Author's review of the monograph: "Introduction to nondegenerate geometry"

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The monograph is an introduction to the most general type of geometry, which can be defined as the metric geometry described in terms of NGO (natural geometric objects). This new type of geometry (called T-geometry, or nondegenerate geometry) may be interpreted also as an alternative way of generalization of the Euclidean geometry other than the Riemannian geometry. The Riemannian geometry is a generalization of the Euclidean geometry founded on extremal property of the Euclidean straight line, i.e. on the property of the straight to be the shortest (extremal) line connecting two points. The T-geometry is a generalization of the same Euclidean geometry based on collimetric property of the Euclidean straight. The term "collimetric" is formed from two terms "collinear" and "metric". The collimetric property is the collinearity condition for two vectors $\mathbf{RP}$ and $\mathbf{RQ}$ expressed via distances between points P, Q, R.

Being applied for description of the space-time, the nondegenerate geometry explains freely quantum effects as geometrical effects. If the nondegenerate geometry is considered to be the space-time geometry, the particle world lines are substituted by stochastic world tubes and the quantum constant $\hbar$ appears to be a geometrical quantity (thickness of the tubes). The particle mass also becomes a geometrical quantity. In the nondegenerate space-time geometry one does not need quantum principles for explanation of quantum effects. The nondegenerate space-time geometry admits one to restrict the field of scope of quantum mechanics in the same way as the statistical physics restricts the field of scope of the axiomatic thermodynamics.

Concept of NGO (natural geometric objects) is a crucial concept of T-geometry. By definition the natural geometric object (NGO) is a set of points determined by geometry and only by geometry. Any NGO depends on several parameters — points. For instance, in the Euclidean geometry the NGO of the first order is a straight which is determined by two parameters – two different points. The NGO of the second order is a plane which is determined by three parameters – three different points not lying on one straight line. Description of geometry in terms of NGO is very old idea. For instance, the proper Euclidean geometry is presented in terms of points, lines and planes (which are NGOs of the proper Euclidean geometry). NGOs of the Euclidean geometry are defined via their properties by means of the Euclidean axioms, because the NGOs are fundamental geometrical objects which cannot be defined through another more fundamental objects. Practically one needs the Euclidean axiomatics only for definition of NGOs. If one defines NGOs by some nonaxiomatic way, the Euclidean axioms turn to theorems and stop to be necessary as axioms.

There is a lot of nonuniform geometries, where it is very difficult to define NGO by means of a system of axioms (for instance, the Riemannian geometries), because any geometry has its own specific NGOs. In the T-geometry all NGOs are determined through the metric (distance between any pair points), and it is the main innovation of the T-geometry. As a result the T-geometry needs no axioms like Euclidean axioms, because it is clear intuitively that the geometry can be formulated in terms of NGOs, and the proper Euclidean geometry is an example of such a kind.

For definition of NGOs via metric one uses the proper Euclidean geometry, where NGOs may be defined very simply. For instance, the first order NGO (straight), depending on points A and B can be defined as the set of points R for which the area of the triangle ABR vanishes. Indeed, three points A, B, R lie on one straight, if and only if the area of the triangle ABR vanishes. The area of the triangle ABR may be expressed via distances between the points A, B, R by means of the Hero's formula. This admits one to formulate the rule for determination of the first order NGO via metric. NGOs of the highest order can be defined by a similar way. In the monograph it is shown that, indeed, the nondegenerate geometry can be constructed in terms of NGOs.

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1 Introduction.

The geometry is defined as a science, investigating spatial properties of real world. There are three different conceptions of geometry:

1. Axiomatic conception of geometry.
2. Conception of differential geometry.
3. Metric conception of geometry.

One compares the three different approach to geometry and considers their advantages and defects.

1.1 Axiomatic conception of geometry

Axiomatic conception of geometry is the oldest kind of geometry (Euclidean geometry). It describes geometry in terms of natural geometric objects (NGO), i.e. in terms of points, straight lines and planes. In general, the nth order NGO is defined by $n+1$ points (parameters of the NGO) and by the geometry. In the proper Euclidean geometry NGOs are defined by their properties by means of Euclidean axioms. Description of geometry in terms of NGOs is very effective. But formulation of geometrical axioms for nonuniform geometries like Riemannian geometries is a very difficult problem. The necessity of formulation of axioms is a defect of this approach which does not admit to use axiomatic conception in nonuniform geometries.

1.2 Conception of differential geometry

Conception of differential geometry uses the concept of manifold as the main tool for the geometry description. Dimension of geometry is determined by the manifold. The geometry is determined by the matrix of the metric tensor. This approach is very adaptable, as far as it does not use axioms. But sometimes the manifold is not adequate to geometry, and some geometries cannot be described in terms of only manifold. NGOs can be introduced in differential geometry, but at this approach they are secondary concepts. The differential geometry is the most developed form of modern geometry. But use of the manifold which is not completely adequate to geometry imposes too strong constraints on geometries, and some very interesting kinds of geometry appear outside the field of scope of the differential geometry.

1.3 Metric conception of geometry

. Metric conception of geometry is the most general approach to geometry. The geometry is determined by the distance between any pair of points. Conventional form of metric geometry does not use NGOs. Instead it uses the triangle axiom which is rather restrictive. It does not permit to apply the metric geometry to the space-time model. In the monograph one suggests to unite advantages of axiomatic approach (adequate description of geometry in terms of NGOs) and those of the metric conception (the most general determination of geometry in terms of distance). The NGOs are defined via metric, and the main defect of axiomatic approach (necessity of the axioms formulation) is removed.

To define the first order NGO determined by two parameters (points) A and B, one considers the proper Euclidean geometry, which is a kind of metric geometry. In the Euclidean geometry the first order NGO (the straight line, passing through points A and B) can be defined as *a set $\mathcal{T}_{AB}$ of points R* as follows

$$\mathcal{T}_{AB} = \left\{R|V_{ABR}^2 = 0\right\} \tag{1}$$

where V_{ABR} is the area of the triangle ABR. This area vanishes, if and only if three points A, B, R lie on one straight line. Let a, b, r be length of sides of the triangle ABR. Using for V_{ABR} the Hero's formula

$$V_{ABR} = \sqrt{p(p-a)(p-b)(p-r)}, \qquad p = (a+b+r)/2, \tag{2}$$

one determines the straight line via metric. If the way of definition of the first order NGO via metric is valid for the Euclidean geometry, it is valid for any metric geometry. But for arbitrary metric geometry the first order NGO $\mathcal{T}_{AB}$ is not a one-dimensional line. If the metric is given on n-dimensional manifold, $\mathcal{T}_{AB}$ is in general $(n-1)$-dimensional surface. This unexpected result cannot be obtained by means of methods of differential geometry. Geometry, where the first order NGOs are tubes (not one-dimensional lines) is a new type of geometry (nondegenerate geometry, or T-geometry). The real space-time geometry appears to be nondegenerate, but tubes are very thin, and macroscopic physical phenomena can be described approximately in the degenerate space-time model of Minkowski. Microscopic physical phenomena, where the thickness of tubes becomes essential, should be described in the nondegenerate space-time model.

Thus, the first order NGO $\mathcal{T}_{AB}$ is defined by the formula (1) The second order NGO $\mathcal{T}_{ABC}$ (two-dimensional plane) is determined by three points A, B, C, which do not lie on one straight ($C \notin \mathcal{T}_{AB}$). It is defined by a similar relation

$$\mathcal{T}_{ABC} = \{R \mid V_{\mathrm{ABCR}}^2 = 0\} \tag{3}$$

where $V_{\rm ABCR}$ is the volume of tetraedr, formed by points A,B,C,R. This volume also can be expressed via distances between points A,B,C,R by means of the relation

$$V_{ABCR}^2=\frac{1}{(3!)^2}\left[\mathbf{abc}\right]^2=\frac{1}{(3!)^2}\left|\begin{array}{ccc}(\mathbf{a.a}) & (\mathbf{a.b}) & (\mathbf{a.c})\\ (\mathbf{b.a}) & (\mathbf{b.b}) & (\mathbf{b.c})\\ (\mathbf{c.a}) & (\mathbf{c.b}) & (\mathbf{c.c})\end{array}\right|, \tag{4}$$

where $\mathbf{a}=\mathbf{RA},\ \mathbf{b}=\mathbf{RB},\ \mathbf{c}=\mathbf{RC}$ are vectors of the length a,b,c relatively. The scalar products in the Gram's determinant (4) are expressed via distances between the points A,B,C,R by means of the relation

$$(\mathbf{a.b})=\frac{1}{2}(a^2+b^2-|\mathbf{a}-\mathbf{b}|^2)=\frac{1}{2}(|\mathbf{RA}|^2+|\mathbf{RB}|^2-|\mathbf{AB}|^2) \tag{5}$$

and other similar relations. Th higher order NGOs are expressed via higher order determinants.

