{11} = \sigma^0_{11}(t), \quad v_2 = v^0_2(t) \qquad \text{at} \quad x = 0. \tag{2.3}$$

We consider small displacements of the medium, neglecting the displacements of the external boundary of the body.

Due to the structure of the applied load and the symmetry of the problem, we have $\sigma_{23} = \sigma_{13} = 0$, $\varepsilon_{22} = \varepsilon_{33} = 0$, $v_3 = 0$, $\sigma_{22} = \sigma_{33}$. Then, in the absence of mass forces, the equations of motion read

$$\rho\frac{\partial v_1}{\partial t} = \frac{\partial \sigma_{11}}{\partial x}, \quad \rho\frac{\partial v_2}{\partial t} = \frac{\partial \sigma_{12}}{\partial x}. \tag{2.4}$$

The constitutive equations of the elastoviscoplastic material [6, 7] take the form

$$\frac{1-\nu}{1-2\nu}\frac{\partial v_1}{\partial x} - \frac{1}{2\mu}\frac{\partial \sigma_{11}}{\partial t} = \frac{2}{3}\frac{\Phi_1}{\tau}, \quad \frac{1}{2}\frac{\partial v_2}{\partial x} - \frac{1}{2\mu}\frac{\partial \sigma_{12}}{\partial t} = \frac{\Phi_2}{\tau}, \quad \frac{\partial \sigma_{22}}{\partial t} = \frac{\nu}{1-\nu}\frac{\partial \sigma_{11}}{\partial t} + \frac{2}{3}\mu\frac{1+\nu}{1-\nu}\frac{\Phi_1}{\tau}. \tag{2.5}$$

The strains $\varepsilon_{ij}$ are related to the velocities $v_1$, $v_2$, and $v_3$ by the compatibility relations

$$\frac{\partial \varepsilon_{11}}{\partial t} = \frac{\partial v_1}{\partial x}, \quad \frac{\partial \varepsilon_{12}}{\partial t} = \frac{1}{2}\frac{\partial v_2}{\partial x}. \tag{2.6}$$

The functions on the right-hand sides of the constitutive equations (2.5) have the form

$$\Phi_1 = \widehat{\Phi}(F)\frac{\sigma_{11}-\sigma_{22}}{J_2}, \quad \Phi_2 = \widehat{\Phi}(F)\frac{\sigma_{12}}{J_2}, \qquad F = \frac{J_2 - \tau_s(W_p)}{\tau_s(W_p)},$$
$$\widehat{\Phi}(F) = \begin{cases} 0, & \text{if } F < 0, \\ \Phi(F), & \text{if } F \geq 0, \end{cases} \quad \widehat{\Phi}(0) \equiv 0, \quad J_2 = \sqrt{\frac{(\sigma_{11}-\sigma_{22})^2}{3} + \sigma_{12}^2},$$

where $\mu$ is the shear modulus, $\nu$ is Poisson's ratio, $\tau_s$ is the shear yield stress of the material, $J_2$ is the second invariant of the stress tensor deviator, and $\tau$ is the relaxation time.

If the medium is capable of hardening, then $\tau_s = \tau_s(W_p)$, where $W_p$ is the work of plastic strain determined by the equation

$$\frac{dW_p}{dt} = \sigma_{ij}\frac{d\varepsilon_{ij}^p}{dt} = 2\frac{\Phi_3}{\tau}, \qquad \Phi_3 = \widehat{\Phi}(F)J_2, \tag{2.7}$$

which makes system (2.4)–(2.6) complete.

Equations (2.6) do not enter the system of equations and are integrated after the basic unknown vector $\mathbf{U} = [v_1, v_2, \sigma_{11}, \sigma_{12}, \sigma_{22}, W^p]$ has been found from equations (2.4), (2.5), and (2.7). For an elastoviscoplastic medium, the yield stress remains constant and $W^p$ is not among the unknown quantities. In that case, the vector $\mathbf{U}$ has only five components.

Introduce the dimensionless variables

$$x = l\bar{x}, \quad t = t_0\bar{t}, \quad \sigma_{ij} = \tau_s^0\bar{\sigma}_{ij}, \quad v_i = v_s\bar{v}_i, \quad \varepsilon_{ij} = \varepsilon_s\bar{\varepsilon}_{ij},$$
$$t_0 = \frac{l}{c_1}, \quad v_s = \frac{\tau_s^0}{c_1\rho}, \quad (c_1)^2 = \frac{\lambda+2\mu}{\rho}, \quad (c_2)^2 = \frac{\mu}{\rho}, \quad \varepsilon_s = \frac{\tau_s^0}{\mu},$$

where $l$ and $t_0$ are characteristic parameters of the problem.

Then the system of equations can be written in the form

$$\frac{\partial v_1}{\partial t} - \frac{\partial \sigma_{11}}{\partial x} = 0, \quad \frac{\partial v_2}{\partial t} - \frac{\partial \sigma_{12}}{\partial x} = 0, \quad \frac{\partial \sigma_{11}}{\partial t} - \frac{\partial v_1}{\partial x} = -\frac{\delta}{3}\Phi_1, \quad \frac{\partial \sigma_{12}}{\partial t} - \left(\frac{c_2}{c_1}\right)^2\frac{\partial v_2}{\partial x} = -\delta\Phi_2,$$
$$(1-\nu)\frac{\partial \sigma_{22}}{\partial t} - \nu\frac{\partial \sigma_{11}}{\partial t} = (1+\nu)\frac{\delta}{3}\Phi_1, \quad \frac{\partial(\tau_s - E_1W^p)}{\partial t} = 0, \quad \frac{\partial W^p}{\partial t} = \delta\Phi_3, \qquad \delta = 2\frac{\mu t_0}{\tau_s^0\tau} = 2\frac{t_0}{\varepsilon_s\tau}. \tag{2.8}$$

The hardening-softening law of (2.8) is written in differential form, which allows one to take into account automatically the initial conditions for load-relief and initial loading.

The system (2.8) depends on the relationship between elastic constants which can be expressed in terms of only Poisson's ratio $\nu$. The right-hand side depends on the dimensionless parameter $\delta$, which is large, $\delta = 10^3$–$10^5$, for most metals and characteristic times of the problems considered here. As $\delta^{-1}$ tends to zero, the equations of an elastoviscoplastic material transform into the equations of an elastoplastic material and the solution of the former equations asymptotically converges to an elastoplastic solution [1–4, 6].

