Comment for physicists, written 30 January 2011

This paper has been published in 1995 in the mathematical journal *Advances in Applied Clifford Algebras.* It was my unique publication on Clifford algebras. Nevertheless I was personally invited to participate in several conferences on Clifford algebras, although I am a physicist, and a use of Clifford algebras is only a mathematical method of the Dirac equation representation. In my paper the γ-matrices are eliminated, and the Dirac equation is presented in the hydrodynamical form, where the γ-matrices are absent. On one hand, the physicists do not use the Sauter - Sommerfeld method of the γ-matrices representation in the form of Clifford algebras. On the other hand, one can eliminate γ-matrices, only if they are not represented in a concrete representation (i.e. they are to be considered as basic elements of a Clifford algebra).

Elimination of γ-matrices admits one to obtain essential physical results: (1) discovery that the Dirac particle has a complicate internal structure (it is not a pointlike particle), (2) discovery that the internal degrees of freedom are described non-relativistically. These statements are very essential, but they are rather unexpected simultaneously, and one cannot understand them without a knowledge of the Sauter - Sommerfeld method. I tried to publish the physically essential parts of the paper [1, 2] in physical reviewed journals. Unfortunately, I failed, because, apparently, the reviewers did not know the Sauter - Sommerfeld method of the γ-matrices representation. They do not trust in mathematical correctness of my results, although experts in Clifford algebras, publishing me paper in the mathematical journal did.

References

[1] Is the Dirac particle composite? *e-print, /physics/0410045.*

[2] Is the Dirac particle completely relativistic? *e-print, /physics/0412032.*

Dirac Equation in Terms of Hydrodynamic Variables

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Abstract

The distributed system $\mathcal{S}_D$ described by the Dirac equation is investigated simply as a dynamic system, i.e. without usage of quantum principles. The Dirac equation is described in terms of hydrodynamic variables: 4-flux j^i, pseudo-vector of the spin S^i, an action $\hbar\varphi$ and a pseudo-scalar κ. In the quasi-uniform approximation, when all transversal derivatives (orthogonal to the flux vector j^i) are small, the system $\mathcal{S}_D$ turns to a statistical ensemble of classical concentrated systems $\mathcal{S}_{dc}$. Under some conditions the classical system $\mathcal{S}_{dc}$ describes a classical pointlike particle moving in a given electromagnetic field. In general, the world line of the particle is a helix, even if the electromagnetic field is absent. Both dynamic systems $\mathcal{S}_D$ and $\mathcal{S}_{dc}$ appear to be non-relativistic in the sense that the dynamic equations written in terms of hydrodynamic variables are not relativistically covariant with respect to them, although all dynamic variables are tensors or pseudo-tensors. They becomes relativistically covariant only after addition of a constant unit timelike vector f^i which should be considered as a dynamic variable describing a space-time property. This "constant" variable arises instead of γ-matrices which are removed by means of zero divizors in the course of the transformation to hydrodynamic variables. It is possible to separate out dynamic variables κ, κ^i responsible for quantum effects. It means that, setting $\kappa, \kappa^i \equiv 0$, the dynamic system $\mathcal{S}_D$ described by the Dirac equation turns to a statistical ensemble $\mathcal{E}_{Dqu}$ of classical dynamic systems $\mathcal{S}_{dc}$.

1 Introduction

The Dirac equation considered as a dynamic equation for a wave function in frames of quantum mechanics principles has been investigated almost completely, and it is hardly possible to add anything new. At the same time the Dirac equation considered as a dynamic equation for a distributed dynamic system $\mathcal{S}_D$ displays a series of such unexpected properties as existence of dynamic variables κ responsible for

quantum effects and appearance of a constant timelike vector describing a split of the space-time into the space and the time.

From mathematical point of view our investigation of the Dirac equation is simply a change of variables, when the Dirac four-component complex wave function is substituted by certain tensor variables having also eight independent real components. The tensor variables are: the 4-flux vector j^i, $i = 0, 1, 2, 3$, the spin pseudo-vector S^i, $i = 0, 1, 2, 3$, the action scalar φ, and a pseudo-scalar κ. These quantities will be referred to as hydrodynamic variables.

From physical standpoint the dynamic system $\mathcal{S}_D$ is considered as a distributed dynamic system describing in some way an electron (or positron) motion. The motion of a single particle is supposed to be stochastic, the variables j^i describing the mean 4-flux of particles. Interpretation of other dynamic variables of $\mathcal{S}_D$ is produced on the base of a comparison with a statistical ensemble of classical particles. The quantal correspondence principle, when a linear operator corresponds to any physical quantity, is not used. Other quantum principles are not used also. Instead of them one uses a more general statistical principle which asserts: *A set $\mathcal{E}$ (statistical ensemble) of many similar independent stochastic systems $\mathcal{S}_s$ is a deterministic dynamic system $\mathcal{S}_d$* [1]. The term "a stochastic system" means that experiments with a single system are irreproducible, and there are no dynamic equations for such a system. If $\mathcal{S}$ means either stochastic system $\mathcal{S}_s$, or a deterministic system $\mathcal{S}_d$, the statistical principle can be formulated in the form

$$\mathcal{E}[\mathcal{S}] \quad \text{is} \quad \mathcal{S}_d$$

The statistical principle can be applied both to quantum and classical systems. It is a more general statement, than the set of quantum principles. In particular, it can be valid for such stochastic systems which are neither quantum, nor classical. The main concept of this approach is the statistical ensemble considered as a dynamic system (not a probability density, or a probability amplitude). This approach will be referred to as *a statistical ensemble technique* (SET).

There is a reason for consideration of the Dirac equation without usage of quantum principles, i.e. simply as a dynamic equation for the system $\mathcal{S}_D$. Recently one suggested a hypothesis that the real space-time is a distorted Minkowski space-time, and the distortion of the space-time is a reason for quantum effects [2]. Distortion is such a deformation of the Minkowski space-time which transform one-dimensional world lines into three-dimensional world tubes. The world tubes of particles appear stochastic, and their statistical description coincides with quantum description provided the Planck constant $\hbar$ determines the space-time distortion (thickness of the tubes). This hypothesis was proved for a non-relativistic free particle. It is interesting to test the hypothesis in the case of a relativistic particle. For such a test it is necessary to separate the action describing the ensemble of Dirac particles into two parts: a classical part and a part responsible for quantum effects.

Another motive is as follows. The conventional quantum dynamics technique (QDT) is an axiomatic construction. QDT relates to SET approximately in the same way, as the axiomatic thermodynamics relates to the statistical physics. For

instance, the Brownian motion cannot be explained and understood from standpoint of thermodynamics. Thermodynamic axioms do not permit to do this. The statistical physics explains the Brownian motion phenomenon and restricts an application of thermodynamic principles, because a statistical approach is more general, than the approach of the axiomatic thermodynamics.

Something like that one can see in the field of the quantum dynamics. Problem of pair production is a principal problem of the high energy physics. In those areas, where the pair production is unessential and can be considered as a correction (for instance, in quantum electrodynamics) the quantum theory succeeds. In those areas, where pair production is a dominating effect, the quantum theory (in particular, QFT) failed.

For last fifty years the quantum field theory has not succeeded in solving the problem of pair production. A suspicion arises that this failure is not accidental. Maybe, the problem of pair production cannot be solved in framework of the quantum principles, and a more general statistical approach is necessary.

Classical limit of the Dirac equation is rather difficult to obtain, because it contains such non-classical quantities as Dirac γ-matrices, which hardly can be considered from the classical point of view.

Usually a quantum electron moving in a given electromagnetic field A_l, $l = 0, 1, 2, 3$ is described by the Dirac equation

$$-i\hbar\gamma^l\partial_l\psi + eA_l\gamma^l\psi + m\psi = 0, \qquad (1.1)$$

where ψ is a four-component complex wave function. The light speed c is chosen to be equal to 1. It is possible to transform the variables ψ and to describe this system in terms of variables φ, j^l, S^l, $(l = 0, 1, 2, 3)$, κ, defined by the relations

$$j^l = \bar{\psi}\gamma^l\psi, \qquad l = 0, 1, 2, 3, \qquad \bar{\psi} = \psi^*\gamma^0;$$
$$S^l = i\bar{\psi}\gamma_5\gamma^l\psi, \qquad l = 0, 1, 2, 3, \qquad \gamma_5 = \gamma^{0123} \equiv \gamma^0\gamma^1\gamma^2\gamma^3;$$
$$\partial_l\varphi = (\bar{\psi}\partial_l\psi - \partial_l\bar{\psi}\psi)(2i\bar{\psi}\psi)^{-1}, \qquad l = 0, 1, 2, 3; \qquad (1.2)$$
$$\cos\kappa = \bar{\psi}\psi(j^l j_l)^{-1/2};$$

Here γ^l, $l = 0, 1, 2, 3$ are the Dirac γ-matrices satisfying the commutation relation

$$\gamma^i\gamma^k + \gamma^k\gamma^i = 2g^{ik}, \qquad i, k = 0, 1, 2, 3, \qquad (1.3)$$

where $g^{ik} = \text{diag}(1, -1, -1, -1)$ is the metric tensor. Only two components of the pseudo-vector S^l are independent, because there are two identities

$$S^l S_l \equiv -j^l j_l, \qquad j^l S_l \equiv 0. \qquad (1.4)$$

Pseudo-vector S^l is treated as a spin pseudo-vector, because it is connected uniquely with the spin tensor $S^{ml,k}$, defined by the relation [3]

$$S^{ml,k} = \frac{1}{4}\bar{\psi}(\gamma^k\sigma^{lm} + \sigma^{lm}\gamma^k)\psi, \qquad \sigma^{lm} = \frac{i}{2}(\gamma^l\gamma^m - \gamma^m\gamma^l) \qquad (1.5)$$

The relation between S^i and $S^{ml,k}$ has the form

$$S^i = \frac{1}{3}g^{ij}\varepsilon_{jmlk}S^{ml,k}, \qquad i = 0, 1, 2, 3;$$

$$S^{ml,k} = -\frac{1}{2}\varepsilon^{imlk}S_i, \qquad m, l, k = 0, 1, 2, 3. \tag{1.6}$$

Here ε_{jmlk} and ε^{imlk} are Levi-Chivita pseudo-tensors ($\varepsilon_{0123} = 1$, $\varepsilon^{0123} = -1$).

Thus, the scalar φ, pseudo-scalar κ, vector j^l, and pseudo-vector S^l have 8 real independent components which are used instead of 8 real components (4 complex components) of the Dirac wave function ψ.

Realization of such a transformation is a goal of the present paper. Such a description can be regarded as a description in terms of hydrodynamic variables. Under some conditions the dynamic system $\mathcal{S}_D$ turns into a statistical ensemble $\mathcal{E}_{Dqu}$ of classical dynamic systems $\mathcal{S}_{dc}$. The $\mathcal{S}_{dc}$ can be interpreted as a classical analog of the Dirac electron. $\mathcal{E}_{Dqu}$ is described usually in terms of hydrodynamic variables j^l, φ, $\boldsymbol{\xi}$. The dynamic systems $\mathcal{S}_D$ and $\mathcal{E}_{Dqu}$ are distinguished by some terms of their Lagrangians, but not conceptually. Thus, one can consider a dynamic distinction between the Dirac dynamic system $\mathcal{S}_D$ and the corresponding classical ensemble.

In the second section some properties of the statistical ensemble $\mathcal{E}_{cl}$ are investigated. The third section is devoted to introduction of hydrodynamic variables. In the fourth section the action for the Dirac equation is written in terms of hydrodynamic variables. The fifth section is devoted to consideration of quasi-uniform state of the system $\mathcal{S}_D$. Relativistic invariance of dynamic equation in terms of hydrodynamic variables is discussed in the sixth section. The seventh section is devoted to investigation of the classical analog $\mathcal{S}_{dc}$ of the Dirac electron.

2 Statistical ensemble of classical dynamic systems

Let there be a classical system $\mathcal{S}$ described by the Lagrangian function $L(t, \mathbf{x}, d\mathbf{x}/dt)$, where $\mathbf{x} = \{x^\alpha\}$, $d\mathbf{x}/dt = \{dx^\alpha/dt\}$, $\alpha = 1, 2, \ldots n$ are generalized coordinates and velocities. Then, by definition a pure statistical ensemble $\mathcal{E}_{cl}$ of systems $\mathcal{S}$ is a set of similar independent systems $\mathcal{S}$. Its Lagrangian is a sum (integral) of Lagrangians L. The action for the ensemble $\mathcal{E}_{cl}$ has the form

$$\mathcal{A}_L[\mathbf{x}] = \int L(t, \mathbf{x}, d\mathbf{x}/dt)dtd\boldsymbol{\xi}, \qquad d\boldsymbol{\xi} = \prod_{\alpha=1}^{\alpha=n} d\xi_\alpha \tag{2.1}$$

where $\boldsymbol{\xi} = \{\xi_\alpha\}$, $\alpha = 1, 2, \ldots n$ are Lagrangian coordinates labelling the systems of the ensemble, and $\mathbf{x} = \mathbf{x}(t, \boldsymbol{\xi})$ is a function of t and $\boldsymbol{\xi}$.

The dynamic system $\mathcal{E}_{cl}$ can be considered as a fluid described in the Lagrangian coordinates (the time t and Lagrangian coordinates $\boldsymbol{\xi}$ are independent variables).

The same action written in the Euler coordinates ($t, \mathbf{x}$ are independent variables) has the form

$$\mathcal{A}_E[j, \varphi, \boldsymbol{\xi}] = \int \{\mathcal{L}(x, j) - \hbar j^i [\partial_i \varphi + g^\alpha(\boldsymbol{\xi}) \partial_i \xi_\alpha]\} d^{n+1}x, \qquad d^{n+1}x = \prod_{i=0}^{n} dx^i, \quad (2.2)$$

where $j = \{j^0, \mathbf{j}\}$, $\mathbf{j} = \{j^\alpha\}$, $\alpha = 1, 2, \ldots n$, φ, $\boldsymbol{\xi} = \{\xi_\alpha\}$, $\alpha = 1, 2, \ldots n$ are functions of $x = \{x^0, \mathbf{x}\}$, $x^0 = t$, $\mathbf{x} = \{x^\alpha\}$, $\alpha = 1, 2, \ldots n$. A summation is made over repeating indices: over Latin ones $(0 - n)$ and over Greek ones $(1 - n)$. All variables $j = \{j^0, \mathbf{j}\}$, φ, $\boldsymbol{\xi}$ are functions of x. The function $\mathcal{L}(x, j)$ is defined by the relation

$$\mathcal{L}(x, j) = j^0 L(x^0, \mathbf{x}, \mathbf{j}/j^0) \qquad (2.3)$$

where L is the Lagrangian of a single system $\mathcal{S}$. The variables $j = \{j^0, \mathbf{j}\}$ describe a flux of particles in the $(n + 1)$-dimensional space V of coordinates x. They are connected with the variables $\mathbf{x}(t, \boldsymbol{\xi})$, $d\mathbf{x}(t, \boldsymbol{\xi})/dt$ by means of relations

$$j^0 = \det \parallel \xi_{\alpha,\beta} \parallel, \qquad \xi_{\beta,\alpha} \equiv \partial_\alpha \xi_\beta, \qquad \alpha, \beta = 1, 2, \ldots n, \qquad \mathbf{j} = j^0 \frac{d\mathbf{x}}{dt} \qquad (2.4)$$

and the functions $\boldsymbol{\xi} = \boldsymbol{\xi}(x)$ are determined as solutions of the equations

$$x^\alpha = x^\alpha(t, \boldsymbol{\xi}), \qquad \alpha = 1, 2, \ldots n.$$

Functions $g^\alpha(\boldsymbol{\xi})$ are arbitrary functions of Lagrangian coordinates ξ_α, $\alpha = 1, 2, \ldots n$. Appearance of arbitrary functions is a result of integration of some dynamic equations arising in the course of transformation from the action (2.1) to the action (2.2) (see Appendix A). In turn a possibility of such an integration is connected with an invariance of dynamic equation with respect to arbitrary transformation of Lagrangian coordinates $\boldsymbol{\xi}$

$$\xi_\alpha \to \tilde{\xi}_\alpha = \tilde{\xi}_\alpha(\boldsymbol{\xi}), \qquad \det \parallel \partial \tilde{\xi}_\beta / \partial \xi_\alpha \parallel = 1, \qquad \alpha, \beta = 1, 2, \ldots \qquad (2.5)$$

Flux $j = \{j^0, \mathbf{j}\}$ is invariant with respect to the renumbering transformation (2.5).