1.4 Difference of geometric conceptions.

Differential (Riemannian) geometry can be formulated mathematically as follows. The geometry on the set Ω of points is given by means of two mappings

$$M_n: \qquad \Omega\rightarrow M_n\subset\mathbb{R}^n, \qquad \mathbb{R}^n\equiv\bigotimes_{i=1}^{i=n}\mathbb{R}, \tag{6}$$

$$g: \qquad M_n\rightarrow G_n, \tag{7}$$

where $\mathbb{R}^n$ is a set of all ordered sets of n real numbers $(x^1,x^2,...x^n)$, M_n is its subset (domain) of $\mathbb{R}^n$. G_n is the set of all symmetric real nonsingular matrices $\mathbf{g}$ of the order n. The mappings (6), (7) are supposed to have such special properties that the mapping (6) determines a n-dimensional manifold. The mapping (7) determines the metric tensor on it. This may be interpreted in the sense that the mapping (7) determines geometry, and the mapping (6) describes labeling of the space points, i.e. the mapping (6) is a mathematical tool for description of geometry. But the mapping (7), describing geometry, is introduced always after the mapping (6), introducing the manifold and dimensionality, and this is a problem. At such an introduction the dimensionality appears as an attribute of the manifold and becomes an attribute of geometry, provided the manifold is considered to be an attribute of geometry.

Let us take into account that the matrix $\mathbf{g}$ determines the quadratic form

$$dS^2=g_{ik}(x)dx^idx^k, \tag{8}$$

determining the distance between two infinitesimally close points. At this determination the order of the matrix $\mathbf{g}$ (i.e. the manifold dimensionality) is of no importance.

One can introduce the finite distance on the basis of (6), (7) defining the world function σ as a half of the squared distance between points P_0 P_1, This distance is measured along the geodesic (the first order NGO) generated by the two points. The world function is defined as the mapping

$$\sigma : \qquad \Omega^2 \to \mathbb{R}, \qquad \Omega^2 \equiv \Omega \otimes \Omega, \tag{9}$$

which does not contain the parameter n, describing the dimensionality of geometry.

The geometry on the set Ω is defined completely by the mapping (9). The mappings (6), (7) can be obtained from the mapping (9) in the cases, when the mapping (6) exists and has necessary properties. In the case, when the mapping (6), having necessary properties, does not exist, the mapping (9) determines geometry which cannot be described in terms of differential geometry.

Representation of geometry in the form of two mappings (6), (7) contains the natural number n (dimensionality), which is absent in the mapping (9). This can be explained as follows. Let us consider the set of mappings depending on the natural number n

$$F_n : \qquad \Omega^{n+1} \to R, \qquad \Omega^{n+1} \equiv \bigotimes_{i=1}^{i=n+1} \Omega, \qquad n = 1, 2, ... \tag{10}$$

where the mapping F_n is defined via the world function σ as follows. $(n!)^{-2} F_n(\mathcal{P}^n)$ is the squared Euclidean volume of $(n+1)$-edr, expressed via distances between its vertices $P_0, P_1, ... P_n$.

$$F_n (\mathcal{P}^n) = \det ||\Gamma (P_0, P_i, P_k)||, \qquad P_0, P_i, P_k \in \Omega, \qquad i, k = 1, 2, ... n$$

$$\Gamma (P_0, P_i, P_k) \equiv \sigma (P_0, P_i) + \sigma (P_0, P_k) - \sigma (P_i, P_k), \qquad i, k = 1, 2, ... n$$

Here $\mathcal{P}^n \equiv \{P_0, P_1, ... P_n\} \in \Omega^{n+1}, \quad P_i \in \Omega, \quad i = 0, 1, ... n$ and one supposes that $\sigma(P, Q), \ P \in \Omega, \ Q \in \Omega$ is a half of the square of the distance between the points $P \in \Omega, \ Q \in \Omega$ in the proper Euclidean space. In particular, $F_0(P_0, P_1) = 2\sigma(P_0, P_1)$. If the geometry described by the mapping (9) is D-dimensional Euclidean geometry, the corresponding necessary conditions may be written in the form of the constraints $F_n(\mathcal{P}^n) = 0, \ \forall \mathcal{P}^n \in \Omega^{n+1}$ for $n > D$, i.e.

$$F_n(\Omega^{n+1}) = 0, \qquad n = D+1, D+2, ... \tag{11}$$

For any value of n the mapping (10) determines the nth order NGO $\mathcal{T}(\mathcal{P}^n) \subset \Omega$, depending on such parameters $\mathcal{P}^n \equiv \{P_0, P_1, ... P_n\} \in \Omega^{n+1}$, for which $F_n(\mathcal{P}^n) \neq 0$.

$\mathcal{T}(\mathcal{P}^n) \subset \Omega$ is such a set of points $P \in \Omega$, that

$$F_{n+1}(\mathcal{P}^n, P) = 0, \qquad \{\mathcal{P}^n, P\} \equiv \{P_0, P_1, ... P_n, P\} \in \Omega^{n+2} \tag{12}$$

т.е.

$$\mathcal{T}(\mathcal{P}^n) = \{P \mid F_{n+1}(\mathcal{P}^n, P) = 0\}, \tag{13}$$

$$P \in \Omega, \qquad \{\mathcal{P}^n, P\} \equiv \{P_0, P_1, ... P_n, P\} \in \Omega^{n+2}.$$

From viewpoint of differential geometry the set (6), is an attribute of geometry in the sense that the geometry cannot be defined without a use of the mapping (6).

From viewpoint of the metric conception of geometry the mapping (9) describes geometry on Ω completely, and the mapping (6) is only a tool for investigation and description of geometry. This mathematical tool is very special. It is suitable only for investigation of special (degenerate) geometries. The set of mappings (10) is also a tool for investigation and description of geometry (9), but this mathematical tool agrees with the geometry (9). It is suitable for investigation of any geometry (9).

Example 1.4.1. Let T-geometry $\{\sigma,\Omega\}$ (9) be given on a set Ω. Let us suppose that this T-geometry describes a unit cube with the proper Euclidean geometry on it. By definition it means that there exists the bijection

$$f: \quad \Omega\rightarrow (0,1)^3\subset \mathbb{R}^3,$$

$$\mathbf{x}=f\left(P\right),\qquad P=f^{-1}\left(\mathbf{x}\right),\qquad \mathbf{x}=\left\{x^1,x^2,x^3\right\}\in (0,1)^3,\qquad P\in\Omega$$

$$\sigma_E:\quad (0,1)^3\times(0,1)^3\rightarrow\mathbb{R},\qquad \sigma_E(\mathbf{x},\mathbf{x}')=\frac{1}{2}|\mathbf{x}-\mathbf{x}'|^2,\qquad \mathbf{x},\mathbf{x}'\in(0,1)^3\subset\mathbb{R}^3$$

$$\sigma\left(P,P'\right)=\sigma_E\left(f\left(P\right),f\left(P'\right)\right),\qquad P,P'\in\Omega,\qquad f(P),f(P')\in(0,1)^3,$$

where σ_E is the world function of the Euclidean space.

Let the form of the world function $\sigma(P,P')$ is known, but the form of the bijection f is unknown. Is it possible to test that the geometry given on Ω is really Euclidean geometry? Is it possible to construct the three-dimensional manifold on Ω? The answer is affirmative. Using the mapping (10) for different values of n, one can determine dimensionality and construct the manifold, even if the form of the bijection f is unknown. One should act as follows.

1. Let us choose two different points $P_0,P_1\in\Omega$, $\sigma(P_0,P_1)\neq0$, form the first order NGO $\mathcal{T}_{P_0P_1}\subset\Omega$, and test that $\Omega\backslash\mathcal{T}_{P_0P_1}\neq\emptyset$. The last inequality is a corollary of the fact that $\mathcal{T}_{P_0P_1}$ is an origin of the straight $\mathcal{L}_{P_0P_1}=f\left(\mathcal{T}_{P_0P_1}\right)\subset(0,1)^3$ in the cube $(0,1)^3$, and this straight does not exhaust the whole cube, i.e. $(0,1)^3\backslash\mathcal{L}_{P_0P_1}\neq\emptyset$.

2. Let us choose the point $P_2\in\Omega\backslash\mathcal{T}_{P_0P_1}$, form the second order NGO $\mathcal{T}_{P_0P_1P_2}\subset\Omega$ and test that $\Omega\backslash\mathcal{T}_{P_0P_1P_2}\neq\emptyset$. This relation is to be fulfilled, because $(0,1)^3\backslash\mathcal{L}_{P_0P_1P_2}\neq\emptyset$, and the plane $\mathcal{L}_{P_0P_1P_2}=f\left(\mathcal{T}_{P_0P_1P_2}\right)\subset(0,1)^3$ does not contain the whole set of points of the cube $(0,1)^3$.

3. Let us choose any point $P_3\in\Omega\backslash\mathcal{T}_{P_0P_1P_2}$, form the third order NGO $\mathcal{T}_{P_0P_1P_2P_3}$, and test that now $\Omega\backslash\mathcal{T}_{P_0P_1P_2P_3}=\emptyset$, because $(0,1)^3\backslash\mathcal{L}_{P_0P_1P_2P_3}=\emptyset$, and there are no points in the cube which do not belong to the three-dimensional plane $\mathcal{L}_{P_0P_1P_2P_3}$, passing through four points P_0,P_1,P_2,P_3. In this case $P_3\notin\mathcal{T}_{P_0P_1P_2}$ and, hence,

$$F_3\left(\mathcal{P}^3\right)\neq0 \tag{14}$$

because if $F_3\left(\mathcal{P}^3\right)=0$, then according to (13) one has $P_3\in\mathcal{T}_{P_0P_1P_2}$.

