Dynamic disquantization of Dirac equation.

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Abstract

Classical model $\mathcal{S}_{\rm dc}$ of Dirac particle $\mathcal{S}_{\rm D}$ is constructed. $\mathcal{S}_{\rm D}$ is the dynamic system described by the Dirac equation. Its classical analog $\mathcal{S}_{\rm dc}$ is described by a system of ordinary differential equations, containing the quantum constant $\hbar$ as a parameter. Dynamic equations for $\mathcal{S}_{\rm dc}$ are determined by the Dirac equation uniquely. Both dynamic systems $\mathcal{S}_{\rm D}$ and $\mathcal{S}_{\rm dc}$ appear to be nonrelativistic. One succeeded in transforming nonrelativistic dynamic system $\mathcal{S}_{\rm dc}$ into relativistic one $\mathcal{S}_{\rm dcr}$. The dynamic system $\mathcal{S}_{\rm dcr}$ appears to be a two-particle structure (special case of a relativistic rotator). It explains freely such properties of $\mathcal{S}_{\rm D}$ as spin and magnetic moment, which are strange for pointlike structure. The rigidity function $f_{\rm r}(a)$, describing rotational part of total mass as a function of the radius a of rotator, has been calculated for $\mathcal{S}_{\rm dcr}$. For investigation of $\mathcal{S}_{\rm D}$ and construction of $\mathcal{S}_{\rm dc}$ one uses new dynamic methods: dynamic quantization and dynamic disquantization. These relativistic pure dynamic procedures do not use principles of quantum mechanics (QM). They generalize and replace conventional quantization and transition to classical approximation. Totality of these methods forms the model conception of quantum phenomena (MCQP). Technique of MCQP is more subtle and effective, than conventional methods of QM. MCQP relates to conventional QM, much as the statistical physics relates to thermodynamics.

Key words: quantization, disquantization, Dirac equation, relativistic rotator

1 Introduction

In this investigation we try to answer the question, whether the Dirac particle has a structure, or it is simply a pointlike particle, which has such unusual for pointlike particle properties as spin and magnetic moment. To answer this question, we use procedure of dynamic disquantization which is a more subtle, than conventional procedure of transition to classical approximation. Dynamic disquantization (D-disquantization) of dynamic system $\mathcal{S}_{\mathrm{D}}$, described by the Dirac equations, results in classical relativistic rotator $\mathcal{S}_{\mathrm{rr}}$, consisting of two coupled charged particles, rotating around their mass center. The more rough classical approximation, being applied to $\mathcal{S}_{\mathrm{D}}$, results in pointlike classical particle, possessing spin and magnetic moment. A characteristic of any rotator is the rigidity function f_{r} which is defined as follows. Let $E_{\mathrm{tot}}(a)$ be total rest energy of the rotator as a function of the radius a. The rigidity function f_{r} connects the relative rotational energy $\gamma = E_{\mathrm{rot}}/E_{\mathrm{tot}}(0)$ with the radius a of the rotator by means of relation

$$\gamma = f_{\mathrm{r}}(a). \tag{1.1}$$

For nonrelativistic rotator $\mathcal{S}_{\mathrm{nr}}$ the rigidity function f_{r} is connected with the potential energy $U(a)$ of the coupling by means of relation

$$f_{\mathrm{r}}(a) = \frac{a}{m_0 c^2}\frac{\partial U(a)}{\partial a}$$

where m_0 is the mass of a rotator particle. The potential energy U of the coupling cannot be introduced for a relativistic rotator, but the rigidity function f_{r} of the rotator coupling may be introduced in any case.

The rigidity function for the rotator $\mathcal{S}_{\mathrm{rr}}$ which is a classical analog of Dirac dynamic system $\mathcal{S}_{\mathrm{D}}$ appears to have the form

$$\gamma = f_{\mathrm{r}}(a) = \frac{\hbar}{\sqrt{\hbar^2 - (4am_0c)^2}} - 1 \tag{1.2}$$

One can see that in (1.2) $\hbar > 4am_0c$, and one can set $\hbar \to 0$ only in the case, if simultaneously $a \to 0$. The magnetic moment and spin are natural properties of any charged rotator.

Conventional disquantization (classical approximation) of the Dirac equation cannot give the result (1.2), because it sets $\hbar = 0$ in the final result. It obtains only a charged pointlike particle with magnetic moment and spin. In general, a classical analog, obtained as a classical approximation of a quantum system, cannot contain the quantum constant $\hbar$.

The more detailed information on classical origin of a quantum system is obtained, because one uses the model conception of quantum phenomena (MCQP), which relates to the conventional (axiomatic) quantum theory in much the same way, that the statistical physics (model conception of thermal phenomena) relates to the axiomatic thermodynamics. Ignoring the internal degrees of freedom of $\mathcal{S}_{\mathrm{D}}$ is justified, when one calculates atom structures. In this case the interaction energy is low, internal degrees of freedom are not excited, and one can neglect them. If one investigates the structure of elementary particles, where the interaction energies are very high, one should use model conception of quantum phenomena (MCQP), which enables one to take into account details of internal structure. Investigation of elementary particles by means of conventional axiomatic quantum theory resembles investigation of crystal structure by means of thermodynamics without having an idea on molecular structure of the matter.

The conventional disquantization (Q-disquantization) of quantum dynamic system $\mathcal{S}_{\rm q}$ results in derivation of classical analog $\mathcal{S}_{\rm c}$ of quantum system $\mathcal{S}_{\rm q}$. The dynamic system $\mathcal{S}_{\rm c}$ has two properties:

(1) Dynamic equations for $\mathcal{S}_{\rm c}$ are ordinary differential equations.

(2) Dynamic equations for $\mathcal{S}_{\rm c}$ do not contain quantum constant $\hbar$.

It is common practice to think that the two properties are connected closely, and there are no such $\mathcal{S}_{\rm c}$, whose dynamic equations be ordinary dynamic equations, containing $\hbar$ as a parameter. Model conception of quantum phenomena (MCQP) separates the two properties of $\mathcal{S}_{\rm c}$. In the scope of MCQP there exist such classical models $\mathcal{S}_{\rm c}$ of quantum system $\mathcal{S}_{\rm q}$, where dynamic equations for $\mathcal{S}_{\rm c}$ are ordinary differential equations, containing $\hbar$ as a parameter. Of course, one can set in this model $\hbar = 0$ and derive a more rough model, which is obtained as a result of conventional disquantization (Q-disquantization). Dynamic models of MCQP are intermediate between pure quantum models and pure classical ones. Practically, models of MCQP are classical models, containing parameter $\hbar$, and these models are more subtle, than conventional classical models.

Let us note that the dynamic disquantization (D-disquantization) is simply a *method of investigation* of dynamic system $\mathcal{S}_{\rm D}$, when the complicated dynamic system $\mathcal{S}_{\rm D}$, described by a system of partial differential equations, associates with a more simple dynamic system $\mathcal{S}_{\rm c}$, described by a system of ordinary differential equations.

For instance, let $\mathcal{S}_{\rm KG}$ be dynamic system described by the free Klein-Gordon equation. Conventional classical analog $\mathcal{S}_{\rm KGc}$ of $\mathcal{S}_{\rm KG}$ is spinless relativistic particle. In the scope of MCQP the classical analog $\mathcal{S}_{\rm KGqu}$ of $\mathcal{S}_{\rm KG}$ is a classical relativistic particle (described by ordinary differential equations) in some force field κ, generated by this particle [1]. The field κ describes a cloud of pairs around the particle. Interaction between the particle and the field κ contains the parameter $\hbar$, and vanishes, if one sets $\hbar = 0$.

The conventional quantum theory states that the classical relativistic particle cannot produce pairs, but quantum one can. Why? Nobody knows. MCQP states that origin of pair production is a special force field, and quantization in itself has nothing to do with pair production.

MCQP uses special procedures: dynamic quantization (D-quantization) and dynamic disquantization (D-disquantization), which are dynamic generalizations of conventional quantum procedures of quantization (Q-quantization) and of the classical approximation derivation (Q-disquantization). Mathematical ground of MCQP is three different methods, used for description of dynamic systems, known as statistical ensembles.

Let $\mathcal{S}_{\rm c}$ be classical dynamic system described by the action

$$\mathcal{S}_{\rm c}: \qquad \mathcal{A}_{\rm c}[\mathbf{x}] = \int L(t, \mathbf{x}, \dot{\mathbf{x}})\, dt \tag{1.3}$$

Here $L(t, \mathbf{x}, \dot{\mathbf{x}}) = L(x, \mathbf{v})$ is its Lagrangian, and $\mathbf{x} = \{x^\alpha\}, \quad \mathbf{v} = \dot{\mathbf{x}} = \left\{\frac{dx^\alpha}{dt}\right\}, \ \alpha = 1, 2, ...n$. The statistical ensemble $\mathcal{E}[\mathcal{S}_{\rm c}]$, consisting of $\mathcal{S}_{\rm c}$, is described by the action

$$\mathcal{E}[\mathcal{S}_{\rm c}]: \qquad \mathcal{A}_{\rm E}[\mathbf{x}] = \int \mathcal{A}_{\rm c}[\mathbf{x}]\, d\boldsymbol{\tau} = \int L(t, \mathbf{x}, \dot{\mathbf{x}})\, dt d\boldsymbol{\tau} \tag{1.4}$$

where $\mathbf{x} = \mathbf{x}(t, \boldsymbol{\tau})$ are considered to be functions of Lagrangian coordinates $\tau = \{t, \boldsymbol{\tau}\}, \quad \boldsymbol{\tau} = \{\tau_\alpha\}, \quad \alpha = 1, 2, ...n$, which label elements (dynamic systems $\mathcal{S}_{\rm c}$) of the statistical ensemble $\mathcal{E}[\mathcal{S}_{\rm c}]$. This labeling is arbitrary, and the action (1.4) is invariant with respect to relabeling of elements $\mathcal{S}_{\rm c}$

$$\tau_\alpha \to \tilde{\tau}_\alpha = \tilde{\tau}_\alpha(\boldsymbol{\tau}), \qquad \alpha = 1, 2, ...n, \qquad \frac{\partial(\tilde{\tau}_1, \tilde{\tau}_2, ...\tilde{\tau}_n,)}{\partial(\tau_1, \tau_2, ...\tau_n,)} = 1 \tag{1.5}$$

This method of description will be referred to as L-description, or L-representation, because it uses Lagrangian coordinates $\boldsymbol{\tau}$ as independent variables.

The statistical ensemble $\mathcal{E}[\mathcal{S}_{\rm c}]$ is a kind of a fluid without pressure. The action (1.4) can be rewritten in the space of variables $x^i = \{t, \mathbf{x}\} = \{x^0, \mathbf{x}\}$ in terms of hydrodynamic variables: flux n-vector $j^i = \{\rho, \mathbf{j}\} = \{\rho, \rho\mathbf{v}\}$ and hydrodynamic potentials (Clebsch potentials [2, 3]) $\tau = \{\varphi, \boldsymbol{\tau}\}$. The action (1.4) transforms to the form

$$\mathcal{E}[\mathcal{S}_{\rm c}]: \qquad \mathcal{A}_{\rm E}[\rho, \mathbf{v}, \varphi, \boldsymbol{\tau}] = \int \rho\{L(x, \mathbf{v}) - \mathbf{v}\mathbf{p} - p_0\} \mathrm{d}^{n+1}x, \tag{1.6}$$

Here dependent variables $\rho, \mathbf{v}, \varphi, \boldsymbol{\tau}$ are considered to be functions of independent variables x. The quantities $p_k = \{p_0, \mathbf{p}\} = \{p_k\}, \quad k = 0, 1, ...n$ are defined by the relations

$$p_k = b_0(\partial_k\varphi + g^\alpha(\boldsymbol{\tau})\partial_k\tau_\alpha), \qquad k = 0, 1, ...n \tag{1.7}$$

where $g^a(\boldsymbol{\tau}), \ \alpha = 1, 2, ...n$ are arbitrary functions of only $\boldsymbol{\tau}$, and b_0 is an arbitrary constant. The action (1.6) is obtained from the action (1.4) by a change of variables. Now the variables x are independent variables, and variables $\boldsymbol{\tau}$ are dependent ones. Besides one introduces designation

$$j^i = \{\rho, \rho\mathbf{v}\} = \frac{\partial(x^i, \tau_1, ...\tau_n)}{\partial(x^0, x^1, ...x^n)}, \qquad i = 0, 1, ...n \tag{1.8}$$

The change of variables is accompanied by integration of some dynamic equations. The arbitrary functions $g^a(\boldsymbol{\tau}), \quad \alpha = 1, 2, ...n$ and the arbitrary constant b_0 are results of this integration. This integration appears to be possible, because the dynamic system $\mathcal{E}[\mathcal{S}_{\rm c}]$ has the symmetry group (1.5), containing n arbitrary functions. We shall not go into details of this change of variables, which is rather complicated.

One can find these details in the paper [4]. This method, using Lagrangian coordinates $\boldsymbol{\tau}$ as hydrodynamic potentials, will be referred to as HDP-description, or HDP-representation.

The action (1.6) can be described also in terms of ψ function (wave function). Let us introduce k-component complex function $\psi = \{\psi_\alpha\}$, $\alpha = 1, 2, \dots k$, defining it by the relations

$$\psi_\alpha = \sqrt{\rho} e^{i\varphi} u_\alpha(\boldsymbol{\tau}), \qquad \psi_\alpha^* = \sqrt{\rho} e^{-i\varphi} u_\alpha^*(\boldsymbol{\tau}), \qquad \alpha = 1, 2, \dots k \tag{1.9}$$

$$\psi^* \psi \equiv \sum_{\alpha=1}^{k} \psi_\alpha^* \psi_\alpha \tag{1.10}$$

where $(^*)$ means the complex conjugate, $u_\alpha(\boldsymbol{\tau})$, $\alpha = 1, 2, \dots k$ are functions of only variables $\boldsymbol{\tau}$. They satisfy the relations

$$-\frac{i}{2} \sum_{\alpha=1}^{k} \left(u_\alpha^* \frac{\partial u_\alpha}{\partial \tau_\beta} - \frac{\partial u_\alpha^*}{\partial \tau_\beta} u_\alpha\right) = g^\beta(\boldsymbol{\tau}), \qquad \beta = 1, 2, ...n, \qquad \sum_{\alpha=1}^{k} u_\alpha^* u_\alpha = 1 \tag{1.11}$$

k is such a natural number that equations (1.11) admit a solution. In general, the minimal possible value of k depends on the form of the arbitrary integration functions $\mathbf{g} = \{g^\beta(\boldsymbol{\tau})\}$, $\beta = 1, 2, ...n$. After such a change of variables the action (1.6) takes the form

$$\mathcal{E}\left[\mathcal{S}_{\mathrm{c}}\right]: \qquad \mathcal{A}_{\mathrm{E}}[\psi, \psi^*] = \int \rho \{L(x, \mathbf{V}) - \mathbf{VP} - P_0\} \mathrm{d}^{n+1} x, \tag{1.12}$$

where

$$\rho = \psi^* \psi, \qquad P_l = P_l(\psi, \psi^*) = -\frac{i b_0}{2 \psi^* \psi} (\psi^* \partial_l \psi - \partial_l \psi^* \cdot \psi), \qquad l = 0, 1, ...n \tag{1.13}$$

and $\mathbf{V} = \{V^\alpha\}$, $\alpha = 1, 2, ...n$ is some function of $\mathbf{P}$ and x, defined by the equations

$$\frac{\partial L(x, \mathbf{V})}{\partial V^\beta} - P_\beta = 0, \qquad \beta = 1, 2, ...n \tag{1.14}$$

If dynamic system $\mathcal{S}_{\mathrm{c}}$ has the Hamilton function

$$H(x, \mathbf{p}) = \mathbf{pv} - \mathbf{L}(x, \mathbf{v}) \tag{1.15}$$

the action (1.12) can be written in the form

$$\mathcal{E}\left[\mathcal{S}_{\mathrm{c}}\right]: \qquad \mathcal{A}_{\mathrm{E}}[\psi, \bar{\psi}] = \int \rho \{-H(x, \mathbf{P}) - P_0\} \mathrm{d}^{n+1} x, \tag{1.16}$$

where P_k and ρ are defined by relations (1.13). This method [4], describing the dynamic system $\mathcal{E}\left[\mathcal{S}_{\mathrm{c}}\right]$ in terms of wave function, will be referred to as WF-description, or WF-representation.

The three different actions (1.4), (1.6), (1.16) can be also used for description of dynamic system $\mathcal{S}\left[\mathcal{S}_{\mathrm{c}}\right]$ which is a set of interacting classical dynamic systems $\mathcal{S}_{\mathrm{c}}$.

In this case the actions (1.4), (1.6), (1.16) contain additional term $\mathcal{A}_{\text{int}}$, describing interaction of dynamic systems $\mathcal{S}_{\text{c}}$.

Application of three different methods of description of the same dynamic system is useful in the relation, that this admits one to distinguish between the properties of dynamic system and those of the description method. It is very important for investigation of the dynamic system properties. All properties, that are common for the three methods of descriptions attribute to dynamic system in itself. If one uses only one method of description, for instance, WF-description, it is rather difficult to distinguish between what attributes to the wave function properties and what attributes to the properties of the dynamic system investigated. For instance, it is common practice to think that the wave function is an attribute of quantum dynamic system, and that the wave function is a fundamental concept, connected with quantum systems, whereas the wave function is only a method of description, which can be applied to any dynamic system (classical or quantum). Use of three different methods of description is useful also in the sense, that L-description is best suited for interpretation. The HDP-description is best suited to realize the D-disquantization and to derive classical analog. The WF-description is best suited to solve dynamic equations, because sometimes they are linear in WF-representation.

Let us list the properties of the three different descriptions in application to the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\text{c}}\right]$.

First, the three representations of the action (1.4), (1.6) and (1.12) for $\mathcal{E}\left[\mathcal{S}_{\text{c}}\right]$ contain derivatives only in one direction. The action (1.4) contains only derivatives with respect to t, the action (1.6) contains derivatives only in the direction of the flux n-vector $j^i = \{\rho, \rho\mathbf{v}\}$.

$$-\rho\left(\mathbf{vp}+p_0\right) = b_0 j^k \left(\partial_k \varphi + g^\alpha\left(\boldsymbol{\tau}\right)\right) \partial_k \tau_\alpha \tag{1.17}$$

In the action (1.16) this fact also takes place, but it is not evident, because it does not contain the variables j^i explicitly.

Second, the state of the ensemble $\mathcal{E}\left[\mathcal{S}_{\text{c}}\right]$ is determined only within the transformation (1.5). The relabeling group is very powerful, and ambiguity of description is very large. It is especially valid for the wave function ψ, whose meaning is very obscure. Obscurity of the wave function ψ is greater, than that of hydrodynamic potentials $\boldsymbol{\tau}$, because the arbitrary functions g^α are "hidden" inside the wave function ψ. To simplify the WF-description, one can impose some constraints on the choice of the wave function. Conventionally one demands that the wave function would satisfy a linear dynamic equation. It is not clear whether such a constraint could be imposed always, but in many physically interesting cases this is possible. Such constraints simplify dealing with wave functions. Nevertheless, meaning of the wave function remains obscure. To work with wave functions, considered to be fundamental objects of theory, one has worked out a system of rules (quantum principles).

Third, the properties of the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\text{c}}\right]$ do not depend on the number of elements $\mathcal{S}_{\text{c}}$ in $\mathcal{E}\left[\mathcal{S}_{\text{c}}\right]$. The actions (1.6), (1.12) have respectively the following property

$$\mathcal{A}_{\mathrm{E}}[\alpha\rho, \mathbf{v}, \varphi, \boldsymbol{\tau}] = \alpha\mathcal{A}_{\mathrm{E}}[\rho, \mathbf{v}, \varphi, \boldsymbol{\tau}], \qquad \mathcal{A}_{\mathrm{E}}[\sqrt{\alpha}\psi, \sqrt{\alpha}\psi^*] = \alpha\mathcal{A}_{\mathrm{E}}[\psi, \psi^*] \tag{1.18}$$

where α is an arbitrary positive constant.