## 3. OBTAINING A SOLUTION BY AN ASYMPTOTIC METHOD

In order to solve the above problem regarding propagation of longitudinal-transverse waves, we use an asymptotic method and seek a solution of system (2.8) in the form of an expansion in powers of $\delta^{-1}$,

$$\mathbf{U} = \mathbf{U}^{(0)} + \delta^{-1}\mathbf{U}^{(1)} + \cdots + o(\delta^{-n}). \tag{3.1}$$

The solution $\mathbf{U}^{(0)}$ can be found from equations (2.8) after passing to the limit as $\delta^{-1} \to 0$. In order to pass to the limit, it is convenient to write the constitutive equations as the plasticity condition and the condition that the plastic strain rate tensor should be proportional to the stress deviator [7]

$$S = S(W_\mathrm{p}) + \Phi^{-1}(\dot{I}_\mathrm{p}\delta^{-1}), \quad \dot{\varepsilon}^\mathrm{p}_{ij} = \dot{\Lambda} s_{ij} = \frac{\dot{I}_\mathrm{p}}{S} s_{ij}, \qquad S = \sqrt{\frac{1}{2} s_{ij} s_{ij}}, \quad \dot{I}_\mathrm{p} = \sqrt{\frac{1}{2}\dot{\varepsilon}^\mathrm{p}_{ij}\dot{\varepsilon}^\mathrm{p}_{ij}}. \tag{3.2}$$

Hence, by letting $\delta^{-1}$ tend to zero, we obtain the plasticity condition $S = S(W_\mathrm{p})$, while the second relation remains unchanged, since it is independent of $\delta$. The scalar quantity $\Lambda$ is determined in a standard way from the plasticity condition after its differentiation, as is the case in the theory of plastic flow [6, 7]. As a result, instead of equations (2.8) we obtain a system of equations of plastic flow for longitudinal-transverse waves,

$$\frac{\partial v_1}{\partial x} - \frac{\partial \sigma_{11}}{\partial t} + \dot{\Lambda} s_{11} = 0, \quad \frac{\partial \sigma_{12}}{\partial t} - \left(\frac{c_2}{c_1}\right)^2 \frac{\partial v_2}{\partial x} + \dot{\Lambda}\sigma_{12} = 0, \quad (1-\nu)\frac{\partial \sigma_2}{\partial t} - \nu\frac{\partial \sigma_1}{\partial t} - (1+\nu)\dot{\Lambda} s_{11} = 0, \tag{3.3}$$
$$\dot{\Lambda} = \frac{3\alpha(K)}{4K^2 E}\frac{\partial F}{\partial \sigma_{ij}}\frac{\partial \sigma_{ij}}{\partial t},$$

where $K$ is the bulk modulus. The function $\alpha(K)$ is determined by means of the stress-strain curve of the material in the uniaxial stress state $\sigma = \sigma(\varepsilon)$ or $\sigma = \sigma(W_\mathrm{p})$:

$$\alpha(K) = \frac{E}{E_\mathrm{p}(3K)} - 1, \qquad E_\mathrm{p}(\sigma) = \frac{d\sigma}{d\varepsilon^\mathrm{p}},$$

where $E_\mathrm{p}(\sigma)$ is the slope of the curve $d\sigma/d\varepsilon^\mathrm{p}$ as a function of $\sigma$ [5, 6]. In the case of perfect plasticity, we have $E_\mathrm{p} \to 0$, $\alpha \to \infty$; in the case of softening, we have $E_\mathrm{p} < 0$, $\alpha < -1$ and the problem has no solutions under the boundary conditions (2.1)–(2.3), if the tangential stress is greater than the ultimate strength on the diagram $\tau_\mathrm{s}(W_\mathrm{p})$.

System (3.3), as well as system (2.8), is of hyperbolic type, but its matrix depends on the solution of the system under the plasticity condition $S = \tau_\mathrm{s}(W_\mathrm{p})$. This implies two essential points of difference in the behavior of solutions of equations (2.8). The first is that their discontinuous solutions have different behavior, and the second is that equations (2.8) have a solution for any intensity in the boundary conditions (2.1)–(2.3), whereas system (3.3) has no solutions in the case of softening diagram $S = \tau_\mathrm{s}(W_\mathrm{p})$ or the ideal diagram of the material, since the constitutive equations turn out to be incompatible with the boundary conditions. These two facts show that the expansion (3.1) in powers of the small parameter becomes incorrect in the region of fast variation rate of solutions of system (2.8) and there is an essential divergence of the solutions of these two systems. This fact should be kept in mind and when constructing the expansion one should take into account fast growth of some components of the solution and slow growth of others.

A method of asymptotic integration was proposed in [1]. Solutions describing the structure of shock waves and contact discontinuities were constructed in [1, 2], and solutions describing plastic strain localization strips were obtained in [3, 4].

In order to construct an asymptotic solution in the boundary layer, it is necessary to have a clear physical picture of the phenomenon. First of all, it should be kept in mind that there is no boundary effect in a half-space subjected to a normal impact whose intensity exceeds the yield stress, $\sigma^0_{11} > \sigma_\mathrm{s}$. This occurs because the elastoplastic problem has a solution for any impact intensity due to hardening accounted for by a geometrical constraint of the material. In the case of skew impact, the distribution of normal stresses $\sigma_{11}$ and $\sigma_{22}$ depends on the tangential stress $\sigma_{12}$ applied at the boundary of the half-space. If $\sigma_{12} < \tau_\mathrm{s}$, then $\sigma_{11} \neq \sigma_{22}$, but we must have $\sigma_{11} = \sigma_{22}$ for the limit value $\sigma_{12} = \tau_\mathrm{s}$ as $\delta^{-1} \to 0$, in view of the static plasticity condition. Then it is easy to understand that for $\sigma_{12} \geq \tau_\mathrm{s}$, we also have $\sigma_{22} \to \sigma_{11}$ in the elastoviscoplastic solution for $t \gg \tau$ in the boundary layer and the normal components vary slowly. For the tangential stresses we have $\sigma_{12} \to \tau_\mathrm{s}(W_\mathrm{p})$, and the shear strain $\varepsilon_{12}$ and the plastic work $W_\mathrm{p}$ change much faster than the longitudinal strain $\varepsilon_{11}$. This fact should be taken into account when expanding the solution in powers of the small parameter $\delta^{-1}$. To that end, we introduce in the equations another small parameter $\lambda$ as the coefficient of extension along the $x$-axis, so that large derivatives of the solution with respect to the stretched coordinate $x_1 = x/\lambda$ would be of the order $O(1)$. The order of $\lambda$ relative to $\delta^{-1}$ is determined in the process of solving the problem. In a neighborhood of the surface $x = 0$, where the material undergoes no hardening, velocities of particles in the transverse wave will be of the same order as in the region of hardening; tangential stresses will be close to the value determined by the stationary plasticity condition $\sigma_{12} \sim \tau_s(W_p)$, and the strain $\varepsilon_{12}$ and $W_p$ will grow rapidly and become much larger than in the region of hardening.

