Euclidean geometry as algorithm for construction of generalized geometries.

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Abstract

It is shown that the generalized geometries may be obtained as a deformation of the proper Euclidean geometry. Algorithm of construction of any proposition $\mathcal{S}$ of the proper Euclidean geometry $\mathcal{E}$ may be described in terms of the Euclidean world function σ_{E} in the form $\mathcal{S}(\sigma_{\mathrm{E}})$. Replacing the Euclidean world function σ_{E} by the world function σ of the geometry $\mathcal{G}$, one obtains the corresponding proposition $\mathcal{S}(\sigma)$ of the generalized geometry $\mathcal{G}$. Such a construction of the generalized geometries (known as T-geometries) uses well known algorithms of the proper Euclidean geometry and nothing besides. This method of the geometry construction is very simple and effective. Using T-geometry as the space-time geometry, one can construct the deterministic space-time geometries with primordially stochastic motion of free particles and geometrized particle mass. Such a space-time geometry defined properly (with quantum constant as an attribute of geometry) allows one to explain quantum effects as a result of the statistical description of the stochastic particle motion (without a use of quantum principles).

1 Introduction

The proper Euclidean geometry has been constructed by Euclid many years ago. The Euclidean geometry may be considered as a set of many algorithms, which are necessary for construction of geometrical objects $\mathcal{O}_{\mathrm{E}}$ and relations $\mathcal{R}_{\mathrm{E}}$ between them. These algorithms of the geometrical objects construction were obtained by means of logical reasonings from fundamental propositions (axioms) of Euclidean geometry. Any algorithm $\mathcal{A}_{\mathrm{E}}$ of construction of some geometrical object $\mathcal{O}_{\mathrm{E}}$ may be considered as an operator O, acting on the set Ω of points P, where the geometry is constructed. The objects, which undergo the action of operators, are called operands. In the given case the points P of the set Ω are operands of algorithms of the Euclidean geometry construction.

There is a necessity in construction of generalized (non-Euclidean) geometries, which are distinguished from the proper Euclidean geometry in different aspects. For instance, a generalized geometry is necessary for description of properties of the real space-time.

In the beginning of 20th century there existed the problem of motion of bodies with very high speed. This problem was solved by a construction of the relativity theory, where the direct product of time and the Euclidean space was substituted by the Minkowski geometry of the space-time. Another problem of the beginning of 20th century: stochastic motion of particle with small mass also should be solved by means of a modification of the space-time geometry. The fact is that the motion of free particles is determined *only by the properties of the space-time* (or by properties of its geometry). Intensity of the stochastic component of the particle motion depends on the particle mass. This intensity is very small for particles of large mass, and it is essential for particles of small mass. It means that in the corresponding space-time geometry the free particle motion is to be primordially stochastic, and the particle mass is to be geometrized, to take into account the influence of the mass upon the free motion. In the beginning of 20th century there was not such a geometry in the framework of the Riemannian geometries. Furthermore, one could not imagine such a deterministic space-time geometry, where the motion of free particles be stochastic. One tried to solve the problem, using stochastic space-time geometry [4, 2]. However, the stochastic geometry is not a geometry in the precise value of the word. The stochastic geometry is the usual space-time geometry (for instance, Minkowski geometry) with some additional structure, given on the space-time.

As a result the problem of the stochastic particle motion had not been solved on the fundamental level (by a modification of the space-time geometry). Instead, the stochastic manifestation of the space-time geometry properties was prescribed to the free particle in the form of quantum properties, generated by the quantum principles. Introduction of quantum principles admitted one to explain all nonrelativistic quantum effects. However, the relativistic quantum effects cannot be described on the basis of the quantum principles, because quantum principles are nonrelativistic. To expand quantum theory to relativistic phenomena, one needs to return on the fundamental level, i. e. to the true space-time geometry, which is valid for both relativistic and nonrelativistic phenomena.

Conventional method of the generalized geometry construction is a modification of the Euclidean algorithms. One changes some Euclidean axioms. Besides, one uses deformation of the Euclidean space. The infinitesimal Euclidean distance $dS_{\mathrm{E}} = \sqrt{g_{\mathrm{E}ik}dx^i dx^k}$ is substituted by means of the Riemannian one $dS_{\mathrm{R}} = \sqrt{g_{\mathrm{R}ik}dx^i dx^k}$. In general, the Euclidean algorithm of the geometry construction is replaced by another one. The new algorithm is used for construction of the generalized geometry. To construct the generalized geometry (for instance, the Riemannian geometry), one repeats many (sometimes almost all) Euclidean logical reasonings . Such a method is very complicated. Besides, it is necessary, that independent modifications of different Euclidean axioms be compatible between themselves.

There is another method of the Euclidean geometry generalization. We do not change the Euclidean algorithms of the geometry construction. However, we replace operands of the Euclidean algorithms. We use the world function σ as the operand of the Euclidean algorithms instead of the set Ω of points P, which are usually is used as operands of the Euclidean algorithms.

The world function $\sigma(P,Q) = \frac{1}{2}\rho^2(P,Q)$, where $\rho(P,Q)$ is the distance between the points P,$Q \in \Omega$. (We do not use the term metric for ρ, because the metric is supposed to be restricted by constraint of positivity and by the triangle axiom, whereas the distance ρ is free of these constraints.) The world function σ defined by the relation

$$\sigma: \qquad \Omega \otimes \Omega \to \mathbb{R}, \qquad \sigma(P,P) = 0, \qquad \forall P \in \Omega \tag{1.1}$$

The world function was introduced by Synge [?], and it is a very important quantity in the proper Euclidean geometry. But the world function σ is not only an important quantity. It is the *only important* quantity in the proper Euclidean geometry. It means that the proper Euclidean geometry and all its algorithms may be described in terms and only in terms of the Euclidean world function $\sigma_{\rm E}$. The property of a geometry to be described completely in terms and only in terms of the world function is called the σ-immanence property. Description of a geometry in terms of the world function is the σ-immanent description of the geometry.