Dynamic quantization (D-quantization) is an addition of a term $\mathcal{A}_{\mathrm{int}}$ to the action $\mathcal{A}_{\mathrm{E}}$ for $\mathcal{E}\left[\mathcal{S}_{\mathrm{c}}\right]$. The term $\mathcal{A}_{\mathrm{int}}$ contains transversal derivatives $\partial_{\perp}^{l}$, defined by relations

$$\partial_{||}^{k} \equiv \frac{j^k j^s}{\rho^2}\partial_s, \qquad \partial_{\perp}^{k} \equiv \partial^k - \frac{j^k j^s}{\rho^2}\partial_s, \qquad \partial^k = \partial_{\perp}^{k} + \partial_{||}^{k}, \qquad \rho^2 \equiv j^s j_s \tag{1.19}$$

The transversal derivatives $\partial_{\perp}^{l}$ describe interaction between the elements $\mathcal{S}_{\mathrm{c}}$ of the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{c}}\right]$. Dynamic equations for $\mathcal{E}\left[\mathcal{S}_{\mathrm{c}}\right]$ stop to be ordinary differential equations and form a system of partial differential equations. As a consequence of D-quantization the ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{c}}\right]$ turns to $\mathcal{E}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{c}}\right] \equiv \mathcal{S}_{\mathrm{q}}\left[\mathcal{S}_{\mathrm{c}}\right]$ which describes a set of interacting dynamic systems $\mathcal{S}_{\mathrm{c}}$. From formal viewpoint the D-quantization is a very general dynamic procedure, which do not contain any conventional attributes of quantization such as linear differential equations, quantum constant $\hbar$, linear operators etc. Nevertheless, under some conditions the D-quantization leads to he same results, as the conventional Q-quantization.

The dynamic system $\mathcal{E}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{c}}\right] \equiv \mathcal{S}_{\mathrm{q}}\left[\mathcal{S}_{\mathrm{c}}\right]$ remains to be a statistical ensemble, because it continues to satisfy the condition (1.18). The dynamic system $\mathcal{S}_{\mathrm{q}}\left[\mathcal{S}_{\mathrm{c}}\right]$ may be considered to be a set of similar independent stochastic systems $\mathcal{S}_{\mathrm{st}}$, i.e. $\mathcal{S}_{\mathrm{q}}\left[\mathcal{S}_{\mathrm{c}}\right] \equiv \mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$. There is no dynamic equations for a single $\mathcal{S}_{\mathrm{st}}$, but dynamic equations exist for statistical ensemble $\mathcal{S}_{\mathrm{q}}\left[\mathcal{S}_{\mathrm{c}}\right] \equiv \mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$, and investigating $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$, one studies the stochastic systems $\mathcal{S}_{\mathrm{st}}$. In fact, investigation of dynamic properties of the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$ is the only way of investigation of stochastic systems $\mathcal{S}_{\mathrm{st}}$. One can study distributions of different physical quantities and their evolution in the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$, and make some conclusions about properties of $\mathcal{S}_{\mathrm{st}}$ on this base.

Let us show properties of D-quantization in the example of the free classical particle, which is described by the Lagrangian L and Hamiltonian H, defined by

$$L\left(x, \mathbf{v}\right) = -mc^2 + \frac{m\mathbf{v}^2}{2}, \qquad H\left(x, \mathbf{p}\right) = mc^2 + \frac{\mathbf{p}^2}{2m} \tag{1.20}$$

The interaction term $\mathcal{A}_{\mathrm{int}}$ is supposed to have the form

$$\mathcal{A}_{\mathrm{int}}[\rho, \mathbf{v}, \varphi, \boldsymbol{\tau}] = -\int \rho\frac{\mathbf{p}_{\mathrm{st}}^2}{2m}\mathrm{d}^{n+1}x, \qquad \mathbf{p}_{\mathrm{st}} = -\frac{\hbar}{2}\boldsymbol{\nabla}\ln\rho \tag{1.21}$$

Formally this term coincides with the Bohm potential energy [5]. Interpretation of this term is as follows. The motion of free particles is supposed to be stochastic, and $\mathbf{p}_{\mathrm{st}} = -\frac{\hbar}{2}\boldsymbol{\nabla}\ln\rho$ describes the mean momentum of the stochastic motion, where ρ is a collective variable

$$\rho = \frac{\partial\left(\tau_1, \tau_2, \tau_3\right)}{\partial\left(x^1, x^2, x^3\right)}, \tag{1.22}$$

describing density of elements $\mathcal{S}_{\rm c}$ in the ensemble $\mathcal{E}\left[\mathcal{S}_{\rm c}\right]$. To take into account the stochasticity influence on the regular motion, the energy of the mean stochastic motion, described by (1.21), should be added to the action (1.6). Let us consider for simplicity the case, when the flow in $\mathcal{S}_{\rm q}\left[\mathcal{S}_{\rm c}\right]$ is irrotational and one can set $g^{\alpha}=0, \quad \alpha=1,2,3$ [4]. In this case the wave function ψ can have one component. Then the actions (1.4), (1.6) and (1.12) for $\mathcal{E}\left[\mathcal{S}_{\rm c}\right]$ take respectively the form

$$\mathcal{S}_{\rm q}\left[\mathcal{S}_{\rm c}\right]: \qquad \mathcal{A}_{\rm q}\left[\mathbf{x},\mathbf{u}_{\rm st}\right]=\int\left\{-mc^2+\frac{m\dot{\mathbf{x}}^2}{2}+\frac{m\mathbf{u}_{\rm st}^2}{2}-\frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}_{\rm st}\right\}dtd\boldsymbol{\tau} \tag{1.23}$$

where $\mathbf{x}=\mathbf{x}\left(t,\boldsymbol{\tau}\right)$ and $\mathbf{u}_{\rm st}=\mathbf{u}_{\rm st}\left(t,\mathbf{x}\right)$ is a new dynamic variable, describing stochastic component of motion.

$$\mathcal{S}_{\rm q}\left[\mathcal{S}_{\rm c}\right]: \qquad \mathcal{A}_{\rm q}[\rho,\varphi]=\int\rho\{-mc^2-\frac{b_0^2\left(\boldsymbol{\nabla}\varphi\right)^2}{2m}-\frac{\hbar^2}{8m}\left(\boldsymbol{\nabla}\ln\rho\right)^2-b_0\partial_0\varphi\}\mathrm{d}^4x, \tag{1.24}$$

$$\mathcal{S}_{\rm q}\left[\mathcal{S}_{\rm c}\right]: \qquad \mathcal{A}_{\rm q}[\psi,\psi^*]=\int\{\frac{ib_0}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-\frac{b_0^2}{2m}\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi$$
$$-mc^2\rho+\frac{b_0^2-\hbar^2}{8\rho m}\left(\nabla\rho\right)^2\}\mathrm{d}^4x, \qquad \rho\equiv\psi^*\psi \tag{1.25}$$

Interaction between the elements $\mathcal{S}_{\rm c}$ of the reduced ensemble $\mathcal{S}_{\rm q}\left[\mathcal{S}_{\rm c}\right]$ is described in actions (1.24), (1.25) by means of terms at the factor $\hbar^2$. In the action (1.23) interaction is described by means of additional dependent dynamic variable $\mathbf{u}_{\rm st}$, which depends on variables $\{t,\mathbf{x}\}$, but not on $\{t,\boldsymbol{\tau}\}$, and varying the action with respect to $\mathbf{u}_{\rm st}$, one has to use $\{t,\mathbf{x}\}$ as independent variables

$$\int(\frac{m\mathbf{u}_{\rm st}^2}{2}-\frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}_{\rm st})dtd\boldsymbol{\tau}\rightarrow\int(\frac{m\mathbf{u}_{\rm st}^2}{2}-\frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}_{\rm st})\rho dtd\mathbf{x}, \qquad \rho\equiv\frac{\partial\left(\tau_1,\tau_2,\tau_3\right)}{\partial\left(x^1,x^2,x^3\right)}$$

Then variation with respect to $\mathbf{u}_{\rm st}$ leads to the dynamic equation

$$m\rho\mathbf{u}_{\rm st}+\frac{\hbar}{2}\boldsymbol{\nabla}\rho=0, \qquad \mathbf{u}_{\rm st}=-\frac{\hbar}{2m}\boldsymbol{\nabla}\ln\rho \tag{1.26}$$

which is equivalent to (1.21). Thus, the term $m\mathbf{u}_{\rm st}^2/2$ in (1.21) describes the energy of stochastic component of motion, whereas the term $-\hbar\boldsymbol{\nabla}\mathbf{u}_{\rm st}/2$ describes interaction of the stochastic component with regular one.

The action (1.25) of WF-description generates, in general, nonlinear dynamic equation for ψ. But if $b_0^2=\hbar^2$, the nonlinear term vanishes, and the dynamic equation becomes linear. The constant b_0 is an arbitrary constant, and nothing prevents from setting $b_0=\hbar$.

Let us set $b_0=\hbar$. The action (1.25) transforms to the action

$$\mathcal{A}_{\rm q}[\psi,\psi^*]=\int\{\frac{i\hbar}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-\frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi-mc^2\psi^*\psi\}\mathrm{d}^4x \tag{1.27}$$

which generates linear dynamic equation

$$i\hbar\partial_0\psi+\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi-mc^2\psi=0 \qquad (1.28)$$

The equation (1.28) turns to conventional Schrödinger equation after transformation $\psi\to\psi\exp\left(-imc^2t/\hbar\right)$. Equivalency of relations (1.24) with $b_0=\hbar$ and (1.27) has been known for a long time [6, 7].

Let us compare the actions (1.25) and (1.27). The action (1.25) contains only one quantum term, i.e. the term, containing quantum constant $\hbar$. If one sets $\hbar=0$, the action (1.25) turns to the action of the form (1.16), describing noninteracting particles $\mathcal{S}_{\mathrm{c}}$. In the action (1.27) all terms (except for the last one) are quantum. If one sets $\hbar=0$ in the action (1.27), it begin to describe nothing, because all terms except for the last one vanish. To obtain the classical description from the action (1.28), one is to apply subtle methods of classical approximation.

The quantum constant $\hbar$ plays different role in conventional quantum mechanics and in MCQP. In QM the quantum constant $\hbar$ is an attribute of Q-quantization procedure in the sense that $\hbar$ appears together with Q-quantization and disappears together with Q-disquantization. In MCQP D-quantization and D-disquantization do not depend on $\hbar$ directly, although D-quantization may contain $\hbar$ as a parameter. In MCQP the constant $\hbar$ is a characteristic of some stochastic agent, which is an origin of stochastic behavior of microparticles. The constant $\hbar$ as well as the speed of the light c may be considered to be some characteristic of space-time [8, 9].

By definition D-quantization is an addition of derivatives transversal to the flux vector j^k, D-disquantization is elimination of derivatives transversal to j^k. Both procedures are possible only in HDP-representation, where the flux vector j^k is an explicit dependent variable. In the relativistic case the Q-quantization may introduce additional degrees of freedom, containing both longitudinal and transversal derivatives. This circumstance is discovered at the D-disquantization.

The Klein-Gordon equation is the dynamic equation for dynamic system $\mathcal{S}_{\mathrm{KG}}$. It is written usually in WF-representation. Being written in L-representation, the action for the dynamic system $\mathcal{S}_{\mathrm{KG}}$ has the form [1]

$$\mathcal{S}_{\mathrm{KG}}:\qquad \mathcal{A}_{\mathrm{L}}\left[x,\kappa\right]=\int -mcK\sqrt{\dot{x}^k\dot{x}_k}d\tau_0 d\boldsymbol{\tau},\qquad K=\sqrt{1+\lambda^2\left(\kappa_l\kappa^l+\partial_l\kappa^l\right)} \qquad (1.29)$$

where $\kappa^l=\kappa^l\left(x\right),\ \ l=0,1,2,3$ are functions of argument $x=\left\{x^k\left(\tau_0,\boldsymbol{\tau}\right)\right\},\ \ k=0,1,2,3,\ \lambda=\hbar/mc$ is the Compton wavelength. The action (1.29) is a relativistic generalization of the relation (1.23). One can verify this, expanding expression for K over degrees of λ^2, neglecting time component of κ^l and setting $\mathbf{u}_{\mathrm{st}}=\hbar\boldsymbol{\kappa}/m$. The field κ in (1.29) as well as the mean stochastic velocity $\mathbf{u}_{\mathrm{st}}$ in (1.23) describes stochastic agent, which is an origin of stochastic behavior of microparticles. But there is an essential difference in the two descriptions. Description in terms of $\mathbf{u}_{\mathrm{st}}$ does not contain time derivatives, and the mean stochastic velocity $\mathbf{u}_{\mathrm{st}}$ describes some field, coupled rigidly with its source (particle). Description in terms of κ contains time derivatives, and the field κ can exist independently of its source (particle). Both fields $\mathbf{u}_{\rm st}$ and κ describe cloud of pairs around the particle. The cloud, described by $\mathbf{u}_{\rm st}$, cannot move away from the particle, whereas the pair cloud, described by the field κ, can. In the case (1.23) one can unite the particle and cloud of pairs, described by $\mathbf{u}_{\rm st}$ into one pointlike object and consider it to be a clothed particle. In the case (1.29) one cannot join particle with the cloud of pairs, because the cloud can exist separately from the particle, taking a part of its energy.

Formally all this follows from dynamic equation for the field κ, which has the form

$$\kappa_l = -\frac{1}{2}\partial_l \ln \rho, \qquad \rho \equiv \sqrt{\frac{\dot{x}^k \dot{x}_k}{1+\lambda^2\left(\kappa_l \kappa^l + \partial_l \kappa^l\right)} \frac{\partial\left(\tau_0, \tau_1, \tau, \tau_3\right)}{\partial\left(x^0, x^1, x^2, x^3\right)}} \tag{1.30}$$

The dynamic equation contains a collective variable ρ, describing distribution of elements in the statistical ensemble, associated with the dynamic system $\mathcal{S}_{\rm KG}$. The field of stochastic velocities $\mathbf{u}_{\rm st}$ is also connected with similar collective variable. One can ignore this collective variable in the nonrelativistic case (1.23), because the cloud of pairs is coupled rigidly with the particle. Besides, the variable ρ, defined by (1.22), contains only transversal derivatives, because in nonrelativistic case all space derivatives are transversal. But one cannot ignore the collective variable ρ in the relativistic case (1.29), because the cloud of pairs can exist separately from the particle. The pair production is a corollary of independent existence of the pair cloud (but not of the quantization procedure). It can be shown also mathematically [1]. Collective (statistical) character of the κ field, responsible for pair production, does not permit one to separate one classical dynamic system from the statistical ensemble, as it is possible in the nonrelativistic case, when the pair production is neglected.

The Dirac particle is the dynamic system $\mathcal{S}_{\rm D}$, described by the Dirac equation. The Dirac dynamic system $\mathcal{S}_{\rm D}$ was investigated by many researchers. There is no possibility to list all them, and we mention only some of them. First, this is transformation of the Dirac equation on the base of quantum mechanics [10, 11]. The complicated structure of Dirac particle was discovered by Schrödinger [12], who interpreted it as some complicated quantum motion (zitterbewegung). Investigation of this quantum motion and different models of Dirac particle can be found in [13, 14, 15, 16, 17] and references therein. Our investigation differs in absence of any suppositions on the Dirac particle model and in a use of only dynamic methods.

The goal of investigation is a construction of dynamic system $\mathcal{S}_{\rm dc}$, whose dynamic equations form a system of ordinary differential equations. Further the dynamic system $\mathcal{S}_{\rm dc}$, will be referred to as classical Dirac particle. It has finite number of degrees of freedom, and is simpler for investigation, than $\mathcal{S}_{\rm D}$. D-disquantization of $\mathcal{S}_{\rm D}$ determines the classical dynamic system $\mathcal{S}_{\rm dc}$ uniquely.

The first step of the investigation is transformation of WF-description into HDP-description (the second and the third sections). In the HDP-representation one realizes D-disquantization, transforming $\mathcal{S}_{\rm D}$ into $\mathcal{S}_{\rm dc}$. It is the second step, described in the fourth section. Writing actions for $\mathcal{S}_{\rm D}$ and $\mathcal{S}_{\rm dc}$ in the relativistically covariant form in the HDP-representation, one discovers that both actions contain constant timelike 4-vector f^k, describing splitting of space-time into space and time. It means that both dynamic systems $\mathcal{S}_{\mathrm{D}}$ and $\mathcal{S}_{\mathrm{dc}}$ are nonrelativistic. This unexpected difficulty is discussed in the fifth section. One succeeded to overcome this obstacle, slightly modifying the dynamic system $\mathcal{S}_{\mathrm{dc}}$. The unit timelike 4-vector f^k is identified with the constant 4-velocity of $\mathcal{S}_{\mathrm{dc}}$ (sixth section). After such a modification the dynamic system $\mathcal{S}_{\mathrm{dc}}$ turns to relativistic dynamic system $\mathcal{S}_{\mathrm{dcr}}$. In the seventh section one solves dynamic equations for $\mathcal{S}_{\mathrm{dcr}}$. In the eighth section one considers relativistic rotator $\mathcal{S}_{\mathrm{rr}}$ and investigates its property. Comparing $\mathcal{S}_{\mathrm{dcr}}$ and $\mathcal{S}_{\mathrm{rr}}$, one discovers that $\mathcal{S}_{\mathrm{dcr}}$ is a relativistic rotator and determines the rigidity function of $\mathcal{S}_{\mathrm{dcr}}$. Calculations, connected with transformation from WF-representation to HDP-representation, are rather bulky. They are presented in mathematical appendices in detail. Unfortunately, detailed presentation of these calculations is necessary, because their result (nonrelativistic character of $\mathcal{S}_{\mathrm{dc}}$) seems to be doubtful for most of readers.

2 Transformation of variables

Considering dynamic system $\mathcal{S}_{\mathrm{D}}$, we set for simplicity that the speed of the light $c = 1$. $\mathcal{S}_{\mathrm{D}}$ is described in WF-representation by the action

$$\mathcal{S}_{\mathrm{D}}: \qquad \mathcal{A}_{\mathrm{D}}[\bar{\psi}, \psi] = \int (-m\bar{\psi}\psi + \frac{i}{2}\hbar\bar{\psi}\gamma^l\partial_l\psi - \frac{i}{2}\hbar\partial_l\bar{\psi}\gamma^l\psi)d^4x \tag{2.1}$$

Here ψ is four-component complex wave function, $\bar{\psi} = \psi^*\gamma^0$ is conjugate wave function, and ψ^* is the Hermitian conjugate one. γ^i, $i = 0,1,2,3$ are 4×4 complex constant matrices, satisfying the relation

$$\gamma^l\gamma^k + \gamma^k\gamma^l = 2g^{kl}I, \qquad k,l = 0,1,2,3. \tag{2.2}$$

where I is unit 4×4 matrix. The action (2.1) generates dynamic equation for the dynamic system $\mathcal{S}_{\mathrm{D}}$, known as Dirac equation

$$i\hbar\gamma^l\partial_l\psi - m\psi = 0 \tag{2.3}$$

and expressions for physical quantities: the 4-flux j^k of particles and energy-momentum tensor T_l^k

$$j^k = \bar{\psi}\gamma^k\psi, \qquad T_l^k = \frac{i}{2}\left(\bar{\psi}\gamma^k\partial_l\psi - \partial_l\bar{\psi}\cdot\gamma^k\psi\right) \tag{2.4}$$

The state of dynamic system $\mathcal{S}_{\mathrm{D}}$ is described by eight real dependent variables (eight real components of four-component complex wave function ψ). Using designations

$$\gamma_5 = \gamma^{0123} \equiv \gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.5}$$

$$\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_3,\} = \{-i\gamma^2\gamma^3, -i\gamma^3\gamma^1, -i\gamma^1\gamma^2\} \tag{2.6}$$

let us make the change of variables

$$\psi = Ae^{i\varphi+\frac{1}{2}\gamma_5\kappa}e^{-\frac{i}{2}\gamma_5\boldsymbol{\sigma\eta}}e^{\frac{i\pi}{2}\boldsymbol{\sigma}}\Pi \tag{2.7}$$

$$\psi^* = A\Pi e^{-\frac{i\pi}{2}\boldsymbol{\sigma}} e^{-\frac{i}{2}\gamma_5\boldsymbol{\sigma}\boldsymbol{\eta}} e^{-i\varphi-\frac{1}{2}\gamma_5\kappa} \tag{2.8}$$

where (*) means the Hermitian conjugation, and

$$\Pi = \frac{1}{4}(1+\gamma^0)(1+\mathbf{z}\boldsymbol{\sigma}), \qquad \mathbf{z} = \{z^\alpha\} = \text{const}, \qquad \alpha = 1,2,3; \qquad \mathbf{z}^2 = 1 \tag{2.9}$$

is a zero divisor. The quantities A, κ, φ, $\boldsymbol{\eta} = \{\eta^\alpha\}$, $\mathbf{n} = \{n^\alpha\}$, $\alpha = 1,2,3$, $\mathbf{n}^2 = 1$ are eight real parameters, determining the wave function ψ. These parameters may be considered as new dependent variables, describing the state of dynamic system $\mathcal{S}_{\rm D}$. The quantity φ is a scalar, and κ is a pseudoscalar. Six remaining variables A, $\boldsymbol{\eta} = \{\eta^\alpha\}$, $\mathbf{n} = \{n^\alpha\}$, $\alpha = 1,2,3$, $\mathbf{n}^2 = 1$ can be expressed through the flux 4-vector $j^l = \bar{\psi}\gamma^l\psi$ and spin 4-pseudovector

$$S^l = i\bar{\psi}\gamma_5\gamma^l\psi, \qquad l = 0,1,2,3 \tag{2.10}$$

Because of two identities

$$S^l S_l \equiv -j^l j_l, \qquad j^l S_l \equiv 0. \tag{2.11}$$

there is only six independent components among eight components of quantities j^l, and S^l. Let us make a change of variables in the action (2.1), using substitution (2.7) – (2.9).