4. Now one can define covariant coordinates $y_i(P)$ of any point P and contravariant components g^{ik} of metric tensor in the basis formed by three vectors $\overrightarrow{\mathbf{PP}}_i$ $(i=1,2,3)$. They are determined by the relations

$$y_i(P)=\left(\overrightarrow{\mathbf{PP}}_i.\overrightarrow{\mathbf{PP}}\right),\qquad i=1,2,3 \tag{15}$$

$$g_{ik} = \left(\overrightarrow{\mathbf{PP}}_i . \overrightarrow{\mathbf{PP}}_k\right), \qquad i,k = 1,2,3, \qquad \det ||g_{ik}|| \neq 0 \tag{16}$$

$$\sum_{l=1}^{3} g^{il} g_{kl} = \delta_i^k, \qquad i,k = 1,2,3$$

Here the scalar product $\left(\overrightarrow{\mathbf{PP}}.\overrightarrow{\mathbf{PP}'}\right)$ is defined by means of the relation

$$\left(\overrightarrow{\mathbf{PP}}.\overrightarrow{\mathbf{PP}'}\right) \equiv \sigma\left(P_0, P\right) + \sigma\left(P_0, P'\right) - \sigma\left(P, P'\right), \qquad P, P' \in \Omega \tag{17}$$

The relations (15) - (17) are valid for images $\mathbf{x} = f\left(P\right) \in (0,1)^3$ of points $P \in \Omega$. These relations are valid also for originals of points $P \in \Omega$.

When coordinates $y_i(P)$ of arbitrary point P has been determined, one can test the validness of the relation

$$\sigma\left(P, P'\right) = \frac{1}{2} \sum_{i,k=1}^{3} g^{ik} \left(y_i(P) - y_i(P')\right)\left(y_k(P) - y_k(P')\right), \qquad \forall P, P' \in \Omega \tag{18}$$

which is to be valid for the world function σ of the Euclidean space. It means that the manifold can be constructed, provided the world function is known, and it is the world function (not manifold) that contains information on geometry.

Example 1.4.2. Let us consider two different T-geometries on unit sphere

$$V = \left\{\mathbf{x} \left| |\mathbf{x}|^2 \leq 1 \right.\right\} \subset \mathbb{R}^3, \qquad \mathbf{x} = \left\{x^1, x^2, x^3\right\} \in \mathbb{R}^3, \qquad |\mathbf{x}|^2 \equiv \sum_{i=1}^{3} \left(x^i\right)^2 \tag{19}$$

One of T-geometries is the proper Euclidean geometry $T_E = \{\sigma_E, V\}$

$$\sigma_E: \quad V \times V \to [0, \infty) \subset \mathbb{R}, \qquad \sigma_E(\mathbf{x}, \mathbf{x}') = \frac{1}{2}|\mathbf{x} - \mathbf{x}'|^2, \qquad \mathbf{x}, \mathbf{x}' \in V, \tag{20}$$

Another T-geometry $T = \{\sigma, V\}$ is defined on the same set V by means of relations

$$\sigma: \quad V \times V \to [0, \infty) \subset \mathbb{R}, \qquad \sigma(\mathbf{x}, \mathbf{x}') = 2\left(\arcsin\sqrt{\frac{\sigma_E(\mathbf{x}, \mathbf{x}')}{2}}\right)^2, \qquad \mathbf{x}, \mathbf{x}' \in V \tag{21}$$

Together with the two T-geometries inside the sphere V, one considers T-geometries on the sphere surface $\Omega = \partial V = \{\mathbf{x} \,||\mathbf{x}|^2 = 1\} \subset V$. As far as Ω is a subset of V, then T-geometries $T_{Es} = \{\sigma_E, \Omega\}$ and $T_s = \{\sigma, \Omega\}$ are generated by T-geometries T_E and T.

Let us denote tubes of T-geometry T_E by symbol $\mathcal{L}$, and those of T-geometry T by $\mathcal{T}$. The first order tube $\mathcal{L}_{AB} \subset V$, $(A, B \in \Omega)$ is a straight line in V and it reduces to two points on Ω. In other words, T-geometry T_E is degenerate in V and it is ultradegenerate on Ω. The first order tube $\mathcal{T}_{AB} \subset V$, $(A, B \in \Omega)$ is non-degenerate tube in V. It may be obtained as a result of rotation of a unit circle, passing through points $A, B \in \Omega$, around the axis $\mathcal{L}_{AB}$. These tube is tangent to the sphere surface Ω along the circle of minimal curvature. The segment $\mathcal{T}_{[AB]}$ of the tube between the points $A, B \in \Omega$ is placed inside the sphere V, whereas the remaining part of the tube $\mathcal{T}_{AB}$ is placed outside the internal part $V\backslash\Omega$ of the sphere V. As a result in the T-geometry $T_s = \{\sigma, \Omega\}$ the segment $\mathcal{T}_{[AB]}$ of the tube $\mathcal{T}_{AB} \subset \Omega$, $(A, B \in \Omega)$ is the shortest line on the sphere surface Ω, connecting the points $A, B \in \Omega$. The remaining part of the tube $\mathcal{T}_{AB} \subset \Omega$ is a circle which is a continuation of the segment $\mathcal{T}_{[AB]} \subset \Omega$. In other words, in the first order the T-geometry $T_s = \{\sigma, \Omega\}$ is a degenerate T-geometry on Ω. Thus, in the first order the T-geometry $T = \{\sigma, V\}$ is non-degenerate in V and it is degenerate on Ω. The second order tube $\mathcal{T}_{ABC} \subset \Omega$ consists of three points $A, B, C \subset \Omega$, and the T-geometry $T_s = \{\sigma, \Omega\}$ is an ultradegenerate T-geometry on Ω in the second order.

On the other hand, T-geometry $T_s = \{\sigma, \Omega\}$ on the sphere surface Ω can be constructed on the basis of the Euclidean geometry $T_E = \{\sigma_E, V\}$ inside the sphere V. To construct $T_s = \{\sigma, \Omega\}$ on the basis of $T_E = \{\sigma_E, V\}$, it is necessary to use the extremal property of geodesics. Let us consider the second order tube $\mathcal{L}_{ABC} \subset V$ in $T_E = \{\sigma_E, V\}$. This tube is a two-dimensional plane passing through points $A, B, C \in \Omega$. In the T-geometry $T_{Es} = \{\sigma_E, \Omega\}$ the tube $\mathcal{L}_{ABC} \subset \Omega$ has the shape of a circle, passing through points $A, B, C \in \Omega$.

To construct T-geometry $T_s = \{\sigma, \Omega\}$, one needs to determine the metric $S(A, B) = \sqrt{2\sigma(A, B)}$, $A, B \in \Omega$ on $\Omega \times \Omega$, as follows

$$S(A, B) = \inf_{C \in \Omega, C \neq A, C \neq B} l_C(A, B), \qquad A, B \in \Omega \tag{22}$$

where $l_C(A, B)$ is the length of the curve interval $\mathcal{L}_{ABC} \subset \Omega$ between the points A, B. Let us parametrize the points $R \in \mathcal{L}_{ABC} \subset \Omega$, solving the relation

$$S_E(A, R)(1 - \tau) = S_E(B, R)\tau, \qquad R \in \mathcal{L}_{ABC} \subset \Omega, \qquad \tau \in \mathbb{R} \tag{23}$$

Solution of this equation determines $R = R_{AB}(\tau, C) \in \mathcal{L}_{ABC} \subset \Omega$ as a function of parameter $\tau \in [0, 1]$ and of the point $C \in \Omega$. The function $R = R_{AB}(\tau, C)$ has two branches $R = R_1(\tau, C)$ and $R = R_2(\tau, C)$. One should choose the branch which has the lesser values of $S_E(A, R_{AB}(\tau, C))$ and define the length $l_C(A, B)$ by means of the relation

$$l_C(A, B) = \int_0^1 \left[\frac{dS_E(R_{AB}(\tau, C), R_{AB}(\tau', C))}{d\tau'}\right]_{\tau'=\tau} d\tau \tag{24}$$

Substituting (24) into (22), one obtains

$$S(A, B) = 2\arcsin\frac{S_E(A, B)}{2}, \qquad A, B \in \Omega \tag{25}$$

that corresponds to (21).

This example shows that, using Euclidean T-geometry inside the sphere V, one can construct internal T-geometry on the sphere surface Ω. Constructing geometry on Ω, one uses essentially the second order tube $\mathcal{L}_{ABC} \subset \Omega$ which introduces a system of curves on Ω and extremal property of geodesic (the shortest line).

These examples show that the manifold is not a necessary attribute or a constituent of the Euclidean geometry. It is only a tool used for investigation of geometry. On the other hand, the pithiness of geometry (i.e. the number of sorts of geometrical objects) depends mainly on the degree of mathematical technique development (but not only on the geometry in itself). T-geometry uses more developed mathematical technique (NGO-technique), and it is more pithy than the conventional metric geometry, although both versions use the same definition of geometry (9).

The conventional metric geometry on the set Ω is defined also by the mapping (9) with some additional constraints on σ. But the conventional metric geometry uses only the first order NGO (the shortest), and the set of mappings (10) is not used. Essentially, an effective mathematical tool for investigation of geometry is absent. As a result the metric geometry appears to be a poor geometry. Practically, connection of metric geometry with Riemannian geometry is exhausted by the statement that Riemannian geometry is a kind of metric geometry. Under what conditions (formulated in terms of metric geometry) does the metric geometry turn to a proper Riemannian geometry? This important question is not considered usually.