A statistical ensemble of classical systems is described conventionally by the distribution function $F(t, \mathbf{x}, \mathbf{p})$ in the phase space of coordinates $\mathbf{x}$ and momenta $\mathbf{p}$. For the pure statistical ensemble the distribution function has a special form

$$F(t, \mathbf{x}, \mathbf{p}) = j^0(t, \mathbf{x})\delta[\mathbf{x} - \mathbf{P}(t, \mathbf{x})] \qquad (2.6)$$

where

$$\mathbf{p} = \{p_\alpha\}, \qquad p_\alpha = \partial L / \partial(dx^\alpha/dt), \qquad \alpha = 1, 2, \ldots n \qquad (2.7)$$

$\mathbf{P}(t, \mathbf{x}) = \{P_\alpha(t, \mathbf{x})\}$, $\alpha = 1, 2, \ldots n$ is a set of functions depending only on t and $\mathbf{x}$, and δ denotes the Dirac δ-function. Evolution of this ensemble is described by the Liouville equation of the form

$$\frac{\partial F}{\partial t} + \frac{\partial H}{\partial p_\alpha} \frac{\partial F}{\partial x^\alpha} - \frac{\partial H}{\partial x^\alpha} \frac{\partial F}{\partial p_\alpha} = 0, \qquad (2.8)$$

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where $H = H(t, \mathbf{x}, \mathbf{p})$ is the Hamiltonian function of the dynamic system $\mathcal{S}$. Dynamic equations for the variables j^0, $\mathbf{P}(t, \mathbf{x})$ can be obtained by means of a substitution of Eq. (2.6) into Eq.(2.8).

Among three ways $[(2.1),(2.2)$ and $(2.8)]$ describing the pure statistical ensemble, the action (2.2) is most convenient for comparison with action of the system $\mathcal{S}_D$.

Dynamic equations, determined by the action (2.2), have the form

$$\frac{\delta\mathcal{A}}{\delta\varphi} = \hbar\partial_i j^i = 0, \tag{2.9}$$

$$\frac{\delta\mathcal{A}}{\delta j^i} = \frac{\partial\mathcal{L}}{\partial j^i} - \hbar\partial_i\varphi - \hbar g^\alpha(\boldsymbol{\xi})\xi_{\alpha,i} = 0, \qquad i = 0, 1, \dots n; \tag{2.10}$$

$$\frac{\delta\mathcal{A}}{\delta\xi_\alpha} = -\hbar\left(\frac{\partial g^\beta(\boldsymbol{\xi})}{\partial\xi_\alpha} - \frac{\partial g^\alpha(\boldsymbol{\xi})}{\partial\xi_\beta}\right) j^i\partial_i\xi_\beta = 0, \qquad \alpha = 1, \dots n; \tag{2.11}$$

j^l, $l = 0, 1, \dots n$ is the current $(n+1)$-vector in the space V. The vector field j^i is tangent to the trajectories of systems in V. According to Eq.(2.3) the vector

$$p_l = \frac{\partial\mathcal{L}}{\partial j^l}, \qquad l = 0, 1, \dots n; \qquad p_0 = \left[L - \frac{\partial L}{\partial(dx^\alpha/dt)}\right]_{dx^\alpha/dt=j^\alpha/j^0};$$

$$p_\alpha = \left[\frac{\partial L}{\partial(dx^\alpha/dt)}\right]_{dx^\alpha/dt=j^\alpha/j^0}, \qquad \alpha = 1, 2, \dots n \tag{2.12}$$

associates with the canonical momentum of a single system of the ensemble. Thus, according to Eq.(2.10)

$$p_i = \hbar\left(\partial_i\varphi + g^\alpha(\xi)\boldsymbol{\xi}_{\alpha,i}\right), \qquad i = 0, 1, 2, \dots n \tag{2.13}$$

where $\hbar$ is a constant having a dimensionality of the action. In this case φ, g^α, and $\boldsymbol{\xi}$ can be considered as dimensionless quantities. $\hbar$ can be treated as the Planck constant, although in the given case it has no quantum meaning.

Eliminating variables φ and $\boldsymbol{\xi}$ from Eqs. (2.10), (2.11), one obtains the equations

$$j^i[\partial_i p_k - \partial_k p_i] = j^i[\partial_i\frac{\partial\mathcal{L}(x, j)}{\partial j^k} - \partial_k\frac{\partial\mathcal{L}(x, j)}{\partial j^i}] = 0, \qquad k = 0, 1, \dots n \tag{2.14}$$

Together with Eq.(2.9) the equations (2.14) form a system of $n+1$ dynamic equations for some kind of a fluid described by the current vector j^i.

At the linear transformation of coordinates x^i, $i = 0, 1, \dots n$ the current vector j^l, $l = 0, 1, \dots n$ transforms as a vector. In this case the Lagrangian coordinates transform as scalars. But the consideration of $\boldsymbol{\xi}$ as scalars is rather conventional, because of the renumbering transformation (2.5).

At the transformation (2.5) one obtains

$$j^l g^\alpha(\boldsymbol{\xi})\partial_l\xi_\alpha \rightarrow j^l\tilde{g}^\alpha(\tilde{\boldsymbol{\xi}})\partial_l\tilde{\xi}_\alpha, \qquad \tilde{g}^\alpha(\tilde{\boldsymbol{\xi}}) = g^\beta(\boldsymbol{\xi})\frac{\partial\xi_\beta}{\partial\tilde{\xi}_\alpha} \tag{2.15}$$

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In other words, $g^\alpha(\boldsymbol{\xi})$ transform at the renumbering transformation as components of a vector in the space of $\boldsymbol{\xi}$. Combining any linear transformation of coordinates x^i with some renumbering transformation, one can ascribe practically arbitrary transformation properties to the Lagrangian coordinates $\boldsymbol{\xi}$.

Any renumbering transformation (2.5) is a kind of a gauge transformation, because the renumbering changes a description of the state of the statistical ensemble without changing the state itself. It is easy to verify that a set of the renumbering transformations forms a group.

For the pure statistical ensemble of classical pointlike charged particles moving in the given electromagnetic field the action (2.2) takes the form

$$\mathcal{A}_{cl}[j,\xi,\varphi] = \int [-m\sqrt{j^l j_l} - eA_l j^l - \hbar j^i(\partial_i\varphi + g^\alpha(\boldsymbol{\xi})\xi_{\alpha,i})]d^4x \qquad (2.16)$$

where the speed of the light $c = 1$, and summation is made over repeated indices (0 - 3) for the Latin indices and (1 - 3) for the Greek ones.

3 Transformation of variables

Transforming the Dirac equation (1.1) to the new variables (1.2), one uses the action for the equation (1.1)

$$\mathcal{A}_D[\bar{\psi},\psi] = \int (-m\bar{\psi}\psi + \frac{i}{2}\hbar\bar{\psi}\gamma^l\partial_l\psi - \frac{i}{2}\hbar\partial_l\bar{\psi}\gamma^l\psi - eA_l\bar{\psi}\gamma^l\psi)d^4x \qquad (3.1)$$

Expressing the variables (1.2) through the wave function ψ, one uses the technique, where the wave function is considered as a Clifford number with 16 base units: I, γ^i, γ^{ik}, γ^{ikl}, γ^{iklm} (all indices are different, and Clifford numbers satisfy (1.3)). Reduction of the Clifford numbers is realized by means of zero divisors [4,5].

Let us introduce matrices γ_5, $\boldsymbol{\sigma} = \{\sigma_\alpha\}$, $\alpha = 1,2,3$

$$\gamma_5 = \gamma^{0123} \equiv \gamma^0\gamma^1\gamma^2\gamma^3, \qquad \sigma_1 = -i\gamma^{23}, \qquad \sigma_2 = -i\gamma^{31}, \qquad \sigma_3 = -i\gamma^{12},$$

$$\gamma_5\sigma_\alpha = \sigma_\alpha\gamma_5, \qquad \gamma^{0\alpha} = -i\gamma_5\sigma_\alpha, \qquad \alpha = 1,2,3; \qquad (3.2)$$

$$\gamma^0\boldsymbol{\sigma} = \boldsymbol{\sigma}\gamma^0, \qquad \gamma^0\gamma_5 = -\gamma_5\gamma^0$$

According to Eqs. (1.3), (3.2) the matrices $\boldsymbol{\sigma} = \{\sigma_\alpha\}$, $\alpha = 1,2,3$ satisfy the relation

$$\sigma_\alpha\sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma, \qquad \alpha,\beta = 1,2,3 \qquad (3.3)$$

where $\varepsilon_{\alpha\beta\gamma}$ is the antisymmetric pseudo-tensor of Levi-Chivita ($\varepsilon_{123} = 1$).

Let us define the wave function ψ in the form

$$\psi = Ae^{i\varphi + \frac{1}{2}\gamma_5\kappa}e^{-\frac{i}{2}\gamma_5\boldsymbol{\sigma}\boldsymbol{\eta}}e^{\frac{i}{2}\boldsymbol{\sigma}\boldsymbol{\zeta}}\Pi$$

$$\bar{\psi} = \Pi\psi^*\gamma^0, \qquad \psi^* = A\Pi e^{-\frac{i}{2}\boldsymbol{\sigma}\boldsymbol{\zeta}}e^{-\frac{i}{2}\gamma_5\boldsymbol{\sigma}\boldsymbol{\eta}}e^{-i\varphi - \frac{1}{2}\gamma_5\kappa} \qquad (3.4)$$

where (*) means the Hermitian conjugation, and

$$\Pi = \frac{1}{4}(1 + \gamma^0)(1 + \mathbf{z}\boldsymbol{\sigma}), \qquad \mathbf{z} = \{z^\alpha\} = \text{const}, \qquad \alpha = 1, 2, 3; \qquad \mathbf{z}^2 = 1 \quad (3.5)$$

is a zero divisor. The quantities A, κ, φ, $\boldsymbol{\eta} = \{\eta^\alpha\}$, $\boldsymbol{\zeta} = \{\zeta^\alpha\}$, $\alpha = 1, 2, 3$ are nine real parameters, determining the wave function ψ.

Using relations (3.2), (3.3), (3.5), it is easy to verify that

$$\Pi^2 = \Pi, \qquad \gamma_0 \Pi = \Pi, \qquad \mathbf{z}\boldsymbol{\sigma}\Pi = \Pi, \qquad \Pi\gamma_5\Pi = 0,$$

$$\Pi\sigma_\alpha\Pi = z^\alpha\Pi, \qquad \alpha = 1, 2, 3. \tag{3.6}$$

Generally, the wave functions ψ, ψ^* defined by Eq.(3.4) are 4×4 complex matrices. In the proper representation, where Π has the form

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.7}$$

the ψ, ψ^* have the form

$$\psi = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix}, \qquad \psi^* = \begin{pmatrix} \psi_1^* & \psi_2^* & \psi_3^* & \psi_4^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.8}$$

Their product $\psi^* O\psi$ has the form

$$\psi^* O\psi = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = a\Pi = \Pi a \tag{3.9}$$

where O is an arbitrary 4×4 matrix and a is a complex quantity. If f is an analytical function having the property $f(0) = 0$, then the function $f(a\Pi)$ of a 4×4 matrix of the type (3.9) is a matrix $f(a)\Pi$ of the same type. For this reason one will not distinguish between the complex quantity a and the complex 4×4 matrix $a\Pi$. In the final expressions of the type $a\Pi$ (a is a complex quantity) the multiplier Π will be omitted.

By means of relations (3.2), (3.6) one can reduce any Clifford number $\Pi O\Pi$ to the form (3.9), without using any concrete form of the γ-matrix representation. This property will be used in our calculations.

By means of (3.4) the variables $\bar{\psi}\psi$, j^l, S^l, $l = 0, 1, 2, 3$ defined by expressions (1.2) can be presented in the form

$$\bar{\psi}\psi = \psi^*\gamma^0\psi = A^2\Pi e^{\gamma_5\kappa}\Pi = A^2\cos\kappa\,\Pi \tag{3.10}$$

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$$j^0\Pi = \bar{\psi}\gamma^0\psi = A^2\Pi e^{-\frac{i}{2}\boldsymbol{\sigma\zeta}}e^{-i\gamma_5\boldsymbol{\sigma\eta}}e^{\frac{i}{2}\boldsymbol{\sigma\zeta}}\Pi = A^2\Pi e^{-i\gamma_5\boldsymbol{\Sigma\eta}}\Pi = A^2\cosh(\eta)\Pi \qquad (3.11)$$

where

$$\boldsymbol{\Sigma} = \{\Sigma_1, \Sigma_2, \Sigma_3\}, \qquad \Sigma_\alpha = e^{-\frac{i}{2}\boldsymbol{\sigma\zeta}}\sigma_\alpha e^{\frac{i}{2}\boldsymbol{\sigma\zeta}}, \qquad \alpha = 1, 2, 3; \qquad (3.12)$$

$$\eta = \sqrt{\boldsymbol{\eta}^2} = \sqrt{\eta^\alpha\eta^\alpha}$$

The matrix Σ_α satisfy the same relations (3.3), as σ_α do.