The σ-immanence property of the Euclidean geometry was discovered rather recently [5, 7]. It has been proved that the Euclidean geometry can be presented in terms and only in terms of the function $\sigma_{\rm E}$, provided the function $\sigma_{\rm E}$ satisfies a series of constraints, written in terms of $\sigma_{\rm E}$. By definition, *any geometry is a totality of all geometric objects $\mathcal{O}$ and of all relations $\mathcal{R}$ between them.* The σ-immanence of the proper Euclidean geometry means that any geometric object $\mathcal{O}_{\rm E}$ and any relation $\mathcal{R}_{\rm E}$ of the Euclidean geometry $\mathcal{G}_{\rm E}$ can be presented in terms of the Euclidean world function $\sigma_{\rm E}$ in the form $\mathcal{O}_{\rm E}(\sigma_{\rm E})$ and $\mathcal{R}_{\rm E}(\sigma_{\rm E})$.

Let us suppose that a geometry $\mathcal{G}$ has the property of the σ-immanence. Then the geometry $\mathcal{G}$ may be constructed as a result of a deformation of the proper Euclidean geometry $\mathcal{G}_{\rm E}$. Indeed, the proper Euclidean geometry $\mathcal{G}_{\rm E}$ is the totality of geometrical objects $\mathcal{O}_{\rm E}(\sigma_{\rm E})$ and relations $\mathcal{R}_{\rm E}(\sigma_{\rm E})$. We produce the change

$$\sigma_{\rm E} \to \sigma, \qquad \mathcal{O}_{\rm E}(\sigma_{\rm E}) \to \mathcal{O}_{\rm E}(\sigma), \qquad \mathcal{R}_{\rm E}(\sigma_{\rm E}) \to \mathcal{R}_{\rm E}(\sigma) \tag{1.2}$$

Then totality of geometrical objects $\mathcal{O}_{\rm E}(\sigma)$, relations $\mathcal{R}_{\rm E}(\sigma)$ and the world function σ form the generalized geometry $\mathcal{G}$. Any generalized geometry, obtained by the method (1.2) will be referred to as a tubular geometry (T-geometry) [5, 7, ?].

Note that the geometrical objects $\mathcal{O}_{\rm E}(\sigma_{\rm E})$, $\mathcal{O}_{\rm E}(\sigma)$ and relations $\mathcal{R}_{\rm E}(\sigma_{\rm E})$, $\mathcal{R}_{\rm E}(\sigma)$ are constructed by means the same algorithm of the Euclidean geometry construction. Only operands $\sigma_{\rm E}$ and σ are different in the two cases. Thus, different generalized geometries $\mathcal{G}_1$ and $\mathcal{G}_2$ are distinguished by the value of their operands σ_1 and σ_2, but not by the algorithms of their construction. It is very convenient, because a change of operand is more simpler, than a change of an algorithm. We may not care for compatibility of different changes of the Euclidean algorithms, as it takes place at the conventional approach to a construction of a generalized geometry, when we change the algorithm at the constant operand. Besides, we can evaluate the power of the set of generalized geometries, which can be considered as a power of the set of functions of two arguments. The power of this set is much more, than the power of the set of all Riemannian geometries, which can be evaluated as the power of the set of several functions of one argument. For instance, the set of all homogeneous isotropic T-geometries is determined by a function of one argument, whereas the set of all homogeneous isotropic Riemannian geometries is labelled by the dimension n and the index $\nu = 1, 2, ...n$.

From practical point of view, the T-geometries are interesting in the relation, that they contains such space-time geometries, where the motion of free particle is primordially stochastic. Furthermore, the space-time T-geometry with stochastic motion of free particle is a general case, whereas the space-time T-geometry with deterministic motion of free particle is a degenerate case.

Further we shall formulate the principal theorem on the σ-immanence of the proper Euclidean geometry. But at first, we put the cases of σ-immanent description of objects of the proper Euclidean geometry

Let $\mathcal{R}_{\mathrm{E}}(\sigma_{\mathrm{E}})$ be the scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_2)_{\mathrm{E}}$ of two vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0\mathbf{P}_2$ in $\mathcal{G}_{\mathrm{E}}$. It can be written in the σ-immanent form (i.e. in terms of the world function σ_{E})

$$\mathcal{R}_{\mathrm{E}}(\sigma_{\mathrm{E}}): \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_2)_{\mathrm{E}} = \sigma_{\mathrm{E}}(P_0, P_1) + \sigma_{\mathrm{E}}(P_0, P_2) - \sigma_{\mathrm{E}}(P_1, P_2) \tag{1.3}$$

where index 'E' shows that the quantity relates to the Euclidean geometry. It is easy to see that (1.3) is a corollary of the Euclidean relations

$$|\mathbf{P}_0\mathbf{P}_1|_{\mathrm{E}}^2 = 2\sigma_{\mathrm{E}}(P_0, P_1) \tag{1.4}$$

$$|\mathbf{P}_1\mathbf{P}_2|_{\mathrm{E}}^2 = |\mathbf{P}_0\mathbf{P}_2 - \mathbf{P}_0\mathbf{P}_1|_{\mathrm{E}}^2 = |\mathbf{P}_0\mathbf{P}_1|_{\mathrm{E}}^2 + |\mathbf{P}_0\mathbf{P}_2|_{\mathrm{E}}^2 - 2(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_2)_{\mathrm{E}} \tag{1.5}$$

According to (1.2) in the generalized geometry $\mathcal{G}$ we obtain instead of (1.3)

$$\mathcal{R}_{\mathrm{E}}(\sigma): \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_2) = \sigma(P_0, P_1) + \sigma(P_0, P_2) - \sigma(P_1, P_2) \tag{1.6}$$

Such a way of the generalized geometry construction is very simple. It does not use any logical reasonings. It is founded on the supposition that *any generalized geometry has the σ-immanence property.* It uses essentially the fact that the algorithms of the proper Euclidean geometry construction are already known, and all necessary logical reasonings has been already produced in the proper Euclidean geometry.