These considerations have been suggested by the asymptotic analysis of the system of equations. Therefore, in the boundary layer we adopt the following expansion in powers of the small parameter $\lambda$ characterizing the scale extension with respect to the coordinate $x = \lambda x_1$:

$$\begin{aligned} &\sigma_{11} = \sigma_{11}^0 + \lambda\sigma_{11}^{(1)} + \lambda^2\sigma_{11}^{(2)} + \cdots, \quad v_1 = v_1^0 + \lambda v_1^{(1)} + \cdots, \\ &\sigma_{22} = \sigma_{11}^0 + \lambda\sigma_{22}^{(1)} + \lambda^2\sigma_{22}^{(2)} + \cdots, \quad \varepsilon_{11} = \varepsilon_{11}^0 + \lambda\varepsilon_{11}^{(1)} + \cdots, \\ &\sigma_{12} = \tau_s(W_p) + \lambda\sigma_{12}^{(1)} + \lambda^2\sigma_{12}^{(2)} + \cdots, \quad v_2 = v_2^{(0)} + \lambda v_2^{(1)} + \cdots. \end{aligned} \tag{3.4}$$

Here, the terms of the zero approximation for the longitudinal wave in the boundary layer are constant and are determined by the boundary condition $\sigma_{11}^{(0)} = \sigma_{11}^0$, and the transverse components $\sigma_{12}^{(0)}$, $v_2^{(0)}$ and $W_p^{(0)}$ change rapidly. The expansions of $\varepsilon_{12}$ and $W_p$ have a singular form,

$$\varepsilon_{12} = \lambda^{-1}\varepsilon_{12}^{(0)} + \varepsilon_{12}^{(1)} + \varepsilon_{12}^{(2)}\lambda + \cdots, \quad W_p = \lambda^{-1}W_p^{(0)} + W_p^{(1)} + \lambda W_p^{(2)} + \cdots. \tag{3.5}$$

Substituting these expansions into equations (2.8), we find that in a neighborhood of the boundary layer the constitutive equations and the equations of motion corresponding to a longitudinal wave are satisfied identically by the zero approximation in the expansion (3.4)

For $\sigma_{11}^{(0)} = \sigma_{22}^{(0)}$, the right-hand sides of the third and the fifth equations of system (2.8) vanish, and the system splits. The normal components are described by homogeneous equations. The invariable homogeneous state $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{11}^0$ and $v_1 = v_2 = \sigma_{11}^0$ (in dimensionless variables) satisfies these equations identically and coincides with the slowly varying solution described by the equations of elastoplastic flow (3.3) obtained in [5].

The tangential components satisfy the second, the fourth, and the sixth equations of system (2.8), which take the form

$$\frac{\partial v_2}{\partial t} - \frac{\partial \sigma_{12}}{\partial x} = 0, \quad \frac{\partial \sigma_{12}}{\partial t} - \left(\frac{c_2}{c_1}\right)^2 \frac{\partial v_2}{\partial x} = -\delta\Phi\left[\frac{\sigma_{12} - \tau_s(W_p)}{\tau_s(W_p)}\right]\sigma_{12}, \quad \frac{dW_p}{dt} = \delta\Phi\left[\frac{\sigma_{12} - \tau_s(W_p)}{\tau_s(W_p)}\right]\sigma_{12}.$$

This system can be written in a simpler form after passing from the hardening parameter $W_p$ directly to the shear strain and using the diagram $\sigma_{12} = \tau_s(\varepsilon_{12})$ obtained in the standard experiment on shear. Then

$$\frac{\partial v_2}{\partial t} - \frac{\partial \sigma_{12}}{\partial x} = 0, \quad \frac{\partial \varepsilon_{12}}{\partial t} = \frac{\partial v_2}{\partial x}, \quad \frac{\partial \sigma_{12}}{\partial t} - \left(\frac{c_2}{c_1}\right)^2 \frac{\partial v_2}{\partial x} = -\delta\Phi[\sigma_{12} - \tau_s(\varepsilon_{12})]. \tag{3.6}$$

Substituting the series (3.4)–(3.5) into (3.6), passing to the stretched coordinate $\beta$, and taking into account that for $z \to 0$ the function $\Phi(z)$ has the form

$$\Phi(z) = z^n + O(z^{n+1}), \qquad n > 0,$$

we find that the zero approximation of the rapidly varying solution is determined by the following system of equations

$$\left(\frac{c_2}{c_1}\right)^2 \frac{\partial v_2^{(0)}}{\partial t} = \frac{\partial \sigma_{12}^{(0)}}{\partial x_1}, \quad \frac{\partial \varepsilon_{12}^{(0)}}{\partial t} = \frac{\partial v_2^{(0)}}{\partial x_1}, \quad \frac{1}{\lambda}\frac{\partial v_2^{(0)}}{\partial x_1} = \frac{1}{\delta}\left(\lambda\sigma_{12}^{(1)}\right)^n.$$

It follows that $\lambda = \delta^{-(n+1)}$. Eliminating $\sigma_{12}^{(1)}$ and letting $c = (c_2/c_1)^{-1/n}$ and $x_1 c = \beta$, we obtain a nonlinear parabolic system for $v_1$ and $\varepsilon_{12}$, namely,

$$\frac{\partial v_2^{(0)}}{\partial t} = \frac{\partial}{\partial \beta}\left(\frac{\partial v_2^{(0)}}{\partial \beta}\right)^{1/n}, \quad \frac{\partial \varepsilon_{12}^{(0)}}{\partial t} = c\frac{\partial v_2^{(0)}}{\partial \beta}. \tag{3.7}$$

Equations (3.7) hold exactly, if the material diagram (stress-strain curve of the material) is perfectly plastic, and if the softening effect is small, then equations (3.7) hold approximately.

The solution of equation (3.7) should be matched with the slowly varying solution of system (3.3). The matching condition reads as follows:

$$v_2^{(0)}\big|_{\beta=0} = v_2^0, \quad v_2^{(0)}\big|_{\beta\to\infty} = v_s^*. \tag{3.8}$$

Here, $v_2^0$ is the velocity value corresponding to the external load applied at $x = 0$, and $v_s^*$ is the velocity value at which softening starts; this value is found from the elastoplastic equations (3.3). Here, the initial conditions may be disregarded, since one seeks a steady-state solution for large $t \gg \tau\varepsilon_s$. The only restriction is that they should be compatible with the boundary conditions (3.8).

Thus, in the boundary layer, the normal stresses express the hydrostatic pressure equal to the stress $\sigma = \sigma^0$ applied at the boundary. The tangential stresses in the boundary layer are determined by the static diagram $\sigma_{12} \approx \tau_s(W_p)$, and the tangential velocities $v_2$ and the strains $\varepsilon_{12}$ change rapidly in accordance with system (3.7). The boundary layer represents a strip of shear strain localization.