In the same way one obtains

$$
\begin{aligned}
j^\alpha\Pi &= \psi^*\gamma^{0\alpha}\psi\Pi = A^2\Pi e^{-\frac{i}{2}\gamma_5\boldsymbol{\Sigma\eta}}(-i\gamma_5\Sigma_\alpha)e^{-\frac{i}{2}\gamma_5\boldsymbol{\Sigma\eta}}\Pi = \\
&= A^2\Pi(\cosh\tfrac{\eta}{2} - i\gamma_5\boldsymbol{\Sigma v}\sinh\tfrac{\eta}{2})(-i\gamma_5\Sigma_\alpha)(\cosh\tfrac{\eta}{2} - i\gamma_5\boldsymbol{\Sigma v}\sinh\tfrac{\eta}{2})\Pi = \\
&= A^2\sinh(\eta)v^\alpha\Pi, \qquad \alpha = 1, 2, 3,
\end{aligned} \qquad (3.13)
$$

where

$$\mathbf{v} = \{v^\alpha\}, \qquad v^\alpha = \eta^\alpha/\eta, \qquad \alpha = 1, 2, 3; \qquad \mathbf{v}^2 = 1. \qquad (3.14)$$

$$
\begin{aligned}
S^0\Pi &= \psi^*(-i\gamma_5)\psi = A^2\Pi(-i\gamma_5)e^{-i\gamma_5\boldsymbol{\Sigma\eta}}\Pi = \\
&= A^2\Pi\sinh(\eta)\boldsymbol{\Sigma v}\Pi = A^2\sinh(\eta)\boldsymbol{\xi}\Pi,
\end{aligned}
$$

$$
\begin{aligned}
S^\alpha\Pi &= \psi^*\sigma_\alpha\psi\Pi = A^2\Pi e^{\frac{i}{2}\gamma_5\boldsymbol{\Sigma\eta}}\Sigma_\alpha e^{\frac{i}{2}\gamma_5\boldsymbol{\Sigma\eta}}\Pi = \\
&\quad A^2[\xi^\alpha + (\cosh\eta - 1)v^\alpha(\mathbf{v}\boldsymbol{\xi})]\Pi, \qquad \alpha = 1, 2, 3.
\end{aligned} \qquad (3.15)
$$

Here $\boldsymbol{\xi} = \{\xi^\alpha\}$, $\alpha = 1, 2, 3$, are determined by the relation

$$\xi^\alpha\Pi = \Pi\Sigma_\alpha\Pi, \qquad \alpha = 1, 2, 3 \qquad (3.16)$$

It follows from Eqs.(3.11), (3.13)

$$j^i j_i\Pi = A^4\Pi, \qquad A = (j^l j_l)^{1/4} \equiv \rho^{1/2} \qquad (3.17)$$

According to Eqs.(3.12), (3.16) one obtains

$$\boldsymbol{\xi}\Pi = \{[\mathbf{z} - \mathbf{n}(\mathbf{nz})]\cos\zeta + (\mathbf{z} \times \mathbf{n})\sin\zeta + \mathbf{n}(\mathbf{nz})\}\Pi \qquad (3.18)$$

where

$$\boldsymbol{\xi} = \{\xi^\alpha\}, \qquad \alpha = 1, 2, 3; \qquad \zeta = \sqrt{\boldsymbol{\zeta}^2} = \sqrt{\zeta^\alpha\zeta^\alpha},$$

$$\mathbf{n} = \boldsymbol{\zeta}/\zeta, \qquad \mathbf{n}^2 = 1 \qquad (3.19)$$

The wave function (3.4) depends on 9 real parameters: A, φ, κ, η_α, ζ_α, $\alpha = 1, 2, 3$. The wave function has 8 real independent components, and not all parameters $A, \kappa, \varphi, \eta^\alpha, \zeta^\alpha$, $\alpha = 1, 2, 3$ are independent. Let us fix one of the parameters, namely let us set

$$\zeta = \pi \qquad (3.20)$$

Then Eq.(3.18) takes the form

$$\boldsymbol{\xi} = 2\mathbf{n}(\mathbf{nz}) - \mathbf{z} \qquad (3.21)$$

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It can be solved with respect to $\mathbf{n} = \boldsymbol{\zeta}/\pi$. One obtains

$$\boldsymbol{\zeta}/\pi = \mathbf{n} = (\boldsymbol{\xi} + \mathbf{z})[2(1 + \mathbf{z}\boldsymbol{\xi})]^{-1/2} \tag{3.22}$$

Using Eqs.(3.11), (3.13), (3.17), (3.22), one can express parameters $A, \varphi, \kappa, \eta^\alpha, \zeta^\alpha$, $\alpha = 1, 2, 3$ describing the wave function, through the variables $j^i, S^i, \varphi, \kappa$. One obtains

$$\xi^\alpha = \rho^{-1}[S^\alpha - \frac{j^\alpha(S^\beta j^\beta)}{j^0(j^0 + \rho)}], \qquad \alpha = 1, 2, 3; \qquad \rho \equiv \sqrt{j^l j_l} \tag{3.23}$$

$$\cosh \eta = j^0/\rho, \qquad v^\alpha = \frac{\eta^\alpha}{\eta} = \frac{j^\alpha}{\sqrt{(j^0)^2 - j^l j_l}}, \qquad \alpha = 1, 2, 3 \tag{3.24}$$

Using Eqs.(3.4), (3.20) - (3.24), one can present the wave function (3.4) in terms of variables $j^i, S^i, \kappa, \varphi$:

$$\psi = \frac{ie^{i\varphi + \frac{1}{2}\gamma_5 \kappa}}{2\sqrt{(1 + \boldsymbol{\xi})(j^0 + \rho)}}[(j^0 + \rho)(1 + \boldsymbol{\xi}\boldsymbol{\sigma}) - i\gamma_5 S^0 - i\gamma_5(\mathbf{j} + i\rho^{-1}\mathbf{j} \times \mathbf{S})\boldsymbol{\sigma}]\Pi \tag{3.25}$$

where $\boldsymbol{\xi}$ is expressed through j^i, S^i by means of Eq.(3.23), the symbol $\times$ means a vector product, and Π is defined by Eq.(3.5).

Any expression of the form $\bar{\psi}O\psi$, where O is an arbitrary combination of γ-matrices, can be expressed through the variables S^i, j^i, κ. For instance,

$$\frac{i}{2}\bar{\psi}(\gamma^l \gamma^k - \gamma^k \gamma^l)\psi = \frac{1}{2\rho}[(j^k S^l - j^l S^k)\sin\kappa + \varepsilon^{lkim} j_i S_m \cos\kappa]\Pi. \tag{3.26}$$

4   Transformation of the action

Now let us calculate the expression (two middle terms of Eq.(3.1))

$$\begin{aligned}\frac{i}{2}\hbar\bar{\psi}\gamma^l \partial_l \psi + \text{h.c.} = \quad & \frac{i}{2}\hbar\psi^* \left[(\partial_0 - i\gamma_5 \boldsymbol{\sigma}\boldsymbol{\nabla})\left(i\varphi + \frac{1}{2}\gamma_5\kappa\right)\right]\psi + \text{h.c.} \\ & + \frac{i}{2}\hbar A^2 \Pi e^{-\frac{i}{2}\boldsymbol{\sigma}\boldsymbol{\zeta}} e^{-\frac{i}{2}\gamma_5 \boldsymbol{\sigma}\boldsymbol{\eta}}(\partial_0 - i\gamma_5 \boldsymbol{\sigma}\boldsymbol{\nabla})(e^{-\frac{i}{2}\gamma_5 \boldsymbol{\sigma}\boldsymbol{\eta}} e^{\frac{i}{2}\boldsymbol{\sigma}\boldsymbol{\zeta}})\Pi + \text{h.c.} \end{aligned} \tag{4.1}$$

where "h.c." means the term obtained from the previous one by the Hermitian conjugation. Using relations (3.11)-(3.15), the expression (4.1) reduces to the form

$$\begin{aligned}\frac{i}{2}\hbar\bar{\psi}\gamma^l \partial_l \psi + \text{h.c.} = \quad & -\hbar j^l \partial_l \varphi - \frac{1}{2}\hbar S^l \partial_l \kappa \\ & + \frac{i}{2}\hbar A^2 \Pi e^{-\frac{i}{2}\gamma_5 \boldsymbol{\Sigma}\boldsymbol{\eta}}(\partial_0 - i\gamma_5 \boldsymbol{\Sigma}\boldsymbol{\nabla})e^{-\frac{i}{2}\gamma_5 \boldsymbol{\Sigma}\boldsymbol{\eta}}\Pi + \text{h.c.} \\ & + \frac{i}{2}\hbar A^2 \Pi e^{-i\gamma_5 \boldsymbol{\Sigma}\boldsymbol{\eta}} e^{-\frac{i}{2}\boldsymbol{\sigma}\boldsymbol{\zeta}} \partial_0 e^{\frac{i}{2}\boldsymbol{\sigma}\boldsymbol{\zeta}}\Pi + \text{h.c.} \\ & + \frac{i}{2}\hbar A^2 \Pi e^{-\frac{i}{2}\gamma_5 \boldsymbol{\Sigma}\boldsymbol{\eta}}(-i\gamma_5 \Sigma_\alpha)e^{-\frac{i}{2}\gamma_5 \boldsymbol{\Sigma}\boldsymbol{\eta}} e^{-\frac{i}{2}\boldsymbol{\sigma}\boldsymbol{\zeta}} \partial_\alpha e^{\frac{i}{2}\boldsymbol{\sigma}\boldsymbol{\zeta}}\Pi + \text{h.c.} \end{aligned} \tag{4.2}$$

where the matrix $\boldsymbol{\Sigma}$ is not differentiated.

Taking into account the fourth Eq.(3.6) and Eqs. (3.11), (3.13), the expression (4.2) reduces to the form

$$\begin{aligned}\frac{i}{2}\hbar\bar{\psi}\gamma^l \partial_l \psi + \text{h.c.} = \quad & -\hbar j^l \partial_l \varphi - \frac{1}{2}\hbar S^l \partial_l \kappa + \frac{i}{2}\hbar j^l \Pi e^{-\frac{i}{2}\boldsymbol{\sigma}\boldsymbol{\zeta}} \partial_l e^{\frac{i}{2}\boldsymbol{\sigma}\boldsymbol{\zeta}}\Pi + \text{h.c.} \\ & + \frac{i}{2}\hbar A^2 \Pi e^{-\frac{i}{2}\gamma_5 \boldsymbol{\Sigma}\boldsymbol{\eta}}(\partial_0 - i\gamma_5 \boldsymbol{\Sigma}\boldsymbol{\nabla})e^{-\frac{i}{2}\gamma_5 \boldsymbol{\Sigma}\boldsymbol{\eta}}\Pi + \text{h.c.} \end{aligned} \tag{4.3}$$

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Substituting the relation (3.20) into the third term of Eq.(4.3), one obtains by means of Eq.(3.22)

$$\frac{i}{2}\hbar j^l \Pi e^{-\frac{i}{2}\boldsymbol{\sigma}\boldsymbol{\varsigma}} \partial_l e^{\frac{i}{2}\boldsymbol{\sigma}\boldsymbol{\varsigma}} \Pi + \text{h.c.} = \frac{i}{2}\hbar j^l \Pi \sigma_\alpha \sigma_\beta n_\alpha \partial_l n_\beta \Pi + \text{h.c.}$$

$$= -\frac{\hbar j^l}{2(1+\boldsymbol{\xi}\mathbf{z})}\varepsilon_{\alpha\beta\gamma}\xi^\alpha \partial_l \xi^\beta z^\gamma \Pi = -\hbar j^l g_\alpha(\boldsymbol{\xi})\partial_l \xi^\alpha \Pi \quad (4.4)$$

where

$$g_\alpha(\boldsymbol{\xi}) = -\frac{1}{2}\varepsilon_{\alpha\beta\gamma}\xi^\beta z^\gamma (1+\boldsymbol{\xi}\mathbf{z})^{-1} \quad (4.5)$$

Calculation of the last term of Eq.(4.3) leads to the following result

$$F_4 = \frac{i}{2}\hbar A^2 \Pi e^{-\frac{i}{2}\gamma_5 \boldsymbol{\Sigma}\boldsymbol{\eta}}(\partial_0 - i\gamma_5 \boldsymbol{\Sigma}\nabla)e^{-\frac{i}{2}\gamma_5 \boldsymbol{\Sigma}\boldsymbol{\eta}}\Pi + \text{h.c.} = \\ -\frac{1}{2}\hbar A^2 \varepsilon_{\alpha\beta\gamma}[\partial_\alpha \eta v^\beta \xi^\gamma + \sinh\eta\,\partial_\alpha v^\beta \xi^\gamma + 2\sinh^2(\frac{\eta}{2})v^\alpha \partial_0 v^\beta \xi^\gamma]\Pi \quad (4.6)$$

Let us introduce two constant vectors

$$f^i = \{1,0,0,0,\}, \qquad z^i = \{0, z^1, z^2, z^3\} \quad (4.7)$$

which satisfy the following conditions

$$f^l f_l = 1, \qquad f^l z_l = 0, \qquad z^l z_l = -1. \quad (4.8)$$

By means of relations (3.11), (3.13) and (4.7) the expression (4.6) reduces to the form

$$F_4 = -\frac{\hbar}{2(\rho + f^s j_s)}\varepsilon_{iklm}[\partial^k(j^i + f^i \rho)](j^l + f^l \rho)[\xi^m - f^m(\xi^s f_s)] \quad (4.9)$$

where ε_{iklm} is the Levi-Chivita pseudo-tensor ($\varepsilon_{0123} = 1$) and $\xi^m = \{\xi^0, \boldsymbol{\xi}\}$. The value of ξ^0 is unessential.

Let us introduce the unit timelike vector

$$q^i \equiv \frac{j^i + f^i \rho}{\sqrt{(j^l + f^l \rho)(j_l + f_l \rho)}} = \frac{j^i + f^i \rho}{\sqrt{2\rho(\rho + j^l f_l)}} \quad (4.10)$$

and two spacelike vectors

$$\nu^i = \xi^i - f^i(\xi^s f_s), \qquad i = 0,1,2,3; \qquad \nu^i \nu_i = -1, \quad (4.11)$$

$$\mu^i \equiv \frac{\nu^i}{\sqrt{-(\nu^l + z^l)(\nu_l + z_l)}} = \frac{\nu^i}{\sqrt{2(1 - \nu^l z_l)}} = \frac{\nu^i}{\sqrt{2(1 + \boldsymbol{\xi}\mathbf{z})}}. \quad (4.12)$$

Then according to Eqs.(4.3), (4.4), (4.9) the relation (4.2) can be presented in the form

$$\frac{i}{2}\hbar\bar{\psi}\gamma^l \partial_l \psi + \text{h.c.} = -\hbar j^l[\partial_l \varphi + g_\alpha(\boldsymbol{\xi})\partial_l \xi^\alpha] - \frac{\hbar}{2}S^l \partial_l \kappa + \hbar\rho\varepsilon_{iklm}q^i(\partial^k q^l)\nu^m \quad (4.13)$$

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where the second term can be written also in the covariant form

$$-\hbar j^l g_\alpha(\xi)\partial_l \xi^\alpha = -\hbar j^l \varepsilon_{jksm}\mu^j(\partial_l\mu^k)f^s z^m \qquad (4.14)$$

Now by means of relations (3.10)-(3.12) one can present the action (3.1) in the form

$$\mathcal{A}_D[j,\varphi,\kappa,\boldsymbol{\xi}] = \int (\mathcal{L}_{cl} + \mathcal{L}_{q1} + \mathcal{L}_{q2})d^4x \qquad (4.15)$$

$$\mathcal{L}_{cl} = -m\rho - eA_l j^l - \hbar j^i[\partial_i\varphi + g_\alpha(\boldsymbol{\xi})\partial_i\xi^\alpha], \qquad \rho \equiv \sqrt{j^l j_l} \qquad (4.16)$$

$$\mathcal{L}_{q1} = 2m\rho\sin^2(\frac{\kappa}{2}) - \frac{\hbar}{2}S^l\partial_l\kappa, \qquad (4.17)$$

$$\mathcal{L}_{q2} = \hbar\rho\varepsilon_{iklm}q^i(\partial^k q^l)\nu^m \qquad (4.18)$$

$g_\alpha(\boldsymbol{\xi})$ is defined by the equation (4.5). S^l, $l = 0,1,2,3$ are considered as functions of j^l and $\boldsymbol{\xi}$, defined by the relations

$$S^0 = \mathbf{j}\boldsymbol{\xi}, \qquad S^\alpha = \rho\xi^\alpha + \frac{(\mathbf{j}\boldsymbol{\xi})j^\alpha}{\rho + j^k f_k}, \qquad \alpha = 1,2,3 \qquad (4.19)$$

obtained from Eqs (1.4), (3.23). Not all variables $\boldsymbol{\xi} = \{\xi^\alpha\}$, $\alpha = 1,2,3$ are independent, because they satisfy the restriction

$$\boldsymbol{\xi}^2 = \xi^\alpha\xi^\alpha = 1 \qquad (4.20)$$

as it follows from Eqs.(3.5), (3.19), (3.21). Variation of the action (4.15) with respect to ξ^α, $\alpha = 1,2,3$ under the condition (4.20) leads to the dynamic equations

$$\frac{\delta\mathcal{A}_D}{\delta\xi^\alpha}(\delta^{\alpha\beta} - \xi^\alpha\xi^\beta) = 0, \qquad \beta = 1,2,3 \qquad (4.21)$$

There are only two independent equations among the equations (4.21), because a contraction of Eq.(4.21) with ξ^β leads to an identity. Note that the term (4.16) of the action (4.15) coincides with the Lagrangian of the action (2.16) for the statistical ensemble of classical pointlike particles.