The application of the replacement (1.2) to the construction of a generalized geometry will be referred to as the deformation principle. Any change of distance ρ, or the world function σ between the points of the space Ω means a deformation of this space. We construe the concept of deformation in a broad sense. The deformation may transform a point into a surface and a surface into a point. The deformation may remove some points of the Euclidean space, violating its continuity, or decreasing its dimension. The deformation may add supplemental points to the space, increasing its dimension. We may interpret any σ-immanent generalization of the Euclidean geometry as its deformation. In other words, the deformation principle is a very general method of the generalized geometry construction.

Construction of a nonhomogeneous geometry on the axiomatic basis is impossible practically, because there is a lot of different nonhomogeneous geometries. It is very difficult to invent axiomatics for a nonhomogeneous geometry, where identical objects have different properties in various places. Besides, one cannot invent axiomatics for each of these geometries. Thus, in reality there is no alternative to application of the deformation principle at the construction of the nonhomogeneous generalized geometry. The real problem consists in the sequential application of the deformation principle. As far as the *deformation principle alone is sufficient for the construction of the T-geometry*, one may not use additional means of the geometry construction. At the T-geometry construction we do not use coordinate system and other means of description.

Mathematicians provide physicists with their geometrical construction, and physicists believe that the space-time is continuous. Continuity of the space-time cannot be tested experimentally, and the only reason of the space-time continuity is the fact that mathematicians are able to construct only continuous geometries, whereas they fail to construct discrete geometries.

We have the same situation with the space dimension. Geometers consider the dimension to be an inherent property of any geometry. They can imagine the n-dimensional Riemannian geometry, but they cannot imagine a geometry without a dimension, or a geometry of an indefinite dimension. The reason of these belief is the fact that the dimension of the manifold and its continuity are the starting points of the Riemannian geometry construction, and at this point one cannot separate the properties of the geometry from the properties of the manifold.

In the T-geometry we deal only with the geometry in itself, because it does not use any means of the description. As a result the T-geometry is insensitive to continuity or discreteness of the space, as well as to its dimension. Application of additional means of description can lead to inconsistency and to a restriction of the list of possible T-geometries.

Any generalization of the proper Euclidean geometry is founded on some property of the Euclidean geometry (or its objects). This property is conserved in all generalized geometries, whereas other properties of the Euclidean geometry are varied. Character and properties of the obtained generalized geometry depend essentially on the choice of the conserved property of the basic Euclidean geometry. For instance, the Riemannian geometry is such a generalization of the Euclidean one, where the one-dimensionality and continuity of the Euclidean straight line are conserved, whereas its curvature and torsion are varied. The straight line is considered to be the principal geometric object of the Euclidean geometry, and one supposes that such properties of the Euclidean straight line as continuity and one-dimensionality (absence of thickness) are to be conserved at the generalization. It means that the continuity and one-dimensionality of the straight line are to be the principal concepts of the generalized geometry (the Riemannian geometry). In accordance with such a choice of the conserved geometrical object one introduces the concept of the curve $\mathcal{L}$ as a continuous mapping of a segment of the real axis onto the space Ω

$$\mathcal{L}: \qquad [0,1] \rightarrow \Omega \tag{1.7}$$

To introduce the concept of the continuity, which is a basic concept of the generalization, one introduces the topological space, the dimension of the space Ω and other basic concepts of the Riemannian geometry, which are necessary for construction of the Riemannian generalization of the Euclidean geometry. However, these properties are not necessary for the T-geometry construction.

The σ-immanence of the Euclidean geometry is a property of the whole Euclidean geometry. Using the σ-immanence for generalization, we do not impose any constraints on the single geometric objects of the Euclidean geometry. As a result the σ-immanent generalization appears to be a very powerful generalization. Besides, from the common point of view the application of the whole geometry property for the generalization seems to be more reasonable, than a use of the properties of a single geometric object. Thus, using the property of the whole Euclidean geometry, the σ-immanent generalization seems to be more reasonable, than the Riemannian generalization, which uses the properties of the Euclidean straight line.

Now we list the most attractive features of the σ-immanent generalization of the Euclidean geometry:

1. The σ-immanent generalization uses the σ-immanence, which is a property of the Euclidean geometry as a whole (but not a property of a single geometric object as it takes place at the Riemannian generalization).

2. The σ-immanent generalization does not use any logical construction, and the σ-immanent generalization is automatically as consistent, as the Euclidean geometry, whose axiomatics is used implicitly. In particular, the T-geometry does not contain any theorems. As a result the main problem of the T-geometry is a correct σ-immanent description of geometrical objects and relations of the Euclidean geometry. There are some subtleties in such a σ-immanent description, which are discussed below.

3. The σ-immanent generalization is a very powerful generalization. It varies practically all properties of the Euclidean geometry, including such ones as the continuity and the parallelism transitivity, which are conserved at the Riemannian generalization.

4. The σ-immanent generalization allows one to use the coordinateless description and to ignore the problems, connected with the coordinate transformations as well as with the transformation of other means of description.

5. The T-geometry may be used as the space-time geometry. In this case the tubular character of straights explains freely the stochastic world lines of quantum microparticles. Considering the quantum constant $\hbar$ as an attribute of the space-time geometry, one can obtain the quantum description as the statistical description of the stochastic world lines [6]. Such a space-time geometry

cannot be obtained in the framework of the Riemannian generalization of the Euclidean geometry.

6. Practically all above mentioned properties of the σ-immanent generalization are corollaries of the fact, that the world function σ is an operand in the algorithms of the proper Euclidean geometry construction.