Matrices γ_5, $\boldsymbol{\sigma} = \{\sigma_\alpha\}$, $\alpha = 1,2,3$ are determined by relations (2.5), (2.6) have the following properties

$$\gamma_5\gamma_5 = -1, \qquad \gamma_5\sigma_\alpha = \sigma_\alpha\gamma_5, \qquad \gamma^{0\alpha} \equiv \gamma^0\gamma^\alpha = -i\gamma_5\sigma_\alpha, \qquad \alpha = 1,2,3; \tag{2.12}$$

$$\left(\gamma^0\right)^* = \gamma^0, \qquad (\gamma^\alpha)^* = -\gamma^\alpha, \qquad \gamma^0\boldsymbol{\sigma} = \boldsymbol{\sigma}\gamma^0, \qquad \gamma^0\gamma_5 = -\gamma_5\gamma^0 \tag{2.13}$$

According to equations (2.2), (2.5), (2.6) the matrices $\boldsymbol{\sigma} = \{\sigma_\alpha\}$, $\alpha = 1,2,3$ satisfy the relation

$$\sigma_\alpha\sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma, \qquad \alpha,\beta = 1,2,3 \tag{2.14}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the antisymmetric pseudo-tensor of Levi-Chivita ($\varepsilon_{123} = 1$).

Using relations (2.12),(2.13), (2.14) and (2.9), it is easy to verify that

$$\Pi^2 = \Pi, \qquad \gamma_0\Pi = \Pi, \qquad \mathbf{z}\boldsymbol{\sigma}\Pi = \Pi, \tag{2.15}$$

$$\Pi\gamma_5\Pi = 0, \qquad \Pi\sigma_\alpha\Pi = z^\alpha\Pi, \qquad \alpha = 1,2,3. \tag{2.16}$$

Generally, the wave functions ψ, ψ^* defined by (2.8) are 4×4 complex matrices. In the proper representation, where Π has the form

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.17}$$

the ψ, ψ^* have the form

$$\psi = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix}, \qquad \psi^* = \begin{pmatrix} \psi_1^* & \psi_2^* & \psi_3^* & \psi_4^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.18}$$

Their product $\psi^* O \psi$, where O is an arbitrary 4×4 matrix, has the form

$$\psi^* O \psi = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = a\Pi = \Pi a \tag{2.19}$$

where a is a complex quantity. If f is an analytical function having the property $f(0) = 0$, then the function $f(\psi^* O \psi) = f(a\Pi)$ of a 4×4 matrix of the type (2.19) is a matrix $f(a)\Pi$ of the same type. For this reason one will not distinguish between the complex quantity a and the complex 4×4 matrix $a\Pi$. In the final expressions of the type $a\Pi$ (a is a complex quantity) the multiplier Π will be omitted.

By means of relations (2.12) – (2.16), one can reduce any Clifford number $\Pi O \Pi$ to the form (2.19), without using any concrete form of the γ-matrix representation. This property will be used in our calculations. Calculating exponents of the type (2.7), (2.8), we shall use the following relations

$$e^{-\frac{i\pi}{2}\boldsymbol{\sigma}} F(\boldsymbol{\sigma})\, e^{\frac{i\pi}{2}\boldsymbol{\sigma}} = F(\boldsymbol{\Sigma})$$

where F is arbitrary function and

$$\boldsymbol{\Sigma} = \{\Sigma_1, \Sigma_2, \Sigma_3\}, \qquad \Sigma_\alpha = e^{-\frac{i\pi}{2}\boldsymbol{\sigma}} \sigma_\alpha e^{\frac{i\pi}{2}\boldsymbol{\sigma}} \qquad \alpha = 1, 2, 3; \tag{2.20}$$

satisfies the same commutation relations (2.14) as the Pauli matrices $\boldsymbol{\sigma}$.

For variables $\bar{\psi}\psi$, j^l, S^l, $l = 0, 1, 2, 3$ one has the following expressions

$$\bar{\psi}\psi = \psi^* \gamma^0 \psi = A^2 \Pi e^{\gamma_5 \kappa} \Pi = A^2 \Pi (\cos\kappa + \gamma_5 \sin\kappa) \Pi = A^2 \cos\kappa \Pi$$

Taking into account the first relation (2.16), the term linear with respect to γ_5 vanishes, and one obtains

$$\bar{\psi}\psi = A^2 \cos\kappa \Pi$$

$$\begin{aligned} j^0 \Pi &= \bar{\psi}\gamma^0\psi = A^2 \Pi e^{-\frac{i\pi}{2}\boldsymbol{\sigma}} e^{-i\gamma_5 \boldsymbol{\sigma}\boldsymbol{\eta}} e^{\frac{i\pi}{2}\boldsymbol{\sigma}} \Pi \\ &= A^2 \Pi e^{-i\gamma_5 \boldsymbol{\Sigma}\boldsymbol{\eta}} \Pi = A^2 \Pi \left(\cosh\eta - \frac{i\gamma_5}{\eta} \boldsymbol{\Sigma}\boldsymbol{\eta} \sinh\eta \right) \Pi \\ &= A^2 \cosh(\eta) \Pi \end{aligned} \tag{2.21}$$

where

$$\eta = \sqrt{\boldsymbol{\eta}^2} = \sqrt{\eta^\alpha \eta^\alpha}$$

Again in force of the first relation (2.16) we omit terms linear with respect to γ_5.

In the same way one obtains

$$\begin{aligned}
j^\alpha\Pi &= \psi^*\gamma^{0\alpha}\psi\Pi = A^2\Pi e^{-\frac{i}{2}\gamma_5\mathbf{\Sigma}\boldsymbol{\eta}}(-i\gamma_5\Sigma_\alpha)e^{-\frac{i}{2}\gamma_5\mathbf{\Sigma}\boldsymbol{\eta}}\Pi = \\
&= A^2\Pi(\cosh\frac{\eta}{2} - i\gamma_5\mathbf{\Sigma v}\sinh\frac{\eta}{2})(-i\gamma_5\Sigma_\alpha)(\cosh\frac{\eta}{2} - i\gamma_5\mathbf{\Sigma v}\sinh\frac{\eta}{2})\Pi = \\
&= A^2\Pi(\cosh\frac{\eta}{2}\sinh\frac{\eta}{2}\left(\Sigma_\beta\Sigma_\alpha + \Sigma_\beta\Sigma_\alpha\right)v^\beta\Pi
\end{aligned}$$

$$j^\alpha\Pi = A^2\sinh(\eta)v^\alpha\Pi, \qquad \alpha = 1,2,3 \tag{2.22}$$

where

$$\mathbf{v} = \{v^\alpha\}, \qquad v^\alpha = \eta^\alpha/\eta, \qquad \alpha = 1,2,3; \qquad \mathbf{v}^2 = 1. \tag{2.23}$$

Let us introduce designation $\boldsymbol{\xi} = \{\xi^\alpha\}$, $\alpha = 1,2,3$ for the expression

$$\xi^\alpha\Pi = \Pi\Sigma_\alpha\Pi, \qquad \alpha = 1,2,3, \qquad \boldsymbol{\xi}^2 = \xi^\alpha\xi^\alpha = 1 \tag{2.24}$$

Then for the spin pseudovector S^l, defined by the relation (2.10), one obtains

$$\begin{aligned}
S^0\Pi &= \psi^*(-i\gamma_5)\psi = A^2\Pi(-i\gamma_5)e^{-i\gamma_5\mathbf{\Sigma}\boldsymbol{\eta}}\Pi = \\
&= A^2\Pi\sinh(\eta)\mathbf{\Sigma v}\Pi = A^2\sinh(\eta)\boldsymbol{\xi}\mathbf{v}\Pi,
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
S^\alpha\Pi &= \psi^*\gamma^0 i\gamma_5\gamma^\alpha\psi = \Pi\psi^*\sigma_\alpha\psi\Pi = A^2\Pi e^{-\frac{i}{2}\gamma_5\mathbf{\Sigma}\boldsymbol{\eta}}\Sigma_\alpha e^{-\frac{i}{2}\gamma_5\mathbf{\Sigma}\boldsymbol{\eta}}\Pi = \\
&= A^2\Pi(\cosh\frac{\eta}{2} - i\gamma_5\mathbf{\Sigma v}\sinh\frac{\eta}{2})\Sigma_\alpha(\cosh\frac{\eta}{2} - i\gamma_5\mathbf{\Sigma v}\sinh\frac{\eta}{2})\Pi = \\
&= A^2\Pi\left(\cosh^2\frac{\eta}{2}\Sigma_\alpha + \sinh^2\frac{\eta}{2}\left(\Sigma_\beta v^\beta\right)\Sigma_\alpha\left(\Sigma_\gamma v^\gamma\right)\right)\Pi
\end{aligned}$$

Now twice using relations (2.14) for Pauli matrices Σ_α, one derives

$$\begin{aligned}
S^\alpha\Pi &= A^2\Pi\left(\cosh^2\frac{\eta}{2}\Sigma_\alpha + \sinh^2\frac{\eta}{2}v^\beta v^\gamma\left(\delta_{\alpha\beta} + i\varepsilon_{\beta\alpha\mu}\Sigma_\mu\right)\Sigma_\gamma\right)\Pi = \\
&= A^2\Pi\left(\cosh^2\frac{\eta}{2}\Sigma_\alpha + \sinh^2\frac{\eta}{2}\left(v^\alpha v^\gamma\Sigma_\gamma + i\varepsilon_{\beta\alpha\mu}v^\beta v^\gamma\left(\delta_{\mu\gamma} + i\varepsilon_{\mu\gamma\nu}\Sigma_\nu\right)\right)\right)\Pi \\
&= A^2\Pi\left(\cosh^2\frac{\eta}{2}\Sigma_\alpha + \sinh^2\frac{\eta}{2}\left(v^\alpha v^\gamma\Sigma_\gamma - v^\beta v^\beta\Sigma_\alpha + v^\beta v^\alpha\Sigma_\beta\right)\right)\Pi
\end{aligned}$$

$$S^\alpha\Pi = A^2[\xi^\alpha + (\cosh\eta - 1)v^\alpha(\mathbf{v}\boldsymbol{\xi})]\Pi, \qquad \alpha = 1,2,3. \tag{2.26}$$

It follows from equations (2.21), (2.22), (2.23)

$$j^i j_i\Pi = A^4\Pi, \qquad A = (j^l j_l)^{1/4} \equiv \rho^{1/2} \tag{2.27}$$

According to the third equation (2.15), (2.20) and (2.24) one obtains

$$\begin{aligned}
\xi^\alpha\Pi &= \Pi\Sigma_\alpha\Pi = \Pi e^{-\frac{i\pi}{2}\boldsymbol{\sigma}}\sigma_\alpha e^{\frac{i\pi}{2}\boldsymbol{\sigma}}\Pi = \\
&= \Pi\left(\cos\frac{\pi}{2} - i\boldsymbol{\sigma}\mathbf{n}\sin\frac{\pi}{2}\right)\sigma_\alpha\left(\cos\frac{\pi}{2} + i\boldsymbol{\sigma}\mathbf{n}\sin\frac{\pi}{2}\right)\Pi = \\
&= \Pi\left(\boldsymbol{\sigma}\mathbf{n}\right)\sigma_\alpha\left(\boldsymbol{\sigma}\mathbf{n}\right)\Pi = \Pi n^\mu n^\nu\sigma_\mu\sigma_\alpha\sigma_\nu\Pi = \\
&= \Pi\left(n^\alpha n^\nu\sigma_\nu + i\varepsilon_{\mu\alpha\gamma}\sigma_\gamma\sigma_\nu n^\mu n^\nu\right)\Pi = \Pi\left(n^\alpha n^\nu\sigma_\nu - \varepsilon_{\mu\alpha\gamma}\varepsilon_{\gamma\nu\beta}\sigma_\beta n^\mu n^\nu\right)\Pi = \\
&= \Pi\left(n^\alpha n^\nu z_\nu - \varepsilon_{\mu\alpha\gamma}\varepsilon_{\gamma\nu\beta}z^\beta n^\mu n^\nu\right)\Pi \\
&= \left(n^\alpha\left(\mathbf{nz}\right) + \left(\mathbf{n}\times\left(\mathbf{n}\times\mathbf{z}\right)\right)^\alpha\right)\Pi, \qquad \alpha = 1,2,3;
\end{aligned}$$

Or

$$\boldsymbol{\xi} = 2\mathbf{n}(\mathbf{nz}) - \mathbf{z} \tag{2.28}$$

3 Transformation of the action

The last two terms of the action (2.1) may be written in the form

$$\begin{aligned}
\frac{i}{2}\hbar\bar{\psi}\gamma^l\partial_l\psi + \text{h.c} &= \frac{i}{2}\hbar\psi^*\left(\partial_0 - i\gamma_5\boldsymbol{\sigma}\boldsymbol{\nabla}\right)\psi + \text{h.c} \\
&= \frac{i}{2}\hbar\psi^*\left(\left(\partial_0 - i\gamma_5\boldsymbol{\sigma}\boldsymbol{\nabla}\right)\left(i\varphi + \frac{1}{2}\gamma_5\kappa\right)\right)\psi + \text{h.c.} \\
&\quad + \frac{i}{2}\hbar A^2\Pi e^{-\frac{i\pi}{2}\boldsymbol{\sigma}}e^{-\frac{i}{2}\gamma_5\boldsymbol{\sigma}\boldsymbol{\eta}}(\partial_0 - i\gamma_5\boldsymbol{\sigma}\nabla)(e^{-\frac{i}{2}\gamma_5\boldsymbol{\sigma}\boldsymbol{\eta}}e^{\frac{i\pi}{2}\boldsymbol{\sigma}})\Pi + \text{h.c}
\end{aligned}$$

where "h.c." means the term obtained from the previous one by the Hermitian conjugation. Calculation of this expression gives the following result (see details of calculation in Appendix A). Let us set

$$\frac{i}{2}\hbar\bar{\psi}\gamma^l\partial_l\psi + \text{h.c} = F_1 + F_2 + F_3 + F_4 \tag{3.1}$$

where

$$F_1 + F_2 = -j^l\partial_l\varphi\Pi - \frac{1}{2}\hbar S^l\partial_l\kappa\Pi \tag{3.2}$$

$$F_3 = -\frac{\hbar j^l}{2\left(1 + \boldsymbol{\xi}\mathbf{z}\right)}\varepsilon_{\alpha\beta\gamma}\xi^\alpha\partial_l\xi^\beta z^\gamma\Pi \tag{3.3}$$

$$F_4 = -\frac{1}{2}\hbar A^2\varepsilon_{\alpha\beta\gamma}\left(\partial_\alpha\eta v^\beta + \sinh\eta\partial_\alpha v^\beta + 2\sinh^2(\frac{\eta}{2})v^\alpha\partial_0 v^\beta\right)\xi^\gamma\Pi \tag{3.4}$$

where $\varepsilon_{\alpha\beta\gamma}$ is 3-dimensional Levi-Chivita pseudotensor.

We see that the expressions (A.6) for F_1 and F_2 as well as the first term of the action (2.1)

$$-m\bar{\psi}\psi = -m\Pi e^{\gamma_5\kappa}\Pi = -mA^2\cos\kappa\Pi = -m\sqrt{j^l j_l}\cos\kappa\Pi \equiv -m\rho\cos\kappa\Pi \tag{3.5}$$

have relativistically covariant form. The terms F_3 and F_4 have non-covariant form, and we try to write them in a covariant form. All these terms contain non-covariant three-dimensional Levi-Chivita pseudotensor $\varepsilon_{\alpha\beta\gamma}$. It can be considered as spatial components of the 4-dimensional Levi-Chivita pseudotensor ε_{iklm} ($\varepsilon_{0123} = 1$), convoluted with the constant timelike unit vector $f^l = \{1,0,0,0\}$. Then only spatial components of $\varepsilon_{iklm}f^m$ do not vanish

$$\varepsilon_{\alpha\beta\gamma} = -\varepsilon_{\alpha\beta\gamma m}f^m, \qquad \alpha,\beta,\gamma = 1,2,3 \tag{3.6}$$

and one may substitute relation (3.6) in expression (3.3) for F_3.

$$F_3 = \frac{\hbar j^l}{2\,(1+\boldsymbol{\xi}\mathbf{z})}\varepsilon_{iklm}\xi^i\partial_l\xi^k z^l f^m \Pi \tag{3.7}$$

where ξ^l, z^l are 4-pseudovectors, whose spatial components in the considered coordinate system are $\boldsymbol{\xi}$ and $\mathbf{z}$, and temporal components are of no importance. To write the scalar product $\boldsymbol{\xi}\mathbf{z}$ in a covariant form, one sets $z^0 = 0$ in the considered coordinate system.

Let us introduce constant 4-vector f^l and constant 4-pseudovector z^l by means of relations

$$f^i = \{1,0,0,0\}, \qquad z^i = \{0, z^1, z^2, z^3\}$$

Then $\boldsymbol{\xi}\mathbf{z} = -\xi_l z^l$, and one can rewrite the relation (3.7) in the covariant form

$$F_3 = \hbar j^l \varepsilon_{iklm}\frac{\xi^i}{\sqrt{2\,(1-\xi^s z_s)}}\partial_l \frac{\xi^k}{\sqrt{2\,(1-\xi^s z_s)}} z^l f^m \Pi \tag{3.8}$$

One may introduce factor $[2\,(1-\xi^s z_s)]^{-1/2}$ under sign of derivative, because differentiation of it gives 0 in virtue of the vanishing factor $\varepsilon_{iklm}\xi^i\xi^k z^l f^m = 0$.