Eliminating κ from the action (4.15) and the dynamic equation $\delta\mathcal{A}_D/\delta\kappa = 0$, one can write the action (4.15) in the form

$$\mathcal{A}_D[j,\varphi,\boldsymbol{\xi}] = \int (\mathcal{L}''_{cl} + \mathcal{L}''_{q1} + \mathcal{L}_{q2})d^4x \qquad (4.22)$$

$$\mathcal{L}''_{cl} + \mathcal{L}''_{q1} = -\sqrt{m^2 j^i j_i - \frac{\hbar^2}{4}(\partial_i S^i)^2} - \frac{\hbar}{2}\partial_i S^i \arcsin\left(\frac{\hbar\partial_i S^i}{2m\rho}\right) -$$

$$-ej^i A_i - \hbar j^i[\partial_i\varphi + g_\alpha(\boldsymbol{\xi})\partial_i\xi^\alpha] \qquad (4.23)$$

S^i, $i = 0,1,2,3$ are functions of $\boldsymbol{\xi}$ and j^i which are determined by Eqs.(4.19). $\mathcal{L}_{q2}$ is determined by Eq. (4.18).

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5 Classical Part of the Action

The classical part of the action $\mathcal{A}_D$ can be separated out either by vanishing $\hbar$, or by a usage of the quasi-uniform state of the system, when all spatial gradients are small, and quantum effects disappear.

In the non-relativistic case a quasi-uniform state satisfies the condition

$$\mid \frac{\hbar}{m}\nabla u \mid \ll \mid u \mid, \qquad u = j^0, j^1, j^2, j^3, \kappa. \tag{5.1}$$

which is written in the coordinate system, where $\mid \mathbf{j} \mid \ll \mid j^0 \mid$. In the general case such a coordinate system does not exist, and the condition (5.1) is written in the form

$$\mid \frac{\hbar}{m}l^i\partial_i u \mid \ll \mid u \mid, \qquad u = j^0, j^1, j^2, j^3, \kappa, \tag{5.2}$$

where l^i is any unit vector orthogonal to j^i

$$l^i j_i = 0, \qquad l^i l_i = -1. \tag{5.3}$$

It means that all derivatives across the direction of the vector j^i are small.

Let us represent all derivatives in the Eq.(4.18) in the form

$$\partial^k q^l = \partial^k_\perp q^l + \frac{j^k j^s}{\rho^2}\partial_s q^l, \qquad \partial^k_\perp \equiv \partial^k - \frac{j^k j^s}{\rho^2}\partial_s \tag{5.4}$$

The first term in rhs of the first equation (5.4) describes a transversal part of the derivative, i.e. a derivative in the direction orthogonal to the vector j^i.

According to Eq.(5.2) all terms containing the transversal derivative $\partial^k_\perp$ are small with respect to the first term of Eq.(4.16). Indeed, an estimation of the transversal part of the Lagrangian (4.18) has the form

$$\mid \hbar\rho\varepsilon_{iklm}q^i(\partial^k_\perp q^l)\nu^m \mid \cong \hbar\rho \mid \sum_{s=0}^{3} l^i_{(s)}\partial_i q^s \mid \ll m\rho, \tag{5.5}$$

where $l^i_{(s)}, s = 0, 1, 2, 3$ are unit vectors orthogonal to j^i

$$l^l_{(s)} = \frac{\varepsilon^{.j}_{i.sm}q^i\nu^m(\delta^l_j - \rho^{-2}j_j j^l)}{\mid \varepsilon_{ijsm}q^i\nu^m(g^{l'j} - \rho^{-2}j^j j^{l'})\varepsilon_{i'l'sm'}q^{i'}\nu^{m'} \mid^{1/2}}, \qquad s = 0, 1, 2, 3 \tag{5.6}$$

(no summation over s)

Neglecting the transversal part and using Eqs.(4.10), (4.11), one can write Eq.(4.18) in the form

$$\mathcal{L}_{qu2} = \hbar\rho^{-1}j^s\varepsilon_{iklm}q^i j^k(\partial_s q^l)\nu^m = \frac{\hbar j^s}{\sqrt{2\rho(\rho + j^j f_j)}}\varepsilon_{iklm}f^i j^k\partial_s q^l\xi^m$$

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$$= \frac{\hbar j^i}{2\rho(\rho + j^j f_j)} \varepsilon_{klsm} j^k \partial_i j^l f^s \xi^m \tag{5.7}$$

The relation (4.17) can be written in the form

$$\mathcal{L}_{q1} = m\rho(2\sin^2 \frac{\kappa}{2} - \frac{\hbar}{2m} w^i \partial_i \kappa), \qquad w^i = \frac{S^i}{\sqrt{-S^l S_l}}, \qquad w^i j_i = 0. \tag{5.8}$$

Let us take into account that the last term in the first equation (5.8) is small with respect to the first term of Eq.(4.16). Then neglecting small terms and taking into account Eq.(5.7), one obtains for the action (4.15)–(4.18) in the quasi-uniform state

$$\mathcal{A}_{Dqu}[j, \varphi, \kappa, \boldsymbol{\xi}] = \int \{-m\rho \cos \kappa - eA_i j^i - \hbar j^i [\partial_i \varphi + g_\alpha(\boldsymbol{\xi}) \partial_i \xi^\alpha]$$

$$+ \frac{\hbar j^i}{2\rho(\rho + j^j f_j)} \varepsilon_{klsm} j^k \partial_i j^l f^s \xi^m \} d^4 x \tag{5.9}$$

where $g_\alpha(\boldsymbol{\xi})$ is determined by Eq.(4.5).

Total derivatives $\rho^{-1} j^i \partial_i$ of variables j^i, κ, $\boldsymbol{\xi}$ in the dynamic system $\mathcal{S}_{Dqu}$ described by the action (5.9) are determined mainly by dynamic equations. Derivatives in the orthogonal directions are determined by the initial conditions which must be such ones, that these derivatives were small enough.

Although the action (5.9) contains a non-classical variable κ, in fact this variable is a constant. Indeed, a variation with respect to κ leads to the dynamic equation

$$\frac{\delta \mathcal{A}_{Dqu}}{\delta \kappa} = m\rho \sin \kappa = 0 \tag{5.10}$$

which has solutions

$$\kappa = n\pi, \qquad n = \text{integer} \tag{5.11}$$

Thus, the effective mass $m_{eff} = m \cos \kappa$ has two values

$$m_{eff} = m \cos \kappa = \pm m \tag{5.12}$$

The value $m_{eff} = m > 0$, $(\kappa = \frac{1}{2} n\pi)$ corresponds to a minimum of the action (5.9), whereas the value $m_{eff} = -m < 0$ corresponds to a maximum. Apparently, $m_{eff} > 0$ corresponds to a stable ensemble state, and $m_{eff} < 0$ does to unstable state.

Eliminating κ by means of the substitution $k = \frac{1}{2} n\pi$ in Eq.(5.9), one obtains the action

$$\mathcal{A}_{Dqu}[j, \varphi, \boldsymbol{\xi}] = \int \{-m\rho - eA_i j^i - \hbar j^i [\partial_i \varphi + g_\alpha(\boldsymbol{\xi}) \partial_i \xi^\alpha]$$

$$+ \frac{\hbar j^i}{2\rho(\rho + j^j f_j)} \varepsilon_{klsm} j^k \partial_i j^l f^s \xi^m \} d^4 x \tag{5.13}$$

Let us introduce Lagrangian coordinates $\tau = \{\tau_i\}$, $i = 0, 1, 2, 3$ by means of relations

$$j^i = \frac{\partial D}{\partial \tau_{0,i}} \equiv \frac{\partial(x^i, \tau_1, \tau_2, \tau_3)}{\partial(x^0, x^1, x^2, x^3)}, \qquad D \equiv \frac{\partial(\tau_0, \tau_1, \tau_2, \tau_3)}{\partial(x^0, x^1, x^2, x^3)}; \tag{5.14}$$

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$$\tau_{k,i} \equiv \partial_i \tau_k, \qquad i, k = 0, 1, 2, 3$$

Taking into account that

$$D^{-1} j^i \partial_i u = D^{-1} \frac{\partial D}{\partial \tau_{0,i}} \partial_i u = \frac{\partial(u, \tau_1, \tau_2, \tau_3)}{\partial(\tau_0, \tau_1, \tau_2, \tau_3)} = \frac{du}{d\tau_0} \tag{5.15}$$

$$d^4 x = D^{-1} d^4 \tau = D^{-1} d\tau_0 d\boldsymbol{\tau} \tag{5.16}$$

$$j^i \partial_i \varphi = \frac{\partial(\varphi, \tau_1, \tau_2, \tau_3)}{\partial(x^0, x^1, x^2, x^3)} \tag{5.17}$$

the action (5.13) can be rewritten in the Lagrangian coordinates in the form

$$\mathcal{A}_{Dqu}[x, \boldsymbol{\xi}] = \int \{ -m\sqrt{\dot{x}^i \dot{x}_i} - eA_i \dot{x}^i + \hbar \frac{(\dot{\boldsymbol{\xi}} \times \boldsymbol{\xi})\mathbf{z}}{2(1 + \boldsymbol{\xi})} + \hbar \frac{(\dot{\mathbf{x}} \times \ddot{\mathbf{x}})\boldsymbol{\xi}}{2\sqrt{\dot{x}^s \dot{x}_s}(\sqrt{\dot{x}^s \dot{x}_s} + \dot{x}^0)} \} d^4 \tau \tag{5.18}$$

where the dot means the total derivative $d/d\tau_0$. $x = \{x^i\}$, $i = 0, 1, 2, 3$, $\boldsymbol{\xi} = \{\xi^\alpha\}$, $\alpha = 1, 2, 3$ are considered as functions of the Lagrangian coordinates τ_0, $\boldsymbol{\tau} = \{\tau_1, \tau_2, \tau_3\}$. $\mathbf{z}$ is a constant unit 3-vector. $A_i = A_i(x)$, $i = 0, 1, 2, 3$ are function of x. The term $j^i \partial_i \varphi$ is omitted, because it reduces to a Jacobian (5.17) and does not contribute into dynamic equations.

The action (5.18) describes a statistical ensemble of deterministic dynamic systems $\mathcal{S}_{dc}$. A state of each system $\mathcal{S}_{dc}$ is described by the variables x^i, $\dot{x}^i$, $\ddot{x}^i$, $\boldsymbol{\xi}$. The variables $\boldsymbol{\xi}$ are connected with the spin by the relation (3.23) which takes the form

$$\boldsymbol{\xi} = \mathbf{s} - \frac{(\mathbf{s}\dot{\mathbf{x}})}{\dot{x}^0(\sqrt{\dot{x}^i \dot{x}_i} + \dot{x}^0)} \dot{\mathbf{x}}, \qquad \mathbf{s} = \boldsymbol{\xi} + \frac{(\boldsymbol{\xi}\dot{\mathbf{x}})}{\sqrt{\dot{x}^i \dot{x}_i}(\dot{x}^0 + \sqrt{\dot{x}^i \dot{x}_i})} \dot{\mathbf{x}} \tag{5.19}$$

Here $\mathbf{s} = \mathbf{S}/\rho$ is the unit spin 3-vector.

All this means that in the quasi-uniform approximation the dynamic system $\mathcal{S}_D$ is a statistical ensemble of some deterministic classical systems $\mathcal{S}_{dc}$. The system $\mathcal{S}_{dc}$ should be treated as a classical analog of the Dirac electron moving in a given electromagnetic field. Under some conditions the classical Dirac electron turns into a classical relativistic pointlike particle, but, in general, $\mathcal{S}_{dc}$ is a more complicated construction than a pointlike particle. As far as under some conditions $\mathcal{S}_D$ is a statistical ensemble of classical relativistic particles, one concludes that, generally, the Dirac equation describes a statistical ensemble of charged quantum particles (not a single particle). Indeed, at first, one knew only that the system $\mathcal{S}_D$ relates to an electron in some way, but one did not know whether $\mathcal{S}_D$ describes a single electron or a statistical ensemble of electrons. One discovers that under some (quasi-uniform) initial conditions the $\mathcal{S}_D$ is a statistical ensemble (of classical systems $\mathcal{S}_{dc}$). It means that $\mathcal{S}_D$ is a statistical ensemble in all other cases. But in the general case $\mathcal{S}_D$ cannot be an statistical ensemble of deterministic classical systems. It means that $\mathcal{S}_D$ is a statistical ensemble of stochastic systems.

As far as the quantum principles are not used, then one uses the statistical principle formulated in the sec.1. This principle permits to determine mean values

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of energy, momentum, angular momentum and other additive quantities for a single electron. Indeed, dividing an additive quantity for $\mathcal{S}_D$ by the number of systems in the ensemble $\mathcal{S}_D$, one obtains corresponding mean value for a single stochastic system.

Let us compare the quasi-uniform approximation with the quasi-classical approximation which is obtained by tending $\hbar \to 0$. Let us go to the limit $\hbar \to 0$ in the action (4.15)-(4.18). One obtains

$$\mathcal{A}_{Dcl}[j, \Phi] = \int [-m\sqrt{j^l j_l} - eA_l j^l - j^i \partial_i \Phi] d^4 x \qquad (5.20)$$

where

$$\Phi = \hbar \varphi \qquad (5.21)$$

The term $-\hbar j^i \partial_i \varphi$ of Eq. (4.16) can be saved at $\hbar \to 0$ by means of the substitution $\varphi \to \Phi = \hbar \varphi$. But the last term of $\mathcal{L}_{cl}$ cannot be conserved by a like substitution, because of the restriction (4.20). For instance, due to Eq.(4.20) the substitution $\boldsymbol{\xi} \to \mathbf{w} = \hbar^{1/2}\boldsymbol{\xi}$ leads to $\mathbf{w} = 0$ at $\hbar \to 0$.

The action (5.20) describes only a part of extremals (solutions) of the action (5.13), namely that part of them which does not contain spin variables (3.23) and describes a potential solution, where the momentum (2.12) forms a potential vector field.

$$p_l = \partial_l \Phi \qquad (5.22)$$

The quasi-uniform approximation (5.13) obtained by a proper choice of rather smooth initial conditions is more realistic, than the quasi-classical approximation (5.20) obtained in the limit $\hbar \to 0$, because in reality nobody can change the quantum constant $\hbar$. Thus, the quasi-uniform approximation is more preferable as a classical approximation of the dynamic system $\mathcal{S}_D$.