2 Euclidean geometry in the σ-immanent form

Definition 1 *The σ-space $V = \{\sigma, \Omega\}$ is the set Ω of points P with the given world function σ*

$$\sigma : \qquad \Omega \times \Omega \to \mathbb{R}, \qquad \sigma(P, P) = 0, \qquad \forall P \in \Omega \tag{2.1}$$

Let the proper Euclidean geometry be given on the set Ω, and the quantity

$$\rho(P_0, P_1) = \sqrt{2\sigma(P_0, P_1)}, \qquad P_0, P_1 \in \Omega \tag{2.2}$$

be the Euclidean distance between the points P_0, P_1.

Let the vector $\mathbf{P}_0\mathbf{P}_1 = \{P_0, P_1\}$ be the ordered set of two points P_0, P_1. The point P_0 is the origin of the vector $\mathbf{P}_0\mathbf{P}_1$, and the point P_1 is its end. The length $|\mathbf{P}_0\mathbf{P}_1|$ of the vector $\mathbf{P}_0\mathbf{P}_1$ is defined by the relation

$$|\mathbf{P}_0\mathbf{P}_1|^2 = 2\sigma(P_0, P_1) \tag{2.3}$$

In the Euclidean geometry the scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_2)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{P}_2$, having the common origin P_0, is expressed by the relation (1.6)

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_2) = \sigma(P_0, P_1) + \sigma(P_1, P_0) - \sigma(P_1, P_2) \tag{2.4}$$

It follows from the expression (2.4), written for scalar products $(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1)$ and $(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_0)$, and from the properties of the scalar product in the Euclidean space, that the scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ can be written in the σ-immanent form

$$\begin{aligned}(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) &= (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1) - (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_0) = \\ &\quad \sigma(P_0, Q_1) + \sigma(P_1, Q_0) - \sigma(P_0, Q_0) - \sigma(P_1, Q_1)\end{aligned} \tag{2.5}$$

Let $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0\mathbf{P}_2$,...$\mathbf{P}_0\mathbf{P}_n$ be n vectors in the Euclidean space. The necessary and sufficient condition of their linear dependence is

$$F_n(\mathcal{P}^n) \equiv \det ||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)|| = 0, \qquad i, k = 1, 2, ..n, \qquad \mathcal{P}^n = \{P_0, P_1, ...P_n\} \tag{2.6}$$

where $F_n(\mathcal{P}^n) \equiv \det ||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)||$ is the Gram's determinant, constructed of the scalar products of vectors.

Let us formulate the theorem on the σ-immanence of the Euclidean geometry.

Theorem 1 *The σ-space $V=\{\sigma,\Omega\}$ is the n-dimensional proper Euclidean space, if and only if the world function σ satisfies the following conditions, written in terms of the world function σ.*

I. Condition of symmetry:

$$\sigma(P,Q)=\sigma(Q,P), \qquad \forall P,Q\in\Omega \tag{2.7}$$

II. Definition of the dimension:

$$\exists\mathcal{P}^n\equiv\{P_0,P_1,...P_n\}\subset\Omega, \qquad F_n(\mathcal{P}^n)\neq 0, \qquad F_k\left(\Omega^{k+1}\right)=0, \qquad k>n \tag{2.8}$$

where $F_n(\mathcal{P}^n)$ is the Gram's determinant (2.6). Vectors $\mathbf{P}_0\mathbf{P}_i$, $i=1,2,...n$ are basic vectors of the rectilinear coordinate system K_n with the origin at the point P_0, and the metric tensors $g_{ik}(\mathcal{P}^n)$, $g^{ik}(\mathcal{P}^n)$, $i,k=1,2,...n$ in K_n are defined by the relations

$$\sum_{k=1}^{k=n}g^{ik}(\mathcal{P}^n)\,g_{lk}(\mathcal{P}^n)=\delta_l^i, \qquad g_{il}(\mathcal{P}^n)=(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_l), \qquad i,l=1,2,...n \tag{2.9}$$

$$F_n(\mathcal{P}^n)=\det||g_{ik}(\mathcal{P}^n)||\neq 0, \qquad i,k=1,2,...n \tag{2.10}$$

III. Linear structure of the Euclidean space:

$$\sigma(P,Q)=\frac{1}{2}\sum_{i,k=1}^{i,k=n}g^{ik}(\mathcal{P}^n)\,(x_i(P)-x_i(Q))\,(x_k(P)-x_k(Q)), \qquad \forall P,Q\in\Omega \tag{2.11}$$

where coordinates $x_i(P)$, $i=1,2,...n$ of the point P are covariant coordinates of the vector $\mathbf{P}_0\mathbf{P}$, defined by the relation

$$x_i(P)=(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}), \qquad i=1,2,...n \tag{2.12}$$

IV: The metric tensor matrix $g_{lk}(\mathcal{P}^n)$ has only positive eigenvalues

$$g_k>0, \qquad k=1,2,...,n \tag{2.13}$$

V. The continuity condition: the system of equations

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P})=y_i\in\mathbb{R}, \qquad i=1,2,...n \tag{2.14}$$

considered to be equations for determination of the point P as a function of coordinates $y=\{y_i\}$, $i=1,2,...n$ has always one and only one solution. Conditions II – V contain a reference to the dimension n of the Euclidean space.

This theorem states that the proper Euclidean space has the property of the σ-immanence, and hence any statement $\mathcal{S}$ of the proper Euclidean geometry can be expressed in terms and only in terms of the world function σ_{E} of the Euclidean geometry in the form $\mathcal{S}(\sigma_{\mathrm{E}})$. Producing the change $\sigma_{\mathrm{E}}\to\sigma$ in the statement $\mathcal{S}$, we obtain corresponding statement $\mathcal{S}(\sigma)$ of another T-geometry $\mathcal{G}$, described by the world function σ.

3 Construction of geometric objects in the σ-immanent form

In the T-geometry the geometric object $\mathcal{O}$ is described by means of the skeleton-envelope method [7]. It means that any geometric object $\mathcal{O}$ is considered to be a set of intersections and joins of elementary geometric objects (EGO).