## 4. STRUCTURE OF LOCALIZATION STRIPS

The solution of problem (3.7)–(3.8) is self-similar. It depends on a single variable and satisfies the following ordinary differential equation

$$\frac{d^2v_2}{dz^2} + \frac{4\nu}{p(p+1)}\left(\frac{dv_2}{dz}\right)^{1-p}, \qquad z = \frac{\beta}{c(2t)^{n/(n+1)}}, \quad p = \frac{1}{n} \quad (n > 0). \tag{4.1}$$

The complete solution of problem (4.1), (3.8) is given in [3].

Strain localization strips (zones where softening has occurred) increase with respect to time, but exponentially decay with respect to the spatial variable. The effective width of this decay can be calculated by the formula

$$\Delta x = \frac{\Delta c_2}{\max(dv_2/dx)} = t^{n/(n+1)}\delta^{1/(n+1)}\Delta z, \qquad \Delta v_2 = v_2^0 - v_s^*, \tag{4.2}$$

where $\Delta z$ is independent of $t$ and $\delta$.

For $n = 1$, the solution of equation (4.1) can be expressed in terms of the probability integral

$$v_2 = \frac{2\Delta v_2}{\sqrt{\pi}}\int_0^z e^{-\xi^2}\,d\xi = \Delta v_2\,\mathrm{erf}(z). \tag{4.3}$$

In this case, the effective width of the localization strip does not depend on $\Delta v$,

$$\Delta x = \sqrt{\delta\pi t}. \tag{4.4}$$

The solution for the strain can be obtained with the help of the second equation of system (3.7),

$$\varepsilon_{12} - \varepsilon_{12}(0) = \frac{\Delta v_2\beta}{2c\sqrt{\pi}}\left[\frac{e^{-z^2}}{z\sqrt{\pi}} - 1 + \mathrm{erf}(z)\right], \tag{4.5}$$

where $\varepsilon_{12}(0)$ is the initial strain at $t = 0$. For $z \gg 1$, the right-hand side of (4.5) has the following asymptotic behavior (4.5):

$$\varepsilon_{12} - \varepsilon_{12}(0) = \Delta v_2\frac{t^{3/2}c^2}{2\pi\beta^2}e^{-\beta^2/4tc^2}[1 + O(z^{-2})]. \tag{4.6}$$

Hence, the strains decay very rapidly with respect to $\beta$ along the thickness of the localization strip.

For small $\beta \ll 1$, from the solution (4.5) we find

$$\varepsilon_{12} - \varepsilon_{12}(0) = \frac{\Delta v_2}{\pi}\sqrt{\frac{tL}{l}}\,[1 + O(z)], \tag{4.7}$$

where $l = \tau\varepsilon_s c_0$ is the internal characteristic length in the elastoviscoplastic material.

From (4.7), it follows that for the classical model of an elastoplastic material regarded as the limit case of an elastoviscoplastic material as $\tau \to 0$, strain becomes infinite in the localization strips.

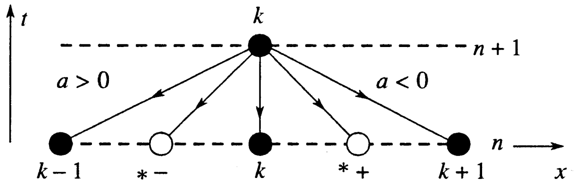


Fig. 1

## 5. A METHOD OF NUMERICAL INTEGRATION

System (2.8) contains a large parameter $\delta$ on the right-hand side or a small coefficient of the highest-order derivatives, i.e., the system is stiff.

There are many papers dedicated to the integration of stiff ODE's, where it is shown that numerical integration of such equations should be carried out on the basis of $A$-stable difference methods [14]. Integration or stiff partial differential equations has been studied much less and there is no mathematical theory of such equations. For each particular class of problems, special methods have to be developed. Some methods of integration of hyperbolic problems for partial differential equations can be found in [16]. One way to approach this problem for hyperbolic equations of elastoviscoplastic media is to reduce them to characteristic relations and integrate the resulting ODE's along the characteristics with the help of an implicit scheme. The $A$-stability is proved for a linearized system of difference equations with their right-hand sides taken into account, since they contain a large parameter and cannot be neglected [15, 17]. Let us apply this method to obtain a solution of system (2.8).

After the reduction to the characteristic form, system (2.8) becomes

$$\begin{aligned} x \pm t = \text{const}: &\quad \mp \frac{dv_1}{dt} + \frac{d\sigma_1}{dt} = -\frac{\delta}{3}\Phi_1, \\ x \pm \frac{c_2}{c_1} t = \text{const}: &\quad \mp \frac{c_2}{c_1}\frac{dv_2}{dt} + \frac{d\sigma_{12}}{dt} = -\delta\Phi_2, \\ x = \text{const}: &\quad (1-\nu)\frac{d\sigma_2}{dt} - \nu\frac{d\sigma_1}{dt} = (1+\nu)\frac{\delta}{3}\Phi_1, \quad \frac{d\tau_\text{s}}{dt} - E_1\frac{dW_\text{p}}{dt} = 0, \quad \frac{dW_\text{p}}{dt} = \delta\Phi_3. \end{aligned} \tag{5.1}$$

The difference scheme is constructed on the basis of the four-point stencil shown in Fig. 1. The characteristics of the longitudinal waves pass through the basic nodes, and those of the transverse waves through the additional nodes, at which the solution is found by interpolation. In the case of an implicit scheme, the right-hand sides are approximated by their values on the upper, $(n+1)$th, layer or as a half-sum of the values on the $n$th and the $(n+1)$th layers,

$$\begin{aligned} &-\frac{v_{1k}^{n+1} - v_{1,k-1}^n}{\Delta t} + \frac{\sigma_{1k}^{n+1} - \sigma_{1,k-1}^n}{\Delta t} = -\frac{\delta}{3}\Phi_{1,k-1}^{n+1}, \quad \frac{v_{1k}^{n+1} - v_{1,k+1}^n}{\Delta t} + \frac{\sigma_{1k}^{n+1} - \sigma_{1,k+1}^n}{\Delta t} = -\frac{\delta}{3}\Phi_{1,k+1}^{n+1}, \\ &\frac{c_2}{c_1}\frac{v_{2k}^{n+1} - v_{2*-}^n}{\Delta t} + \frac{\sigma_{12k}^{n+1} - \sigma_{12*-}^n}{\Delta t} = -\delta\Phi_{2*-}^{n+1}, \quad \frac{c_2}{c_1}\frac{v_{2k}^{n+1} - v_{2*+}^n}{\Delta t} + \frac{\sigma_{12k}^{n+1} - \sigma_{12*+}^n}{\Delta t} = -\delta\Phi_{2*+}^{n+1}, \\ &(1-\nu)\frac{\sigma_{2k}^{n+1} - \sigma_{2k}^n}{\Delta t} - \nu\frac{\sigma_{1k}^{n+1} - \sigma_{1k}^n}{\Delta t} = (1+\nu)\frac{\delta}{3}\Phi_{1k}^{n+1}, \\ &\frac{\tau_{\text{s}k}^{n+1} - \tau_{\text{s}k}^n}{\Delta t} - E_1\frac{W_k^{\text{p},n+1} - W_k^{\text{p}n}}{\Delta t} = 0, \quad \frac{W_k^{\text{p},n+1} - W_k^{\text{p}n}}{\Delta t} = \delta\Phi_{3k}^{n+1}. \end{aligned} \tag{5.2}$$

The subscripts $*-$ and $*+$ mark the nodes through which the characteristics of the transverse wave pass.