Thus, a usage of the quasi-uniform approximation permits to separate out a classical part of the action (4.15)-(4.18). The quantum part $\mathcal{L}_q = \mathcal{L}_{q1} + \mathcal{L}_{q2}$ of the action (4.15) is determined by Eqs.(4.17), (4.18). $\mathcal{L}_{q1}$ contains a specific variable κ which can be treated neither as a current, nor as a Lagrangian coordinate $\boldsymbol{\xi}$. Suppressing κ (i.e. setting $\kappa \equiv 0$), the term $\mathcal{L}_{q1}$ vanishes. The term $\mathcal{L}_{q2}$ is a quantum term, in general, although it contains a classical part which can be separated out by means of introduction of specific quantum variables.

Let us introduce new variables

$$\kappa^i = q^i = \frac{j^i + \rho f^i}{\sqrt{2\rho(\rho + j^s f_s)}} = \frac{j^i + \rho f^i}{\sqrt{(j^s + \rho f^s)(j_s + \rho f_s)}}, \qquad i = 0, 1, 2, 3; \qquad (5.23)$$

$$\rho \equiv \sqrt{j^l j_l},$$

using the Lagrangian multipliers λ_i. Then $\mathcal{L}_{q2}$ is substituted by

$$\mathcal{L}'_{q2} = \hbar\rho\varepsilon_{iklm}(\partial_\perp^k \kappa^l)\kappa^i \nu^m + \frac{\hbar}{\rho}j^s\varepsilon_{iklm}q^i j^k \partial_s q^l \nu^m +$$

$$+\lambda_i[j^i + \rho f^i - \kappa^i\sqrt{2\rho(\rho + j^s f_s)}] \tag{5.24}$$

$$\rho \equiv \sqrt{j^l j_l}, \qquad \nu^m = [\xi^m - f^m(\xi^s f_s)], \qquad m = 0, 1, 2, 3 \tag{5.25}$$

where $\partial_\perp^k$ is defined by Eq.(5.4). Then variation of the action with respect to κ^i leads to the dynamic equations

$$-\lambda_i\sqrt{2\rho(\rho + j^s f_s)} + \frac{\delta\mathcal{A}_{q2}''}{\delta\kappa^i} = 0, \tag{5.26}$$

$$\mathcal{A}_{q2}''[j, \xi, \kappa^i] = \int \hbar\rho\varepsilon_{iklm}(\partial_\perp^k \kappa^i)\kappa^l\nu^m d^4x \tag{5.27}$$

Resolving the equation (5.26) with respect to λ_i and substituting the λ_i into Eq.(5.24), one obtains instead of $\mathcal{L}_{q2}'$

$$\mathcal{L}_{q3} = \hbar\rho\varepsilon_{iklm}(\kappa^l\partial_\perp^k\kappa^i - q^l\partial_\perp^k\kappa^i + \kappa^i\partial_\perp^k q^l)\nu^m + \frac{\hbar j^i}{2\rho(\rho + j^j f_j)}\varepsilon_{klsm}j^k\partial_i j^l f^s\xi^m \tag{5.28}$$

Now the action has the form

$$\mathcal{A}_D[j, \varphi, \kappa, \xi, \kappa^i] = \int(\mathcal{L}_{cl} + \mathcal{L}_{q1} + \mathcal{L}_{q3})d^4x \tag{5.29}$$

Dynamic equations generated by actions (3.1), (4.15) and (5.29) are equivalent. The quantities ξ^α cannot be treated as Lagrangian coordinates (numbers labelling systems of the ensemble), because they are not constant along world lines of particles, and relations (2.10) do not take place. According to Eq.(3.23) the quantities ξ^α should be treated as some functions of the spin $\mathbf{S} = \{2S^{23,0}, 2S^{31,0}, 2S^{12,0}\} = \{S^1, S^2, S^3\}$.

The variables κ, κ^i, $i = 0, 1, 2, 3$ are special quantum variables which are responsible for quantum effects described by the Dirac equation. Indeed, let us set $\kappa \equiv 0$, $\kappa^i \equiv 0$, $i = 0, 1, 2, 3$ in the action (5.29). Then the action (5.29) turns to Eq.(5.13) with $g_\alpha(\boldsymbol{\xi})$ defined by Eq.(4.5). The action (5.29) generates the dynamic equation

$$\frac{\delta\mathcal{A}_D}{\delta\kappa^i} = \hbar\varepsilon_{iklm}\{\rho\nu^m\partial_\perp^k(q^l - \kappa^l) + \partial_\perp^{*k}[\rho\nu^m(q^l - \kappa^l)]\} = 0 \tag{5.30}$$

where the operator $\partial_\perp^{*k}$ is defined by the relation

$$\partial_\perp^{*k}u = \partial^k u - \partial_s(\frac{j^k j^s}{\rho^2}u) \tag{5.31}$$

Resolving Eq.(5.30) with respect to κ^i and substituting the κ^i into Eq.(5.29), one returns to the action (4.15)-(4.18). The fact that the solution (5.23) of Eq.(5.30) is not sole is of no importance, because Eq.(5.28) reduces to Eq.(4.18) by virtue of Eq.(5.30). Indeed, convoluting Eq.(5.30) with κ^i and using the obtained relation for eliminating κ^i from Eq.(5.24), one obtains Eq.(4.18).

Let us note that κ^l are not rigorous dynamic variables, because the dynamic equations (5.30) for κ^l contain derivatives only along spacelike directions orthogonal to j^i. Rather the introduction of κ^i is an invariant (with respect to a change of variables) way of separating out the classical part of the action.

The concentrated dynamic system $\mathcal{S}_{dc}$ has eleven degrees of freedom. It is associated with the distributed dynamic system $\mathcal{S}_D$. It is described by the action

$$\mathcal{A}_{dc}[x,\boldsymbol{\xi}] = \int \{-m\sqrt{\dot{x}^i \dot{x}_i} - eA_i\dot{x}^i + \hbar\frac{(\dot{\boldsymbol{\xi}} \times \boldsymbol{\xi})\mathbf{z}}{2(1+\boldsymbol{\xi}\mathbf{z})} + \hbar\frac{(\dot{\mathbf{x}} \times \ddot{\mathbf{x}})\boldsymbol{\xi}}{2\sqrt{\dot{x}^s\dot{x}_s}(\sqrt{\dot{x}^s\dot{x}_s} + \dot{x}^0)}\}d\tau_0 \quad (5.32)$$

with non-relativistically invariant Lagrangian. It is a very surprising fact which needs a special investigation.

6 Relativistical Invariance

At our consideration of the relativistic invariance of the Dirac equation written in hydrodynamic terms we shall follow the approach of Anderson [6] with the modification that the definition of the relativistic covariance is provided by an explicit reference to the quantities with respect to which the dynamic equations are relativistically covariant. Let us consider a simple example which is relevant to the Dirac equation.

One considers a system of differential equations consisting of the Maxwell equations for the electromagnetic tensor F^{ik} in some inertial coordinates x

$$\partial_k F^{ik} = 4\pi J^i, \qquad \varepsilon_{iklm}g^{ij}\partial_j F^{kl} = 0 \qquad (6.1)$$

and equations

$$m\frac{d}{d\tau}[(l_k\dot{q}^k)^{-1}\dot{q}^i - \frac{1}{2}g^{ik}l_k(l_j\dot{q}^j)^{-2}\dot{q}^s g_{sl}\dot{q}^l] = eF^{il}g_{lk}(q)\dot{q}^k; \qquad i = 0,1,2,3 \qquad (6.2)$$

$$\dot{q}^k \equiv \frac{dq^k}{d\tau}$$

where $q^i = q^i(\tau)$, $i = 0,1,2,3$ describe coordinates of a pointlike charged particle as functions of a parameter τ, l_i is a constant timelike unit vector,

$$g^{ik}l_i l_k = 1; \qquad (6.3)$$

and the speed of the light $c = 1$.

This system of equations is relativistically covariant with respect to quantities q^i, F^{ik}, J^i, l_i, g_{ik}, i.e. it does not change its form at any infinitesimal Lorentz transformation

$$x^i \rightarrow \tilde{x}^i = x^i + \omega^i_{.k}x^k + o(\omega); \qquad \omega^i_{.k} = g^{il}\omega_{lk}; \qquad \omega_{ik} = -\omega_{ki} \qquad (6.4)$$

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which is accompanied by corresponding transformation of quantities q^i, F^{ik}, J^i, l_i, g_{ik},

$$q^i(\tau) \rightarrow \tilde{q}^i(\tau) = \frac{\partial \tilde{x}^i}{\partial x^k} q^k(\tau) = q^i + \omega^i_{.k} q^k + o(\omega) \tag{6.5}$$

$$F^{ik}(x) \rightarrow \tilde{F}^{ik}(\tilde{x}) = \frac{\partial \tilde{x}^i}{\partial x^l} \frac{\partial \tilde{x}^k}{\partial x^j} F^{lj}(x) = F^{ik} + \omega^i_{.l} F^{lk} + \omega^k_{.l} F^{il} + o(\omega) \tag{6.6}$$

$$J^i(x) \rightarrow \tilde{J}^i(\tilde{x}) = \frac{\partial \tilde{x}^i}{\partial x^k} J^k(x) = J^i + \omega^i_{.k} J^k + o(\omega) \tag{6.7}$$

$$l_i \rightarrow \tilde{l}_i = \frac{\partial x^k}{\partial \tilde{x}^i} l_k = l_i + \omega^{.k}_i l_k + o(\omega), \qquad \omega^{.k}_i = g^{kl} \omega_{il} \tag{6.8}$$

$$g_{ik}(x) \rightarrow \tilde{g}_{ik}(\tilde{x}) = \frac{\partial x^l}{\partial \tilde{x}^i} \frac{\partial x^j}{\partial \tilde{x}^k} g_{lj}(x) = g_{ik} + \omega^{.l}_i g_{lk} + \omega^{.l}_k g_{il} + o(\omega) \tag{6.9}$$

The reference to the quantities q^i, F^{ik}, J^i, l_i, g_{ik} means that all these quantities are considered as formal dependent variables, when one compares the form of dynamic equations written in two different coordinate systems. For instance, if a reference to J^i is omitted in the formulation of the relativistic covariance, it means that J^i are considered as some functions of the coordinates x. If $J^i \neq 0$, then according to Eq.(6.7) J^i and $\tilde{J}^i$ are different functions of the arguments x and $\tilde{x}$ respectively, and the first equation (6.1) has different form in different coordinate systems. In other words, the dynamic equations (6.1)–(6.2) are not relativistically covariant with respect to quantities q^i, F^{ik}, l_i, g_{ik}. Thus, for the relativistic covariance it is important both the laws of transformation (6.5)–(6.9) and how each of quantities is considered as a formal variable, or as some function of coordinates.

Following Anderson [6] we divide the quantities q^i, F^{ik}, J^i, l_i, g_{ik} into two parts: dynamical objects (variables) q^i, F^{ik} and absolute objects J^i, l_i, g_{ik}. By definition of absolute objects they have the same value for all solutions of the dynamic equations, whereas dynamic variables are different, in general, for different solutions. If the dynamic equations are written in the relativistically covariant form, their symmetry group (and a compatibility with the relativity principles) is determined by the symmetry group of the absolute objects J^i, l_i, g_{ik}.

Let for simplicity $J^i \equiv 0$. A symmetry group of the constant timelike vector l_i is a group of rotations in the 3-plane orthogonal to the vector l_i. The Lorentz group is a symmetry group of the metric tensor $g_{ik} =$diag $\{1, -1, -1, -1\}$. Thus, the symmetry group of all absolute objects l_i, g_{ik}, $J^i \equiv 0$ is a subgroup of the Lorentz group (rotations in the 3-plane orthogonal to l_i). As far as the symmetry group is a subgroup of the Lorentz group and does not coincide with it, the system of equations (6.1)–(6.2) is non-relativistic (incompatible with the relativity principles).

Of course, the compatibility with the relativity principles does not depend on the fact with respect to which quantities the relativistic covariance is considered. For instance, let us consider a covariance of Eqs. (6.1), (6.2) with respect to quantities q^i, F^{ik}, $J^i \equiv 0$. It means that now l_i are to be considered as functions of x (in the given case these functions are constants), because a reference to l_i as a formal

variables is absent. After the transformations (6.4)–(6.9) the equation (6.2) takes the form

$$m\frac{d}{d\tau}[(\tilde{l}_k\frac{d\tilde{q}^k}{d\tau})^{-1}\frac{d\tilde{q}^i}{d\tau} - \frac{\tilde{l}^i}{2}(\tilde{l}_k\frac{d\tilde{q}^k}{d\tau})^{-2}\frac{d\tilde{q}^s}{d\tau}\frac{d\tilde{q}_s}{d\tau}] = e\tilde{F}^i_{.k}\frac{d\tilde{q}^k}{d\tau} \qquad (6.10)$$

Here $\tilde{l}_i$ are considered as functions of $\tilde{x}$. But $\tilde{l}_i$ are other functions of $\tilde{x}$, than l_i of x (other numerical constants $\tilde{l}_k = l_j\partial x^j/\partial\tilde{x}^k$ instead of l_k), and equations (6.2) and (6.10) have different forms with respect to quantities q^i, F^{ik}, $J^i \equiv 0$. It means that Eq.(6.2) is not relativistically covariant with respect to q^i, F^{ik}, $J^i \equiv 0$, although it is relativistically covariant with respect to q^i, F^{ik}, l_i, $J^i \equiv 0$.

Setting $l_i = \{1,0,0,0\}$, $t = q^0(\tau)$ in Eq.(6.2), one obtains

$$m\frac{d^2q^\alpha}{dt^2} = eF^\alpha_{.0} + eF^\alpha_{.\beta}\frac{dq^\beta}{dt}, \qquad i = \alpha = 1,2,3;$$

$$\frac{m}{2}\frac{d}{dt}(\frac{dq^\alpha}{dt}\frac{dq^\alpha}{dt}) = eF^\alpha_{.0}\frac{dq^\alpha}{dt}, \qquad i = 0. \qquad (6.11)$$

It is easy to see that this equation describes a non-relativistic motion of a charged particle in a given electromagnetic field F^{ik}. The fact that the equations (6.2) or (6.11) are non-relativistic is connected with the space-time split into space and time that is characteristic for Newtonian mechanics. This space-time split is described in different ways in Eqs. (6.2) and (6.11). It is described by the constant timelike vector l_i in Eq.(6.2). In the equation (6.11) the space-time split is described by a special choice of the coordinate system whose time axis is directed along the vector l^i.

Thus, a relativistic covariance in itself does not mean a compatibility with the special relativity principles. It is important with respect to which quantities the dynamic equations are relativistically covariant

There is something like that in the case of the Dirac equation. The Dirac equation (1.1) is relativistically covariant with respect to variables ψ, A_i, which are transformed at the Lorentz transformation (6.4) as follows

$$A_i(x) \to \tilde{A}_i(\tilde{x}) = \frac{\partial x^k}{\partial\tilde{x}^i}A_k(x) = A_i + \omega^{.k}_i A_k + o(\omega) \qquad (6.12)$$

$$\psi(x) \to \tilde{\psi}(\tilde{x}) = \exp(\frac{1}{4}\gamma^{ik}\omega_{ik})\psi(x) + o(\omega), \qquad (6.13)$$

$$\gamma^i \to \tilde{\gamma}^i = \gamma^i, \qquad i = 0,1,2,3.$$

A direct physical meaning of the variables ψ is unclear. Only such quantities as the current j^i and spin (1.6) have a direct physical meaning. The action (4.15)-(4.18) is a result of the Dirac action (3.1) transformation to variables j^i, S^i having a physical meaning.