Finally, introducing unit 4-pseudovector

$$\nu^i = \xi^i - (\xi^s f_s) f^i, \qquad i = 0,1,2,3; \qquad \nu^i\nu_i = -1,$$

and 4-pseudovector

$$\mu^i \equiv \frac{\nu^i}{\sqrt{-(\nu^l+z^l)(\nu_l+z_l)}} = \frac{\nu^i}{\sqrt{2(1-\nu^l z_l)}} = \frac{\nu^i}{\sqrt{2(1+\boldsymbol{\xi}\mathbf{z})}}.$$

one can rewrite the expression for F_3 in the form

$$F_3 = \hbar j^l \varepsilon_{iklm}\mu^i\partial_l\mu^k z^l f^m \Pi \tag{3.9}$$

Transformation of the expression (3.4) for F_4 to covariant form is rather complicated. As a result of this transformation the expression (3.4) takes the form (See proof of this fact in Appendix B)

$$F_4 = -\frac{\hbar}{2(\rho + f^s j_s)}\varepsilon_{iklm}[\partial^k(j^i + f^i\rho)](j^l + f^l\rho)[\xi^m - (\xi^s f_s) f^m] \tag{3.10}$$

Let us introduce the unit timelike vector

$$q^i \equiv \frac{j^i + f^i\rho}{\sqrt{(j^l+f^l\rho)(j_l+f_l\rho)}} = \frac{j^i + f^i\rho}{\sqrt{2\rho(\rho + j^l f_l)}}, \qquad q_s q^s = 1$$

Then the relation (3.10) can be written shortly in a covariant form

$$F_4 = \hbar\rho\varepsilon_{iklm} q^i(\partial^k q^l)\nu^m \tag{3.11}$$

Now one can write the action (2.1) in the covariant form

$$\mathcal{S}_{\mathrm{D}}: \qquad \mathcal{A}_D[j,\varphi,\kappa,\boldsymbol{\xi}] = \int \mathcal{L} d^4x, \qquad \mathcal{L} = \mathcal{L}_{cl} + \mathcal{L}_{q1} + \mathcal{L}_{q2} \tag{3.12}$$

$$\mathcal{L}_{cl} = -m\rho - \hbar j^i \partial_i \varphi + \hbar j^s \varepsilon_{iklm} \mu^i \partial_s \mu^k z^l f^m, \qquad \rho \equiv \sqrt{j^l j_l} \tag{3.13}$$

$$\mathcal{L}_{q1} = 2m\rho \sin^2(\frac{\kappa}{2}) - \frac{\hbar}{2} S^l \partial_l \kappa, \tag{3.14}$$

$$\mathcal{L}_{q2} = \hbar\rho \varepsilon_{iklm} q^i (\partial^k q^l) \nu^m \tag{3.15}$$

Lagrangian is a function of 4-vector j^l, scalar φ, pseudoscalar κ, and unit 3-pseudovector $\boldsymbol{\xi}$, which is connected with the spin 4-pseudovector S^l by means of the relation

$$\xi^\alpha = \rho^{-1} \left[S^\alpha - \frac{j^\alpha S^0}{(j^0+\rho)} \right], \qquad \alpha = 1,2,3; \qquad \rho \equiv \sqrt{j^l j_l} \tag{3.16}$$

$$S^0 = \mathbf{j}\boldsymbol{\xi}, \qquad S^\alpha = \rho \xi^\alpha + \frac{(\mathbf{j}\boldsymbol{\xi}) j^\alpha}{\rho + j^0}, \qquad \alpha = 1,2,3 \tag{3.17}$$

4 Dynamic disquantization and dynamic quantization

Let $\mathcal{S}_{\mathrm{st}}$ be some stochastic system, and stochasticity of $\mathcal{S}_{\mathrm{st}}$ be a result of action of some stochastic agent on deterministic dynamic system $\mathcal{S}_{\mathrm{d}}$. Let $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ and $\mathcal{E}[\mathcal{S}_{\mathrm{d}}]$ be statistical ensembles respectively of stochastic systems $\mathcal{S}_{\mathrm{st}}$ and deterministic dynamic systems $\mathcal{S}_{\mathrm{d}}$. Any statistical ensemble is a distributed system, and all physical quantities $u = u(\mathbf{x})$ are, in general, some functions of coordinates $\mathbf{x}$ of the configuration space, where $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ is described. If the state of the statistical ensemble is such one, that the quantities u do not depend on $\mathbf{x}$, such a state will be referred to as uniform state. If the quantities u depend on $\mathbf{x}$ very slightly, such a state will be referred to as quasi-uniform.

Dynamic equations for the statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ describe the regular component of motion of stochastic system $\mathcal{S}_{\mathrm{st}}$, constituting the statistical ensemble. Stochastic component of motion of $\mathcal{S}_{\mathrm{st}}$ influences the regular component of motion, provided the the ensemble state is not uniform, and there are gradients of physical quantities u. Dynamic equations for ensembles $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ and $\mathcal{E}[\mathcal{S}_{\mathrm{d}}]$ coincide, if their state is uniform and gradients of u in dynamic equations for $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ vaniish. For instance, dynamic equations for the statistical ensemble of Brownian particles have the form

$$\frac{\partial w}{\partial t} = \boldsymbol{\nabla}(D\boldsymbol{\nabla} w - \mathbf{u}w), \qquad \mathbf{v} = \mathbf{u} - D\boldsymbol{\nabla}\ln w$$

where distribution $w = w(\mathbf{x})$ describes the state of the ensemble. The velocity $\mathbf{u} = \mathbf{u}(\mathbf{x})$ describes motion of the viscous medium, where the Brownian particles move. If the state w is uniform, and w =const, then $\mathbf{v} = \mathbf{u}$, and the velocity $\mathbf{v}$ of regular motion of stochastic Brownian particles coincide with the velocity $\mathbf{u}$ of free particles, moving with a friction in the viscous medium.

In the case of non-relativistic quantum system $\mathcal{S}_{\mathrm{S}}$, described by the Schrödinger equation, the system $\mathcal{S}_{\mathrm{S}}$ may be considered to be a dynamic system of the type $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$. The state of $\mathcal{S}_{\mathrm{S}}$ is quasi-uniform, if all physical quantities change slightly at the de Broglie wavelength. In this case there exists such a coordinate system and such a state of $\mathcal{S}_{\mathrm{S}}$, where

$$\lambda_{\mathrm{B}}\left|\frac{\partial u}{\partial x^{\alpha}}\right| \ll |u|, \qquad \alpha=1,2,3, \qquad \lambda_{\mathrm{B}}=\frac{\hbar}{mv} \tag{4.1}$$

and the quantum dynamic system may be considered to be a classical one, in the sense that dynamic equations for $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$ coicide with those for $\mathcal{E}\left[\mathcal{S}_{\mathrm{d}}\right]$. Here u is any physical quantity and λ_{B} is the de Broglie wavelength. In the relativistic case one cannot be sure, in general, that there are such a coordinate system and such a state of the dynamic system, where the condition (4.1) is fulfilled.

Let us introduce a local version of the condition (4.1), which is written in the form

$$\frac{\hbar}{mc}\sqrt{\frac{\left(j_k\partial_{\perp}^k u\right)^2}{j^s j_s}}=\frac{\hbar}{mc}\sqrt{\partial_k u\cdot\partial^k u-\frac{(j^k\partial_k u)^2}{j^s j_s}} \ll |u| \tag{4.2}$$

where c is the speed of the light, j^k is the 4-vector of the particle flux, and $\partial_{\perp}^k$ is transversal component of derivative, defined by the relation (1.19). The condition (4.2) is somewhat slighter, than the condition (4.1), because the Compton wavelength $\hbar/mc \leq \lambda_{\mathrm{B}}$. At the same time the condition (4.2) is relativistically covariant and local.

Let us introduce the procedure of dynamic disquantization (D-disquantization). It is a special relativistically covariant procedure which realizes the constraint (4.2). Any derivative ∂^k is separated into longitudinal component $\partial_{||}^k$ and transversal one $\partial_{\perp}^k$, where $\partial_{||}^k$ and $\partial_{\perp}^k$ are defined by the relation (1.19). The transversal component is neglected, and one obtains the dynamic system in the quasi-uniform state.

For the quasi-uniform state, when one can neglect transversal derivatives, the action (3.12)–(3.14) takes the form

$$\begin{aligned}\mathcal{A}_{\mathrm{Dqu}}[j,\varphi,\kappa,\boldsymbol{\xi}] \;=\; & \int\{-m\rho\cos\kappa-\hbar j^i\left(\partial_i\varphi-\frac{\varepsilon_{jklm}\xi^j\partial_i\xi^k z^l f^m}{2\left(1+\boldsymbol{\xi}\mathbf{z}\right)}\right) \\ & -\frac{\hbar j^i}{2\rho(\rho+j^j f_j)}\varepsilon_{klsm}j^k\partial_i j^l f^s\xi^m\}d^4x\end{aligned} \tag{4.3}$$

Note that the second term $-\frac{\hbar}{2}S^l\partial_l\kappa$ in the relation (3.14) is neglected, because 4-pseudovector S^k is orthogonal to 4-vector j^k, and the derivative is transversal.

Although the action (4.3) contains a non-classical variable κ, but in fact this variable is a constant. Indeed, a variation with respect to κ leads to the dynamic equation

$$\frac{\delta\mathcal{A}_{Dqu}}{\delta\kappa}=m\rho\sin\kappa=0 \tag{4.4}$$

which has solutions

$$\kappa = n\pi, \qquad n = \text{integer} \tag{4.5}$$

Thus, the effective mass $m_{\text{eff}} = m \cos\kappa$ has two values

$$m_{\text{eff}} = m \cos\kappa = \pm m \tag{4.6}$$

The value $m_{\text{eff}} = m > 0$ ($\kappa = \frac{\pi}{2} + 2n\pi$) corresponds to a minimum of the action (4.3), whereas the value $m_{\text{eff}} = -m < 0$ corresponds to a maximum. Apparently, $m_{\text{eff}} > 0$ corresponds to a stable ensemble state, and $m_{\text{eff}} < 0$ does to unstable state.

Eliminating κ by means of the substitution $k = \frac{\pi}{2} + 2n\pi$ in (4.3), one obtains the action

$$\mathcal{A}_{\text{Dqu}}[j,\varphi,\boldsymbol{\xi}] = \int \{-m\rho - \hbar j^i \left(\partial_i\varphi - \frac{\varepsilon_{jklm}\xi^j\partial_i\xi^k z^l f^m}{2\,(1+\boldsymbol{\xi}\mathbf{z})}\right)$$

$$-\frac{\hbar j^i}{2\rho(\rho + j^j f_j)}\varepsilon_{klsm} j^k \partial_i j^l f^s \xi^m\} d^4x \tag{4.7}$$

Let us introduce Lagrangian coordinates $\tau = \{\tau_0, \boldsymbol{\tau}\} = \{\tau_i(x)\}$, $i = 0,1,2,3$ as functions of coordinates x in such a way that only coordinate τ_0 changes along the direction j^l, i.e.

$$j^k\partial_k\tau_\mu = 0, \qquad \mu = 1,2,3 \tag{4.8}$$

Considering coordinates x to be a functions of $\tau = \{\tau_0, \boldsymbol{\tau}\}$, one has the following identities

$$\frac{\partial D}{\partial \tau_{0,k}}\tau_{i,k} \equiv \delta_i^0 D, \qquad i = 0,1,2,3 \qquad \tau_{i,k} \equiv \partial_k\tau_i, \qquad i,k = 0,1,2,3 \tag{4.9}$$

where

$$D \equiv \frac{\partial(\tau_0,\tau_1,\tau_2,\tau_3)}{\partial(x^0,x^1,x^2,x^3)}, \qquad \frac{\partial D}{\partial\tau_{0,i}} \equiv \frac{\partial(x^i,\tau_1,\tau_2,\tau_3)}{\partial(x^0,x^1,x^2,x^3)}. \tag{4.10}$$

Comparing (4.8) with (4.9), one concludes that it is possible to set

$$j^i = \frac{\partial D}{\partial\tau_{0,i}} \equiv \frac{\partial(x^i,\tau_1,\tau_2,\tau_3)}{\partial(x^0,x^1,x^2,x^3)}, \qquad i = 0,1,2,3 \tag{4.11}$$

because the dynamic equation

$$\frac{\delta\mathcal{A}_{\text{Dqu}}}{\delta\varphi} = \hbar\partial_l j^l = 0 \tag{4.12}$$

is satisfied by the relation (4.11) identically in force of identity

$$\partial_i \frac{\partial D}{\partial\tau_{k,i}} \equiv 0, \qquad k = 0,1,2,3.$$

Let us take into account that for any variable u

$$D^{-1}j^i\partial_i u = D^{-1}\frac{\partial D}{\partial \tau_{0,i}}\partial_i u = \frac{\partial(u,\tau_1,\tau_2,\tau_3)}{\partial(\tau_0,\tau_1,\tau_2,\tau_3)} = \frac{du}{d\tau_0} \tag{4.13}$$

and in particular,

$$D^{-1}j^i = D^{-1}\frac{\partial D}{\partial \tau_{0,i}} \equiv \frac{\partial(x^i,\tau_1,\tau_2,\tau_3)}{\partial(\tau_0,\tau_1,\tau_2,\tau_3)} = \frac{dx^i}{d\tau_0} \equiv \dot{x}^i, \qquad i = 0,1,2,3 \tag{4.14}$$

Besides

$$d^4x = D^{-1}d^4\tau = D^{-1}d\tau_0 d\boldsymbol{\tau} \tag{4.15}$$

$$j^i\partial_i\varphi = \frac{\partial(\varphi,\tau_1,\tau_2,\tau_3)}{\partial(x^0,x^1,x^2,x^3)} \tag{4.16}$$

The action (4.7) can be rewritten in the Lagrangian coordinates τ in the form

$$\mathcal{A}_{\mathrm{Dqu}}[x,\boldsymbol{\xi}] = \int \{-m\sqrt{\dot{x}^i\dot{x}_i} + \hbar\frac{(\dot{\boldsymbol{\xi}}\times\boldsymbol{\xi})\mathbf{z}}{2(1+\boldsymbol{\xi}\mathbf{z})} + \hbar\frac{(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\boldsymbol{\xi}}{2\sqrt{\dot{x}^s\dot{x}_s}(\sqrt{\dot{x}^s\dot{x}_s}+\dot{x}^0)}\}d^4\tau \tag{4.17}$$

where the dot means the total derivative $\dot{x}^s \equiv dx^s/d\tau_0$. $x = \{x^0,\mathbf{x}\} = \{x^i\}$, $i = 0,1,2,3$, $\boldsymbol{\xi} = \{\xi^\alpha\}$, $\alpha = 1,2,3$ are considered to be functions of the Lagrangian coordinates τ_0, $\boldsymbol{\tau} = \{\tau_1,\tau_2,\tau_3\}$. $\mathbf{z}$ is the constant unit 3-vector (2.9). The term $j^i\partial_i\varphi$ is omitted, because it reduces to a Jacobian (4.16), which does not contribute to dynamic equations. In fact, variables x depend on $\boldsymbol{\tau}$ as on parameters, because the action (4.17) does not contain derivatives with respect to τ_α, $\alpha = 1,2,3$. Lagrangian density of the action (4.17) does not contain independent variables τ explicitly. Hence, it may be written in the form

$$\mathcal{A}_{\mathrm{Dqu}}[x,\boldsymbol{\xi}] = \int \mathcal{A}_{\mathrm{dc}}[x,\boldsymbol{\xi}]d\boldsymbol{\tau}, \qquad \mathbf{d}\boldsymbol{\tau} = d\tau_1 d\tau_2 d\tau_3 \tag{4.18}$$

where

$$\mathcal{A}_{\mathrm{dc}}[x,\boldsymbol{\xi}] = \int \{-m\sqrt{\dot{x}^i\dot{x}_i} + \hbar\frac{(\dot{\boldsymbol{\xi}}\times\boldsymbol{\xi})\mathbf{z}}{2(1+\boldsymbol{\xi}\mathbf{z})} + \hbar\frac{(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\boldsymbol{\xi}}{2\sqrt{\dot{x}^s\dot{x}_s}(\sqrt{\dot{x}^s\dot{x}_s}+\dot{x}^0)}\}d\tau_0 \tag{4.19}$$

The action (4.18) is the action for the dynamic system $\mathcal{S}_{\mathrm{Dqu}}$, which is a set of similar independent dynamic systems $\mathcal{S}_{\mathrm{dc}}$. Such a dynamic system is called a statistical ensemble. Dynamic systems $\mathcal{S}_{\mathrm{dc}}$ are elements (constituents) of the statistical ensemble $\mathcal{E}_{\mathrm{Dqu}}$. Dynamic equations for each $\mathcal{S}_{\mathrm{dc}}$ form a system of ordinary differential equations. It may be interpreted in the sense that the dynamic system $\mathcal{S}_{\mathrm{dc}}$ may be considered to be a classical one, although its Lagrangian contains the quantum constant $\hbar$.

The first term in the action (4.19) is relativistic. It describes a motion of classical Dirac particle as a whole. The last two terms in the action (4.19) are nonrelativistic. They describe some internal degrees of freedom of the classical Dirac particle. This internal motion (classical zitterbewegung) means that the classical Dirac particle has some internal structure which is described by a method incompatible with relativity principles. Maybe, the classical Dirac particle should be considered to be consisting of several pointlike particles. At any rate the classical Dirac particle is not a pointlike particle. It has a more complicated structure which is described by the variable $\boldsymbol{\xi}$ and by the second order derivative $\ddot{\mathbf{x}}$.

It is easy to see that the action (4.19) is invariant with respect to transformation $\tau_0 \to \tilde{\tau}_0 = F(\tau_0)$, where F is an arbitrary monotone function. This transformation admits one to choose the variable $t = x^0$ as a parameter τ_0, or to choose the parameter τ_0 in such a way that $\dot{x}^l \dot{x}_l = \dot{x}_0^2 - \dot{\mathbf{x}}^2 = 1$. In the last case the parameter τ_0 is the proper time along the world line of classical Dirac particle.

One can introduce the dynamic quantization (D-quantization) as a procedure reciprocal to the D-disquantization. From mathematical viewpoint D-quantization means introduction of transversal derivatives into the action. Such an introduction of transversal derivatives means appearance in the action of some additional terms, describing interaction between the independent systems $\mathcal{S}_{\rm dc}$ of the statistical ensemble $\mathcal{E}_{\rm Dqu}$. There is a lot of different ways to introduce terms, containing transversal derivatives. Each such introduction means D-quantization of classical system. It may be considered to be an alternative to canonical quantization, which is determined uniquely by the Hamiltonian of the system and can be performed only for Hamiltonian dynamic system. D-quantization is not unique. It is chosen on the base of some model (physical) reasoning.

Application of D-quantization to the statistical ensemble $\mathcal{E}_{\rm Dqu}$ means that elements $\mathcal{S}_{\rm dc}$ of the statistical ensemble $\mathcal{E}_{\rm Dqu}$ becomes to interact between themselves through some "quantum vector field" κ^i and "scalar quantum field" κ. Of course, the dynamic quantization is not unique, because one can introduce quantum fields κ and κ^i in different ways. But the quantum fields κ^i and κ can be introduced in such a way, that the dynamic system, formed by the set of interacting identical dynamic systems $\mathcal{S}_{\rm dc}$, coincides with the dynamic system $\mathcal{S}_{\rm D}$. The D-quantization is a purely dynamic procedure which is relativistically covariant. It admits one to reduce a statistical ensemble $\mathcal{S}_{\rm Dqu}$ of classical dynamic systems to a quantum system $\mathcal{S}_{\rm D}$ by means of dynamic methods, i.e. without a reference to attributes of quantum mechanics.