1.5 Generalizations of the Euclidean geometry.

There are two possible generalizations of the n-dimensional proper Euclidean geometry. In the first method the proper Euclidean geometry is defined by means of the quadratic form

$$dS^2 = g_{ik}dx^i dx^k, \qquad g_{ik} = \mathrm{diag}\{1, 1..., 1\}, \qquad i, k = 1, 2, ...n, \tag{26}$$

on the manifold $\mathbb{R}^n$, where $x = \{x^1, x^2, \ldots, x^n\} \in \mathbb{R}^n$, Summation over repeated indices is produced $(1 - n)$. The straight line $\mathcal{L}$ is defined as an extremal of the functional

$$S[x, x', \mathcal{L}] = \int_x^{x'} \sqrt{g_{ik}dx^i dx^k}, \qquad \mathcal{L}:\ x^i = x^i(\tau), \qquad i = 1, 2, ...n \tag{27}$$

where integration is produced along the line $\mathcal{L}$, connecting the points x and x'. $\tau \in \mathbb{R}$ is a parameter along the $\mathcal{L}$.

Substituting the set $\mathbb{R}^n$ by the manifold $M_n \subset \mathbb{R}^n$ and the quadratic form (26) by the a symmetric nonsingular matrix g_{ik} $i, k = 1, 2, ...n$ which is a function of $x \in \mathbb{R}^n$, one obtains the Riemannian geometry. Extremals $\mathcal{L}$ of the functional (27), called geodesics, play the same role in the Riemannian geometry as the straight lines do in the proper Euclidean geometry. A possibility of the Riemannian geometry construction is restricted by the basic statements (26), (27). These constructions (26), (27) need that the manifold $M_n \subset \mathbb{R}^n$ be smooth. The concept of continuity is a crucial concept of the Riemannian geometry.

One can present the proper Euclidean geometry as follows. The relation

$$\sigma(x, x') = \frac{1}{2}S^2(x, x') = \frac{1}{2}g_{ik}(x^i - x'^i)(x^k - x'^k), \qquad g_{ik} = \mathrm{diag}\{1, 1, ..., 1\} \tag{28}$$

determines the finite distance $S(x,x')$ between the two points $x\in\mathbb{R}^n$ and $x'\in\mathbb{R}^n$ of the proper Euclidean space. The relation

$$\mathcal{L}_{xx'}=\left\{y\left|\overrightarrow{xx'}||\overrightarrow{xy}\right.\right\} \tag{29}$$

determines the straight $\mathcal{L}_{xx'}$ passing through points x x' as *the set of such points* y that vectors $\overrightarrow{xx'}$ and $\overrightarrow{xy}$ are collinear. The collinearity $\overrightarrow{xx'}||\overrightarrow{xy}$ of vectors $\overrightarrow{xx'}$ and $\overrightarrow{xx}$ is defined via the scalar product

$$\left(\overrightarrow{xx'}.\overrightarrow{xy}\right)=\sigma(x,x')+\sigma(x,y)-\sigma(y,x'),\qquad |\overrightarrow{xx'}|^2\equiv\left(\overrightarrow{xx'}.\overrightarrow{xx'}\right)=2\sigma(x,x') \tag{30}$$

which is expressed only through the world function $\sigma=\frac{1}{2}S^2(x,x')$ defined by the relation (28). $|\overrightarrow{xx'}|$ is the length of the vector $\overrightarrow{xx'}$. Vector $\overrightarrow{xx'}$ and vector $\overrightarrow{xy}$ are collinear provided that they satisfy the relation

$$\left(\overrightarrow{xx'}.\overrightarrow{xy}\right)^2=\left(\overrightarrow{xx'}.\overrightarrow{xx'}\right)\left(\overrightarrow{xy}.\overrightarrow{xy}\right)\equiv|\overrightarrow{xx'}|^2\cdot|\overrightarrow{xy}|^2, \tag{31}$$

By virtue of (30) this relation contains only distances between the points x, x' y.

To obtain T–geometry, one substitutes the set $\mathbb{R}^n$ of real numbers by an arbitrary set Ω of points $x\in\Omega$ and generalizes the relation (29), substituting the metric (28), by a function $\sigma:\ \Omega\times\Omega\to\mathbb{R}$, satisfying only the condition

$$\sigma(x,x')=\sigma(x',x),\qquad \sigma(x,x)=0,\qquad x,x'\in\Omega. \tag{32}$$

As far as all relations defining the T-geometry are algebraic, they do not generate any constraints on continuity or smoothness of the set Ω.

Ideas of construction of Riemannian geometry and that of T-geometry are similar in the sense that in both cases one defines fundamental objects: distance and straight line. The relations (26), (27) and (28),(29) are similar in the sense that both relations determines the proper Euclidean geometry. But they use different methods of determination. The relations (26), (27) determine the Euclidean straight line via its extremal property, whereas the relations (28),(29) determine the Euclidean straight line via collimetric property. The Riemannian geometry is constructed on the basis of two structures: the topological structure and the metric one. The T-geometry is constructed on the basis of only metric structure. It is formal difference between the two geometries.

In the application to the space-time model the T-geometry defined by the collimetric property appears to be preferable. There are two arguments in favour of T-geometry. Let us consider the Minkowski geometry as the simplest uniform and isotropic space-time model. It appears if one substitutes the metric of the proper Euclidean space E_4 by the metric

$$g_{ik}=\mathrm{diag}\{1,-1,-1,-1\},\qquad i,k=0,1,2,3 \tag{33}$$

of the Minkowski space-time E_{1+3}. The substitution of the metric $g_{ik}=\mathrm{diag}\{1,1,1,1\}$, $i,k=0,1,2,3$ in (28) by (33) generates corresponding version T_{1+3} of T-geometry.

Timelike straights are NGOs of both models E_{1+3} and T_{1+3}. But spacelike straights are NGO of the model E_{1+3}, but they are not NGOs of T_{1+3}. In other words, from viewpoint of T_{1+3} the spacelike straights has no relation to real objects of geometry. The real particles which are described by spacelike straights are not yet discovered. It is an argument in favour of T-geometry as an adequate space-time model. Another argument is as follows. It is known that world lines of free microparticles (electron, proton, etc.) are extremals of the functional (27) only approximately. It is an argument in favour of T-geometry, which is constructed on the basis of the collinear–metric (not extremal) property.

2 World function of the Riemannian space.

The second chapter is devoted to investigation of properties of the world function (a half of the distance square between any two points x and x') of the Riemannian space. Such an investigation generates prerequisite for a transition to the nondegenerate geometry.

2.1 Riemannian geometry in terms of the world function.

It is shown that the n dimensional Riemannian geometry can be described in terms of the world function σ which is defined by the formula

$$\sigma(x,x') = \frac{1}{2}S^2(x,x') = \frac{1}{2}\left[\int\limits_{x'}^{x}\sqrt{g_{ik}\frac{dx^i}{d\tau}\frac{dx^k}{d\tau}}d\tau\right]^2 \tag{34}$$

where the integral is taken along the geodesic $x^k = x^k(\tau)$ passing through the points x' and x. If the world function σ is known, one can describe geodesic passing through the point x' in the direction of the vector $b_{i'}$ by means of the system of algebraic equations

$$\sigma_{i'}(x,x') = b_{i'}\tau, \qquad i' = 1,2,...n \tag{35}$$

which determines x as a function of the parameter τ.

2.2 Two-point formalism

Two-point formalism is generated by the world function σ, which carries out a geodesic mapping of the n-dimensional Riemannian space V on a set of flat n-dimensional spaces $E_{x'}$ tangent to V at the point x'. The quantity

$$\Gamma^i_{kl} = \Gamma^i_{kl}(x,x') = \sigma_{kls'}\sigma^{is'}, \qquad \sigma_{ik'} \equiv \frac{\partial^2\sigma}{\partial x^i \partial x'^{k'}}, \qquad \sigma_{ik'}\sigma^{lk'} = \delta^l_k \tag{36}$$

$$\sigma_{ilk'} \equiv \frac{\partial^3\sigma}{\partial x^i \partial x^l \partial x'^{k'}}$$

constructed only from the world function σ and their derivatives, is the Christoffel coefficient in $E_{x'}$. Covariant derivatives with the Christoffel coefficient $\Gamma^i_{kl}(x,x')$ are covariant derivatives in the spaces $E_{x'}$. It is shown that any world function of the Riemannian space V satisfies the system of equations

$$\begin{array}{llll} (1) & \sigma_l \sigma^{lj'} \sigma_{j'} = 2\sigma & (4) & \det \parallel G_{ik} \parallel \neq 0 \\ (2) & \sigma(x,x') = \sigma(x',x) & (5) & \det \parallel \sigma_{ik'} \parallel \neq 0 \\ (3) & \sigma(x,x) = 0 & (6) & G_{i'k'\parallel l'} = 0 \end{array} \tag{37}$$

where

$$\sigma_i \equiv \frac{\partial \sigma}{\partial x^i}, \qquad \sigma_{i'} \equiv \frac{\partial \sigma}{\partial x'^{i'}}, \qquad G_{ik} \equiv \sigma_{i\parallel k} \equiv \frac{\partial^2 \sigma}{\partial x^i \partial x^k} - \Gamma^l_{ik} \sigma_l \tag{38}$$

and two vertical strokes mean the covariant derivative in the flat space $E_{x'}$. This system of equations contains only world function σ and its derivatives (it does not contain metric tensor).