Let us replace the relation (3.23) by the relation

$$\xi^l = \rho^{-1}[S^l - \frac{j^l S^k f_k}{\rho + j^s f_s}], \qquad l = 0,1,2,3; \qquad (6.14)$$

where the vector f^l is defined by Eq.(4.7). According to relations (1.4) the equation (6.14) is equivalent to Eq.(3.23), if the first relation (4.7) takes place.

Let f^k be transformed as a vector, then according to Eq.(6.14) ξ^l is a pseudo-vector, because j^l is a vector, and S^l is a pseudo-vector. Substituting Eq.(6.14) into Eqs.(5.29), (4.16)-(4.17), (5.28), one obtains the action for the Dirac equation in terms of variables $j, S, \kappa, \varphi, \kappa^i$

$$\mathcal{A}_D[j, S, \kappa, \varphi, \kappa^i] = \int (\mathcal{L}'_{cl} + \mathcal{L}_{q1} + \mathcal{L}_{q3}) d^4 x \tag{6.15}$$

$$\mathcal{L}'_{cl} = -m\rho - e j^i A_i - \hbar j^i (\partial_i \varphi + \varepsilon_{mjkl} \mu^j \partial_i \mu^k f^m z^l) \tag{6.16}$$

where $\mathcal{L}_{q1}$ and $\mathcal{L}_{q3}$ are defined by Eqs.(4.17), (5.28), μ^j is defined by Eqs. (4.12), (4.11), (6.14), and ν^i is defined by Eq.(4.11)

$$\mu^k = [w^k - \frac{j^k w^m f_m}{\rho + j^s f_s}][2(1 + w^l z_l - \frac{j_l z^l w^m f_m}{\rho + j^s f_s})]^{-1/2}, \qquad w^i \equiv \frac{S^i}{\sqrt{-S^l S_i}} \tag{6.17}$$

$$\nu^k = w^k - \frac{j^k w^m f_m}{\rho + j^s f_s} \tag{6.18}$$

For obtaining dynamic equations the action (6.15) is varied with respect to variables j^i, S^i, κ^i, φ, κ under additional constraints (1.4). Vector f^i and pseudo-vector z^i are constant and satisfy the conditions (4.8).

The obtained dynamic equations are equivalent to the Dirac equation (1.1). They are relativistically covariant with respect to quantities j^i, S^i, κ^i, φ, κ, A_i, f^i, z^i. But they are not relativistically covariant, generally, with respect to variables j^i, S^i, κ^i, φ, κ A_i. It is easy to verify, setting values (4.7) of f^i, z^i into the action (6.15)-(6.18).

Thus, a relativistical covariance of dynamic equations described in terms of A_i, j^i, S^i, κ^i, φ, κ depends essentially on introduction of additional quantities f^i, z^i. These quantities are constant. They arise from the zero divisor (3.5) which can be written in the covariant form

$$\Pi = \frac{1}{2}\left(1 + f_l \gamma^l\right) \frac{1}{2}\left(1 - i\gamma_5 f_i z_k \gamma^{ik}\right) \tag{6.19}$$

where f^l is a constant timelike unit vector, and z^l is a constant unit pseudo-vector.

It is easy to verify that under the conditions (4.8) the factors of the matrix (6.19) commute, and each factor is a zero divisor. Using designations (3.2), one obtains that under conditions (4.7) the expression (6.19) coincides with Eq.(3.5). Thus, variables f^l, z^l are present in a hidden form inside the zero divisor Π of the expression (3.4) for the wave function. When one says that the Dirac equation (1.1) is relativistically covariant with respect to variables A_i, ψ, one implies also the absolute objects f^l, z^l, hidden inside ψ. It is impossible to construct a proper zero divisor without using absolute objects.

Indeed, if all components of the wave function ψ are given, the relations (1.2) determine the quantities j^l and S^l uniquely. The quantities φ, κ, are determined to

within an additive constant and to within an additive (2π)fold constant respectively. The quantities do not depend on the form of the zero divisor Π. If the quantities j^l, S^l, φ, κ are given, the wave function components are determined by the relation (3.25) with the zero divisor determined by Eq.(3.5). The wave function components depend on parameters determining the form of the zero divisor. In other words, the transformation from the variables j^l, S^l, φ, κ to the wave function ψ is not single-valued. Substitution of ψ from Eq.(3.4) into Eq.(1.1) leads to appearance of constant parameters f^k, z^k in dynamic equations, provided they are written in the relativistically covariant form.

A common continuous symmetry group of both vectors f^k and z^k is a group of rotation inside the 2-plane orthogonal to the bivector $f^k z^l - f^l z^k$. The Lorentz group is not a symmetry group of the dynamic equations. If the peculiar directions determined by parameters f^k, z^k are not fictitious, then the dynamic equations generated by the action (6.15) are incompatible with the relativity principles.

The variable f^l describes the space-time split into space and time, what is characteristic for the Newtonian mechanics. In this sense the Dirac equation is not more relativistic, than the non-relativistic equation (6.2).

7 Concentrated Dynamic System Associated with the Dirac Equation

Let us return to an investigation of the dynamic system $\mathcal{S}_{dc}$. The action (5.32) can be presented in the relativistically covariant form as follows

$$\mathcal{A}_{dc}[x,\xi] = \int \{-m\sqrt{\dot{x}^i \dot{x}_i} - eA_i \dot{x}^i - \hbar \frac{\varepsilon_{iklm} \xi^i \dot{\xi}^k f^l z^m}{2(1 - \xi^s z_s)}$$

$$+\hbar \frac{\varepsilon_{iklm} \dot{x}^i \ddot{x}^k f^l \xi^m}{2\sqrt{\dot{x}^s \dot{x}_s}(\dot{x}^l f_l + \sqrt{\dot{x}^l \dot{x}_l})} \} d\tau \tag{7.1}$$

where z^i and f^i satisfy Eqs.(4.8), and the dot means a differentiation with respect to invariant parameter τ. ξ^i is a unit spacelike vector orthogonal to the vector f^i. It can be presented in the form

$$\xi^i = \frac{\eta^i - f^i \eta^s f_s}{\sqrt{-\eta^l \eta_l + (\eta^l f_l)^2}} \tag{7.2}$$

where η^i is an arbitrary spacelike vector.

It is easy to verify that

$$\delta \xi^i = \frac{(\delta^i_l + \xi^i \xi_l)(\delta^l_k - f^l f_k)\delta\eta^k}{\sqrt{-\eta^s \eta_s + (\eta^s f_s)^2}} \tag{7.3}$$

Then varying Eq.(7.1) with respect to ξ^i and using that $\delta\eta^i$ is an arbitrary vector, one obtains dynamic equations

$$\frac{\delta\mathcal{A}_{dc}}{\delta\xi^i}(\delta^i_l + \xi^i\xi_l)(\delta^l_k - f^l f_k) = \frac{\delta\mathcal{A}_{dc}}{\delta\xi^i}(\delta^i_k + \xi^i\xi_k - f^i f_k) = 0, \qquad k = 0, 1, 2, 3 \quad (7.4)$$

There are only two independent dynamic equations among the equations (7.4), because a convolution of Eq.(7.4) with vectors ξ^k and f^k converts them into identities.

The action (7.1) is invariant with respect to a change of the invariant parameter τ

$$\tau \rightarrow \tilde{\tau} = f(\tau) \tag{7.5}$$

where f is an arbitrary monotone function. In particular, τ can be chosen in such a way that

$$\dot{x}^i \dot{x}_i = 1 \tag{7.6}$$

In this case τ can be interpreted as a proper time.

Let us written down the dynamic equations in the case of absence of the electromagnetic field $A_i = 0$.

$$\frac{\delta\mathcal{A}_{dc}}{\delta x^i} = \frac{d}{d\tau}\{\frac{m\dot{x}_i}{\sqrt{\dot{x}^s\dot{x}_s}} - \hbar\varepsilon_{iklm}[Qf^l\ddot{x}^k\xi^m + \frac{1}{2}\dot{x}^k\frac{d}{d\tau}(Qf^l\xi^m)]-$$

$$-\frac{\hbar}{2}\varepsilon_{jklm}f^l\dot{x}^j\ddot{x}^k\xi^m\frac{\partial Q}{\partial\dot{x}^i}\} = 0, \qquad i = 0, 1, 2, 3 \tag{7.7}$$

$$\delta\xi^i: \qquad -\{\varepsilon_{sklm}[Pf^l\dot{\xi}^k z^m + \frac{1}{2}\xi^k\frac{d}{d\tau}(Pf^l z^m) + \frac{1}{2}Q\dot{x}^k\ddot{x}^l f^m]+$$

$$\frac{1}{2}\varepsilon_{jklm}f^l\xi^j\dot{\xi}^k z^m\frac{\partial P}{\partial\xi^s}\}(\delta^s_i + \xi^s\xi_i - f^s f_i) = 0 \tag{7.8}$$

where P and Q are defined by the relations

$$Q = Q(\dot{x}) = [\sqrt{\dot{x}^s\dot{x}_s}(\dot{x}^l f_l + \sqrt{\dot{x}^s\dot{x}_s})]^{-1} \tag{7.9}$$

$$P = P(\xi) = (1 - z^l\xi_l)^{-1} \tag{7.10}$$

Equation (7.7) is integrated in the form

$$\frac{m\dot{x}_i}{\sqrt{\dot{x}^s\dot{x}_s}} - \hbar\varepsilon_{iklm}[Qf^l\ddot{x}^k\xi^m + \frac{1}{2}\dot{x}^k\frac{d}{d\tau}(Qf^l\xi^m)] - \frac{\hbar}{2}\varepsilon_{jklm}f^l\dot{x}^j\ddot{x}^k\xi^m\frac{\partial Q}{\partial\dot{x}^i} = p_i \tag{7.11}$$

$$p_i = \text{const}, \qquad i = 0, 1, 2, 3;$$

Let τ be the proper time, and the condition (7.6) takes place. Convoluting Eq.(7.11) with f^i and eliminating the last term in lhs of Eq.(7.11), one obtains

$$-\varepsilon_{iklm}[\ddot{x}^k f^l\xi^m + \frac{1}{2}\dot{x}^k f^l\dot{\xi}^m - \frac{\dot{x}^k f^l\xi^m(\ddot{x}^j f_j)}{2(1 + \dot{x}^s f_s)}] = (\delta^s_i - f_i f^s)(u_s - \dot{x}_s)\frac{1 + \dot{x}^l f_l}{\lambda}, \tag{7.12}$$

$$\lambda = \frac{\hbar}{m}, \qquad u_i = \frac{p_i}{m}, \qquad i = 0, 1, 2, 3$$

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One can see from Eqs.(7.8), (7.12) that only directions orthogonal to the timelike vector f^i are essential.

Let us choose f^i in the form (4.7). Then using designations

$$x^i = \{x^0, \mathbf{x}\}, \qquad \xi^i = \{0, \boldsymbol{\xi}\}, \qquad z^i = \{0, \mathbf{z}\}, \qquad u^i = \{u^0, \mathbf{u}\}, \qquad (7.13)$$

Eqs. (7.8), (7.12) reduce to the form

$$[-\dot{\boldsymbol{\xi}} + \frac{(\mathbf{z}\dot{\boldsymbol{\xi}})}{2(1 + \mathbf{z}\boldsymbol{\xi})}\boldsymbol{\xi}] \times \mathbf{z} + \frac{\boldsymbol{\xi}(\dot{\boldsymbol{\xi}} \times \mathbf{z})}{2(1 + \mathbf{z}\boldsymbol{\xi})}\mathbf{z} + \frac{(1 + \mathbf{z}\boldsymbol{\xi})}{2(1 + \dot{x}^0)}\dot{\mathbf{x}} \times \ddot{\mathbf{x}} = C\boldsymbol{\xi} \qquad (7.14)$$

$$\frac{d}{d\tau}\left(\frac{\dot{\mathbf{x}}}{\sqrt{1 + \dot{x}_0}}\right) \times \boldsymbol{\xi} + \frac{\dot{\mathbf{x}} \times \dot{\boldsymbol{\xi}}}{2\sqrt{1 + \dot{x}_0}} = \frac{\sqrt{1 + \dot{x}_0}}{\lambda}(\dot{\mathbf{x}} - \mathbf{u}) \qquad (7.15)$$

where C is some indefinite function of τ.

According to Eqs.(7.2), (4.7)

$$\boldsymbol{\xi}^2 = 1, \qquad \boldsymbol{\xi}\dot{\boldsymbol{\xi}} = 0. \qquad (7.16)$$

Using these relations, the equation (7.14) reduces to the form (see Appendix B)

$$\dot{\boldsymbol{\xi}} = -\frac{(\dot{\mathbf{x}} \times \ddot{\mathbf{x}})}{1 + \dot{x}_0} \times \boldsymbol{\xi} \qquad (7.17)$$

which does not contain the vector $\mathbf{z}$. It means that $\mathbf{z}$ determines a fictitious direction in the space-time. Note that $\mathbf{z}$ in the action (4.15) for the system $\mathcal{S}_D$ is fictitious also, because the term containing $\mathbf{z}$ is the same in the actions (4.15) and (5.13) for $\mathcal{S}_D$ and $\mathcal{S}_{Dqu}$ respectively.

Eq.(7.17) can be written in the relativistically covariant form

$$\dot{\xi}_i = -\frac{\varepsilon_{iklm}\xi^l f^m \varepsilon^k_{.k'l'm'}\dot{x}^{k'}\ddot{x}^{l'}f^{m'}}{1 + f_s\dot{x}^s} \qquad (7.18)$$

The system of Eqs.(7.12), (7.18) is relativistically covariant with respect to the quantities x^i, ξ^i, p_i, f^i considered as 4-vectors (x^i, p_i, f^i) and pseudo-vectors (ξ^i). But it is not relativistically covariant with respect to the dynamic variables x^i, ξ^i, p_i. Under the Lorentz transformation (6.4) the set $\mathcal{S}_f$ of all solutions $\{x^i, \xi^i, p_i\}$ of Eqs.(7.12), (7.18) at fixed parameters f^i is transformed into another set $\mathcal{S}_{\tilde{f}}$ of solutions $\{\tilde{x}^i, \tilde{\xi}^i, \tilde{p}_i\}$. But the set $\mathcal{S} = \{\mathcal{S}_f\}$ of all sets $\mathcal{S}_f$ is transformed into itself, provided f^i satisfies Eq.(4.8).

Formally the integration constants p_i can be considered as some parameters of the system of differential equations (7.18), (7.12). At the same time the parameters p_i, f^i of Eqs. (7.18), (7.12) can be considered as some integration constants of some system $\mathcal{D}$ of differential equations which is relativistically covariant with respect to its dynamic variables. The set of all solutions of $\mathcal{D}$ can be obtained as a result of the Lorentz transformation (6.4) of the set $\mathcal{S}_p$ of all solutions with fixed values of p_i, or as that of the set $\mathcal{S}_f$ of all solutions with fixed values of f^i.

It is sufficient to investigate solutions $\mathcal{S}_{f_0}$ of equations (7.12), (7.18) at $f^i_{(0)} = \{1, 0, 0, 0\}$, because solutions at other values of f^i can be obtained from solutions at $f^i = f^i_{(0)}$ by means of a proper Lorentz transformation.