The finite set $\mathcal{P}^n \equiv \{P_0, P_1, ..., P_n\} \subset \Omega$ of parameters of the envelope function $f_{\mathcal{P}^n}$ is the skeleton of elementary geometric object (EGO) $\mathcal{E} \subset \Omega$. The set $\mathcal{E} \subset \Omega$ of points forming EGO is called the envelope of its skeleton $\mathcal{P}^n$. In the continuous generalized geometry the envelope $\mathcal{E}$ is usually a continual set of points. The envelope function $f_{\mathcal{P}^n}$

$$f_{\mathcal{P}^n}: \qquad \Omega \rightarrow \mathbb{R}, \tag{3.1}$$

determining EGO is a function of the running point $R \in \Omega$ and of parameters $\mathcal{P}^n \subset \Omega$. The envelope function $f_{\mathcal{P}^n}$ is supposed to be an algebraic function of s arguments $w = \{w_1, w_2, ...w_s\}$, $s = (n+2)(n+1)/2$. Each of arguments $w_k = \sigma(Q_k, L_k)$ is a σ-function of two arguments $Q_k, L_k \in \{R, \mathcal{P}^n\}$, either belonging to skeleton $\mathcal{P}^n$, or coinciding with the running point R. Thus, any elementary geometric object $\mathcal{E}$ is determined by its skeleton and its envelope function as the set of zeros of the envelope function

$$\mathcal{E} = \{R | f_{\mathcal{P}^n}(R) = 0\} \tag{3.2}$$

For instance, the cylinder $\mathcal{C}(P_0, P_1, Q)$ with the points P_0, P_1 on the cylinder axis and the point Q on its surface is determined by the relation

$$\begin{aligned} \mathcal{C}(P_0, P_1, Q) &= \{R | f_{P_0 P_1 Q}(R) = 0\}, \\ f_{P_0 P_1 Q}(R) &= F_2(P_0, P_1, Q) - F_2(P_0, P_1, R) \end{aligned} \tag{3.3}$$

$$F_2(P_0, P_1, Q) = \left| \begin{array}{cc} (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}) \\ (\mathbf{P}_0\mathbf{Q}.\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{Q}.\mathbf{P}_0\mathbf{Q}) \end{array} \right| \tag{3.4}$$

Here $\frac{1}{2}\sqrt{F_2(P_0, P_1, Q)}$ is the area of triangle with vertices at the points P_0, P_1, Q. The equality $F_2(P_0, P_1, Q) = F_2(P_0, P_1, R)$ means that the distance between the point Q and the axis, determined by the vector $\mathbf{P}_0\mathbf{P}_1$ is equal to the distance between R and the axis.

The elementary geometrical object $\mathcal{E}$ is determined in all T-geometries at once. In particular, it is determined in the proper Euclidean geometry, where we can obtain its meaning. We interpret the elementary geometrical object $\mathcal{E}$, using our knowledge of the proper Euclidean geometry. Thus, the proper Euclidean geometry is used as a sample geometry for interpretation of any generalized geometry. In particular, the cylinder (3.3) is determined uniquely in any T-geometry with any world function σ.

In the Euclidean geometry the points P_0 and P_1 determine the cylinder axis. The shape of a cylinder depends on its axis and radius, but not on the disposition of points P_0, P_1 on the cylinder axis. As a result in the Euclidean geometry the cylinders $\mathcal{C}(P_0, P_1, Q)$ and $\mathcal{C}(P_0, P_2, Q)$ coincide, provided vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{P}_2$ are collinear. In the general case of T-geometry the cylinders $\mathcal{C}(P_0, P_1, Q)$ and

$\mathcal{C}(P_0, P_2, Q)$ do not coincide, in general, even if vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{P}_2$ are collinear. Thus, in general, a deformation of the Euclidean geometry splits the Euclidean geometrical objects.

At construction of a generalized geometry we do not try to repeat derivation of Euclidean algorithms from other axioms. We take the geometrical objects and relations between them, prepared in the framework of the Euclidean geometry and describe them in terms of the world function. Thereafter we deform them, replacing the Euclidean world function σ_{E} by the world function σ of the geometry in question. In practice the construction of the elementary geometric object is reduced to the representation of the corresponding Euclidean geometrical object in the σ-immanent form, i.e. in terms of the Euclidean world function. The last problem is the problem of the proper Euclidean geometry. The problem of representation of the geometrical object (or relation between objects) in the σ-immanent form is a real problem of the T-geometry construction.

Application of the deformation principle is restricted by two constraints.

1. The deformation principle is to be applied separately from other methods of the geometry construction. In particular, one may not use topological structures in construction of a T-geometry, because for effective application of the deformation principle the obtained T-geometry must be determined only by the world function.

2. Describing Euclidean geometric objects $\mathcal{O}(\sigma_{\mathrm{E}})$ and Euclidean relation $\mathcal{R}(\sigma_{\mathrm{E}})$ in terms of σ_{E}, we are not to use special properties of Euclidean world function σ_{E}. In particular, definitions of $\mathcal{O}(\sigma_{\mathrm{E}})$ and $\mathcal{R}(\sigma_{\mathrm{E}})$ are to have similar form in Euclidean geometries of different dimensions. They must not depend on the dimension of the Euclidean space.

The T-geometry construction is not to use coordinates and other methods of description, because the application of the means of description imposes constraints on the constructed geometry. Any means of description is a structure St given on the basic Euclidean geometry with the world function σ_{E}. Replacement $\sigma_{\mathrm{E}} \to \sigma$ is sufficient for construction of unique generalized geometry $\mathcal{G}_\sigma$. If we use an additional structure St for the T-geometry construction, we obtain, in general, other geometry $\mathcal{G}_{St}$, which coincides with $\mathcal{G}_\sigma$ not for all σ, but only for some of world functions σ. Thus, a use of additional means of description restricts the list of possible generalized geometries. For instance, if we use the coordinate description at construction of the generalized geometry, the obtained geometry appears to be continuous, because description by means of the coordinates is effective only for continuous geometries, where the number of coordinates coincides with the geometry dimension.