Let us derive κ and κ^i fields for the dynamic system $\mathcal{S}_{\rm D}$. If one adds the omitted terms

$$m\rho\left(1-\cos\kappa\right) - \frac{\hbar}{2} S^l \partial_l \kappa, \qquad \hbar\rho\varepsilon_{iklm} q^i \left(\partial_\perp^k q^l\right) \nu^m$$

to the Lagrangian density of (4.7), one returns to the action (3.12) – (3.15) for $\mathcal{S}_{\rm D}$. Let us make this. One obtains the action (3.12) – (3.15). Let us introduce new variables

$$\kappa^i = q^i = \frac{j^i + \rho f^i}{\sqrt{2\rho(\rho + j^s f_s)}} = \frac{j^i + \rho f^i}{\sqrt{(j^s + \rho f^s)(j_s + \rho f_s)}}, \qquad i = 0, 1, 2, 3; \qquad (4.20)$$

and introduce them in the Lagrangian density (3.15) by means of the Lagrangian multipliers λ_i. Then $\mathcal{L}_{q2}$ is substituted by

$$\mathcal{L}'_{q2} = -\hbar\rho\varepsilon_{iklm}(\partial_\perp^k\kappa^l)\kappa^i\nu^m - \frac{\hbar}{\rho}j^s\varepsilon_{iklm}q^ij^k\partial_s q^l\nu^m + \lambda_i\left(q^i - \kappa^i\right) \tag{4.21}$$

where $\partial_\perp^k$ is defined by (1.19) and

$$\rho \equiv \sqrt{j^l j_l}, \qquad \nu^m = [\xi^m - (\xi^s f_s)f^m], \qquad m = 0,1,2,3 \tag{4.22}$$

The derivative ∂^k in (4.21) is separated into longitudinal component $\partial_{||}^k$ and transversal one $\partial_\perp^k$. The quantities q^i in the coefficient at $\partial_\perp^k$ are replaced by κ^i, in the coefficient at $\partial_{||}^k$ they are not changed. According to (4.20) $\kappa^i = q^i$, and such a replacement does not change anything essentially.

Variation of the action with respect to κ^i leads to the dynamic equations

$$-\hbar\rho\varepsilon_{iklm}(\partial_\perp^k\kappa^l)\nu^m - \hbar\partial_\perp^{*k}\left(\rho\varepsilon_{iklm}\kappa^l\nu^m\right) + \lambda_i = 0 \tag{4.23}$$

where the operator $\partial_\perp^{*k}$ is defined by the relation

$$\partial_\perp^{*k}u = \partial^k u - \partial_s(\frac{j^k j^s}{\rho^2}u) \tag{4.24}$$

Eliminating λ_i from (4.21) by means of (4.23), one obtains instead of $\mathcal{L}'_{q2}$

$$\mathcal{L}_{q3} = -\hbar\rho\varepsilon_{iklm}(\kappa^l\partial_\perp^k\kappa^i - q^l\partial_\perp^k\kappa^i + \kappa^i\partial_\perp^k q^l)\nu^m - \frac{\hbar j^i}{2\rho(\rho + j^j f_j)}\varepsilon_{klsm}j^k\partial_i j^l f^s\xi^m \tag{4.25}$$

Now the action has the form

$$\mathcal{A}_D[j,\varphi,\kappa,\xi,\kappa^i] = \int(\mathcal{L}_{cl} + \mathcal{L}_{q1} + \mathcal{L}_{q3})d^4x \tag{4.26}$$

where the Lagrangian densities $\mathcal{L}_{cl}$, $\mathcal{L}_{q1}$, $\mathcal{L}_{q3}$ are determined by (3.13), (3.14), (4.25) respectively. Dynamic equations generated by action (4.26) are equivalent to dynamic equations, generated by actions (2.1) and (3.12) – (3.15). The action (4.26) generates the dynamic equation

$$\frac{\delta\mathcal{A}_D}{\delta\kappa^i} = -\hbar\varepsilon_{iklm}\{\rho\nu^m\partial_\perp^k(q^l - \kappa^l) + \partial_\perp^{*k}[\rho\nu^m(q^l - \kappa^l)]\} = 0 \tag{4.27}$$

where the operator $\partial_\perp^{*k}$ is defined by the relation (4.24). Resolving (4.27) with respect to κ^i and substituting the κ^i into (4.26), one returns to the action (3.12) – (3.15). The fact that the solution (4.20) of (4.27) is not unique is of no importance, because (4.25) reduces to (3.15) by virtue of (4.27). Indeed, convoluting equation.(4.27) with κ^i and using the obtained relation for eliminating κ^i from (4.21), one obtains (3.15).

Let us note that κ^l are not rigorous dynamic variables, because the dynamic equations (4.27) for κ^l contain derivatives only along spacelike directions orthogonal to j^i. Rather the introduction of κ^i is an invariant (with respect to a change of variables) way of separating out the classical part of the action.

The variables κ, κ^i, $i = 0, 1, 2, 3$ are special quantum variables, which are responsible for quantum effects, described by the Dirac equation. They are introduced in such a way that, when they vanish, all terms, containing transversal component of derivative vanish also, and $\mathcal{S}_{\mathrm{D}}$ reduces to $\mathcal{S}_{\mathrm{Dqu}}$. Indeed, let us suppress quantum variables κ, κ^i, $i = 0, 1, 2, 3$ in the action (4.25), (4.26). Then the action (4.26) reduces to the action (4.7).

5 Relativistic invariance

It is a common practice to think that if dynamic equations of a system can be written in the relativistically covariant form, such a possibility provides automatically relativistic character of considered dynamic system, described by these equations. In general, it is valid only in the case, when dynamic equations do not contain absolute objects, or these absolute objects has the Poincare group as a group of their symmetry [18]. The absolute object is one or several quantities, which are the same for all states of the dynamic system [18]. A given external field, or metric tensor (when it is given, but not determined from the gravitational equations) are examples of absolute objects. In the case of dynamic system $\mathcal{S}_{\mathrm{D}}$, described by the Dirac equation the Dirac γ-matrices are absolute objects.

Anderson [18] investigated in details the role of absolute objects for symmetry of dynamic systems. His conclusion is as follows. If a dynamic system is described by dynamic equations, written in covariant form, the symmetry group of dynamic system is determined by the symmetry group of these absolute objects. Here we confirm this result in a simple example, when dynamic equations of certainly nonrelativistic dynamic system are written in a relativistically covariant form.

Let us consider a system of differential equations, consisting of the Maxwell equations for the electromagnetic tensor F^{ik} in some inertial coordinates x

$$\partial_k F^{ik}(x) = 4\pi J^i, \qquad \varepsilon_{iklm} g^{jm} \partial_j F^{kl}(x) = 0, \qquad \partial_k \equiv \frac{\partial}{\partial x^k} \tag{5.1}$$

and equations

$$m\frac{d}{d\tau}[(l_k \dot{q}^k)^{-1} \dot{q}^i - \frac{1}{2} g^{ik} l_k (l_j \dot{q}^j)^{-2} \dot{q}^s g_{sl} \dot{q}^l] = eF^{il}(q) g_{lk} \dot{q}^k; \qquad i = 0, 1, 2, 3 \tag{5.2}$$

$$\dot{q}^k \equiv \frac{dq^k}{d\tau} \tag{5.3}$$

where $q^i = q^i(\tau)$, $i = 0, 1, 2, 3$ describe coordinates of a pointlike charged particle as functions of a parameter τ, l_i is a constant timelike unit vector,

$$g^{ik} l_i l_k = 1; \tag{5.4}$$

and the speed of the light $c = 1$.

This system of equations is relativistically covariant with respect to quantities q^i, F^{ik}, J^i, l_i, g_{ik}, i.e. it does not change its form at any Lorentz transformation, which is accompanied by corresponding transformation of quantities q^i, F^{ik}, J^i, l_i, g_{ik}, where the quantities q^i, J^i, l_i transform as 4-vectors and the quantities F^{ik}, g_{ik} as 4-tensors

The reference to the quantities q^i, F^{ik}, J^i, l_i, g_{ik} means that all these quantities are considered as formal dependent variables, when one compares the form of dynamic equations written in two different coordinate systems. For instance, if a reference to J^i is omitted in the formulation of the relativistic covariance, it means that components of J^i are considered as some functions of the coordinates x. If $J^i \neq 0$, then J^i and $\tilde{J}^i$ in other coordinate system are different functions of the arguments x and $\tilde{x}$ respectively, and the first equation (5.1) has different form in different coordinate systems. In other words, the dynamic equations (5.1)–(5.2) are not relativistically covariant with respect to quantities q^i, F^{ik}, l_i, g_{ik}. Thus, for the relativistic covariance it is important both the laws of transformation and how each of quantities is considered as a formal variable, or as some function of coordinates.

Following Anderson [18] we divide the quantities q^i, F^{ik}, J^i, l_i, g_{ik} into two parts: dynamic objects (variables) q^i, F^{ik} and absolute objects J^i, l_i, g_{ik}. By definition of absolute objects they have the same value for all solutions of the dynamic equations, whereas dynamic variables are different, in general, for different solutions. If the dynamic equations are written in the relativistically covariant form, their symmetry group (and a compatibility with the relativity principles) is determined by the symmetry group of the absolute objects J^i, l_i, g_{ik}.

Let for simplicity $J^i \equiv 0$. A symmetry group of the constant timelike vector l_i is a group of rotations in the 3-plane orthogonal to the vector l_i. The Lorentz group is a symmetry group of the metric tensor g_{ik} =diag $\{1,-1,-1,-1\}$. Thus, the symmetry group of all absolute objects l_i, g_{ik}, $J^i \equiv 0$ is a subgroup of the Lorentz group (rotations in the 3-plane orthogonal to l_i). As far as the symmetry group is a subgroup of the Lorentz group and does not coincide with it, the system of equations (5.1)–(5.2) is nonrelativistic (incompatible with the relativity principles).

Of course, the compatibility with the relativity principles does not depend on the fact with respect to which quantities the relativistic covariance is considered. For instance, let us consider a covariance of equations (5.1), (5.2) with respect to quantities q^i, F^{ik}, $J^i \equiv 0$. It means that now l_i are to be considered as functions of x (in the given case these functions are constants), because a reference to l_i as a formal variables is absent. After the transformation to another coordinate system the equation (5.2) takes the form

$$m\frac{d}{d\tau}[(\tilde{l}_k\frac{d\tilde{q}^k}{d\tau})^{-1}\frac{d\tilde{q}^i}{d\tau}-\frac{1}{2}g^{ik}\tilde{l}_k(\tilde{l}_j\frac{d\tilde{q}^j}{d\tau})^{-2}\frac{d\tilde{q}^s}{d\tau}\frac{d\tilde{q}_s}{d\tau}]=e\tilde{F}^{il}(\tilde{q})g_{lk}\frac{d\tilde{q}^k}{d\tau} \qquad (5.5)$$

Here $\tilde{l}_i$ are considered as functions of $\tilde{x}$. But $\tilde{l}_i$ are other functions of $\tilde{x}$, than l_i of x (other numerical constants $\tilde{l}_k = l_j\partial x^j/\partial\tilde{x}^k$ instead of l_k), and equations (5.2) and (5.5) have different forms with respect to quantities q^i, F^{ik}, $J^i \equiv 0$. It means that (5.2) is not relativistically covariant with respect to q^i, F^{ik}, $J^i \equiv 0$, although it is relativistically covariant with respect to q^i, F^{ik}, l_i, $J^i \equiv 0$.

Setting $l_i = \{1, 0, 0, 0\}$, $t = q^0(\tau)$ in (5.2), one obtains

$$m\frac{d^2 q^\alpha}{dt^2} = eF^\alpha_{.0} + eF^\alpha_{.\beta}\frac{dq^\beta}{dt}, \qquad i = \alpha = 1, 2, 3; \tag{5.6}$$

$$\frac{m}{2}\frac{d}{dt}\left(\frac{dq^\alpha}{dt}\frac{dq^\alpha}{dt}\right) = eF^\alpha_{.0}\frac{dq^\alpha}{dt}, \qquad i = 0.$$

It is easy to see that this equation describes a nonrelativistic motion of a charged particle in a given electromagnetic field F^{ik}. The fact that the equations (5.2) or (5.6) are nonrelativistic is connected with the space-time splitting into space and time that is characteristic for Newtonian mechanics. This space-time splitting is described in different ways in equations (5.2) and (5.6). It is described by the constant timelike vector l_i in (5.2). In the equation (5.6) the space-time splitting is described by a special choice of the coordinate system whose time axis is directed along the vector l^i.

This example shows that nonrelativistic equation (5.6) can be written in a relativistically covariant form (5.2), provided one introduces an absolute object l_i, describing space-time splitting.

Dirac matrices γ^k are absolute objects, as well as the metric tensor g^{kl}, which may be considered as a derivative absolute object determined by the relation (2.2).

There are two approaches to the Dirac equation. In the first approach [19, 20] the wave function ψ is considered to be a scalar function defined on the field of Clifford numbers γ^l,

$$\psi = \psi(x, \gamma)\Gamma, \qquad \overline{\psi} = \Gamma\overline{\psi}(x, \gamma), \tag{5.7}$$

where Γ is a constant nilpotent factor which has the property $\Gamma f(\gamma)\Gamma = a\Gamma$. Here $f(\gamma)$ is arbitrary function of γ^l and a is a complex number, depending on the form of the function f. Within such an approach ψ, $\bar{\psi}$ transform as scalars and γ^l transform as components of a 4-vector under the Lorentz transformations. In this case the symmetry group of γ^l is a subgroup of the Lorentz group, and $\mathcal{S}_{\rm D}$ is nonrelativistic dynamic system. Then the matrix vector γ^l describes some preferred direction in the space-time.

In the second (conventional) approach ψ is considered to be a spinor, and γ^l, $\;l = 0, 1, 2, 3$ are scalars with respect to the transformations of the Lorentz group. In this case the symmetry group of the absolute objects γ^l is the Lorentz group, and dynamic system $\mathcal{S}_{\rm D}$ is considered to be a relativistic dynamic system.

Of course, the approaches leading to incompatible conclusions cannot be both valid. At least, one of them is wrong. Analyzing the two approaches, Sommerfeld [20] considered the first approach to be more reasonable. In the second case the analysis is rather difficult due to non-standard transformations of γ^l and ψ under linear coordinate transformations T. Indeed, the transformation T for the vector $j^l = \bar{\psi}\gamma^l\psi$ has the form

$$\tilde{\bar{\psi}}\tilde{\gamma}^l\tilde{\psi} = \frac{\partial \tilde{x}^l}{\partial x^s}\bar{\psi}\gamma^s\psi, \tag{5.8}$$

where quantities marked by tilde mean quantities in the transformed coordinate system. This transformation can carried out by two different ways

$$1: \quad \tilde{\psi} = \psi, \qquad \tilde{\overline{\psi}} = \overline{\psi}, \qquad \tilde{\gamma}^l = \frac{\partial \tilde{x}^l}{\partial x^s}\gamma^s, \qquad l = 0,1,2,3 \tag{5.9}$$

$$2: \quad \tilde{\gamma}^l = \gamma^l, \qquad l = 0,1,2,3, \qquad \tilde{\psi} = S(\gamma,T)\psi, \qquad \tilde{\overline{\psi}} = \overline{\psi}S^{-1}(\gamma,T), \tag{5.10}$$

$$S^*(\gamma,T)\gamma^0 = \gamma^0 S^{-1}(\gamma,T) \tag{5.11}$$

The relations (5.9) correspond to the first approach and the relations (5.10) correspond to the second one. Both ways (5.9) and (5.10) lead to the same result, provided

$$S^{-1}(\gamma,T)\gamma^l S(\gamma,T) = \frac{\partial \tilde{x}^l}{\partial x^s}\gamma^s \tag{5.12}$$

In particular, for infinitesimal Lorentz transformation $x^i \to x^i + \delta\omega^i_{.k}x^k$ $S(\gamma,T)$ has the form [21]

$$S(\gamma,T) = \exp\left(\frac{\delta\omega_{ik}}{8}\left(\gamma^i\gamma^k - \gamma^k\gamma^i\right)\right) \tag{5.13}$$

The second way (5.10) has two defects. First, the transformation law of ψ depends on γ, i.e. under linear transformation T of coordinates the components of ψ transform through ψ and γ^l, but not only through ψ. Note that tensor components in a coordinate system transform only through tensor components in other coordinate system, and this transformation does not contain any absolute objects. (for instance, the relation (5.8)). Second, the relation (5.12) is compatible with (2.2) only under transformations T between orthogonal coordinate systems, when components $g^{lk} = \{1,-1,-1,-1\}$ of the metric tensor are invariant. In other words, at the second approach the relation (2.2) is not covariant, in general, with respect to arbitrary linear transformations of coordinates. In this case one cannot be sure that the symmetry group of the dynamic system coincides with the symmetry group of absolute objects.

The fact that the symmetry group of dynamic system coincides with the symmetry group of absolute objects was derived with the supposition, that under the coordinate transformation any object transforms only via its components. This condition is violated in the second case, and one cannot be sure that the symmetry group of dynamic system coincides with that of absolute objects.

After change of variables the action (2.1) transforms to the form (3.12) – (3.15), the γ-matrices being eliminated. But after reduction of the action to the relativistically covariant form two new absolute objects appear: constant 4-vector f^i and constant 4-pseudovector z^i. We shall see further that the 4-pseudovector z^i is fictitious. But the action is really depends on 4-vector f^i, which resembles the vector l_i in the considered example (5.2). It means that the dynamic system $\mathcal{S}_{\mathrm{D}}$ is nonrelativistic, because it supposes an absolute separation of the space-time into the space and the time.

The fact that the Dirac γ-matrices contain a constant timelike vector seems to be rather unexpected. As far as calculations leading to this result are rather bulky, they can cast some doubt upon their correctness. To remove this doubt, we present detailed calculations in the mathematical appendices. Besides a constant timelike vector appears to be "hidden" in Dirac γ-matrices in the case of two-dimensional Dirac equation, where corresponding calculations are essentially simpler [22].

6 Relativization of dynamic equations

Let us return to the action (4.19), which is written in the covariant form

$$\mathcal{S}_{\rm dc}:\mathcal{A}_{\rm dc}\left[x,\xi\right]=\int\{-m\sqrt{\dot{x}^i\dot{x}_i}-\hbar\frac{\varepsilon_{iklm}\xi^i\dot{\xi}^k f^l z^m}{2(1-\xi^s z_s)}-\frac{\hbar}{2}Q\varepsilon_{iklm}\dot{x}^i\ddot{x}^k f^l\xi^m\}d\tau_0 \tag{6.1}$$

where

$$Q=Q\left(\dot{x}\right)=\frac{1}{\sqrt{\dot{x}^s\dot{x}_s}(\dot{x}^l f_l+\sqrt{\dot{x}^l\dot{x}_l})} \tag{6.2}$$

To relativize this action, it is sufficient to make the 4-vector f^i to be a dynamic variable instead of an absolute object. Let us identify f^i with the constant 4-velocity vector

$$u_i=-\frac{P_i}{M},\qquad M=\sqrt{P_k P^k} \tag{6.3}$$

where P_k is the total momentum of the dynamic system $\mathcal{S}_{\rm dc}$, and M is the total mass of $\mathcal{S}_{\rm dc}$. Let L be Lagrangian of the action (6.1). Then the momentum has the form

$$\begin{aligned}P_i\ &=\ \frac{\partial L}{\partial\dot{x}^i}-\frac{d}{d\tau_0}\frac{\partial L}{\partial\ddot{x}^i}=-m\frac{\dot{x}_i}{\sqrt{\dot{x}^s\dot{x}_s}}-\frac{\hbar}{2}Q\varepsilon_{iklm}\ddot{x}^k f^l\xi^m\\ &-\frac{\hbar}{2}\frac{d}{d\tau_0}\left(Q\varepsilon_{iklm}\dot{x}^k\xi^m\right)f^l-\frac{\hbar}{2}\varepsilon_{sklm}\dot{x}^s\ddot{x}^k f^l\xi^m\frac{\partial Q}{\partial\dot{x}^i}\end{aligned} \tag{6.4}$$

Let us introduce designations

$$\begin{aligned}P_i\ &\equiv\ P_i\left(\dot{x},\xi,\dot{\xi},u\right)=-m\frac{\dot{x}_i}{\sqrt{\dot{x}^s\dot{x}_s}}-\hbar Q\varepsilon_{iklm}\ddot{x}^k u^l\xi^m\\ &-\frac{\hbar}{2}Q_s\ddot{x}^s\varepsilon_{iklm}\xi^m\dot{x}^k u^l-\frac{\hbar}{2}Q\varepsilon_{iklm}\dot{\xi}^m\dot{x}^k u^l-\frac{\hbar}{2}Q_i\varepsilon_{sklm}\dot{x}^s\ddot{x}^k u^l\xi^m\end{aligned} \tag{6.5}$$

where

$$Q\equiv Q\left(\dot{x},u\right)=\frac{1}{\sqrt{\dot{x}^s\dot{x}_s}(\dot{x}^l u_l+\sqrt{\dot{x}^l\dot{x}_l})} \tag{6.6}$$

$$Q_i\equiv Q_i\left(\dot{x},u\right)=\frac{\partial Q}{\partial\dot{x}^i}=-\frac{\dot{x}_i\left(\dot{x}^l u_l+2\sqrt{\dot{x}^l\dot{x}_l}\right)+\dot{x}_l\dot{x}^l u_i}{\left(\dot{x}^s\dot{x}_s\right)^{3/2}\left(\dot{x}^l u_l+\sqrt{\dot{x}^l\dot{x}_l}\right)^2},\qquad i=0,1,2,3 \tag{6.7}$$

Variation of the action (6.1) with respect to x^i leads to the dynamic equations

$$\frac{d}{d\tau_0}P_i\left(\dot{x},\xi,\dot{\xi},f\right)=0,\qquad i=0,1,2,3 \tag{6.8}$$

It means that the quantity P_i, defined by the equation (6.5) is constant P_i =const., $i=0,1,2,3$.