The system of nonlinear equations (5.2) should be solved by linearizing the right-hand sides for each characteristic,

$$\Phi_{ik}^{n+1} = \Phi_i^n + \Phi_{i,\sigma_1}^n\Delta\sigma_1 + \Phi_{i,\sigma_{12}}^n\Delta\sigma_{12} + \Phi_{i,\sigma_2}^n\Delta\sigma_2 + \Phi_{i,\tau_\text{s}}^n\Delta\tau_\text{s}, \qquad \Phi_{i,\sigma_{kl}} = \frac{\partial\Phi_i}{\partial\sigma_{kl}}. \tag{5.3}$$

The difference scheme can be reduced to a system of linear equations. It is convenient to write this system in matrix form, arranging the known quantities on the right-hand side according to the nodes of the pattern,

$$[B]\mathbf{V}_k^{n+1} = [C_{-1}]\mathbf{V}_{k-1}^n + [C_{*-}]\mathbf{V}_{*-}^n + [C_0]\mathbf{V}_k^n + [C_{*+}]\mathbf{V}_{*+}^n + [C_{+1}]\mathbf{V}_{k+1}^n + \mathbf{F}, \tag{5.4}$$

$$\mathbf{F} = \left[z_1\Phi^n_{1,k-1},\ z_1\Phi^n_{1,k+1},\ z_2\Phi^n_{2*-},\ z_2\Phi^n_{2*+},\ z_3\Phi^n_{1k},\ 0,\ z_4\Phi^n_{3k}\right]^{\mathrm{T}},$$

$$\mathbf{V}(t,x) = \left[v_1, v_2, \sigma_{11}, \sigma_{12}, \sigma_{22}, \tau_{\mathrm{s}}, W^{\mathrm{p}}\right]^{\mathrm{T}}, \qquad z_1 = -\frac{\delta}{3}, \quad z_2 = -\delta, \quad z_3 = (1+\nu)\frac{\delta}{3}, \quad z_4 = \delta,$$

$$[B] = \begin{bmatrix} -\dfrac{1}{\Delta t} & 0 & \dfrac{1}{\Delta t} - z_1\Phi^n_{1,\sigma_{1,k-1}} & -z_1\Phi^n_{1,\sigma_{12,k-1}} & -z_1\Phi^n_{1,\sigma_{2,k-1}} & -z_1\Phi^n_{1,\tau_{\mathrm{s},k-1}} & 0 \\ \dfrac{1}{\Delta t} & 0 & \dfrac{1}{\Delta t} - z_1\Phi^n_{1,\sigma_{1,k+1}} & -z_1\Phi^n_{1,\sigma_{12,k+1}} & -z_1\Phi^n_{1,\sigma_{2,k+1}} & -z_1\Phi^n_{1,\tau_{\mathrm{s},k+1}} & 0 \\ 0 & -\dfrac{c_2}{c_1}\dfrac{1}{\Delta t} & -z_2\Phi^n_{2,\sigma_{1*-}} & \dfrac{1}{\Delta t} - z_2\Phi^n_{2,\sigma_{12*-}} & -z_2\Phi^n_{2,\sigma_{2*-}} & -z_2\Phi^n_{2,\tau_{\mathrm{s}*-}} & 0 \\ 0 & \dfrac{c_2}{c_1}\dfrac{1}{\Delta t} & -z_2\Phi^n_{2,\sigma_{1*+}} & \dfrac{1}{\Delta t} - z_2\Phi^n_{2,\sigma_{12*+}} & -z_2\Phi^n_{2,\sigma_{2*+}} & -z_2\Phi^n_{2,\tau_{\mathrm{s}*+}} & 0 \\ 0 & 0 & -\dfrac{\nu}{\Delta t} - z_3\Phi^n_{1,\sigma_{1k}} & -z_3\Phi^n_{1,\sigma_{12k}} & \dfrac{1-\nu}{\Delta t} - z_3\Phi^n_{1,\sigma_{2k}} & -z_3\Phi^n_{1,\tau_{\mathrm{s}k}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1}{\Delta t} & -\dfrac{E_1}{\Delta t} \\ 0 & 0 & -z_4\Phi^n_{3,\sigma_{1k}} & -z_4\Phi^n_{3,\sigma_{12k}} & -z_4\Phi^n_{3,\sigma_{2k}} & -z_4\Phi^n_{3,\tau_{\mathrm{s}k}} & \dfrac{1}{\Delta t} \end{bmatrix}.$$

The matrices $[C_i]$ coincide with the $i$th row of the matrix $[B]$ and their other elements are equal to zero.

Total strains are calculated on the basis of the three-point Lax scheme, with the central differences in $x$ being

$$\frac{\varepsilon^{n+1}_{1k} - \widetilde{\varepsilon}^n_1}{\Delta t} = \left(\frac{c_2}{c_1}\right)^2 \frac{v^n_{1,k+1} - v^n_{1,k-1}}{2\Delta x}, \qquad \frac{\varepsilon^{n+1}_{12k} - \widetilde{\varepsilon}^n_{12}}{\Delta t} = \frac{1}{2}\left(\frac{c_2}{c_1}\right)^2 \frac{v^n_{2,k+1} - v^n_{2,k-1}}{2\Delta x}, \qquad \widetilde{\varepsilon}^n = \frac{\varepsilon^n_{k+1} + \varepsilon^n_{k-1}}{2}.$$

Note that solving the system of nonlinear equations (2.5) for large $\delta$ by the method of iterations leads to instability.

## 6. STABILITY ANALYSIS

When studying the stability of stiff systems of ODE's, the $A$-stability is usually proved for model problems. In the case under consideration, it is natural to use as a model problem that of tangential impact on an elastoviscoplastic half-space. The above asymptotic analysis shows that this problem contains all main features of the general problem. The equations of this problem can be obtained from system (5.2) if we take $\sigma_{11}$, $\sigma_{22}$, and $v_1$ equal to zero. Writing these equations in terms of the Riemann invariants, we obtain a system of two nonlinear equations whose right-hand sides contain the large parameter $\delta$. Therefore, these right-hand sides cannot be neglected, unlike the case of the stability analysis of difference schemes that are not stiff.