Eq.(7.17) describes a rotation of the unit 3-vector $\boldsymbol{\xi}$ around the vector $\dot{\mathbf{x}} \times \ddot{\mathbf{x}}$ with the angular frequency

$$\boldsymbol{\Omega}_\xi = \frac{\dot{\mathbf{x}} \times \ddot{\mathbf{x}}}{1 + \dot{x}_0} \tag{7.19}$$

Eliminating $\dot{\boldsymbol{\xi}}$ by means of a substitution of Eq.(7.17) into Eq.(7.15), and introducing a new variable $\mathbf{y}$ instead of $\dot{\mathbf{x}}$

$$\mathbf{y} = \frac{\dot{\mathbf{x}}}{\sqrt{1 + \dot{x}^0}}, \qquad \dot{\mathbf{x}} = \mathbf{y}\sqrt{2 + \mathbf{y}^2}, \qquad \dot{x}^0 \equiv \sqrt{1 + \dot{\mathbf{x}}^2}, \tag{7.20}$$

Eq.(7.15) reduces to the form

$$\lambda \dot{\mathbf{y}} \times (\boldsymbol{\xi} + \frac{(\boldsymbol{\xi})}{2\sqrt{2 + \mathbf{y}^2}}\mathbf{y}) = (\mathbf{y}\sqrt{2 + \mathbf{y}^2} - \mathbf{u})\sqrt{2 + \mathbf{y}^2} \tag{7.21}$$

Let us introduce a dimensionless variable $\mathbf{w}$ which is supposed to be small

$$\mathbf{w} = \mathbf{y} - \mathbf{b}, \qquad \mathbf{b} = \mathbf{u}(1 + \sqrt{1 + \mathbf{u}^2})^{-1/2}, \qquad |\mathbf{w}| \ll 1 \tag{7.22}$$

Then Eq.(7.21) takes the form

$$\lambda \dot{\mathbf{w}} \times \mathbf{a} = [G\mathbf{w} + \mathbf{b}(\mathbf{w}\mathbf{b})] + O(\mathbf{w}^2), \tag{7.23}$$

$$G = 2 + \mathbf{b}^2, \qquad \mathbf{a} = \boldsymbol{\xi} + \frac{(\boldsymbol{\xi}\mathbf{b})}{2\sqrt{2 + \mathbf{b}^2}}\mathbf{b} \tag{7.24}$$

It follows from Eq.(7.23) that $\mathbf{a}$ is orthogonal to rhs of Eq.(7.23), or

$$\mathbf{w}\mathbf{d} = 0, \qquad \mathbf{d} \equiv G\mathbf{a} + (\mathbf{b}\mathbf{a})\mathbf{b} \tag{7.25}$$

Characteristic frequency for $\mathbf{w}$ is of the order of λ^{-1}, whereas that for $\boldsymbol{\xi}$ is of the order of $\lambda^{-1}|\mathbf{w}| \ll \boldsymbol{\lambda}^{-1}$. It means that, integrating Eq.(7.23), one can consider $\boldsymbol{\xi}$ as a constant vector.

If $\boldsymbol{\xi} \parallel \mathbf{b}$, $\mathbf{w}$ is orthogonal to $\mathbf{b}$, and Eq. (7.23) reduces to the form

$$\lambda \dot{\mathbf{w}} = \frac{G}{\mathbf{a}^2}\mathbf{a} \times \mathbf{w} + O(\mathbf{w}^2), \qquad \mathbf{w}^2 = \text{const} \tag{7.26}$$

which describes a rotation of the vector $\mathbf{w}$ with the angular velocity $G\mathbf{a}/\mathbf{a}^2$.

If $\boldsymbol{\xi} \times \mathbf{b} \neq 0$, let us choose orthonormal vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ in such a way that $\mathbf{e}_1, \mathbf{e}_2$ lie on the plane $\mathcal{P}$ spanned by vectors $\boldsymbol{\xi}$ and $\mathbf{b}$. Let the vector $\mathbf{e}_1$ be chosen on $\mathcal{P}$ in such a way that $\mathbf{e}_1 \parallel \mathbf{d}$. Then one has the following decomposition of vectors $\mathbf{w}, \mathbf{a}, \mathbf{b}$

$$\mathbf{w} = w_2\mathbf{e}_2 + w_3\mathbf{e}_3, \qquad \mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2, \qquad \mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 \tag{7.27}$$

According to Eqs.(7.23), (7.25), (7.27) the first equation (7.25) reduces to the form

$$w_2[a_1b_1b_2 + a_2(G + b_2^2)] = 0. \tag{7.28}$$

Substituting Eqs. (7.27) into Eq.(7.26), one obtains

$$-\lambda \dot{w}_3 a_2 = b_1 b_2 w_2 + O(\mathbf{w}^2),$$

$$\lambda \dot{w}_3 a_1 = (G + b_2^2) w_2 + O(\mathbf{w}^2), \tag{7.29}$$

$$-\lambda \dot{w}_2 a_1 = G w_3 + O(\mathbf{w}^2),$$

Due to Eq.(7.28) two first equations (7.29) are equivalent. The system (7.29) can be easily solved. The solution has the form.

$$\mathbf{w} = A[\mathbf{e}_2 \cos \phi + (1 + b_2^2/G)^{1/2} \mathbf{e}_3 \cos \phi] + O(A^2) \tag{7.30}$$

where

$$\mathbf{e}_1 = \mathbf{d}/ \mid \mathbf{d} \mid, \qquad \mathbf{e}_2 = [1 - (\mathbf{e}_1 \boldsymbol{\xi})^2]^{-1/2} \mathbf{e}_1 \times \boldsymbol{\xi}, \qquad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2,$$

$$\phi = \Omega_x(\tau - \tau_{(0)}) + \phi_{(0)} \tag{7.31}$$

$$\Omega_x = \frac{G}{\lambda a_1}[1 + b_2^2/G]^{1/2}, \qquad a_1 = (\mathbf{a}\mathbf{e}_1), \qquad b_2 = (\mathbf{b}\mathbf{e}_2) \tag{7.32}$$

Using Eqs.(7.20), (7.22), one obtains

$$\dot{\mathbf{x}} = u + A[(G\mathbf{e}_2 + \frac{b_2}{G}b) \cos \phi + (G + b_2^2)^{1/2} e_3 \sin \phi] + O(A^2)$$

$$\dot{x}^0 = \sqrt{1 + \dot{\mathbf{x}}^2} \tag{7.33}$$

$$\mathbf{x} = x_{(0)} + u(\tau - \tau_{(0)}) + A\Omega_x^{-1}[(G\mathbf{e}_2 + \frac{b_2}{G}b) \sin \phi - (G + b_2^2)^{1/2} e_3 \cos \phi] + O(A^2)$$

$$x^0 = x_{(0)}^0 + \int\limits_{\tau_{(0)}}^{\tau} \sqrt{1 + \dot{\mathbf{x}}^2(\tau')} d\tau' \tag{7.34}$$

$A, \phi_{(0)}, x_{(0)}^0, \mathbf{x}_{(0)}, \mathbf{u}, \tau_{(0)}$ are independent integration constants. $\mathbf{a}, \mathbf{b}, G$ and Ω_x are determined by relations (7.22), (7.24), (7.32) through these constants.

According to Eq.(7.19) the angular velocity Ω_ξ of the vector $\boldsymbol{\xi}$ rotation reduces to the form

$$\Omega_\xi = \frac{A\Omega_x}{1 + \dot{x}^0}[-u \times (G\mathbf{e}_2 + \frac{b_2}{G}b) \sin \phi - (1 + b_2^2 G)^{1/2} \mathbf{u} \times \mathbf{e}_3 \cos \phi] + O(A^2) \tag{7.35}$$

Ω_ξ oscillates rapidly around the mean value $< \Omega_\xi > \simeq O(a^2)$. It means that

$$\boldsymbol{\xi} = \boldsymbol{\xi}_{(0)} + O(A^2) \tag{7.36}$$

$$\boldsymbol{\xi}_{(0)} = \text{const}, \qquad \boldsymbol{\xi}_{(0))}\boldsymbol{\xi}_{(0)} = 1 \tag{7.37}$$

In order to show that f^i is not a fictitious parameter, let us consider the exact solution of equations (7.12), (7.18) at f^i defined by Eq.(4.7) .

$$x^i = \{\sqrt{1+a^2}\,\tau, \ -a\Omega^{-1}\sin\phi, \ a\Omega^{-1}\cos\phi, \ 0\}; \qquad \xi^i = \{0,0,0,1\}$$

$$u^i = \{1,0,0,0\}, \qquad \Omega = \frac{1+\sqrt{1+a^2}}{\lambda}, \qquad \phi = \Omega\tau + \phi_{(0)}, \qquad a = \text{const} \quad (7.38)$$

If f^i is a fictitious, the system of equations (7.12), (7.18) is relativistically covariant with respect to the vectors x^i, ξ^i, u^i, and the vectors $\tilde{x}^i$, $\tilde{\xi}^i$, $\tilde{u}^i$, obtained from Eqs.(7.38) by means of a Lorentz transformation are to form a solution of Eqs.(7.12), (7.18).

In the coordinate system moving with the velocity $V = \tanh\vartheta$ in the direction of the axis x^3 the vectors (7.38) have the form

$$\tilde{x}^i = \{\tau\sqrt{1+a^2}\cosh\vartheta, \ -a\Omega^{-1}\sin\phi, \ a\Omega^{-1}\cos\phi, \ \tau\sqrt{1+a^2}\sinh\vartheta\};$$

$$\tilde{\xi}^i = \{\sinh\vartheta, \ 0, \ 0, \ \cosh\vartheta\}; \qquad \tilde{u}^i = \{\cosh\vartheta, \ 0, \ 0, \ \sinh\vartheta\}; \qquad (7.39)$$

$$\Omega = \frac{1+\sqrt{1+a^2}}{\lambda} \qquad \phi = \Omega\tau + \phi_{(0)}, \qquad a = \text{const}$$

It is easy to verify that (7.39) is not a solution of Eqs.(7.12), (7.18), if $\vartheta \neq 0$ and f^i is determined by the equation (4.7). (Of course, Eq.(7.39) is a solution of Eqs. (7.12), (7.18), if $f^i = \tilde{f}^i = \{\cosh\vartheta, 0, 0, \sinh\vartheta\}$). It means that the vector f^i is an essential parameter, and the system of equations (7.12), (7.18) is incompatible with the special relativity principle.

The fact that z^i is fictitious, but f^i is not, is explained, apparently, by incomplete symmetry of the Dirac equation with respect to the time and the space. The matrix γ^0 is used for constructing the Dirac conjugate spinor $\bar{\psi} = \psi^*\gamma^0$. It separates out the matrix γ^0 among matrices γ^i

A space-time split generated by γ^0 appears in the Space-Time Algebra (STA) suggested by Hestenes [7]. STA is a kind of Clifford Algebra describing space-time properties [8]. This space-time split is connected with a use of the matrix γ^0.

In the case, when the electromagnetic field does not vanish, the quantities p_i determined by Eq.(7.11) are not constant. They satisfy the equations

$$\dot{p}_i \equiv m\dot{u}_i = eF_{ik}(x)\dot{x}^k, \qquad F_{ik} \equiv \partial_i A_k - \partial_k A_i. \qquad (7.40)$$

There are only three independent equations among them

$$m\dot{\mathbf{u}} = e\mathbf{E}\dot{x}^0 + e[\dot{\mathbf{x}} \times \mathbf{H}], \qquad \mathbf{E} = \mathbf{E}(x^0, \mathbf{x}), \qquad \mathbf{H} = \mathbf{H}(x^0, \mathbf{x}) \qquad (7.41)$$

where $\mathbf{E} = \{E_\alpha\} = \{-F_{\alpha 0}\}$, $\alpha = 1,2,3$; $\mathbf{H} = \{H_\alpha\} = \{-\frac{1}{2}\varepsilon_{\alpha\beta\gamma}F_{\beta\gamma}\}$, $\alpha = 1,2,3$. The system of equations (7.6), (7.16), (7.17), (7.20), (7.41) is a complete system of dynamic equations for the dynamic variables $\boldsymbol{\xi}$, $\mathbf{x}$, x^0, $\mathbf{u}$.

For not too strong electromagnetic field, when the Larmor frequency is much less, than Ω_x ($eH/m \ll \Omega_x \simeq m/\hbar$, i.e. if $H \ll 10^{12}G$), $\mathbf{u}$ in Eq.(7.21) can be considered

approximately as a constant. Then the world line $x^i = x^i(\tau)$, $i = 0, 1, 2, 3$ describing a solution of the system (7.6), (7.16), (7.17), (7.20), (7.21), (7.41) spirals round the world line $X^i = X^i(\tau)$ which describes a motion of the guiding center and satisfies the equation

$$m\ddot{X}^i = eF^i_{.k}(X)\dot{X}^k, \qquad i = 0, 1, 2, 3 \tag{7.42}$$

In general, a charged particle moving in such a way must intensively radiate electromagnetic waves. As a result its world line is to approach to the world line of the guiding center.

Such a helical motion of the electron associates with the classical model [9] of the Schrödinger zitterbewegung [10]. Another approaches to the interpretation of the zitterbewegung can be found in papers [11-14] and references therein.

8 Discussion

It seems that p_i and $\dot{x}^i$ should be interpreted respectively as a 4-momentum and 4-velocity of the system $\mathcal{S}_{dc}$. But then the system $\mathcal{S}_{dc}$ can be hardly interpreted as a pointlike classical particle. There are two reasons. First, p_i and $\dot{x}^i$ of a pointlike particle are connected by an algebraic relation of the kind of $p_i = m\dot{x}_i$. For the dynamic system $\mathcal{S}_{dc}$ a distinction between $y^i = \dot{x}^i(1 + \dot{x}^0)^{-1/2}$ and $u^i = p^i/m$ is described by the dynamic variable w^i which is introduced by Eq.(7.22). According to Eq.(7.21) this distinction is proportional to the quantum constant $\hbar$. Second, it is very difficult to understand, why the world line of a free particle is a helix spiralling round the straight line

$$X^i = \frac{p^i}{m}(\tau - \tau_{(0)}) + X^i_{(0)}. \tag{8.1}$$

even in an absence of the electromagnetic field.

All this can be explained by means of the supposition that x^i describes an observable part of a complicated bound system whose center of inertia X^i moves according to Eq.(8.1) in the case of $A_i \equiv 0$ and according to Eq.(7.42) in the case of not too strong electromagnetic field. But under such an interpretation the system $\mathcal{S}_{dc}$ is not a pointlike particle with some inner degrees of freedom, but a dynamic system consisting of a few particles interacting through a distance. But such an interaction through a distance is incompatible with the relativity principle.

One can try to save the compatibility with the relativity by a consideration of some distributed classical system $\mathcal{S}_{\phi}$ instead of the concentrated dynamic system $\mathcal{S}_{dc}$. The system $\mathcal{S}_{dc}$ arises as an element of the statistical ensemble $\mathcal{S}_{Dqu}$. But an one-parameter statistical ensemble of dynamic systems $\mathcal{S}_{dc}$ can be also considered as an element of the statistical ensemble $\mathcal{S}_{Dqu}$. For instance, let us consider a set $\mathcal{S}_{\phi}$ of systems $\mathcal{S}_{dc}$, having all similar integration constants $\{a, x^0_{(0)}, \mathbf{x}_{(0)}, \mathbf{u}, \tau_{(0)}, \boldsymbol{\xi}_{(0)}\}$ except of $\phi_{(0)}$. Let values of $\phi_{(0)}$ be distributed uniformly among the systems $\mathcal{S}_{dc}$. (For instance, in Eq.(7.38) x^i is considered as a function of two variables τ and $\phi_{(0)}$ with fixed parameter a). Then $\mathcal{S}_{\phi}$ can be considered as a ring. World lines of the ring particles form a two-dimensional world tube in the space-time. Such

a construction is more symmetric, than helically moving particle. Besides such a ring does not radiate electromagnetic waves. The statistical ensemble $\mathcal{S}_{Dqu}$ can be considered as consisted of rings $\mathcal{S}_\phi$ which are classical distributed dynamic systems. Unfortunately, rings $\mathcal{S}_\phi$ and their two-dimensional world tubes are incompatible with the concept of a pointlike particle.