Let us introduce new dynamic variables u_i and M, defining them by the relations

$$Mu_i=-P_i\left(\dot{x},\xi,\dot{\xi},u\right),\qquad u_iu^i-1=0 \tag{6.9}$$

These constraints are added to the action (6.1) by means of the Lagrangian multipliers λ^i and η. Simultaneously we identify the constant 4-vector f^k with the 4-vector u^k, which is dynamic variable and at the same time u^k =const in force of dynamic equations (6.8) and designations (6.9). After identification $f^k=u^k$, the dynamic system $\mathcal{S}_{\rm dc}$ turns to a relativistic dynamic system $\mathcal{S}_{\rm dcr}$, which is described by the action

$$\begin{aligned}\mathcal{S}_{\rm dcr}\quad:\quad\mathcal{A}_{\rm dcr}\left[x,\xi,u,M,\lambda,\eta\right]=\int\Big\{-m\sqrt{\dot{x}^i\dot{x}_i}-\hbar\frac{\varepsilon_{iklm}\xi^i\dot{\xi}^ku^lz^m}{2(1-\xi^sz_s)}\\ -\frac{\hbar}{2}Q\varepsilon_{iklm}\dot{x}^i\ddot{x}^ku^lz^m+\lambda^i\left(Mu_i-P_i\right)+\eta\left(u_iu^i-1\right)\Big\}d\tau_0\end{aligned} \tag{6.10}$$

where P_i and Q are known functions (6.5), (6.6) of variables $\dot{x},u,\xi,\dot{\xi}$. Now the action (6.10) for the dynamic system $\mathcal{S}_{\rm dcr}$ do not contain absolute objects, and dynamic equations for $\mathcal{S}_{\rm dcr}$ appear to be compatible with the relativity principles. Dynamic equations for $\mathcal{S}_{\rm dcr}$ coincide with dynamic equations $\mathcal{S}_{\rm dc}$ in the special case, when the momentum P_i of the dynamic system $\mathcal{S}_{\rm dc}$, determined by the relations (6.4), is chosen in such a way, that $P_i=-\sqrt{P_sP^s}f_i$. It means that the systems $\mathcal{S}_{\rm dcr}$ and $\mathcal{S}_{\rm dc}$ coincide in the nonrelativistic approximation, when $f^i=\{1,0,0,0\}$. The procedure of elimination of the absolute object f^i and transformation of nonrelativistic dynamic system $\mathcal{S}_{\rm dc}$ to the relativistic dynamic system $\mathcal{S}_{\rm dcr}$ will be referred to as a relativization of classical Dirac particle $\mathcal{S}_{\rm dc}$.

Note that the relativization procedure may be applied to the quantum Dirac particle $\mathcal{S}_{\rm D}$ also. To make this, one should write the action $\mathcal{A}_{\rm Dqur}$ for the statistical ensemble $\mathcal{E}_{\rm Dqur}$ of dynamic systems $\mathcal{S}_{\rm dcr}$

$$\mathcal{S}_{\rm Dqur}:\quad\mathcal{A}_{\rm Dqur}\left[x,\xi,u,M,\lambda,\eta\right]=\int L_{\rm dcr}d\tau_0d\boldsymbol{\tau} \tag{6.11}$$

where $L_{\rm dcr}$ is Lagrangian in the action (6.10), and dependent dynamic variables x,ξ,u,M,λ,η are considered to be functions of independent variables $\tau=\{\tau_0,\boldsymbol{\tau}\}$. Thereafter the variables x^k are considered to be independent variables, and all other variables are considered to be dependent ones. The action (6.11) transforms to the form

$$\mathcal{S}_{\rm Dqur}:\quad\mathcal{A}_{\rm Dqur}\left[j,\xi,u,M,\lambda,\eta\right]=\int\mathcal{L}_{\rm dcr}d^4x \tag{6.12}$$

The action (6.12) differs from the action (6.11) as well as the action (4.7) differs from (4.17). Thereafter one can apply D-quantization to the action (6.12). It means that one should add Lagrangian densities $\mathcal{L}_{\mathrm{q1}}$ and $\mathcal{L}_{\mathrm{q3}}$, defined respectively by relations (3.14) and (4.25) (with f^k substituted by u^k), to the Lagrangian density $\mathcal{L}_{\mathrm{dcr}}$ in (6.12). The Lagrangian densities $\mathcal{L}_{\mathrm{q1}}$ and $\mathcal{L}_{\mathrm{q3}}$ contain transversal derivatives and describe quantum effects. One may return to description in terms of wave functions, but one cannot be sure that dynamic equations appear to be linear in terms of wave functions ψ.

7 Solution of dynamic equations for $\mathcal{S}_{\mathrm{dcr}}$

Having solved dynamic equations for $\mathcal{S}_{\mathrm{dcr}}$ at the coordinate system, where

$$u^k = \{1, 0, 0, 0\} \tag{7.1}$$

one can obtain solutions for other values of u^k =const by means of proper Lorentz transformations. At the values (7.1) dynamic equations for $\mathcal{S}_{\mathrm{dcr}}$ coincide with dynamic equations for $\mathcal{S}_{\mathrm{dc}}$, provided they are written for the case $P_i = \{P_0, 0, 0, 0\}$, $f^k = \{1, 0, 0, 0\}$. Thus, we shall solve dynamic equations, derived from the action (4.19), which is written for the case $f^k = \{1, 0, 0, 0\}$.

Variation of the action (4.19) with respect to $\mathbf{x}$ gives the dynamic equation

$$\frac{d}{d\tau_0}\left(-m\frac{\dot{\mathbf{x}}}{\sqrt{\dot{x}^s\dot{x}_s}} + \frac{\hbar Q}{2}(\boldsymbol{\xi}\times\ddot{\mathbf{x}}) - \frac{\hbar}{2}\frac{\partial Q}{\partial\dot{\mathbf{x}}}(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\boldsymbol{\xi} + \frac{\hbar}{2}\frac{d}{d\tau_0}\left(Q(\boldsymbol{\xi}\times\dot{\mathbf{x}})\right)\right) = 0 \tag{7.2}$$

where

$$Q = Q\left(\dot{x}\right) = \left(\sqrt{\dot{x}^s\dot{x}_s}(\sqrt{\dot{x}^s\dot{x}_s} + \dot{x}^0)\right)^{-1}, \qquad \dot{x}^s\dot{x}_s = \dot{x}_0^2 - \dot{\mathbf{x}}^2 \tag{7.3}$$

Varying the action (4.19) with respect to x^0, one obtains

$$\frac{d}{d\tau_0}\left(m\frac{\dot{x}^0}{\sqrt{\dot{x}^s\dot{x}_s}} - \frac{\hbar}{2}\frac{\partial Q}{\partial\dot{x}^0}(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\boldsymbol{\xi}\right) = 0 \tag{7.4}$$

Varying the action (4.19) with respect to $\boldsymbol{\xi}$, one should take into account the side constraint $\boldsymbol{\xi}^2 = 1$. Setting

$$\xi^\alpha = \frac{\zeta^\alpha}{\sqrt{\boldsymbol{\zeta}^2}}, \qquad \alpha = 1, 2, 3 \tag{7.5}$$

where $\boldsymbol{\zeta}$ is an arbitrary 3-pseudovector, one obtains

$$\frac{\delta\mathcal{A}_{\mathrm{dc}}}{\delta\zeta^\mu} = \frac{\delta\mathcal{A}_{\mathrm{dc}}}{\delta\xi^\alpha}\frac{\delta\xi^\alpha}{\delta\zeta^\mu} = \frac{\delta\mathcal{A}_{\mathrm{dc}}}{\delta\xi^\alpha}\frac{\delta^{\alpha\mu} - \xi^\alpha\xi^\mu}{\sqrt{\boldsymbol{\zeta}^2}} = 0 \tag{7.6}$$

It means that there are only two independent components of dynamic equation (7.6). These components are orthogonal to 3-pseudovector $\boldsymbol{\xi}$ and can be obtained from equation $\delta\mathcal{A}_{\rm dc}/\delta\xi^\alpha=0$ by means of vector product with $\boldsymbol{\xi}$.

$$-\hbar\frac{\left(\dot{\boldsymbol{\xi}}\times\mathbf{z}\right)\times\boldsymbol{\xi}}{2(1+\mathbf{z}\boldsymbol{\xi})}+\hbar\left(-\frac{d}{d\tau_0}\frac{(\boldsymbol{\xi}\times\mathbf{z})}{2(1+\mathbf{z}\boldsymbol{\xi})}-\frac{(\dot{\boldsymbol{\xi}}\times\boldsymbol{\xi})\mathbf{z}}{2(1+\mathbf{z}\boldsymbol{\xi})^2}\mathbf{z}\right)(\mathbf{z}\times\boldsymbol{\xi})+\hbar\frac{(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\times\boldsymbol{\xi}}{2}Q=0 \tag{7.7}$$

After transformations this equation reduces to the equation (see Appendix C)

$$\dot{\boldsymbol{\xi}}=-(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\times\boldsymbol{\xi}Q, \tag{7.8}$$

which does not contain the vector $\mathbf{z}$. It means that $\mathbf{z}$ determines a fictitious direction in the space-time. Note that $\mathbf{z}$ in the action (3.12) for the system $\mathcal{S}_{\rm D}$ is fictitious also, because the term containing $\mathbf{z}$ is the same in both actions (3.12) and (4.7) for $\mathcal{S}_{\rm D}$ and $\mathcal{S}_{\rm Dqu}$ respectively.

Let us choose the parameter τ_0 in such a way, that

$$\sqrt{\dot{x}^s\dot{x}_s}=\sqrt{\dot{x}_0^2-\dot{\mathbf{x}}^2}=1,\qquad \dot{x}_0=\sqrt{1+\dot{\mathbf{x}}^2} \tag{7.9}$$

Then, using the condition (7.9), one obtains

$$Q=\frac{1}{1+\dot{x}_0},\qquad \frac{\partial Q}{\partial\dot{x}_0}=-1,\qquad \frac{\partial Q}{\partial\dot{\mathbf{x}}}=\frac{\dot{\mathbf{x}}\,(2+\dot{x}_0)}{(1+\dot{x}_0)^2} \tag{7.10}$$

Integration of equation (7.4) leads to

$$m\dot{x}_0+\frac{\hbar}{2}\,(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\,\boldsymbol{\xi}=-p_0 \tag{7.11}$$

where p_0 is an integration constant.

Eliminating $(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\,\boldsymbol{\xi}$ from (7.2) by means of (7.11) and integrating it, one obtains after simplification

$$\begin{aligned}&\frac{d}{d\tau}\left(\frac{\dot{\mathbf{x}}}{\sqrt{1+\dot{x}_0}}\right)\times\boldsymbol{\xi}+\frac{1}{2}\frac{\dot{\mathbf{x}}\times\dot{\boldsymbol{\xi}}}{\sqrt{1+\dot{x}_0}}\\ =\;&(\mathbf{p}-m\dot{\mathbf{x}})\frac{\sqrt{(1+\dot{x}_0)}}{\hbar}+\frac{\dot{\mathbf{x}}\,(2+\dot{x}_0)}{\hbar\,(\dot{x}_0+1)^{3/2}}\,(p_0+m\dot{x}_0)\end{aligned} \tag{7.12}$$

where $p_0,\mathbf{p}$ are integration constants, which are constant values of 4-momentum $\{P_0,\mathbf{P}\}$ of the classical Dirac particle defined by (6.4)

Let us set $\mathbf{p}=0$ and introduce new designations

$$w_0=\frac{p_0}{m},\qquad \lambda=\frac{\hbar}{m} \tag{7.13}$$

$$\mathbf{y}=\frac{\dot{\mathbf{x}}}{\sqrt{1+\dot{x}_0}}=\frac{\dot{\mathbf{x}}}{\sqrt{1+\sqrt{1+\dot{\mathbf{x}}^2}}},\qquad \dot{\mathbf{x}}=\mathbf{y}\sqrt{(\mathbf{y}^2+2)} \tag{7.14}$$

$$\dot{x}_0=\sqrt{1+\mathbf{y}^2\,(\mathbf{y}^2+2)}=\mathbf{y}^2+1 \tag{7.15}$$

Dynamic equations (7.12), (7.11), (7.8) for dynamic system $\mathcal{S}_{\rm dc}$ reduce to the form

$$\lambda\dot{\mathbf{y}}\times\left(\boldsymbol{\xi}+\frac{1}{2}\mathbf{y}\left(\mathbf{y}\boldsymbol{\xi}\right)\right)=-\mathbf{y}\left(\frac{1-w_0}{\left(\mathbf{y}^2+2\right)}-w_0\right) \tag{7.16}$$

$$\lambda\dot{\mathbf{y}}\left(\mathbf{y}\times\boldsymbol{\xi}\right)=2\left(1-\frac{1-w_0}{\left(\mathbf{y}^2+2\right)}\right) \tag{7.17}$$

$$\dot{\boldsymbol{\xi}}=\left(\mathbf{y}\times\dot{\mathbf{y}}\right)\times\boldsymbol{\xi} \tag{7.18}$$

$$\dot{x}_0=\sqrt{1+\mathbf{y}^2\left(\mathbf{y}^2+2\right)}=\mathbf{y}^2+1 \tag{7.19}$$

It follows from (7.16) that $\mathbf{y}$ is orthogonal to $\dot{\mathbf{y}}$ and to $\boldsymbol{\xi}+\frac{1}{2}\mathbf{y}\left(\mathbf{y}\boldsymbol{\xi}\right)$

$$\left(\mathbf{y}\dot{\mathbf{y}}\right)=0,\qquad\left(\mathbf{y}\boldsymbol{\xi}\right)\left(1+\frac{1}{2}\mathbf{y}^2\right)=0 \tag{7.20}$$

Hence,

$$\mathbf{y}^2=b=\text{const},\qquad\left(\mathbf{y}\boldsymbol{\xi}\right)=0 \tag{7.21}$$

and equation (7.16) has the form

$$\lambda\dot{\mathbf{y}}\times\boldsymbol{\xi}=\left(-\frac{1-w_0}{b+2}+w_0\right)\mathbf{y} \tag{7.22}$$

Although it does not follow from (7.16) – (7.20) directly that $(\dot{\mathbf{y}}\boldsymbol{\xi})=0$ and $\dot{\boldsymbol{\xi}}=0$, but special investigation results, that $\boldsymbol{\xi}=$const, and one can find such a coordinate system, where $\boldsymbol{\xi}=\{0,0,1\}$. In this coordinate system

$$\lambda\dot{y}_2=\left(-\frac{1-w_0}{b+2}+w_0\right)y_1,\qquad-\lambda\dot{y}_1=\left(-\frac{1-w_0}{b+2}+w_0\right)y_2 \tag{7.23}$$

Solution of this system of equations has the form

$$y_1=\sqrt{b}\cos\left(\omega\tau_0\right),\quad y_2=\sqrt{b}\sin\left(\omega\tau_0\right), \tag{7.24}$$

where

$$\omega=\frac{1}{\lambda}\left(-\frac{1-w_0}{b+2}+w_0\right) \tag{7.25}$$

Substituting (7.24) into (7.17), one obtains

$$-\left(-\frac{1-w_0}{b+2}+w_0\right)b=2\left(1-\frac{1-w_0}{(b+2)}\right) \tag{7.26}$$

It follows from (7.26) that integration constants b and w_0 are connected by the relation

$$w_0=-\frac{1}{b+1} \tag{7.27}$$

As far as the vector $\boldsymbol{\xi}$ is orthogonal to both vectors $\mathbf{y}$ and $\dot{\mathbf{y}}$, the equation (7.18) is satisfied by $\boldsymbol{\xi}$ =const. Combining equations (7.24), (7.19) and (7.14), one obtains

$$\frac{d\mathbf{x}}{dt}=\mathbf{y}\frac{\sqrt{b+2}}{b+1}=\left\{\frac{\sqrt{b\left(b+2\right)}}{b+1}\cos\left(\Omega t\right),-\frac{\sqrt{b\left(b+2\right)}}{b+1}\sin\left(\Omega t\right),0\right\} \tag{7.28}$$

where

$$\Omega=\frac{\omega}{b+1}=\frac{2}{\lambda\left(b+1\right)^{2}} \tag{7.29}$$

Integration of equation (7.28) results

$$\mathbf{x}=\left\{\lambda\frac{\left(b+1\right)\sqrt{b\left(b+2\right)}}{2}\sin\left(\Omega t\right),\lambda\frac{\left(b+1\right)\sqrt{b\left(b+2\right)}}{2}\cos\left(\Omega t\right),0\right\} \tag{7.30}$$

In this special case the world line of the classical Dirac particle is a helix. The total particle mass $m_{\mathrm{dcr}}=m|w_0|$, radius a_{dcr} of the helix and the particle velocity v are determined by the relations

$$m_{\mathrm{dcr}} = \frac{m}{b+1},\qquad a_{\mathrm{dcr}}=\hbar\frac{\left(b+1\right)\sqrt{b\left(b+2\right)}}{2m}=\hbar\frac{\sqrt{b\left(b+2\right)}}{2m_{\mathrm{dcr}}}, \tag{7.31}$$

$$v = \left|\frac{d\mathbf{x}}{dt}\right|=\frac{\sqrt{b\left(b+2\right)}}{\left(b+1\right)} \tag{7.32}$$

Resolving the second equation (7.31) with respect to b, one expresses b as a function of the radius a_{dcr}

$$b=\frac{1}{\sqrt{2}}\sqrt{1+\sqrt{1+\zeta^{2}}}-1,\qquad \zeta=4a_{\mathrm{dcr}}\frac{mc}{\hbar}$$

One can express the observable mass m_{dcr}, velocity v and the angular velocity ω_{dcr} as functions of the helix radius a_{dcr}. These relations have the following form

$$m_{\mathrm{dcr}}=\frac{m\sqrt{2}}{\sqrt{\left(\sqrt{1+\zeta^{2}}+1\right)}}=\frac{m}{\cosh\beta},\qquad \zeta=4a_{\mathrm{dcr}}\frac{mc}{\hbar}=\sinh\left(2\beta\right) \tag{7.33}$$

$$v=c\frac{\zeta}{\left(\sqrt{1+\zeta^{2}}+1\right)}=c\tanh\beta,\qquad \omega_{\mathrm{dcr}}=\frac{4mc^{2}}{\hbar}\frac{1}{\left(\sqrt{1+\zeta^{2}}+1\right)}=\frac{2mc^{2}}{\hbar\cosh^{2}\beta} \tag{7.34}$$

Here c is the speed of the light.

Note that v is the Dirac particle velocity in the coordinate system, where the particle momentum $\mathbf{p}=0$. It seems strange that the world line of a free Dirac particle is a helix and the Dirac particle rotates. All this looks as if there were two coupled particles rotating around their center of mass, and we observe only one of them. Let us consider relativistic rotator and test this hypothesis.