In order to prove the $A$-stability, one has to show that system (5.4) is absolutely stable with respect to $\delta\Delta t$ (the step multiplied by the large parameter). Applying the Fourier transformation to the third and the fourth equations of in (5.2), we obtain an equation for the characteristic numbers of the transition matrix,

$$p_1(\lambda - e^{ik\Delta t}) = \tfrac{1}{4}\delta\Delta t(\lambda + e^{-ik\Delta t})(p_2 - p_1), \quad p_2(\lambda - e^{-ik\Delta t}) = \tfrac{1}{4}\delta\Delta t(\lambda + e^{ik\Delta t})(p_2 - p_1), \tag{6.1}$$
$$p_1 = v_2 - (\sigma_{12} - \tau_{\mathrm{s}}), \quad p_2 = v_2 + (\sigma_{12} - \tau_{\mathrm{s}}),$$

where $\lambda$ is found from the condition that the determinant of system (6.1) vanishes,

$$\begin{vmatrix} \lambda - e^{-ik\Delta t} + A(\lambda + e^{-ik\Delta t}) & -A(\lambda + e^{-ik\Delta t}) \\ -A(\lambda + e^{ik\Delta t}) & \lambda - e^{ik\Delta t} + A(\lambda + e^{ik\Delta t}) \end{vmatrix} = 0,$$
$$\lambda^2 + \frac{2\lambda}{1+2A}\cos(k\Delta t) + \frac{1-2A}{1+2A} = 0, \qquad 2A = \frac{\delta\Delta t}{2}, \tag{6.2}$$
$$\lambda_{1,2} = \frac{1}{1+2A}\left[\cos(k\Delta t) \pm \sqrt{4A^2 - \sin^2(k\Delta t)}\,\right].$$

**1.** For $\frac{1}{2}\delta\Delta t \geq 1$, the radicand has the form $4A^2 - \sin^2(k\Delta t) > 0$, and $\lambda$ is real. If $\cos(k\Delta t) > 0$, then

$$|\lambda_1| = \left|\frac{1}{1+2A}\left[\cos(k\Delta t) \pm \sqrt{4A^2 - \sin^2(k\Delta t)}\,\right]\right| < \left|\frac{1}{1+2A}[\cos(k\Delta t) + 2A]\right| \leq 1.$$

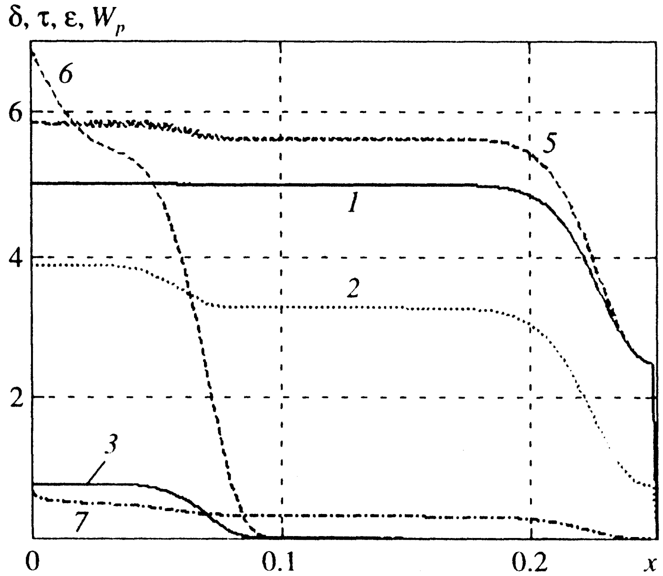


Fig. 2

If $\cos(k\Delta t) < 0$, then

$$|\lambda_1| < |\lambda_2| \le \frac{1}{1+2A}\left[|\cos(k\Delta t)| + \sqrt{4A^2 - \sin^2(k\Delta t)}\right] \le 1.$$

**2.** For $\frac{1}{2}\delta\Delta t < 1$, we have $\max[4A^2 - \sin^2(k\Delta t)] < 0$ and $\lambda$ is complex,

$$\operatorname{Im}\sqrt{4A^2 - \sin^2(k\Delta t)} < i\sin(k\Delta t), \quad |\lambda| < |\cos(k\Delta t) \pm i\sin(k\Delta t)| = 1.$$

It follows that the scheme is $A$-stable for any $\delta\Delta t = \text{const}$.

It can be shown that the purely implicit scheme (the right-hand side is taken only from the upper layer) is also $A$-stable. For the explicit scheme, stability will take place only if $\delta\Delta t \le 1$.

In order to describe the structure of the boundary layer in stiff problems, the equations should be integrated with a small step $\Delta t$ such that $\delta\Delta t \ll 1$ (practically, $\delta\Delta t \lesssim 0.1$–$0.01$), and it is inessential whether the method is explicit or implicit. The explicit method is preferable because of its simplicity. If detailed information about solutions in the boundary layer is of no interest and the aim is to obtain a solution for large $t$, then the equations should be integrated with a much larger step on the basis of the implicit scheme.

In this case, the stability of the solution and the required accuracy are guaranteed in the region of slow variation of the functions, although there might be large errors in the boundary layer.

## 7. NUMERICAL RESULTS

The numerical results are given in Figs. 2–4 which show the quantities versus the variable $x$ for a fixed instant $t \gg \tau$. Curve *1* presents the stress component $\sigma_{11}$, curve *2* shows $\sigma_{22}$, curve *3* shows $\sigma_{12}$, curve *4* gives the yield stress $\tau_s$, curve *5* presents the strain component $\varepsilon_{11}$, curve *6* corresponds to $\varepsilon_{12}$, and curve *7* shows the plastic work $W_p$. Our calculations were performed for $\delta = 100$, $\nu = 0.3$. The curves in Figs. 2–4 were obtained on the basis of the explicit scheme with step $\Delta t = 0.00125$.

**7.1.** Consider first the longitudinal-transverse impact of constant intensity applied to the elastoviscoplastic half-space. This case corresponds to the boundary conditions $\sigma_{11} = \sigma_{11}^0 H(t)$ and $\sigma_{12} = \sigma_{12}^0 H(t)$, where $H(t)$ is the Heaviside unit step function.