As far as the σ-immanent description of the proper Euclidean geometry is possible, it is possible for any T-geometry, because any geometrical object $\mathcal{O}$ and any relation $\mathcal{R}$ in the physical geometry $\mathcal{G}$ is obtained from the corresponding geometrical object $\mathcal{O}_{\mathrm{E}}$ and from the corresponding relation $\mathcal{R}_{\mathrm{E}}$ in the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ by means of the replacement $\sigma_{\mathrm{E}} \to \sigma$ in description of $\mathcal{O}_{\mathrm{E}}$ and $\mathcal{R}_{\mathrm{E}}$. For such a replacement be possible, the description of $\mathcal{O}_{\mathrm{E}}$ and $\mathcal{R}_{\mathrm{E}}$ is not to refer to special properties of σ_{E}, described by conditions II – V. A formal indicator of the conditions II – V application is a reference to the dimension n, because any of conditions II – V contains a reference to the dimension n of the proper Euclidean space.

Let us suppose that some geometrical object $\mathcal{O}_{\mathrm{E}_n}(\sigma_{\mathrm{E}_n}, n)$ is defined in the n-dimensional Euclidean space, and this definition refers explicitly to the dimension of the Euclidean space n. We deform the n-dimensional Euclidean space E_n in the m-dimensional Euclidean space E_m. Then we must make the change

$$\mathcal{O}_{\mathrm{E}_n}(\sigma_{\mathrm{E}_n}, n) \rightarrow \mathcal{O}_{\mathrm{E}_n}(\sigma_{\mathrm{E}_m}, n) \tag{3.5}$$

On the other hand, we may define the same geometrical object directly in the m-dimensional Euclidean space E_m in the form $\mathcal{O}_{\mathrm{E}_m}(\sigma_{\mathrm{E}_m}, m)$. Equating this expression to (3.5), we obtain

$$\mathcal{O}_{\mathrm{E}_n}(\sigma_{\mathrm{E}_m}, n) = \mathcal{O}_{\mathrm{E}_m}(\sigma_{\mathrm{E}_m}, m), \qquad \forall m, n \in \mathbb{N} \tag{3.6}$$

It means that the definition of the geometrical object $\mathcal{O}$ is to be independent on the dimension of the Euclidean space.

If nevertheless we use one of special properties II – V of the Euclidean space in the σ-immanent description of a geometrical object $\mathcal{O}$, or relation $\mathcal{R}$, we refer to the dimension n and, ultimately, to the coordinate system, which is only a means of description.

Let us show this in the example of the determination of the straight in the Euclidean space. The straight $\mathcal{T}_{P_0Q}$ in the proper Euclidean space is defined by two its points P_0 and Q $(P_0 \neq Q)$ as the set of points R

$$\mathcal{T}_{P_0Q} = \{R \mid \mathbf{P}_0\mathbf{Q} || \mathbf{P}_0\mathbf{R}\} \tag{3.7}$$

where condition $\mathbf{P}_0\mathbf{Q}||\mathbf{P}_0\mathbf{R}$ means that vectors $\mathbf{P}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$ are collinear, i.e. the scalar product $(\mathbf{P}_0\mathbf{Q}.\mathbf{P}_0\mathbf{R})$ of these two vectors satisfies the relation

$$\mathbf{P}_0\mathbf{Q}||\mathbf{P}_0\mathbf{R}: \qquad (\mathbf{P}_0\mathbf{Q}.\mathbf{P}_0\mathbf{R})^2 = (\mathbf{P}_0\mathbf{Q}.\mathbf{P}_0\mathbf{Q})(\mathbf{P}_0\mathbf{R}.\mathbf{P}_0\mathbf{R}) \tag{3.8}$$

where the scalar product is defined by the relation (2.5). Thus, the straight line $\mathcal{T}_{P_0Q}$ is defined σ-immanently, i.e. in terms of the world function σ. We shall use two different names (straight and tube) for the geometric object $\mathcal{T}_{P_0Q}$. We shall use the term "straight", when we want to stress that $\mathcal{T}_{P_0Q}$ is a result of deformation of the Euclidean straight. We shall use the term "tube", when we want to stress that $\mathcal{T}_{P_0Q}$ may be a many-dimensional surface.

In the Euclidean geometry one can use another definition of collinearity. Vectors $\mathbf{P}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$ are collinear, if components of vectors $\mathbf{P}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$ are proportional in some rectilinear coordinate system. For instance, in the n-dimensional Euclidean space one can introduce rectilinear coordinate system, choosing $n+1$ points $\mathcal{P}^n = \{P_0, P_1, ...P_n\}$ and forming n basic vectors $\mathbf{P}_0\mathbf{P}_i$, $i = 1, 2, ...n$. Then the collinearity condition can be written in the form of n equations

$$\mathbf{P}_0\mathbf{Q}||\mathbf{P}_0\mathbf{R}: \qquad (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{Q}) = a(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{R}), \qquad i = 1, 2, ...n, \qquad a \in \mathbb{R}\setminus\{0\} \tag{3.9}$$

where $a \neq 0$ is some real constant. Relations (3.9) are relations for covariant components of vectors $\mathbf{P}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$ in the considered coordinate system with basic vectors $\mathbf{P}_0\mathbf{P}_i$, $i = 1, 2, ...n$. The definition of collinearity (3.9) depends on the dimension n of the Euclidean space. Let points $\mathcal{P}^n$ be chosen in such a way, that $(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}) \neq 0$. Then eliminating the parameter a from relations (3.9), we obtain $n-1$ independent relations, and the geometrical object

$$\mathcal{T}_{Q\mathcal{P}^n} = \{R \mid \mathbf{P}_0\mathbf{Q} || \mathbf{P}_0\mathbf{R}\} = \bigcap_{i=2}^{i=n} \mathcal{S}_i, \tag{3.10}$$

$$\mathcal{S}_i = \left\{ R \left| \frac{(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{Q})}{(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q})} = \frac{(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{R})}{(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R})} \right. \right\}, \qquad i = 2, 3, ...n \tag{3.11}$$