8 Relativistic rotator

Rotator is a dynamic system $\mathcal{S}_\mathrm{r}$, consisting of two coupled particles of mass m_0, which can rotate around their center of mass. The distance between particles is to be constant, i.e. the particles are not to vibrate. Rigid nonrelativistic rotator $\mathcal{S}_\mathrm{nr}$ is described by the action

$$\mathcal{S}_\mathrm{nr}: \qquad \mathcal{A}[\mathbf{x}_1, \mathbf{x}_2, \mu] = \int \left(\sum_{k=1}^{k=2} \frac{m_0 \dot{\mathbf{x}}_k^2}{2} + \mu \left((\mathbf{x}_1 - \mathbf{x}_2)^2 - 4a^2 \right) \right) dt \tag{8.1}$$

where $2a$ is the distance (length of string) between the particles. The parameter a is determined by the length of rigid coupling between two particles. It does not depend on initial conditions.

If the coupling is elastic, the action for $\mathcal{S}_\mathrm{nr}$ should be written in the form

$$\mathcal{S}_\mathrm{nr}: \qquad \mathcal{A}[\mathbf{x}_1, \mathbf{x}_2, \mu] = \int \left(\sum_{k=1}^{k=2} \frac{m_0 \dot{\mathbf{x}}_k^2}{2} - U\left((\mathbf{x}_1 - \mathbf{x}_2)^2 \right) + \dot{\mu} (\mathbf{x}_1 - \mathbf{x}_2)^2 \right) dt \tag{8.2}$$

where U is the potential energy, describing interaction energy between two particles. This energy is constant for dynamic system (8.2), because of dynamic equation

$$\frac{\delta \mathcal{A}}{\delta \mu} = -\frac{d}{dt} (\mathbf{x}_1 - \mathbf{x}_2)^2 = 0, \qquad (\mathbf{x}_1 - \mathbf{x}_2)^2 = 4a^2 = \mathrm{const} \tag{8.3}$$

Evolution of variables $\mathbf{x}_1$, $\mathbf{x}_2$ does not depend on the form of U. Only μ depend on U. The last relation (8.3) appears as integral of motion. Interesting only in evolution of $\mathbf{x}_1$, $\mathbf{x}_2$, one can omit the potential energy U in the expression for the action.

In the relativity theory a rigid coupling is impossible. Also there are no reasons for introduction of a potential energy of interaction between the particles, because in the relativity theory a long-range action is absent. We are forced to choose another way.

Let us consider established relativistic motion of two particles of mass m_0, coupled between themselves by a massless elastic string. The condition of established motion means that the particles move in such a way that the length of the string does not change, and one may neglect degrees of freedom, connected with the string. Mathematically it means, that there is such a coordinate system K (maybe, rotating), where particles are at rest.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$, be world lines of particles

$$\mathcal{L}_k: \qquad x_{(k)}^i = x_{(k)}^i(\tau_k), \qquad i = 0, 1, 2, 3; \quad k = 1, 2 \tag{8.4}$$

where τ_k, $k = 1, 2$ are parameters along these world lines. From geometrical viewpoint the steady (established) motion of particles means that any spacelike 3-plane $\mathcal{S}$, crossing $\mathcal{L}_1$ orthogonally, crosses $\mathcal{L}_2$ also orthogonally. This circumstance permits one to synchronize events on $\mathcal{L}_1$ and $\mathcal{L}_2$

Let parameters τ_1 and τ_2 are chosen in such a way that they have the same value τ at points $\mathcal{L}_1 \cap \mathcal{S}$ and $\mathcal{L}_2 \cap \mathcal{S}$. The steady state conditions are written in the form

$$\frac{dx^i_{(1)}(\tau)}{d\tau}\left(x_{(1)i}(\tau) - x_{(2)i}(\tau)\right) = \frac{dx^i_{(2)}(\tau)}{d\tau}\left(x_{(1)i}(\tau) - x_{(2)i}(\tau)\right) = 0 \tag{8.5}$$

Let us describe motion of the two particles by the action

$$\mathcal{S}_{\mathrm{rr}}: \qquad \mathcal{A}_{\mathrm{rr}}\left[x_{(1)}, x_{(2)}\right] = -\int \sum_{k=1}^{k=2} m_0 \sqrt{\dot{x}^l_{(k)}\dot{x}_{(k)l}}\, d\tau \tag{8.6}$$

where variables $x_{(k)}$ are considered to be functions of the same parameter τ, and the period denotes differentiation with respect to τ. World lines of particles are determined as extremals of the functional (8.6) with side constraints (8.5).

It is convenient to introduce new variables

$$X^k = \frac{1}{2}\left(x^k_{(1)} + x^k_{(2)}\right), \quad x^k = \frac{1}{2}\left(x^k_{(1)} - x^k_{(2)}\right), \tag{8.7}$$

$$x^k_{(1)} = X^k + x^k, \qquad x^k_{(2)} = X^k - x^k, \tag{8.8}$$

and rewrite the constraints (8.5) in the equivalent form

$$\frac{d}{d\tau}\left(x^k x_k\right) = 0, \qquad \dot{X}^k x_k = 0 \tag{8.9}$$

Here the first condition describes constancy of the string length. This condition is used in the nonrelativistic case also. The second condition is the synchronization condition, which is possible only for established motion of two rotator particles. This condition is not used in the nonrelativistic case in explicit form.

Let us add conditions (8.9) to the action by means of Lagrangian multipliers μ and ν. Then one obtains the following expression for the action

$$\mathcal{S}_{\mathrm{rr}}: \qquad \mathcal{A}_{\mathrm{rr}}[X, x, \mu, \nu] = \int \left\{-m_0 R - m_0 r + \dot{\mu} x^k x_k + \nu \dot{X}^k x_k\right\} d\tau \tag{8.10}$$

where

$$R = \sqrt{\left(\dot{X}^k + \dot{x}^k\right)\left(\dot{X}_k + \dot{x}_k\right)}, \qquad r = \sqrt{\left(\dot{X}^k - \dot{x}^k\right)\left(\dot{X}_k - \dot{x}_k\right)} \tag{8.11}$$

Momenta are expressed as follows

$$P_k = \frac{\partial L}{\partial \dot{X}^k} = -\frac{m_0}{R}\left(\dot{X}_k + \dot{x}_k\right) - \frac{m_0}{r}\left(\dot{X}_k - \dot{x}_k\right) + \nu x_k, \tag{8.12}$$

$$p_k = \frac{\partial L}{\partial x^k} = -\frac{m_0}{R}\left(\dot{X}_k + \dot{x}_k\right) + \frac{m_0}{r}\left(\dot{X}_k - \dot{x}_k\right), \tag{8.13}$$

The action (8.10) is invariant with respect to transformation $\tau \to f(\tau)$. Using this, one chooses the parameter τ in such a way that $R + r = 1$. Introducing designation

$$\beta = R - r, \qquad R + r = 1, \qquad R = \frac{1+\beta}{2}, \qquad r = \frac{1-\beta}{2}, \tag{8.14}$$

and resolving relations (8.12), (8.13) with respect to $\dot{X}$ and $\dot{x}$, one obtains dynamic equations

$$\begin{aligned}
\dot{X}_k &= -\frac{1}{4m_0}\left(P_k + \beta p_k - \nu x_k\right) && (8.15)\\
\dot{x}_k &= -\frac{1}{4m_0}\left(\beta P_k + p_k - \beta\nu x_k\right) && (8.16)
\end{aligned}$$

Variation with respect to X^k, x^k, μ, ν gives respectively

$$\begin{aligned}
-\dot{P}_k &= 0, \qquad \dot{p}_k = 2\dot{\mu}x_k + \nu\dot{X}_k && (8.17)\\
2\left(\dot{x}^k x_k\right) &= 0, \qquad \dot{X}^k x_k = 0, && (8.18)
\end{aligned}$$

Equations (8.15) – (8.18) form the complete system of dynamic equations for determination of variables $X^k, P_k, x^k, p_k, \mu, \nu$ as functions of parameter τ.

It follows from the first equations (8.17) and (8.18)

$$P_k = \text{const}, \qquad k = 0,1,2,3, \qquad x^k x_k = -a^2 = \text{const} \tag{8.19}$$

Convoluting equations (8.15), (8.16) and the second equation (8.17) with x^k and using equations (8.18), one obtains the relations

$$\nu = -\frac{P_k x^k}{a^2}, \qquad p_k x^k = 0, \quad \dot{\mu} = -\frac{x^k\dot{p}_k}{2a^2} \tag{8.20}$$

$$\dot{p}_s x^s = -p_s\dot{x}^s = \frac{p^k}{4m_0}\left(\beta P_k + p_k\right) \tag{8.21}$$

$\dot{\mu}$ is expressed as follows

$$\dot{\mu} = -\frac{\dot{p}_s x^s}{2a^2} = -\frac{p^k}{8m_0 a^2}\left(\beta P_k + p_k\right) \tag{8.22}$$

Let us now substitute equations (8.15), (8.16) in relations (8.11). Using relations (8.20), one obtains the conditions

$$\begin{aligned}
\left(2m_0\right)^2 &= \left(P_k + p_k\right)\left(P^k + p^k\right) + \nu^2 a^2\\
\left(2m_0\right)^2 &= \left(P_k - p_k\right)\left(P^k - p^k\right) + \nu^2 a^2
\end{aligned}$$

Combining them, one obtains

$$p^k p_k = -\left(P^k P_k - 4m_0^2\right) - a^2\nu^2, \qquad P_k p^k = 0 \tag{8.23}$$

Let us substitute (8.22) into the second equation (8.17) and take into account constraint (8.23) and dynamic equation (8.15). The second equation (8.17) reduces to

$$\dot{p}_k = -x_k\frac{1}{4m_0 a^2}\left(4m_0^2 - P^s P_s\right) - \frac{\nu}{4m_0}\left(P_k + \beta p_k\right) \tag{8.24}$$

Convoluting (8.24) with p^k and taking into account (8.20), one obtains

$$p^k \dot{p}_k = -\beta\nu \frac{p^k p_k}{4m_0} \tag{8.25}$$

Differentiating the first equation (8.23) and taking into account (8.20), (8.16), one obtains

$$p^k \dot{p}_k = -a^2 \nu\dot{\nu} = -\frac{\beta\nu}{4m_0}\left(P_k P^k + \nu^2 a^2\right) \tag{8.26}$$

Comparing relations (8.25), (8.26) and taking into account (8.23), one concludes that

$$\beta\nu\left(p_k p^k - 2m_0^2\right) = 0 \tag{8.27}$$

If $p^k p_k = 2m_0^2$ =const, then it follows from (8.25) that

$$\nu\beta = 0 \tag{8.28}$$

and constraint (8.27) reduces to (8.28)

It follows from (8.28), that

$$\beta = 0 \wedge \nu = 0 \tag{8.29}$$

To show this, let us introduce spacelike vector

$$\zeta_i = \varepsilon_{iklm} x^k p^l P^m. \tag{8.30}$$

It follows from dynamic equations (8.16), (8.24) that

$$\dot{\zeta}_i = \varepsilon_{iklm} \dot{x}^k p^l P^m + \varepsilon_{iklm} x^k \dot{p}^l P^m = 0 \tag{8.31}$$

Let

$$\xi_i = \frac{\zeta_i}{\sqrt{-\zeta_k \zeta^k}}, \qquad \xi_i \xi^i = -1 \tag{8.32}$$

Thus, ξ_i is a unit constant spacelike vector, which is orthogonal to P_k, p_k, x_k. 4-vector p_k is orthogonal to vectors P_i and ξ_i.

Let us choose such a coordinate system, where

$$P_i = \{P_0, 0, 0, 0\}, \quad \xi_i = \{0, 0, 0, 1\}, \tag{8.33}$$

Then

$$x_i = \{x_0, x_1, x_2, 0\}, \qquad p_i = \{0, p_1, p_2, 0\}, \tag{8.34}$$

Let $\nu = 0$ be a solution of (8.28). For $k = 0$ the equations (8.16), (8.24) have respectively the form

$$\dot{x}_0 = -\frac{1}{4m_0}\beta P_0, \qquad \dot{p}_0 = 0 = \frac{P_0^2 - 4m_0^2}{4m_0 a^2} x_0 \tag{8.35}$$

It follows from (8.35) that $x_0 = 0$, and $\beta = 0$.

Let now $\beta = 0$. Then for $k = 0$ the equations (8.16), (8.24) have respectively the form

$$\dot{x}_0 = 0, \qquad 0 = \frac{P_0^2 - 4m_0^2}{4m_0 a^2} x_0 - \frac{1}{4m_0} \nu P_0 \tag{8.36}$$

It follows from (8.36) that x_0 =const,

$$\nu = \frac{P_0^2 - 4m_0^2}{P_0 a^2} x_0 = \text{const} \tag{8.37}$$

The condition (8.37) appears to be compatible with (8.23) and the second constraint (8.19), provided $\nu = 0$ or $m_0 = 0$. As far as $m_0 \neq 0$, the relation (8.29) takes place and $x_0 = 0$. The dynamic equations (8.16), (8.24) take the form

$$\dot{x}_\alpha = -\frac{p_\alpha}{4m_0}, \qquad \dot{p}_\alpha = \frac{P_0^2 - 4m_0^2}{4m_0 a^2} x_\alpha, \qquad \alpha = 1, 2 \tag{8.38}$$

Solution of dynamic equations (8.38), (8.15) is written in the form

$$x^i = \{0,\ a\cos(\omega\tau + \phi),\ a\sin(\omega\tau + \phi),\ 0\} \tag{8.39}$$

$$p_i = \{0,\ -4am_0\omega\sin(\omega\tau + \phi),\ 4am_0\omega\cos(\omega\tau + \phi),\ 0\} \tag{8.40}$$

$$X^k = \left\{-\frac{P_0}{4m_0}\tau, 0, 0, 0\right\}, \quad X_0 = t = -\frac{P_0}{4m_0}\tau \tag{8.41}$$

where

$$\omega = \frac{\sqrt{P_0^2 - 4m_0^2}}{4m_0 a}, \tag{8.42}$$

and ϕ is an arbitrary constant.

Let us substitute the independent variable τ by $t = -\frac{P_0}{4m_0}\tau$. Then one obtains

$$\omega\tau = \omega_0 t, \qquad \omega_0 = -\frac{\sqrt{P_0^2 - 4m_0^2}}{aP_0} \tag{8.43}$$

$$x_{(1)}^i = \{t,\ a\cos(\omega_0 t + \phi),\ a\sin(\omega_0 t + \phi),\ 0\} \tag{8.44}$$

$$x_{(2)}^i = \{t,\ -a\cos(\omega_0 t + \phi),\ -a\sin(\omega_0 t + \phi),\ 0\} \tag{8.45}$$

One can see that both world lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are helixes. They describe rotation of two particles around their common center of mass along a circle of radius a. Such a dynamic system $\mathcal{S}_{\rm rr}$ may be qualified as a relativistic rotator. This rotator may be described by the relative mass increase $\gamma = (M - 2m_0)/2m_0$, where

$$\gamma = \frac{(M - 2m_0)}{2m_0} = \frac{1}{\sqrt{1 - v^2}} - 1 = \frac{v^2}{\sqrt{1 - v^2}\left(\sqrt{1 - v^2} + 1\right)} \tag{8.46}$$

Here $M = \sqrt{P_i P^i}$ is the total mass of the rotator, and $v = a\omega_0$ is the velocity of a particle in the coordinate system, where center of mass is at rest. γ is a part of total mass, conditioned by rotation.

The distance $2a$ between particles appears as an integration constant. In the solution (8.44) the radius a and angular frequency ω_0 of rotation are independent integration constants.

For any real rotator these quantities cannot be independent. Elastic properties of the coupling between two particles determine the relative mass increase γ. The elastic properties can be described by the rigidity function $\gamma = f_{\mathrm{r}}(a)$, which determines relation between quantities a and ω_0, or between a and γ for any real rotator. For nonrelativistic rotator the rigidity function $f_{\mathrm{r}}(a)$ is connected with the potential energy $U(a)$ of elastic coupling by means of relation

$$f_{\mathrm{r}}(a) = \frac{a}{mc^2}\frac{\partial U(a)}{\partial a}$$

In the relativistic case one cannot introduce potential energy of elastic coupling. It is replaced by the rigidity function $f_{\mathrm{r}}(a)$.

The dynamic system $\mathcal{S}_{\mathrm{dcr}}$, where angular velocity is coupled with the radius of helix, is a special case of relativistic rotator $\mathcal{S}_{\mathrm{rr}}$. To show this, let us compare relations (8.43), (8.44) with relations (7.29), (7.30) and identify the quantities ω_0, M, a of dynamic system $\mathcal{S}_{\mathrm{rr}}$ respectively with ω_{dcr}, m_{dcr}, a_{dcr} of dynamic system $\mathcal{S}_{\mathrm{dcr}}$. One derives

$$\frac{\sqrt{M^2 - 4m_0^2}}{aM} = \frac{4m}{\hbar}\frac{1}{\left(\sqrt{1+\zeta^2}+1\right)}, \qquad M = \frac{m\sqrt{2}}{\sqrt{\left(\sqrt{1+\zeta^2}+1\right)}} \tag{8.47}$$

$$a = a_{\mathrm{dcr}}, \qquad \zeta = 4a_{\mathrm{dcr}}\frac{m}{\hbar}, \qquad c = 1 \tag{8.48}$$

Resolving relations (8.47) with respect to variables M and m_0, one obtains all quantities of the relativistic rotator $\mathcal{S}_{\mathrm{rr}}$ in terms of parameters $\zeta = 4\hbar a_{\mathrm{dcr}}/m$ and m of the Dirac particle $\mathcal{S}_{\mathrm{dcr}}$

$$M = m_{\mathrm{dcr}} = \frac{m\sqrt{2}}{\sqrt{\left(\sqrt{1+\zeta^2}+1\right)}}, \qquad m_0 = \frac{m}{\left(\sqrt{1+\zeta^2}+1\right)} \tag{8.49}$$

One can express parameters m, m_{dcr}, ω_{dcr}, v_{dcr} of $\mathcal{S}_{\mathrm{dcr}}$ in terms of parameters $v = \zeta_0 = 4am_0/\hbar$, m_0 of the relativistic rotator $\mathcal{S}_{\mathrm{rr}}$. After some calculations one obtains

$$m = 2m_0\frac{1}{1-v^2}, \qquad m_{\mathrm{dcr}} = \frac{2m_0}{\sqrt{1-v^2}}, \qquad \omega_{\mathrm{dcr}} = \frac{4m_0c^2}{\hbar}, \tag{8.50}$$

where

$$v = \zeta_0 = 4a\frac{m_0c^2}{\hbar} \tag{8.51}$$

The relativistic angular momentum A and magnetic moment μ_0 have the form

$$A = 2m_0 \frac{av}{\sqrt{1-v^2}}, \qquad \mu_0 = \frac{eav}{2\sqrt{1-v^2}}$$

$$\frac{\mu_0}{A} = \frac{e}{4m_0} = \frac{e}{2m_{\mathrm{dcr}}}\sqrt{1-v^2}$$

The rigidity function $\gamma = f_{\mathrm{r}}\left(a_{\mathrm{dcr}}\right)$ is described by relations (8.46). As it follows from relations (8.50), (8.51) the rigidity function has the form

$$\gamma = f_{\mathrm{r}}\left(a\right) = \frac{\hbar}{\sqrt{\hbar^2 - \left(4am_0c\right)^2}} - 1 \tag{8.52}$$

It follows from (8.52) that $a \leq \frac{\hbar}{4m_0c}$. Apparently, this points to quantum origin of interaction between the rotator particles.

9 Discussion

It is a common practice to consider the Dirac particle to be a simple pointlike construction, which has its proper characteristics such as mass, charge, spin and magnetic moment. Spin and magnetic moment seem to be extraneous for pointlike construction. Directions of momentum and velocity do not coincide for Dirac particle. It also seems rather strange for pointlike structure. The classical Dirac particle has ten degrees of freedom in the sense, that integration of dynamic equations leads to appearance of ten integration constants. It is too much for pointlike structure.