Figure 2 shows stress and strain distributions in the half-space at the instant $t = 0.25$ for $\sigma_{11}^0 = 5\sigma_s$, and $\sigma_{12}^0 = 0.75\tau_s$, i.e., $\sigma_{12}^0$ is smaller than the yield stress. Near the plane $x = 0$, there is a region of constant normal stresses, strains, and velocities. The transverse strain in the boundary region does not remain constant, because there is no shear hardening (in the presence of hardening, the strain component $\varepsilon_{12}$ will also remain constant). Therefore, for $\sigma_{12}^0 < \tau_s$, there is no

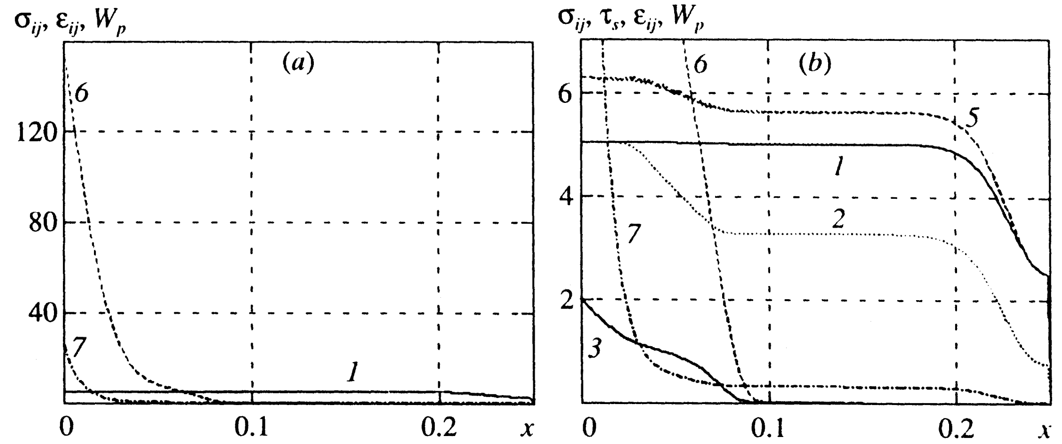


Fig. 3

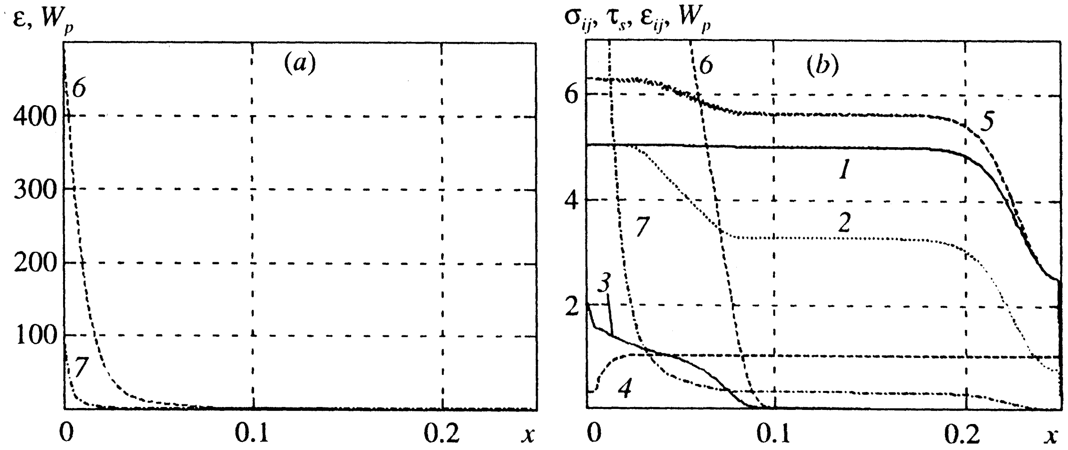


Fig. 4

boundary effect, and near the boundary there appears a region of constant values of the solution—a plateau—and we have $\sigma_{11} \neq \sigma_{22}$.

Figure 3 shows the behavior of the solution in the boundary region for $\sigma_{12}^0 > \tau_s$ ($\sigma_{11}^0 = 5\sigma_s$, $\sigma_{12}^0 = 2\tau_s$). For the normal components, the region of constant values is preserved and the stress state in that region corresponds to the state of hydrostatic pressure $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma^0$, while the tangential stresses $\sigma_{12}$, the strains $\varepsilon_{12}$, and the plastic work $W_p$ change rapidly and form a boundary layer.

Figure 3 shows how plastic longitudinal and combined longitudinal-transverse waves propagate. The elastic forerunner and the longitudinal plastic wave are moving with the respective velocities $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_p = \sqrt{K/\rho}$. The combined wave is linked to the transverse wave and propagates with the velocity $c_2 = \sqrt{\mu/\rho}$. Tangential stress variation is accompanied by variation of the stress component $\sigma_{22}$ (curve *2*) and $\varepsilon_{11}$ (curve *6*), with $\sigma_{11}$ being constant on the front of this wave. Hence, in the combined wave, interaction occurs only between the stress components $\sigma_{12}$ and $\sigma_{22}$, and there is loading with respect to $\sigma_{22}$.

This fully confirms the asymptotic solution obtained above. The distribution $\varepsilon_{12}(x)$, the time variation of the localization strip thickness, and other characteristics coincide with those obtained by the numerical solution.

Figure 4 illustrates the problem with the same external load as in Fig. 3, but the material has the softening diagram shown in Fig. 5 ($W_p^* = 2$). Here, the normal components have the same behavior as in the previous example and form a region of constant hydrostatic pressure near the boundary. The rate of change of the transverse components is substantially higher, as compared with the case of perfect plasticity. From Fig. 4 it is apparent that the increase in the shear strain during the same time interval is 3 times as large as that corresponding to Fig. 3. This also pertains to the work $W_p$.