The system (37) generates the following problem. On one hand, the world function σ describes the Riemannian geometry completely. On the other hand, there exists such a world function which does not satisfy the system (37) and, hence, does not describe a Riemannian geometry. What does describe this world function? Some new sort of geometry, or no geometry?

2.3 Curvature of Riemannian space

Curvature of Riemannian space can be described by three different two-point tensors

$$B_{ik} = g_{ik} - G_{ik} \tag{39}$$

$$Q^i_{kl} = \gamma^i_{kl} - \Gamma^i_{kl} = -\sigma^{sj'} \sigma_{i|l|j'} \tag{40}$$

$$F_{ilk'j'} = \sigma_{ilj'\parallel k'} = \sigma_{ilj',k'} - \sigma_{sj'k'} \sigma^{sm'} \sigma_{ilm'} \tag{41}$$

where the vertical stroke means the covariant derivative in V. For the flat Riemannian space V all three two-point tensors B_{ik}, Q^i_{kl}, $F_{ilk'j'}$ vanish.

3 T-geometry.

Here one considers the most general type of metric geometry described by symmetric real function σ of two points P and Q

$$\sigma(P,P) = 0, \qquad \sigma(P,Q) = \sigma(Q,P) \tag{42}$$

3.1 σ-space and its properties.

The world function σ is given on a set of points Ω, which is referred to as σ-space. The main object of the σ-space is a nth order multivector $\overrightarrow{P_0P_1...P_n} \equiv \overrightarrow{\mathcal{P}^n}$ defined as an ordered set of $n+1$ points $P_0, P_1, ...P_n \in \Omega$. The vector $\mathbf{P}_0\mathbf{P}_1 \equiv \overrightarrow{P_0P_1}$ is the first order multivector. The scalar product can be defined for any two nth order multivectors, having a common point. The scalar product $(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})$ of two multivectors $\overrightarrow{\mathcal{P}^n}$, $\overrightarrow{\mathcal{Q}^n}$ with $P_0 = Q_0$

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = \det \|\Gamma(P_0, P_i, Q_k)\|, \qquad i,k = 1,2,...n, \qquad P_0 = Q_0 \tag{43}$$

is defined via scalar product $\Gamma(P_0, P_1, P_2)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{P}_2$

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_2) \equiv \Gamma(P_0, P_1, P_2) \equiv \sigma(P_0, P_1) + \sigma(P_0, P_2) - \sigma(P_1, P_2), \tag{44}$$

The length $|\overrightarrow{\mathcal{P}^n}|$ of the multivector $\overrightarrow{\mathcal{P}^n}$ is defined by the relation

$$|\overrightarrow{\mathcal{P}^n}| = \begin{cases} \mid \sqrt{(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n})} \mid, & (\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n}) \geq 0 \\ i \mid \sqrt{(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n})} \mid, & (\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n}) < 0 \end{cases} \tag{45}$$

Multivectors are parallel $\overrightarrow{\mathcal{P}^n} \uparrow\uparrow \overrightarrow{\mathcal{Q}^n}$ if

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = |\overrightarrow{\mathcal{P}^n}| \cdot |\overrightarrow{\mathcal{Q}^n}| \tag{46}$$

and they are antiparallel $\overrightarrow{\mathcal{P}^n} \uparrow\downarrow \overrightarrow{\mathcal{Q}^n}$ if

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = -|\overrightarrow{\mathcal{P}^n}| \cdot |\overrightarrow{\mathcal{Q}^n}| \tag{47}$$

Any nth order multivector of nonvanishing length generates nth order NGO (nth order tube) $\mathcal{T}_{\mathcal{P}^n}$, defined as a set of points P_{n+1}

$$\mathcal{T}_{\mathcal{P}^n} = \left\{ P_{n+1} \left| (\overrightarrow{\mathcal{P}^{n+1}}.\overrightarrow{\mathcal{P}^{n+1}}) = 0 \right. \right\} \tag{48}$$

If the function σ is the world function of the Euclidean space the scalar product coincides with the conventionally defined scalar product of vectors and multivectors. In the proper Euclidean space the nth order tube $\mathcal{T}_{\mathcal{P}^n}$ coincides with the n-dimensional plane of the Euclidean space.

In general, if the relation $\mathbf{P}_0\mathbf{Q} \uparrow\uparrow \mathbf{P}_0\mathbf{P}$, considered to be an equation for P at fixed P_0, Q $\quad (P_0 \neq Q)$ and fixed length $|\mathbf{P}_0\mathbf{P}| = a$, has not more than one solution for $\forall P_0, Q \in \Omega$ and $\forall a \neq 0$, the T-geometry is degenerate. In the nondegenerate T-geometry there are many vectors $\mathbf{P}_0\mathbf{P}$ of fixed length which are parallel to the vector $\mathbf{P}_0\mathbf{Q}$.

3.2 Euclidean σ-space.

It is shown that in the n-dimensional Euclidean space the world function satisfies the relation

$$\Gamma(P_0, P, Q) = \sum_{i,k=1}^{n} g^{ik}(\mathcal{P}^n)\Gamma(P_0, P_i, P)\Gamma(P_0, P_k, Q), \qquad \forall P, Q \in \Omega \tag{49}$$

where $\overrightarrow{\mathcal{P}^n}$ is any nonvanishing nth order multivector, and the metric tensor $g^{ik}(\mathcal{P}^n)$ is determined by the relations

$$g_{ik}(\mathcal{P}^n) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k) = \Gamma(P_0, P_i, P_k), \qquad i,k = 1,2,\dots n \tag{50}$$

$$\sum_{k=1}^{n} g_{ik}(\mathcal{P}^n)g^{kl}(\mathcal{P}^n) = \delta_i^l, \qquad i,l = 1,2,\dots n \tag{51}$$

Thus relations (49), containing only world function σ, are necessary conditions of the Euclideness of the σ-space.

3.3 Separation of tube into parts and concept "inside"

It is shown that it is possible to divide any tube $\mathcal{T}_{\mathcal{P}^n}$ into parts and to select the part $\mathcal{T}_{[\mathcal{P}^n]}$ which is placed "between" $n+1$ points $\mathcal{P}^n$. For the first order tube $\mathcal{T}_{P_0P_1}$ this segment $\mathcal{T}_{[P_0P_1]}$ has the form

$$\mathcal{T}_{[P_0P_1]} = \{P \mid S(P_0,P) + S(P,P_1) = S(P_0,P_1)\}, \qquad S(P_0,P) = \sqrt{2\sigma(P_0,P)} \tag{52}$$

3.4 Euclideness and extremality.

Connection between the Euclideness and extremality is investigated. Extremality of a tube describes degeneration of the first order tube into a line, of the second order tube into a two-dimensional plane, etc.. Formally the extremality is defined by two properties of $\mathcal{T}_{\mathcal{P}^n}$:

1. The nth order tube $\mathcal{T}_{\mathcal{P}^n}$ has the property of definiteness, if any nth order multivector $\overrightarrow{\mathcal{Q}^n}$, having the properties $\mathcal{Q}^n \subset \mathcal{T}_{\mathcal{P}^n}$, $|\overrightarrow{\mathcal{Q}^n}| \neq 0$, generates the tube $\mathcal{T}_{\mathcal{Q}^n} = \mathcal{T}_{\mathcal{P}^n}$

2. The nth order tube $\mathcal{T}_{\mathcal{P}^n}$ has the property of minimal section, if the section of $\mathcal{T}_{\mathcal{P}^n}$ at any point $Q \in \mathcal{T}_{\mathcal{P}^n}$ is minimal. It means that the section of the tube is only the point of the section

$$\mathcal{S}_{n;Q}(\mathcal{T}(\mathcal{P}^n)) = \{Q\} \tag{53}$$

where the section $\mathcal{S}_{n;Q}(\mathcal{T}(\mathcal{P}^n))$ of the tube $\mathcal{T}_{\mathcal{P}^n}$ at the point Q is defined by the relation

$$\mathcal{S}_{n;Q}(\mathcal{T}(\mathcal{P}^n)) = \{P_{n+1} \mid (\overrightarrow{\mathcal{P}^{n+1}}.\overrightarrow{\mathcal{P}^{n+1}}) = 0 \bigwedge_{l=0}^{l=n} \sigma(P_l, P_{n+1}) = \sigma(P_l, Q)\}, \tag{54}$$

The σ-space is extremal on the tube $\mathcal{T}_{\mathcal{P}^n}$, if the tube $\mathcal{T}_{\mathcal{P}^n}$ has the two properties. The proper Euclidean σ-space is extremal on tubes of all orders. The σ-space extremal on tubes of all orders is the proper Euclidean space with some exceptions. The proper Riemannian space is extremal on the first order tubes

3.5 Extremality and curtailed tubes.

There are such σ-spaces, where the zeroth order tube $\mathcal{T}_P$, determined by one point P is a surface (not a point). For instance, in the Minkowski space the zeroth order NGO $\mathcal{T}_P$ is the light cone with the vortex at the point P. As a consequence, if the vector $\mathbf{P}_0\mathbf{P}_1$ is spacelike, the first order tube $\mathcal{T}_{P_0P_1}$ in the Minkowski space is two three-dimensional planes, passing through the spacelike straight line, determined by the vector $\mathbf{P}_0\mathbf{P}_1$, because $\mathcal{T}_{P_0P_1}$ is constituted of $\mathcal{T}_P$, but not of points. Curtailed tubes $\mathcal{T}^c_{P_0P_1}$ are one-dimensional lines derived from $\mathcal{T}_{P_0P_1}$ by some artificial operations (curtailing). Different kinds of the curtailing are investigated.