defined according to (3.9), depends on $n+2$ points $Q, \mathcal{P}^n$. This geometrical object $\mathcal{T}_{Q\mathcal{P}^n}$ is defined σ-immanently. It is a complex, consisting of the straight line and of the coordinate system, represented by $n+1$ points $\mathcal{P}^n = \{P_0, P_1, ...P_n\}$. In the Euclidean space the dependence on the choice of the coordinate system and on n points $\{P_1, ...P_n\}$, determining this system, is fictitious. The geometrical object $\mathcal{T}_{Q\mathcal{P}^n}$ depends essentially only on two points P_0, Q and coincides with the straight line $\mathcal{T}_{P_0Q}$ in the Euclidean space. But at deformations of the Euclidean space the geometrical objects $\mathcal{T}_{Q\mathcal{P}^n}$ and $\mathcal{T}_{P_0Q}$ are deformed differently. The points $P_1, P_2, ...P_n$ cease to be fictitious in definition of $\mathcal{T}_{Q\mathcal{P}^n}$, and geometrical objects $\mathcal{T}_{Q\mathcal{P}^n}$ and $\mathcal{T}_{P_0Q}$ become to be different geometric objects, in general. But being different, in general, they may coincide in some special cases.

What of the two geometrical objects in the deformed geometry $\mathcal{G}$ should be interpreted as a straight line, passing through the points P_0 and Q in the geometry $\mathcal{G}$? Of course, it is $\mathcal{T}_{P_0Q}$, because its definition does not contain a reference to a coordinate system, whereas definition of $\mathcal{T}_{Q\mathcal{P}^n}$ depends on the choice of the coordinate system, represented by points $\mathcal{P}^n$. In general, definitions of geometric objects and relations between them are not to refer to the means of description. Otherwise, the points determining the coordinate system are to be included in definition of the geometrical object.

But in the given case the geometrical object $\mathcal{T}_{P_0Q}$ is a $(n-1)$-dimensional surface, in general, whereas $\mathcal{T}_{Q\mathcal{P}^n}$ is an intersection of $(n-1)$ $(n-1)$-dimensional surfaces, i.e. $\mathcal{T}_{Q\mathcal{P}^n}$ is a one-dimensional curve, in general. The one-dimensional curve $\mathcal{T}_{Q\mathcal{P}^n}$ corresponds better to our ideas on the straight line, than the $(n-1)$-dimensional surface $\mathcal{T}_{P_0Q}$. Nevertheless, in T-geometry $\mathcal{G}$ it is $\mathcal{T}_{P_0Q}$, that is an analog of the Euclidean straight line.

It is very difficult to overcome our conventional idea that the Euclidean straight line cannot be deformed into many-dimensional surface, and *this idea has been prevent for years from construction of T-geometries.* Practically one uses such generalized geometries, where deformation of the Euclidean space transforms the Euclidean straight lines into one-dimensional lines. It means that one chooses such geometries, where geometrical objects $\mathcal{T}_{P_0Q}$ and $\mathcal{T}_{Q\mathcal{P}^n}$ coincide.

$$\mathcal{T}_{P_0Q} = \mathcal{T}_{Q\mathcal{P}^n} \tag{3.12}$$

Condition (3.12) of coincidence of the objects $\mathcal{T}_{P_0Q}$ and $\mathcal{T}_{Q\mathcal{P}^n}$, imposed on the T-geometry, restricts the list of possible T-geometries.

In general, the condition (3.12) cannot be fulfilled, because lhs does not depend on points $\{P_1, P_2, ...P_n\}$, whereas rhs of (3.12) depends, in general. The tube $\mathcal{T}_{Q\mathcal{P}^n}$ does not depend on the points $\{P_1, P_2, ...P_n\}$, provided the distance $\sqrt{2\sigma(P_i, P_k)}$ between any two points $P_i, P_k \in \mathcal{P}^n$ is infinitesimal. In the Riemannian geometry the constraint (3.12) is fulfilled at the additional restriction.

$$\sqrt{2\sigma(P_i, P_k)} = \text{infinitesimal}, \qquad i, k = 1, 2, ...n \tag{3.13}$$

4 Interplay between metric geometry and T-geometry

Let us consider the metric geometry, given on the set Ω of points. The metric space $M = \{\rho, \Omega\}$ is given by the metric (distance) ρ.

$$\rho \quad : \quad \Omega \times \Omega \to [0, \infty) \subset \mathbb{R} \tag{4.1}$$

$$\rho(P, P) = 0, \qquad \rho(P, Q) = \rho(Q, P), \qquad \forall P, Q \in \Omega \tag{4.2}$$

$$\rho(P, Q) \geq 0, \qquad \rho(P, Q) = 0, \quad \text{iff } P = Q, \qquad \forall P, Q \in \Omega \tag{4.3}$$

$$0 \leq \rho(P, R) + \rho(R, Q) - \rho(P, Q), \qquad \forall P, Q, R \in \Omega \tag{4.4}$$

At first sight the metric space is a special case of the σ-space (2.1), and the metric geometry is a special case of the T-geometry with additional constraints (4.3), (4.4) imposed on the world function $\sigma = \frac{1}{2}\rho^2$. However it is not so, because the metric geometry is not equipped by the deformation principle. The metric geometry does not use the algorithms of the Euclidean geometry construction. In the metric geometry the deformation principle can be used only in its coordinate form

$$g_{\mathrm{E}ik} dx^i dx^k \to g_{ik} dx^i dx^k \tag{4.5}$$

because the coordinateless form (1.2) of the deformation principle as well as the σ-immanence of the Euclidean geometry and complex of conditions (2.7) - (2.14) were not known until 1990, although each of relations (2.7) - (2.14) was well known. But the metric geometry is described in the coordinateless form, and application of the deformation (4.5) is impossible in the metric geometry.