Dynamic analysis of dynamic system $\mathcal{S}_{\mathrm{D}}$ shows that the Dirac particle is not a pointlike structure. It is a more complicated structure, consisting of two pointlike coupled particles. Such properties as spin and magnetic moment are quite reasonable for such a dynamic system. Discrepancy between the direction of velocity and that of momentum is explained also, if one compares alternating velocity of a particle with the constant momentum of the whole rotator. The number of degrees of freedom is explained freely also by existence of two coupled particles. Six external degrees of freedom, describing motion of the particle as a whole, are relativistic in the sense that the global velocity is less, than the speed of light. If one neglects the internal degrees of freedom, described by two last terms in the action (4.17), the dynamic system $\mathcal{S}_{\mathrm{dc}}$ becomes relativistic. The nonrelativistic character of the dynamic system $\mathcal{S}_{\mathrm{dc}}$ is connected with incorrect description of internal degrees of freedom. The dynamic system $\mathcal{S}_{\mathrm{dc}}$ can be made relativistic, if one supposes that $\mathcal{S}_{\mathrm{dc}}$ is a special case of $\mathcal{S}_{\mathrm{rr}}$ with the rigidity function $\gamma = f_{\mathrm{r}}\left(a\right)$, described by the relation (8.52). The rigidity function $\gamma = f_{\mathrm{r}}\left(a\right)$, is a relativistically invariant characteristic of the relativistic rotator $\mathcal{S}_{\mathrm{rr}}$.

Explaining freely all strange properties of the Dirac particle, the two-particle model poses a very important question. What is an origin of the coupling between two particles? Appearance of quantum constant $\hbar$ in the relation (8.52), describing character of coupling, shows, that the coupling origin is, apparently, quantum.

Conventional quantum mechanics failed to discover internal structure of Dirac particle. Methods of QM are too rough. Internal structure of Dirac particle is discovered by dynamic methods of MCQP which appear to be more subtle. Internal structure of the Dirac particle is of no importance for calculations of stationary atom states, because characteristic energies are too small, and internal degrees of freedom of the Dirac particle are not excited. But the Dirac particle structure may appear to be important at investigation of elementary particles structure, because characteristic energies are large enough, and internal degrees of freedom can be excited.

Mathematical Appendices

A Calculation of Lagrangian

Let us calculate the expression

$$\frac{i}{2}\hbar\bar{\psi}\gamma^l\partial_l\psi + \text{h.c} = F_1 + F_2 + F_3 + F_4 \tag{A.1}$$

where the following designations are used

$$F_1 = \frac{i}{2}\hbar\psi^*\left(\left(\partial_0 - i\gamma_5\boldsymbol{\sigma}\boldsymbol{\nabla}\right)i\varphi\right)\psi + \text{h.c.} \tag{A.2}$$

$$F_2 = \frac{i}{2}\hbar\psi^*\left(\left(\partial_0 - i\gamma_5\boldsymbol{\sigma}\boldsymbol{\nabla}\right)\left(\frac{1}{2}\gamma_5\kappa\right)\right)\psi + \text{h.c.} \tag{A.3}$$

$$F_3 = +\frac{i}{2}\hbar A^2\Pi\left(\left(\boldsymbol{\sigma}\mathbf{n}\right)e^{-i\gamma_5\boldsymbol{\sigma}\boldsymbol{\eta}}\left(\boldsymbol{\sigma}\mathbf{n}\right)\right)e^{-\frac{i\pi}{2}\boldsymbol{\sigma}}(\partial_0 - i\gamma_5\boldsymbol{\sigma}\boldsymbol{\nabla})(e^{\frac{i\pi}{2}\boldsymbol{\sigma}})\Pi + \text{h.c.} \tag{A.4}$$

$$F_4 = \frac{i}{2}\hbar A^2\Pi e^{-\frac{i}{2}\gamma_5\boldsymbol{\Sigma}\boldsymbol{\eta}}(\partial_0 - i\gamma_5\boldsymbol{\Sigma}\boldsymbol{\nabla})e^{-\frac{i}{2}\gamma_5\boldsymbol{\Sigma}\boldsymbol{\eta}}\Pi + \text{h.c.} \tag{A.5}$$

In the last relation the matrix $\boldsymbol{\Sigma}$ is not differentiated.

Using definitions of j^l and S^l, the expression F_1 and F_2 reduce to the form

$$F_1 = \frac{i}{2}\hbar\psi^*\left(\left(\partial_0 - i\gamma_5\boldsymbol{\sigma}\boldsymbol{\nabla}\right)i\varphi\right)\psi + \text{h.c.} = -j^l\partial_l\varphi\Pi \tag{A.6}$$

$$\begin{aligned} F_2 &= \frac{i}{2}\hbar\psi^*\left(\left(\partial_0 - i\gamma_5\boldsymbol{\sigma}\boldsymbol{\nabla}\right)\left(\frac{1}{2}\gamma_5\kappa\right)\right)\psi + \text{h.c} \\ &= \frac{i}{2}\hbar\psi^*\gamma_5\gamma^l\partial_l\left(\frac{1}{2}\gamma_5\kappa\right)\psi + \text{h.c} \end{aligned}$$

$$F_2 = -\frac{1}{2}\hbar S^l\partial_l\kappa\Pi \tag{A.7}$$

$$\begin{aligned} F_3 &= \frac{i}{2}\hbar A^2\Pi\left(e^{-\frac{i\pi}{2}\boldsymbol{\sigma}}e^{-i\gamma_5\boldsymbol{\sigma}\boldsymbol{\eta}}e^{\frac{i\pi}{2}\boldsymbol{\sigma}}\right)e^{-\frac{i\pi}{2}\boldsymbol{\sigma}}(\partial_0 - i\gamma_5\boldsymbol{\sigma}\nabla)(e^{\frac{i\pi}{2}\boldsymbol{\sigma}})\Pi + \text{h.c.} \\ &= \frac{i}{2}\hbar j^l\Pi\sigma_\alpha\sigma_\beta n^\alpha\partial_l n^\beta\Pi + \text{h.c.} = \frac{i}{2}\hbar j^l n^\alpha\partial_l n^\beta\Pi\left(\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma\right)\Pi + \text{h.c.} \\ &= \frac{i}{2}\hbar j^l\left(n^\alpha\partial_l n^\alpha + i\varepsilon_{\alpha\beta\gamma}n^\alpha\partial_l n^\beta z^\gamma\right)\Pi + \text{h.c.} \end{aligned}$$

As far as $\mathbf{n}^2 = 1$, one obtains

$$n^\alpha\partial_l n^\alpha = 0 \tag{A.8}$$

Besides it follows from (2.28) that

$$\mathbf{n} = \frac{\boldsymbol{\sigma} + \mathbf{z}}{\sqrt{2\left(1 + \boldsymbol{\sigma}\mathbf{z}\right)}} \tag{A.9}$$

Then

$$F_3 = -\hbar j^l \left(\varepsilon_{\alpha\beta\gamma} n^\alpha \partial_l n^\beta z^\gamma\right) \Pi = -\frac{\hbar j^l}{2\left(1+\boldsymbol{\xi}\mathbf{z}\right)} \varepsilon_{\alpha\beta\gamma} \xi^\alpha \partial_l \xi^\beta z^\gamma \Pi \qquad \text{(A.10)}$$

Calculation of F_4 leads to the following result

$$\begin{aligned}
F_4 &= \frac{i}{2}\hbar A^2 \Pi e^{-\frac{i}{2}\gamma_5 \boldsymbol{\Sigma\eta}} (\partial_0 - i\gamma_5 \boldsymbol{\Sigma\nabla}) e^{-\frac{i}{2}\gamma_5 \boldsymbol{\Sigma\eta}} \Pi + \text{h.c.} \\
&= \frac{i}{2}\hbar A^2 \Pi \left(\cosh\frac{\eta}{2} - i\gamma_5 v^\alpha \Sigma_\alpha \sinh\frac{\eta}{2}\right) \\
&\quad \times (\partial_0 - i\gamma_5 \boldsymbol{\Sigma\nabla}) \left(\cosh\frac{\eta}{2} - i\gamma_5 v^\alpha \Sigma_\alpha \sinh\frac{\eta}{2}\right) \Pi + \text{h.c.} \\
&= \frac{i}{2}\hbar A^2 \Pi \left(\cosh\frac{\eta}{2}\sinh\frac{\eta}{2}\partial_0\eta + \sinh^2\frac{\eta}{2} v^\alpha \partial_0 v^\beta \Sigma_\alpha \Sigma_\beta\right) \Pi + \text{h.c.} \\
&\quad + \frac{i}{2}\hbar A^2 \Pi \left(\cosh^2\frac{\eta}{2}\Sigma_\alpha\Sigma_\beta v^\beta + \sinh^2\frac{\eta}{2}\Sigma_\beta\Sigma_\alpha v^\beta\right) \partial_\alpha \frac{\eta}{2} \Pi + \text{h.c.} \\
&\quad + \frac{i}{2}\hbar A^2 \Pi \cosh\frac{\eta}{2}\sinh\frac{\eta}{2}\Sigma_\alpha\Sigma_\beta \partial_\alpha v^\beta \Pi + \text{h.c.}
\end{aligned}$$

$$\begin{aligned}
F_4 &= \frac{i}{2}\hbar A^2 \Pi \left(\frac{1}{2}\sinh\eta\partial_0\eta + \sinh^2\frac{\eta}{2} v^\alpha \partial_0 v^\beta i\varepsilon_{\alpha\beta\gamma}\Sigma_\gamma\right) \Pi + \text{h.c.} \\
&\quad + \frac{i}{2}\hbar A^2 \Pi \left(\cosh\eta v^\alpha + i\varepsilon_{\beta\alpha\gamma} v^\beta \Sigma_\gamma\right) \partial_\alpha \frac{\eta}{2} \Pi + \text{h.c.} \\
&\quad + \frac{i}{4}\hbar A^2 \Pi \sinh\eta \left(\partial_\alpha v^\alpha + i\varepsilon_{\alpha\beta\gamma}\partial_\alpha v^\beta \Sigma_\gamma\right) \Pi + \text{h.c.}
\end{aligned}$$

$$\begin{aligned}
F_4 &= -\hbar A^2 \left(\sinh^2\frac{\eta}{2} v^\alpha \partial_0 v^\beta \varepsilon_{\alpha\beta\gamma}\xi^\gamma\right) \Pi \\
&\quad - \frac{1}{2}\hbar A^2 \Pi \left(\varepsilon_{\beta\alpha\gamma} v^\beta \xi^\gamma \partial_\alpha \eta + \sinh\eta \varepsilon_{\alpha\beta\gamma} \partial_\alpha v^\beta \xi^\gamma\right) \Pi
\end{aligned}$$

$$F_4 = -\frac{1}{2}\hbar A^2 \varepsilon_{\alpha\beta\gamma} \left(\partial_\alpha \eta v^\beta + \sinh\eta \partial_\alpha v^\beta + 2\sinh^2(\frac{\eta}{2}) v^\alpha \partial_0 v^\beta\right) \xi^\gamma \Pi \qquad \text{(A.11)}$$

B Transformation of Lagrangian to covariant form

Let us show that the expression (3.4) is equivalent to expression (3.10)

$$F_4 = -\frac{\hbar}{2(\rho + f^s j_s)} \varepsilon_{iklm} [\partial^k (j^i + f^i \rho)](j^l + f^l \rho)[\xi^m - f^m (\xi^s f_s)] \qquad \text{(B.1)}$$

$$\mathcal{L}_{\text{q2}} = \frac{\hbar(\rho + j_0)}{2} \varepsilon_{\alpha\beta\gamma} \partial^\alpha \frac{j^\beta}{(j^0 + \rho)} \xi^\gamma - \frac{\hbar}{2(\rho + j_0)} \varepsilon_{\alpha\beta\gamma} \left(\partial^0 j^\beta\right) j^\alpha \xi^\gamma \qquad \text{(B.2)}$$

$$\mathcal{L}_{\text{q2}} = -\frac{\hbar(\rho f^0 + j_0)}{2} \varepsilon_{\alpha\beta\gamma 0} \partial^\alpha \frac{j^\beta}{(j^0 + \rho)} \xi^\gamma + \frac{\hbar}{2} \varepsilon_{\alpha\beta\gamma 0} \left(\partial^0 \frac{j^\beta}{(\rho + j_0)}\right) (j^\alpha + f^\alpha \rho) \xi^\gamma \qquad \text{(B.3)}$$

$$\mathcal{L}_{q2} = \frac{\hbar(\rho f^0 + j^0)}{2}\varepsilon_{0\alpha\beta\gamma}\partial^\alpha \frac{j^\beta + f^\beta\rho}{(j^0+\rho)}\xi^\gamma + \frac{\hbar}{2}\left(j^\alpha + f^\alpha\rho\right)\varepsilon_{\alpha 0\beta\gamma}\left(\partial^0\frac{j^\beta + f^\beta\rho}{(\rho + j_0)}\right)\xi^\gamma \quad \text{(B.4)}$$

$$\mathcal{L}_{q2} = \frac{\hbar(\rho f^0 + j^0)}{2}\varepsilon_{0k\beta\gamma}\partial^\alpha \frac{j^\beta + f^\beta\rho}{(j^0+\rho)}\xi^\gamma + \frac{\hbar}{2}\left(j^\alpha + f^\alpha\rho\right)\varepsilon_{\alpha 0\beta\gamma}\left(\partial^0\frac{j^\beta + f^\beta\rho}{(\rho + j_0)}\right)\xi^\gamma \quad \text{(B.5)}$$

To prove this statement, one sets $f^0 = 1, \quad f^\alpha = 0$ in the relation (B.1) and expands it

$$\begin{aligned}
F_4 &= -\frac{\hbar}{2(\rho + f^s j_s)}\varepsilon_{iklm}[\partial^k(j^i + f^i\rho)](j^l + f^l\rho)[\xi^m - f^m(\xi^s f_s)] \\
&= -\frac{\hbar\xi^\mu}{2(\rho + j^0)}\left(\varepsilon_{i0l\mu}\partial^0(j^i + f^i\rho)(j^l + f^l\rho) + \varepsilon_{i\beta l\mu}\partial^\beta(j^i + f^i\rho)(j^l + f^l\rho)\right) \\
&= -\frac{\hbar\xi^\mu}{2(\rho + j^0)}\left(\varepsilon_{\alpha 0\beta\mu}\partial_0 j^\alpha j^\beta + \varepsilon_{0\beta\alpha\mu}\partial^\beta(j^0+\rho)j^\alpha + \varepsilon_{\alpha\beta 0\mu}\partial^\beta j^\alpha(j^0 + \rho)\right)
\end{aligned}$$

One substitutes the expression of j^i via variables $\mathbf{v}$, η

$$j^0 = \rho\cosh\eta, \qquad j^\alpha = \rho\sinh\eta v^\alpha \quad \text{(B.6)}$$

in this expression and obtains

$$\begin{aligned}
F_4 &= -\frac{\hbar\xi^\mu\rho}{2(1+\cosh\eta)}\sinh^2\eta\varepsilon_{\alpha 0\beta\mu}\partial_0 v^\alpha v^\beta + \frac{\hbar\xi^\mu}{2}\varepsilon_{\alpha\beta 0\mu}\partial_\beta\left(\rho\sinh\eta v^\alpha\right) \\
&\quad -\frac{\hbar\xi^\mu\sinh\eta}{2(1+\cosh\eta)}\varepsilon_{0\beta\alpha\mu}\partial_\beta\left(\rho(\cosh\eta + 1)\right)v^\alpha \\
&= \frac{\hbar\xi^\mu}{2}\frac{\rho\sinh^2\eta}{(1+\cosh\eta)}\varepsilon_{0\alpha\beta\mu}\partial_0 v^\alpha v^\beta + \frac{\hbar\xi^\mu}{2}\rho\varepsilon_{0\alpha\beta\mu}\partial_\beta\left(\sinh\eta v^\alpha\right) \\
&\quad +\frac{\hbar\xi^\mu}{2}\frac{\rho\sinh\eta}{(1+\cosh\eta)}\varepsilon_{0\alpha\beta\mu}\partial_\beta(\cosh\eta + 1)v^\alpha \\
&= -\frac{\hbar\xi^\mu}{2}\rho\varepsilon_{0\alpha\beta\mu}\left(-\frac{\sinh^2\eta}{(1+\cosh\eta)}\partial_0 v^\alpha v^\beta + \partial_\beta\left(\sinh\eta v^\alpha\right)\right) \\
&\quad +\frac{\hbar\xi^\mu}{2}\rho\varepsilon_{0\alpha\beta\mu}\frac{\sinh\eta}{(1+\cosh\eta)}\partial_\beta(\cosh\eta + 1)v^\alpha \\
&= -\frac{\hbar\xi^\mu}{2}\rho\varepsilon_{0\alpha\beta\mu}\left(-2\sinh^2\frac{\eta}{2}\partial_0 v^\alpha v^\beta + \cosh\eta v^\alpha\partial_\beta\eta + \sinh\eta\partial_\beta v^\alpha\right) \\
&\quad +\frac{\hbar\xi^\mu}{2}\rho\varepsilon_{0\alpha\beta\mu}\frac{\sinh^2\eta}{(1+\cosh\eta)}v^\alpha\partial_\beta\eta \\
&= -\frac{\hbar\xi^\mu}{2}\rho\varepsilon_{0\alpha\beta\mu}\left(-2\sinh^2\frac{\eta}{2}\partial_0 v^\alpha v^\beta + \sinh\eta\partial_\beta v^\alpha\right) \\
&\quad +\frac{\hbar\xi^\mu}{2}\rho\varepsilon_{0\alpha\beta\mu}\frac{\sinh^2\eta - \cosh\eta - \cosh^2\eta}{(1+\cosh\eta)}v^\alpha\partial_\beta\eta \\
&= -\frac{\hbar\xi^\mu}{2}\rho\varepsilon_{0\alpha\beta\mu}\left(-2\sinh^2\frac{\eta}{2}\partial_0 v^\alpha v^\beta + \sinh\eta\partial_\beta v^\alpha + v^\alpha\partial_\beta\eta\right)
\end{aligned}$$

The obtained relation coincides with the expression (3.4) for F_4, that proves correctness of expression (3.10).

C Transformation of equation for variable $\boldsymbol{\xi}$

Let us transform equation (7.7)

$$\boldsymbol{\xi}\times\left(-\dot{\boldsymbol{\xi}}\times\mathbf{z}+\frac{(\mathbf{z}\dot{\boldsymbol{\xi}})}{2(1+\mathbf{z}\ \boldsymbol{\xi})}\boldsymbol{\xi}\times\mathbf{z}+\frac{\boldsymbol{\xi}(\dot{\boldsymbol{\xi}}\times\mathbf{z})}{2(1+\mathbf{z}\ \boldsymbol{\xi})}\mathbf{z}-\frac{(1+\mathbf{z}\ \boldsymbol{\xi})}{2}\mathbf{b}\right)=0 \tag{C.1}$$

keeping in mind that $\boldsymbol{\xi}^2=1$ and $\mathbf{z}^2=1$. Two middle terms could be represented as the double vector product

$$\boldsymbol{\xi}\times\left(-\dot{\boldsymbol{\xi}}\times\mathbf{z}+\frac{1}{2(1+\mathbf{z}\ \boldsymbol{\xi})}\left(\dot{\boldsymbol{\xi}}\times((\boldsymbol{\xi}\times\mathbf{z})\times\mathbf{z})\right)-\frac{(1+\mathbf{z}\ \boldsymbol{\xi})}{2}\mathbf{b}\right)=0 \tag{C.2}$$

Or in the form

$$\boldsymbol{\xi}\times\left(\dot{\boldsymbol{\xi}}\times\left(-\mathbf{z}+\frac{(\mathbf{z}\boldsymbol{\xi})\,\mathbf{z}-\boldsymbol{\xi}}{2(1+\mathbf{z}\ \boldsymbol{\xi})}\right)-\frac{(1+\mathbf{z}\ \boldsymbol{\xi})}{2}\mathbf{b}\right)=0 \tag{C.3}$$

Now calculating double vector products and taking into account that $\boldsymbol{\xi}\dot{\boldsymbol{\xi}}=0$, one obtains

$$-\dot{\boldsymbol{\xi}}-(\boldsymbol{\xi}\times\mathbf{b})=0 \tag{C.4}$$

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