4 Geodesic mapping.

In this chapter one considers a possibility of constructing the degenerate (Riemannian) geometry on the basis of only T-geometry. It is possible only in the case, when the world function σ satisfies some special constraints (formulated in terms of the world function). If these constraints are fulfilled, one can construct a manifold and Riemannian geometry on this manifold. Geodesic mapping determined by the world function is the main mathematical tool of this construction (compare geodesic mapping generated by the world function considered in the second chapter). Consideration of the geodesic mapping contains definitions of the vector σ space and geodesic rays, definition of a vector determined in the vector space, but not belonging to this vector space, and a lot of other mathematical subtleties. They are not considered in this review. Constructing a manifold for the Minkowski space, only timelike first order tubes are taken into account. In other words, only world lines of real particles are of importance.

4.1 Vector σ-space. Geodesic mapping.

4.2 Dense σ-space.

4.3 Riemannian space.

5 T-geometry on manifold.

5.1 World function on a manifold. The simplest generalizations of the Riemannian space.

In this case two mathematical tools (manifold and formalism of NGOs, based on the world function properties) are used in geometry. Formalism of NGOs is adequate to the nondegenerate geometry considered. Differential geometry technique is not adequate to the geometry, but it is also important, because differential geometry technique is the principal technique of modern geometry and physics, whereas the NGO technique is not yet developed sufficiently.

One considers two different world functions $\tilde{\sigma}$ and σ_R on the manifold M, where σ_R is the world function of a Riemannian space. The are connected by the relation

$$\tilde{\sigma} = \begin{cases} \sigma_R + d, & \sigma_0 < \sigma_R \\ \sigma_R + D(\sigma_R), & 0 \leq \sigma_R < \sigma_0 \\ \sigma_R, & \sigma_R < 0 \end{cases}, \tag{55}$$

where $D(\sigma_R)$ is the distortion function, which changes only inside a small region $(0, \sigma_0)$. The constant is the global distortion $d = D(\sigma_0)$ =const. For $d \ll \sigma_R$ the world function $\tilde{\sigma}$.

$$(2\tilde{\sigma})^{1/2} = [2(\sigma_R + d)]^{1/2} \cong (2\sigma_R)^{1/2}\left(1 + \frac{d}{2\sigma_R}\right) \tag{56}$$

distinguishes from σ_R slightly.

Let a uniform isotropic space-time $V = \{\sigma, M\}$ and Minkowski space-time $E = \{\sigma_E, M\}$ be defined on the same manifold M. Their world functions are connected by the relation

$$\sigma = \sigma_E + D(\sigma_E) \tag{57}$$

where σ_E is the world function of the Minkowski space-time, σ is the world function in the space-time V between the same points, and D is the distortion function.

Let us calculate the shape of the timelike tube segment $\mathcal{T}_{[P_0P_1]}$, determined by σ. Let the interval $S = (2\sigma)^{1/2}$ in the σ-space V be connected with the interval $S_E = (2\sigma_E)^{1/2}$ of the Minkowski space-time E by means of the relation

$$S_E = g(S) \tag{58}$$

According to (57) the function g is connected with the distortion function D by means of the relation

$$\frac{1}{2}S^2 = \frac{1}{2}g^2(S) + D[\frac{1}{2}g^2(S)] \tag{59}$$

The shape of the timelike tube segment $\mathcal{T}_{[P_0P_1]}$ is described in the Minkowski space-time E by its radius $R_E(\tau)$, which is measured in E in the coordinate system, where the vector $\mathbf{P}_0\mathbf{P}_1$ has vanishing spatial coordinates. τ is a parameter along the straight $\mathcal{L}_{[P_0P_1]}$ in E, which is equal to 0 and 1 at points P_0 and P_1 respectively. The result of calculation looks as follows

$$R_E^2(\tau) = \frac{1}{4}g^{-2}(S)[g(S) + g(S\tau) + g(S - \tau S)][g(S) - g(S\tau) - g(S - \tau S)]\times$$
$$[g(S) + g(S\tau) - g(S - \tau S)][g(S) - g(S\tau) + g(S - \tau S)] \tag{60}$$

where S is the length of $\mathcal{T}_{[P_0P_1]}$ measured in V, The radius $R_{\text{mid}} = R_E(1/2)$ of the segment $\mathcal{T}_{[P_0P_1]}$ at the middle has the form

$$R_{\text{mid}} = R_E(1/2) = \sqrt{\frac{1}{4}g^2(S) - g^2(S/2)} \tag{61}$$

The shape of the segment $\mathcal{T}_{[P_0P_1]}$ depends essentially on the form of the function g, describing distortion of V.

5.2 Tubular model of the space-time and free particle in this model.

Here the uniform and isotropic space-time geometry is supposed to be nondegenerate. It is described by the world function.

$$\sigma = \sigma(P, P') = \sigma_E(P, P') + D[\sigma_E(P, P')], \qquad P, P' \in \mathbb{M} \tag{62}$$

where D is a restricted function

$$D(0) = 0, \qquad D(\sigma_E) < d, \qquad d > 0. \tag{63}$$

Here d =const is the integral distortion. The Minkowski space-time E_4 is an auxiliary space-time, associated with the real space-time V_4. The timelike vector $\mathbf{P}_0\mathbf{P}_1$, $P_0, P \in M$ has two lengths

$$\mu = \mid \mathbf{P}_0\mathbf{P}_1 \mid = \sqrt{2\sigma(P_0, P_1)}, \qquad \mu_E = \mid \mathbf{P}_0\mathbf{P}_1 \mid_E = \sqrt{2\sigma_E(P_0, P_1)}, \tag{64}$$

in V_4 and in E_4 respectively.

It is supposed that the real world tube of a particle with the mass μ is a broken tube

$$\mathcal{T}_{br} = \bigcup_i \mathcal{T}_{[P_i P_{i+1}]} \tag{65}$$

consisted of similar segments $\mathcal{T}_{[P_i P_{i+1}]}$ of the same length μ.

$$S(P_i, P_{i+1}) = \sqrt{2\sigma(P_i, P_{i+1})} = \mu, \qquad i = 0, \pm 1, \pm 2, \ldots \tag{66}$$

Vector $\mathbf{P}_i\mathbf{P}_{i+1}$, $i = 0, \pm 1, \pm 2, \ldots$ is the particle momentum on the segment $\mathcal{T}_{[P_i P_{i+1}]}$. The length $\mu = \mid \mathbf{P}_i\mathbf{P}_{i+1} \mid$ of the momentum vector $\mathbf{P}_i\mathbf{P}_{i+1}$ is regarded as the mass. The particle mass is a geometric quantity, because it can be determined by the shape of the broken tube $\mathcal{T}_{br}$, given by the relation (65). Indeed, the section of the tube (65) is a sphere of the radius R_E, which is determined by the relation (60). The sphere degenerates to a point only at the points P_i, $i = 0, \pm 1, \pm 2, \ldots$and interval μ between these points determines the particle mass. In the case of the Minkowski space-time $\mathcal{T}_{br}$ degenerates into a broken line. Section of this line is a point everywhere. Looking at this broken line, one cannot determine the points P_i, $i = 0, \pm 1, \pm 2, \ldots$, interval between which determines the particle mass μ. This geometric mass μ is connected with the usual mass by the relation

$$m = b\mu, \qquad [b] = \mathrm{g/cm} \tag{67}$$

where b is some universal constant.

The world tube of a free particle is determined by the relation

$$\mathbf{P}_i\mathbf{P}_{i+1} \uparrow\downarrow \mathbf{P}_i\mathbf{P}_{i-1}, \qquad i = 0, \pm 1, \pm 2 \ldots \tag{68}$$

Under the distortion D tending to 0, the relation (68) transforms to the relation

$$dp_i/d\tau = 0, \qquad i = 0, 1, 2, 3 \tag{69}$$

where p_i is the particle momentum components in the Minkowski space-time.