One can try to associate the world tubes with the space-time properties (but not with the structure of the dynamic system $\mathcal{S}_\phi$). In this case it is necessary to admit that the distortion of the space-time does not vanish, i.e. the characteristic geometric structures of the space-time are three-dimensional world-tubes (but not straight lines) [2,15]. Then one can hope to remove incompatibility between the concept of a pointlike particle and the cylindrical space-time structures generated by the Dirac equation. Note that in the distorted space-time the momentum of a particle and its velocity are independent quantities which are connected by no algebraic relation.

Thus, the well known dynamic system $\mathcal{S}_D$ described by the Dirac equation has been investigated simply as a dynamic system without using any additional suppositions. The obtained results can be summed as follows.

1. Describing the dynamic system $\mathcal{S}_D$ in terms of hydrodynamic variables, one discovers that in the quasi-uniform approximation the $\mathcal{S}_D$ turns to a statistical ensemble $\mathcal{S}_{Dqu}$ of classical systems $\mathcal{S}_{dc}$. It means that, in general, the dynamic system $\mathcal{S}_D$ is a statistical ensemble of stochastic systems, and it permits to separate out a classical part $\mathcal{L}_{Dqu}$ of the total Lagrangian $\mathcal{L}_D$.

2. It is possible to separate out the classical part $\mathcal{L}_{Dqu}$ of the Lagrangian by means of special quantum variables κ, κ^i responsible for quantum effects. Suppressing the quantum variables (setting $\kappa, \kappa^i \equiv 0$), one transforms the Lagrangian $\mathcal{L}_D$ of the system $\mathcal{S}_D$ into its classical part $\mathcal{L}_{Dqu}$.

3. The transformation of the Dirac equation to hydrodynamic variables separates out a peculiar direction in the space-time. This direction is described by the timelike unit vector f^k. Existence of a preferred direction in the space-time is incompatible with the special relativity principle. It means that the Dirac equation is incompatible with the special relativity principle and needs a modification for such a compatibility.

4. The dynamic system $\mathcal{S}_{dc}$ is a classical analog of the quantum Dirac electron. $\mathcal{S}_{dc}$ cannot be interpreted as a pointlike charged particle in the Minkowski space-time. Rather it is either a complicated non-local dynamic system in the Minkowski space-time, or a pointlike particle, but in a non-Riemannian (distorted) space-time.

APPENDIX A
Classical Ensemble Described in Lagrangian and Eulerian Coordinates

Let us show equivalency of two form (2.1) and (2.2) of the action for the statistical ensemble of classical systems. Let us introduce the time Lagrangian coordinate ξ_0, supposing that $t = t(\xi_0)$, $x = \{t, \mathbf{x}\} = x(\xi)$, $\xi = \{\xi_0, \boldsymbol{\xi}\}$. Then the action (2.1) can be rewritten in the form

$$A_L[x] = \int \mathcal{L}(x, \dot{x}) \, d^{n+1}\boldsymbol{\xi}, \quad \dot{x} \equiv \partial x / \partial \xi_0, \quad d^{n+1}\boldsymbol{\xi} = d\xi_0 d\boldsymbol{\xi} \qquad (A.1)$$

$$\mathcal{L}(x, \dot{x}) = L(x^0, \mathbf{x}, \dot{\mathbf{x}}/\dot{x}^0)\dot{x}^0. \qquad (A.2)$$

According to Eq.(A.2) the Lagrangian $\mathcal{L}(x, \dot{x})$ is a first order homogeneous function of $\dot{x}$, i.e.

$$\mathcal{L}(x, a\dot{x}) = a\mathcal{L}(x, \dot{x}), \qquad (A.3)$$

where a is an arbitrary parameter.

Let the variational problem with the functional (A.1) take place. Let us show that it is equivalent to the variational problem with the functional (2.2).

Considering $\xi = \{\xi_i\}$, $i = 0, 1, \ldots n$ as functions of $x = \{x^i\}$, $i = 0, 1, \ldots n$, the variational problem with the action (A.1) can be formulated as a variational problem with the action

$$\mathcal{A}[\xi] = \int \mathcal{L}(x, \frac{\partial J}{\partial \xi_{0,i}}) d^{n+1}x, \qquad \xi = \xi(x) \qquad (A.4)$$

where J is a Jacobian

$$J \equiv \frac{\partial(\xi_0, \xi_1, \ldots \xi_n)}{\partial(x^0, x^1, \ldots x^n)} \equiv \det \parallel \xi_{i,k} \parallel,$$

$$\xi_{i,k} \equiv \partial\xi_i/\partial x^k, \qquad i, k = 0, 1, \ldots n \qquad (A.5)$$

and

$$\frac{\partial J}{\partial \xi_{0,i}} \equiv \frac{\partial(x^i, \xi_1, \xi_2, \ldots \xi_n)}{\partial(x^0, x^1, \ldots x^n)} \equiv \frac{\partial x^i}{\partial \xi_0} J, \qquad i = 0, 1, \ldots n \qquad (A.6)$$

Deriving Eq.(A.4) from Eq.(A.1), the relations (A.3), (A.6) were used.

In the variational problem with the action (A.4) the Lagrangian coordinates $\xi = \{\xi_i\}$, $i = 0, 1, \ldots n$ are dynamic variables considered as functions of coordinates $x = \{x^i\}$, $i = 0, 1, \ldots n$.

Let us use designations

$$j^i = \frac{\partial J}{\partial \xi_{0,i}}, \qquad i = 0, 1, \ldots n \qquad (A.7)$$

Adding constraints (A.7) to the action (A.4), one does not change the variational problem, because Eq.(A.4) does not contain variables j^i, and Eq. (A.7) is not a constraint in reality. Let us introduce Eq.(A.7) into the action (A.4) by means of Lagrange multipliers $p = \{p_i\}$, $i = 0, 1, \ldots n$. j^i and p_i, $i = 0, 1, \ldots n$ mean respectively a current density and a momentum density in the space $\{t, \mathbf{x}\}$.

Then one obtains

$$\mathcal{A}[j, p, \xi] = \int [L(x, j) - p_i(j^i - \frac{\partial J}{\partial \xi_{0,i}})]dx^{n+1}. \qquad (A.8)$$

Dynamic equations generated by the action (A.8) have the form

$$\frac{\delta \mathcal{A}}{\delta \xi_\alpha} = -\partial_k(p_l \frac{\partial^2 J}{\partial \xi_{0,l} \partial \xi_{\alpha,k}}) = 0, \qquad \alpha = 1, 2, \ldots n; \qquad (A.9)$$

$$\frac{\delta \mathcal{A}}{\delta j^i} = \frac{\partial \mathcal{L}}{\partial j^i} - p_i = 0, \qquad i = 0, 1, \ldots n \qquad (A.10)$$

Variation with respect to p_i leads to Eq.(A.7).

Using identities

$$\partial_k(\frac{\partial^2 J}{\partial \xi_{0,l} \partial \xi_{i,k}}) \equiv 0, \qquad i, l = 0, 1, \ldots n, \qquad \partial_k j^k = \partial_k \frac{\delta J}{\delta \xi_{0,i}} \equiv 0, \qquad (A.11)$$

$$\frac{\partial^2 J}{\partial \xi_{0,i} \partial \xi_{j,k}} \xi_{j,l} \equiv \delta_l^k \frac{\partial J}{\partial \xi_{0,i}} - \delta_l^i \frac{\partial J}{\partial \xi_{0,k}}, \qquad i, k, l = 0, 1, \ldots n, \qquad (A.12)$$

and eliminating the variables ξ, one can reduce equations (A.7), (A.9), (A.10) to the conventional hydrodynamic form (2.9), (2.14)

Instead of elimination of variables ξ Eqs. (A.9), (A.10) can be integrated in the form

$$p_k = \frac{\partial \mathcal{L}(x, j)}{\partial j^k} = \hbar \partial_k \varphi + \hbar g^\beta(\boldsymbol{\xi}) \partial_k \xi_\beta; \qquad k = 0, 1, \ldots n, \qquad (A.13)$$

where $g^\beta(\boldsymbol{\xi})$, $\beta = 1, 2, \ldots n$ are arbitrary functions of the argument $\boldsymbol{\xi} = \{\xi_\alpha\}$, $\alpha = 1, 2, \ldots n$. φ is some new variable. $\hbar$ is the Planck constant which is introduced, for variables φ, $\boldsymbol{\xi} = \{\xi_\alpha\}$, $\alpha = 1, 2, \ldots n$ and the functions g^β were dimensionless. In this context $\hbar$ has no quantum meaning.

Indeed, let us substitute relation (A.13) into Eq.(A.9) and use the first Eq.(A.11), antisymmetry of $\partial^2 J/\partial \xi_{0,l} \partial \xi_{\alpha,k}$ and symmetry of $\partial_k \partial_l \varphi$ with respect to indices l, k. Then one obtains

$$\partial_k[\frac{\partial^2 J}{\partial \xi_{0,l} \partial \xi_{\alpha,k}} g^\beta(\boldsymbol{\xi}) \xi_{\beta,l}] = 0, \qquad \alpha = 1, 2, \ldots n. \qquad (A.14)$$

Now using the identity

$$\frac{\partial^2 J}{\partial \xi_{0,l} \partial \xi_{\alpha,k}} \xi_{\beta,l} \equiv -\delta_\beta^\alpha \frac{\partial J}{\partial \xi_{0,k}}, \qquad \alpha, \beta = 1, 2 \ldots n, \qquad k = 0, 1, \ldots n, \qquad (A.15)$$

and the second relation (A.11), one can reduce Eq.(A.14) to the form

$$-\delta_\beta^\alpha \frac{\partial J}{\partial \xi_{0,k}} \xi_{\gamma,k} \frac{\partial g^\beta(\boldsymbol{\xi})}{\partial \xi_\gamma} = 0, \qquad \alpha = 1, 2, \ldots n. \qquad (A.16)$$

Due to the identity

$$\frac{\partial J}{\partial \xi_{0,k}} \xi_{l,k} \equiv \delta_l^0 J, \qquad l = 0, 1, \ldots n \qquad (A.17)$$

the equation (A.16) is satisfied for any functions $g^\beta(\boldsymbol{\xi})$, $\beta = 1, 2, \ldots$ n. Thus, the relation (A.13) is a general solution of the system of equations (A.9).

Let us substitute Eq.(A.13) into the action (A.8) and use that due to Eq.(A.17)

$$\frac{\partial J}{\partial \xi_{0,k}} p_k = \hbar \frac{\partial(\varphi, \xi_1, \xi_2, \ldots \xi_n)}{\partial(x^0, x^1, \ldots x^n)}. \qquad (A.18)$$

Then the action (A.8) turns to Eq.(2.2). The term (A.18) is omitted, because it does not give any contribution into dynamic equations. Thus, Eq.(2.2) is a corollary of Eq.(2.1).

Now let the variational problem with the functional (2.2) take place. Corresponding dynamic equations have the form (2.9)–(2.11). The relations (A.7) satisfy Eqs. (2.9), (2.11) identically, and an usage of them as additional constraints does not change the variational problem. Substituting (A.7) into Eq.(2.2) and using identities (A.17), (A.18), one obtains Eq.(A.4) which is equivalent to Eq.(2.1). Thus, the variational problems with actions (2.1) and (2.2) are equivalent.

APPENDIX B

Let us show here that under conditions

$$\boldsymbol{\xi}^2 = 1, \qquad \mathbf{z}^2 = 1 \qquad (B.1)$$

the equation

$$-\dot{\boldsymbol{\xi}} \times \mathbf{z} + \frac{(\mathbf{z}\dot{\boldsymbol{\xi}})}{2(1+\mathbf{z}\boldsymbol{\xi})}\boldsymbol{\xi} \times \mathbf{z} + \frac{\boldsymbol{\xi}(\dot{\boldsymbol{\xi}} \times \mathbf{z})}{2(1+\mathbf{z}\boldsymbol{\xi})}\mathbf{z} - \frac{(1+\mathbf{z}\boldsymbol{\xi})}{2}\mathbf{b} = C\boldsymbol{\xi}, \qquad (B.2)$$

where $\mathbf{b}$ is a given vector, and C is an indefinite function of τ, reduces to the form

$$\dot{\boldsymbol{\xi}} = \mathbf{b} \times \boldsymbol{\xi} \qquad (B.3)$$

Let $\mathbf{z}\boldsymbol{\xi} = \cos \alpha \neq \pm 1$. Let us introduce an orthonormal basis $\{\mathbf{e}_\alpha\}$, $\alpha = 1, 2, 3$

$$\mathbf{e}_3 = \boldsymbol{\xi}, \qquad \mathbf{e}_2 = \frac{(\boldsymbol{\xi} \times \mathbf{z})}{\sin \alpha}, \qquad \boldsymbol{\xi} = \cos \alpha \qquad (B.4)$$

$$\mathbf{e}_1 = \mathbf{e}_2 \times \mathbf{e}_3 = \frac{\boldsymbol{\xi} \times (\boldsymbol{\xi} \times \mathbf{z})}{\sin \alpha} = \frac{\mathbf{z} - \boldsymbol{\xi} \cos \alpha}{\sin \alpha} \qquad (B.5)$$

Then

$$\mathbf{z} = \mathbf{e}_1 \sin \alpha + \mathbf{e}_3 \cos \alpha, \qquad \dot{\boldsymbol{\xi}} = \dot{\xi}_\alpha \mathbf{e}_\alpha, \qquad \mathbf{b} = b_\alpha \mathbf{e}_\alpha \qquad (B.6)$$

It follows from (B.1) and (B.6) that

$$\boldsymbol{\xi}\dot{\boldsymbol{\xi}} = 0, \qquad \dot{\xi}_3 = 0, \qquad z_2 = 0. \qquad (B.7)$$

Substituting Eqs.(B.6), (B.7) into Eq.(B.2) and equating coefficients before the basis vectors $\mathbf{e}_\alpha$, one obtains

$$\mathbf{e}_1 : \qquad \dot{\xi}_2 = -b_1 \qquad (B.8)$$

$$\mathbf{e}_2 : \qquad \dot{\xi}_1 = b_2 \qquad (B.9)$$

$$\mathbf{e}_3: \qquad C = \frac{b_3}{2}(1 + \cos\alpha) - \frac{2 + \cos\alpha}{2(1 + \cos\alpha)}\sin\alpha \dot{\xi}_2. \qquad (B.10)$$

Eq.(B.3) follows from Eqs.(B.7)-(B.9). Eqs.(B.8)–(B.10) determine the indefinite function C through the given vector $\mathbf{b}$ and the scalar product $(\mathbf{z}\boldsymbol{\xi})$.

If $\mathbf{z} = \boldsymbol{\xi}$, the equation (B.2) reduces to the form

$$-\dot{\boldsymbol{\xi}} \times \boldsymbol{\xi} - \mathbf{b} = C\boldsymbol{\xi}. \qquad (B.11)$$

Forming the vector product of Eq.(B.11) with $\boldsymbol{\xi}$ and using (B.7), one obtains Eq.(B.3).

If $\mathbf{z} = -\boldsymbol{\xi}$, the equation (B.2) becomes indefinite, and it is necessary to use the limit $\boldsymbol{\xi} \to -\mathbf{z}$. It leads to the result (B.3).

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