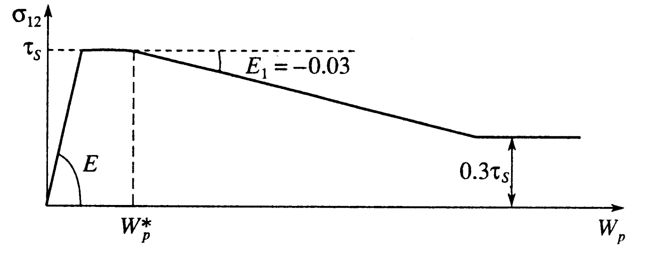


Fig. 5

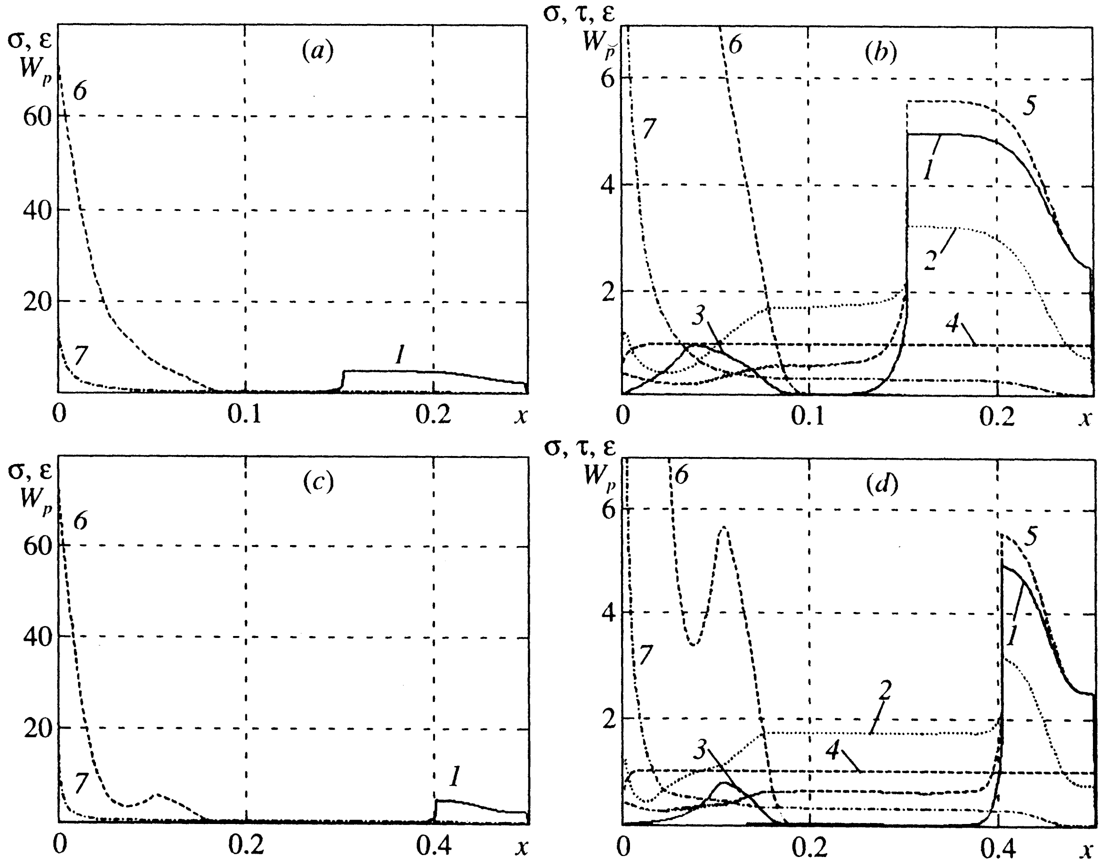


Fig. 6

Figure 4 shows the variation of $\tau_s(W_p)$ (curve *4*). We see that softening takes place only in the localization strip, where $W_p$ increase rapidly. Although $\partial\tau_s/\partial W_p = -0.03$ is sufficiently small, the derivative $\partial\tau_s/\partial x$ is large. For a steeper stress drop on the softening curve, the derivatives $\varepsilon'_{12}(x)$ and $W'_p(x)$ become so large that numerical calculations become unstable and the integration step has to be substantially reduced.

It should be emphasized that the importance of the above solution in the boundary layer is not restricted to the problem considered here. It is of general importance, because the effect described by this solution takes place for any body with a tangential load applied to its surface, provided that the load intensity exceeds the shear yield stress and its duration satisfies the inequality $t_0 \gg \tau$. This can be explained by the fact that an arbitrary wave can be locally regarded as a plane wave, for which the solution obtained above is valid. Moreover, the above solution has steady-state character, and therefore, at large distances away from the wave front the accelerations $\partial v_1/\partial t$ and $\partial v_2/\partial t$ tend to zero (in particular, in the region of boundary layer) and the solution coincides with the quasi-static solution.

**7.2.** Consider the case of longitudinal-transverse impact with a pulse of normal and tangential stresses of finite duration $t = t_0$ (Fig. 6). The material diagram is assumed to have a softening portion (Fig. 5) and the applied load has the form $\sigma_{11} = \sigma^0_{11}[H(t) - H(t - t_0)]$, $\sigma_{12} = \sigma^0_{12}[H(t) - H(t - t_0)]$, $\sigma^0_{12} > \tau_s$. At $t = 0$, the half-space is unperturbed, $\mathbf{U}\big|_{t=0} \equiv 0$.

For an impact of medium duration $t_0 \sim 10\tau = 0.1$, the elastoviscoplastic solution relaxes into the elastoplastic solution during the time $t \approx t_0$, and one can trace propagation not only of elastoviscoplastic waves, but also of plastic waves, although the plastic wave front is fairly diffuse compared with the fronts of the forerunner wave and the elastic load-relief wave.

From Fig. 6, we see that in a plastic material elastic load-relief waves propagate with the elastic velocity $c_1$, catch up with the plastic wave $c_p$, and reduce the plastic part of the pulse, so that plastic strains penetrate the half-space to a finite depth determined by the duration of the applied pressure intensity.

Load-relief takes place only up to the neutral state corresponding to the material yield stress, i.e., $\sigma_{11} = 0$, $\sigma_{12} = \tau_s$.

For $\sigma^0_{12} > \tau_s$ after load removal, the combined wave propagates over the material in the state of neutral loading. Owing to the residual stresses $\sigma_{22}$, the arrival of that wave leads to stress redistribution ($\sigma_{22}$ decreases and $\sigma_{12}$ increases), in accordance with the diagram of perfect plasticity or softening (depending on the state of the material at a point under consideration).

The combined wave lags far behind the longitudinal wave and moves with the velocity $c_2 = \sqrt{\mu/\rho}$. The tangential stress intensity decays rapidly and tends to the intensity of the elastic wave. The transverse wave causes load-relief with respect to $\sigma_{22}$ (curve *2*), but as a whole, the action of the combined wave brings about loading, and since before the front there has been the neutral state, plastic reloading takes place in the material (Fig. 8). The material in this zone has not yet reached the softening state, but is in the state of perfect plasticity (see curve *4*).

Near the surface $x = 0$, a region of large shear plastic strains is formed, the so-called adiabatic shear strip, which represents the main distinction between the elastoviscoplastic and the elastoplastic solutions in the case of material softening.

In the case of a pulse of finite duration, the strains in the boundary layer region increase up to a certain value depending on the duration of the tangential stress intensity $\sigma^0_{12}$ applied at the boundary.

Comparing the curves in Fig. 6(*a*), (*b*) and Fig. 6(*c*), (*d*), constructed for different time instants, $t = 0.25$ and $t = 0.5$, one can see that the boundary layer has formed during a relatively small time interval, $t = t_0 = 0.1$, and after that remains virtually unchanged. The longitudinal strain $\varepsilon_{11} \sim \varepsilon_s$ is small, and the shear strain $\varepsilon_{12}$ is scores of times greater, although softening is relatively low in this example, since it had no time to develop, because of short duration of the applied pulse. In Fig. 4, for unlimited action of the pulse, $\varepsilon_{12}$ is by a factor of 100 larger than $\varepsilon_{11}$ and continues to grow in time as $t^{1/2}$, according to formula (4.7) obtained by the asymptotic method.

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