In the Minkowski space-time the system of equations (66), (68) determines all points P_i, $i = 0, \pm 1, \pm 2 \ldots$unambiguously, provided position of two adjacent points, for instance, P_0 P_1, is given, i.e. the momentum $\mathbf{p}$, given at one point of the world line, determines the whole world line. In the tubular model, fixing position of one link of the world tube and using system of equations (66), (68), one cannot determine position of the links of the broken tube, because there are many vectors $\mathbf{P}_i\mathbf{P}_{i+1}$ of the length μ, which are antiparallel to the vector $\mathbf{P}_i\mathbf{P}_{i-1}$. In this sense the broken tube of a free particle is stochastic. This stochasticity can be evaluated by the dispersion $\Delta\vartheta$ of the angle between the links, measured in the Minkowski space-time.

$$\Delta\vartheta = \frac{R_E(1/2)}{|\mathbf{P}_i\mathbf{P}_{i-1}|} = \frac{\sqrt{g^2(\mu) - 4g^2(\mu/2)}}{4\mu} \qquad (70)$$

Using the relation (59) and supposing that for large μ $\quad g(\mu) \simeq \mu$ and $D(\mu)$ tends to the integral distortion d, the relation (70) for large μ turns to

$$\Delta\vartheta = \frac{\sqrt{8D(\mu^2/8) - 2D(\mu^2/2)}}{4\mu} = \frac{\sqrt{6d}}{4\mu} \qquad (71)$$

It follows from the relation (71), that the stochasticity is proportional to $\sqrt{d}/\mu$. It vanishes for large particle mass μ or for small integral distortion d. Stochasticity of quantum particle is proportional to the Compton wavelength $\hbar/mc$. If one supposes that the integral distortion d is proportional to $\hbar$, the dependence of the quantum stochasticity on the particle mass becomes to be close to the natural geometric stochsticity, conditioned by the space-time distortion. Choosing the distortion D in proper way one may hope to derive the quantum stochasticity as a result of the natural geometric stochasticity.

5.3 Dynamical conception of statistical description.

The natural geometric stochasticity of particle world tubes needs a statistical description. Essentially the statistical description is a constituent of the nondegenerate geometry. In the monograph the statistical description is carried out on a dynamic basis and it does not use the probability theory. It may be explained as follows.

The result of an experiment with a single stochastic particle $\mathcal{S}_{\mathrm{st}}$ is irreproducible. But distributions of results of similar experiments with N independent stochastic particles are reproducible (N is very large). Projecting many independent identical stochastic particles $\mathcal{S}_{\mathrm{st}}$ to the same space–time region, one obtains a cloud $\mathcal{E}[N, \mathcal{S}_{\mathrm{st}}]$ of N independent identical particles $\mathcal{S}_{\mathrm{st}}$ moving randomly. With the number N of particles tending to ∞, this cloud $\mathcal{E}[\infty, \mathcal{S}_{\mathrm{st}}]$ may be considered as a continuous medium, or as a fluid. This fluid is a deterministic dynamic system, because experiments with the fluid $\mathcal{E}[\infty, \mathcal{S}_{\mathrm{st}}]$ are reproducible. Besides, any reproducible series of experiments with the stochastic particle can be described in terms of the fluid $\mathcal{E}[\infty, \mathcal{S}_{\mathrm{st}}]$ without reference to any probabilistic construction (i.e. without a reference to the statistical ensemble's property of being a tool for calculation of average values). The probabilistic constructions are effective only if the statistical ensemble $\mathcal{E}[\infty, \mathcal{S}_{\rm d}]$ consists of deterministic particles $\mathcal{S}_{\rm d}$, whose properties can be determined independent of $\mathcal{E}[\infty, \mathcal{S}_{\rm d}]$.

For instance, let us consider a single electron $\mathcal{S}_{\rm st}$, flying from an electron gun, passing through a narrow slit in a diaphragm and hitting a screen at a point x_1. Another electron $\mathcal{S}_{\rm st}$, prepared in the same way, hits the screen at other point x_2 which does not coincide with x_1. In other words, an experiment with single electron is irreproducible in general. It means that a single electron is a stochastic particle. Let us consider a series of $N \quad (N \to \infty)$ experiments with identically prepared independent electrons. The distribution of N impact points over the screen is reproducible, i.e., it is approximately the same for other series of N experiments. It means that a set $\mathcal{E}[N, \mathcal{S}_{\rm st}]$ of $N \quad (N \to \infty)$ independent identical electrons $\mathcal{S}_{\rm st}$ is a deterministic dynamic system, although a single electron $\mathcal{S}_{\rm st}$ is a stochastic system. If $\mathcal{E}[\infty, \mathcal{S}_{\rm st}]$ can be considered to be a fluid, then by solving dynamic equations for this fluid and calculating the flux of the fluid $\mathcal{E}[\infty, \mathcal{S}_{\rm st}]$ through the screen, one can calculate the diffraction picture (the distribution of the impact points over the screen). For such a calculation one needs only characteristics of the dynamic system $\mathcal{E}[\infty, \mathcal{S}_{\rm st}]$ (dynamic equations, expressions for the particle flux and the energy-momentum tensor). Any quantum axiomatics and corresponding probabilistic constructions (wave function, linear operators, commutation relations, etc.) are not needed. It means that quantum effects can be explained and calculated as purely dynamical effects.

5.4 Connection of statistical description with the quantum one. General conception of statistical description.

The conventional quantum description contains specific quantum quantities such as wave function and spin. These quantities are fundamental in the sense that they are defined by their properties by means of the system of specific quantum axioms, known as quantum principles, or quantum axiomatics. These principles dictate the rules of working with the wave function and spin. If one succeeded in defining the wave function and spin by some way, the quantum axiomatics stops to be necessary. The wave function and spin have been shown to be attributes of any nondissipative fluid [1]. The wave function is a variety of a description of a fluid in terms of hydrodynamic potentials. There are many different forms of these hydrodynamic potentials. One sort of them is known as Clebsch potentials [2, 3].

Statistical description of stochastic world tubes reduces to introduction of a fluid $\mathcal{E}[\infty, \mathcal{S}_{\rm st}]$, and the wave function appears as a way of description of this fluid. In quantum mechanics some states (mixed ones) cannot be described by one wave function. They are described by the density matrix. In the statistical description the mixed states are described as follows. Any statistical ensemble $\mathcal{E}$ is a dynamic system and may be an element of other ensemble $\mathcal{E}_1$, the ensemble $\mathcal{E}_1$ may be an element of the ensemble $\mathcal{E}_2$ etc.. In this hierarchy $\mathcal{S}_{\rm st}$, $\mathcal{E}$, $\mathcal{E}_1$,... the stochastic system $\mathcal{S}_{\rm st}$ has no dynamic equations, the statistical ensemble $\mathcal{E}$ is described by means of the wave function, and the statistical ensemble $\mathcal{E}_1$ is described by means of the density matrix.

Practically the real problem of statistical description reduces to determination of the form of the dynamic system $\mathcal{E}[\infty, \mathcal{S}_{\mathrm{st}}]$. It is solved as follows. The statistical ensemble $\mathcal{E}[\infty, \mathcal{S}_{\mathrm{st}}]$ of stochastic particles is described by the Hamilton functional of deterministic (classical) particles

$$\mathcal{H}[\rho, \varphi] = \int \rho c \sqrt{m_q^2 c^2 + (\nabla\varphi)^2} d\mathbf{x}, \tag{72}$$

where φ, ρ are functions of $t, \mathbf{x}$, and the mass m_q is considered as a function of the ensemble state φ, ρ. If m_q =const the functional (72) describes the fluid without vanishing pressure. In this case the Hamiltonian (72) describes a statistical ensemble of relativistic classical particles.

If the statistical description by means of the Hamilton functional is equivalent to the quantum description with the Schrödinger equation, m_q is to have the form

$$m_q^2 = m^2 + (\frac{\hbar}{2c}\nabla \log\rho)^2, \qquad m^2 = \mathrm{const} \tag{73}$$

where

$$(\frac{\hbar}{2c}\nabla \log\rho)^2 \ll m^2 \tag{74}$$

Indeed, in this case introducing new variables

$$\psi = \sqrt{\rho}\exp(i\varphi/\hbar) \qquad \psi^* = \sqrt{\rho}\exp(-i\varphi/\hbar), \tag{75}$$

the relation (72) can be written in the form

$$\mathcal{H}_q[\psi, \psi^*] = \int mc^2\psi^*\psi\sqrt{1 + \frac{\hbar^2}{m^2c^2}\frac{\nabla\psi^*\nabla\psi}{\psi^*\psi}}\, \mathrm{d}\mathbf{x} \tag{76}$$

Taking into account Eq. (74), one can transform the functional (76) to the form

$$\mathcal{H}_q[\psi, \psi^*] = \int (mc^2\psi^*\psi + \frac{\hbar^2}{2m}\nabla\psi^*\nabla\psi) d\mathbf{x} \tag{77}$$

Dynamic equations generated by the Hamiltonian (77) have the form

$$\frac{\partial\psi}{\partial t} = -\frac{i}{\hbar}\frac{\delta\mathcal{H}}{\delta\psi^*}, \qquad \frac{\partial\psi^*}{\partial t} = \frac{i\delta\mathcal{H}}{\hbar\delta\psi}, \tag{78}$$

It follows from Eqs. (77) (78)

$$i\hbar\frac{\partial\psi}{\partial t} = mc^2\psi - \frac{\hbar^2}{2m}\nabla^2\psi \tag{79}$$

After the change of variables $\psi \to \psi\exp(-imc^2t/\hbar)$ the equation (79) takes the form of the Schrödinger equation for a free particle

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi \tag{80}$$

Let us go to geometric variables in the Hamiltonian functional (72). The change of variables has the form

$$\varphi = bc\tilde{\varphi}, \qquad m = b\tilde{\mu}, \qquad t = \tau/c, \qquad \hbar = bc\sigma_1 \tag{81}$$

where b is the universal constant defined by the relation (67), and σ_1 is some constant of dimensionality [cm^2] introduced instead of the Planck constant. One obtains the following result

$$\mathcal{H}[\rho, \tilde{\varphi}] = \int \sqrt{\mu_q^2 + (\nabla\tilde{\varphi})^2}\rho d\mathbf{x} \tag{82}$$

$$\mu_q^2 = \tilde{\mu}^2 + [(\sigma_1/2)\nabla \log \rho]^2, \qquad \tilde{\mu} = \text{const} \tag{83}$$

Quantum effects in the statistical ensemble, described by the Hamiltonian (82) appear by virtue of the μ_q dependence on the ensemble state (ρ, φ).

5.5 Oriented mass

There are two different particle masses μ, and μ_E, defined by Eq. (64). Besides one can define two oriented masses, defined by the relations

$$\mu_o = (\mathbf{p}.\mathbf{u}). \tag{84}$$

$$\mu_{oE} = (\mathbf{p}.\mathbf{u})_E \tag{85}$$

where $\mathbf{u}$ is the four-velocity, and index "E" means that the scalar product is calculated in the Minkowski space-time E_4. In the real space-time V_4 either $\mathbf{u} \uparrow\uparrow \mathbf{p}$, or $\mathbf{u} \uparrow\downarrow \mathbf{p}$, and $\mu_o = \pm\mu$. In the Minkowski space-time $\mathbf{u}$ is not collinear with $\mathbf{p}$, and $\mu_{oE} = (\mathbf{p}.\mathbf{u})_E$ depends on the point of the segment $\mathcal{T}_{[P_i P_{i+1}]}$, where the four velocity $\mathbf{u}$ is calculated.

Let us suppose that in the statistical ensemble of stochastic tubes described by the Hamiltonian (72) one should use the oriented associated mass $m_{oE} = b\mu_{oE}$. In the Minkowski model $m_{oE}^2 = m_o^2 = m^2$, and such a supposition does not lead to any change in the conventional presentation of mechanics.

Let us determine the form of the four-velocity which is needed for μ_{oE} to have the form (83). Let us choose the coordinate system in such a way that the spatial components of the momentum p_i=$\mathbf{P}_0\mathbf{P}_1$ vanish at the point P_0. In this coordinate system

$$p_i = (\mu_E, \mathbf{0}), \qquad u^i = \left(\frac{1}{\sqrt{1-\beta^2}}, \frac{\beta}{\sqrt{1-\beta^2}}\right) \tag{86}$$

where β=$\mathbf{v}/c$, and $\mathbf{v}$ is the usual three-velocity at the point P_0, in the coordinate system where $\mathbf{p}$ =0. One obtains for the oriented associated mass μ_{oE} at this point

$$\mu_{oE} = p_i u^i = \mu_E\sqrt{1-\beta^2} \cong \mu_E(1+\beta^2/2), \qquad \beta \ll 1 \tag{87}$$

The angle ϑ = arcthβ between the vectors p_i and u^i does not vanish, although it is small, $\vartheta \ll 1$.

Comparing (87) and (83), one obtains

$$\beta = \pm \frac{\sigma_1}{2\mu_E} \frac{\nabla \rho}{\rho}, \qquad \mu_E = \tilde{\mu}. \tag{88}$$

Thus, the velocity in the coordinate system, where the three-momentum $\mathbf{p} = 0$, is determined by the formula (88). The mean velocity in the ensemble of Brownian particles has the same form (the sign is minus in (88)) in the coordinate system where the medium is fixed and the diffusion coefficient is equal to $\sigma_1/2\mu_E$.

The mean velocity $\mathbf{u}$ at some point P is formed by contribution of different tubes of the statistical ensemble. Averaging these contributions, one obtains

$$\beta = -\frac{2R_E^2}{3\mu_E} \nabla \log \rho, \qquad j^0 = \rho. \tag{89}$$

where $R_E^2 = R_E^2(1/2)$ is determined by Eq. (61). Comparing (89) with (88) and taking into account (61), one obtains the equation for determining the function g

$$R_E^2 = \frac{1}{4} g^2(\mu) - g^2(\mu/2) = \frac{3}{4}\sigma_1. \tag{90}$$

This relation is valid in the region $\sigma_E = g^2(\mu)/2 > \sigma_0$, where σ_0 is a constant. Solution of the equation has the form

$$\mu_E^2 = g^2(\mu) = \mu^2 - \sigma_1, \qquad \sigma_E = \frac{1}{2}\mu_E^2 \geq \sigma_0 > 0, \tag{91}$$

Finally, taking into account (81), one obtains the relation between the world function σ of V and the world function σ_E, of E, as well the connection between the integral distortion d and the Planck constant $\hbar$.

$$\sigma = \sigma_E + d, \qquad d = \frac{\hbar}{2bc}, \qquad \sigma_E \geq \sigma_0 \tag{92}$$

Constraint upon the argument σ_E is connected with the fact that both σ_E and σ are to vanish simultaneously, that is impossible at the constant distortion d. The distortion must vanish simultaneously with σ_E and σ.

Unfortunately, one cannot calculate the value of the universal constant b and that of d. But one can evaluate supremum of b and infinum of d. The evaluation is founded on the supposition that the considered procedure may be applied to all massive particles, i.e. to all particles with the mass larger, than the electron mass. Let us take that $\sigma_0 = \alpha d$, where α is some positive constant of the order 1. Supposing that the constraint (92) is valid for the electron mass $m_\mathrm{e} = \mu_{\mathrm{e}E}/b$, one obtains

$$m_\mathrm{e}^2/2b^2 \geq (1+\alpha)d = \frac{(1+\alpha)\hbar}{2bc} \tag{93}$$

where m_e is the usual electron mass. The mass $m_\mathrm{e} = b\mu_{\mathrm{e}E}$, appearing in dynamic equations and in the Hamiltonian (82), (83), is connected with $\mu_{\mathrm{e}E}$ but not with μ_e. One obtains the following estimation of b

$$b \leq \frac{m_e^2 c}{\hbar(1+\alpha)} \cong \frac{3 \cdot 10^{-16}}{(1+\alpha)} \mathrm{g/cm} \tag{94}$$

Using expression (92), for d, one obtains from (94) the following estimation for the space-time distortion

$$d=\frac{\hbar}{2bc}\geq\frac{\hbar^2(1+\alpha)}{2m_e^2c^2}=\frac{1+\alpha}{2}\lambda_e^2\cong 7\cdot 10^{-22}\mathrm{cm}^2\,(1+\alpha) \tag{95}$$

where λ_e is the electron Compton wavelength. Setting $\alpha=2$, one obtains for the characteristic radius of the tube the following estimation

$$R_E=(3d/2)^{1/2}>0.5\cdot 10^{-10}\mathrm{cm} \tag{96}$$

For the geometric mass of the electron μ_{eE} one obtains the following estimation

$$\mu_{eE}>10^{-10}\mathrm{cm}, \tag{97}$$

what agrees with (96).

Thus, the world function of the space-time can be approximated by the expression

$$\sigma=\sigma_E+\frac{\hbar}{2bc},\qquad \sigma>\sigma_0=\frac{\hbar}{bc} \tag{98}$$

where c is the speed of the light. b and σ_0 are universal constants. $b<10^{-17}\mathrm{g/cm}$.

5.6 Statistically averaged dynamic system and reduction of its state

One considers interrelation between the individual stochastic system $\mathcal{S}_{\mathrm{st}}$ and the statistically averaged dynamic system $\langle S\rangle$. This interrelation is important in the measurement of stochastic systems, when dynamic equations describe behaviour of the statistically averaged dynamic system $\langle S\rangle$, whereas in any individual measurement one deals with the individual stochastic system $\mathcal{S}_{\mathrm{st}}$. In application to quantum systems the problem of reduction of the quantum system state as a result of the measurement is considered. Any individual measurement of some quantity $\mathcal{R}$ is produced under the individual stochastic system $\mathcal{S}_{\mathrm{st}}$. Such a measurement always leads to some stochastic, but definite result R'. It is impossible to speak about a change of the system state (wave function ψ), because the wave function ψ is an attribute of the statistically averaged system $\langle S\rangle$, but not of the individual system $\mathcal{S}_{\mathrm{st}}$. On the other hand, any mass measurement under the statistically averaged system $\langle S\rangle$ leads to a distribution $d(R')$ of results, because individual measurement under the statistically averaged system $\langle S\rangle$ is impossible. In this case it is difficult to speak about influence of obtained definite result R' upon the reduction of the wave function ψ. The problem is solved by means of introduction of special "mass measurement leading to definite result" (MMLDR). The mass measurement leading to definite result R' is a mass measurement of the quantity $\mathcal{R}$ in the system $\langle S\rangle$ which is accompanied by a selection operation. In the process of this selection one forms a new $\langle S\rangle$ (statistical ensemble normalized on unity), where only those systems $\mathcal{S}_{\mathrm{st}}$ are selected for which the value of the measured quantity $\mathcal{R}$ is R'. It is the reduction of the state of the statistically averaged system $\langle S\rangle$ (reduction of the wave function). One discusses how the measuring device carries out this reduction.

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* See Web site: $http://rsfq1.physics.sunysb.edu/\tilde{}rylov/yrylov.htm.$
or mirror Web site: $http://194.190.131.172/\tilde{}rylov/yrylov.htm.$