Additional (with respect to the σ-space) constraints (4.3), (4.4) are imposed to provide one-dimensionality of the straight lines. In the metric geometry the shortest (straight) line can be constructed only in the case, when it is one-dimensional.

Let us consider the set $\mathcal{EL}(P, Q, a)$ of points R

$$\mathcal{EL}(P, Q, 2a) = \{R | f_{P,Q,2a}(R) = 0\}, \qquad f_{P,Q,2a}(R) = \rho(P, R) + \rho(R, Q) - 2a \tag{4.6}$$

If the metric space coincides with the proper Euclidean space, this set of points is an ellipsoid with focuses at the points P, Q and the large semiaxis a. The relations $f_{P,Q,2a}(R) > 0$, $f_{P,Q,2a}(R) = 0$, $f_{P,Q,2a}(R) < 0$ determine respectively external points, boundary points and internal points of the ellipsoid. If $\rho(P,Q)=2a$, we obtain the degenerate ellipsoid, which coincides with the segment $\mathcal{T}_{[PQ]}$ of the straight line, passing through the points P, Q. In the proper Euclidean geometry, the degenerate ellipsoid is one-dimensional segment of the straight line, but it is not evident that it is one-dimensional in the case of arbitrary metric geometry. For such a degenerate ellipsoid be one-dimensional in the arbitrary metric space, it is necessary that any degenerate ellipsoid $\mathcal{EL}(P,Q,\rho(P,Q))$ have no internal points. This constraint is written in the form

$$f_{P,Q,\rho(P,Q)}(R)=\rho(P,R)+\rho(R,Q)-\rho(P,Q)\geq 0 \tag{4.7}$$

Comparing relation (4.7) with (4.4), we see that the constraint (4.4) is introduced to provide the straight (shortest) line one-dimensionality (absence of internal points in the geometrical object determined by two points).

As far as the metric geometry does not use the deformation principle, it is a poor geometry, because in the framework of this geometry one cannot construct the scalar product of two vectors, define linear independence of vectors and construct such geometrical objects as planes. All these objects as well as other are constructed on the basis of the deformation of the proper Euclidean geometry.

Generalizing the metric geometry, Menger [3] and Blumenthal [1] removed the triangle axiom (4.4). They tried to construct the distance geometry, which would be a more general geometry, than the metric one. As far as they did not use the deformation principle, they could not determine the shortest (straight) line without a reference to the topological concept of the curve $\mathcal{L}$, defined as a continuous mapping (1.7), which cannot be expressed only via the distance. As a result the distance geometry appeared to be not a pure metric geometry (i.e. the geometry determined only by the distance).

Note that the Riemannian geometry uses the deformation principle in the coordinate form. The distance geometry cannot use it in such a form, because the metric and distance geometries are formulated in the coordinateless form. It is to use the deformation principle in the coordinateless form. But application of the deformation principle in the coordinateless form needs a use of the Euclidean geometry σ-immanence. K. Menger went to the concept of the σ-immanence, but he stopped in one step before the σ-immanence. Look at the Menger's theorem [3], written in our designations

Theorem 2 *The σ-space $V=\{\sigma,\Omega\}$ is isometrically embeddable in n-dimensional proper Euclidean space E_n, if and only if any set of $n+3$ points of Ω is isometrically embeddable in E_n.*

The theorem on the σ-immanence of the Euclidean geometry is obtained from the Menger's theorem, if instead of the condition "any set of $n+3$ points of Ω is isometrically embeddable in E_n" one writes the condition (2.11), which also contains $n+3$ points: $P,Q,\mathcal{P}^n$ and describes the fact that $\{P,Q,\mathcal{P}^n\}\subset E_n$. In this case the theorem condition contains only a reference to the properties of the world function of the Euclidean space, but not to the Euclideaness of the space. (continuity of the σ-space V is neglected in such a formulation.)

5 Conditions of the deformation principle application

Riemannian geometries satisfy the condition (3.12). The Riemannian geometry is a kind of inhomogeneous generalized geometry, and, hence, it uses the deformation principle. Constructing the Riemannian geometry, the infinitesimal Euclidean distance is deformed into the Riemannian distance. The deformation is chosen in such a way that any Euclidean straight line $\mathcal{T}_{\mathrm{E}P_0Q}$, passing through the point P_0, collinear to the vector $\mathbf{P}_0\mathbf{Q}$, is transformed into the geodesic $\mathcal{T}_{P_0Q}$, passing through the point P_0, collinear to the vector $\mathbf{P}_0\mathbf{Q}$ in the Riemannian space.

Note that in T-geometries, satisfying the condition (3.12) for all points $Q, \mathcal{P}^n$, the straight line

$$\mathcal{T}_{Q_0;P_0Q} = \{R \mid \mathbf{P}_0\mathbf{Q}||\mathbf{Q}_0\mathbf{R}\} \qquad (5.1)$$

passing through the point Q_0 collinear to the vector $\mathbf{P}_0\mathbf{Q}$, is not a one-dimensional line, in general. If the Riemannian geometries be T-geometries, they would contain non-one-dimensional geodesics (straight lines). But the Riemannian geometries are not T-geometries, because at their construction one uses not only the deformation principle, but some other methods, containing a reference to the means of description. In particular, in the Riemannian geometries the absolute parallelism is absent, and one cannot define a straight line (5.1), because the collinearity relation $\mathbf{P}_0\mathbf{Q}||\mathbf{Q}_0\mathbf{R}$ is not defined, if points P_0 and Q_0 do not coincide. On one hand, a lack of absolute parallelism allows one to go around the problem of non-one-dimensional straight lines. On the other hand, it makes the Riemannian geometries to be inconsistent, because they cease to be T-geometries, which are consistent by the construction (see for details [8]).

The fact is that the application of *only deformation principle* is sufficient for construction of a generalized geometry. Besides, such a construction is consistent, because the original Euclidean geometry is consistent and, deforming it, we do not use any logical reasonings. If we introduce additional structure (for instance, a topological structure) we obtain a fortified geometry, i.e. a generalized geometry with additional structure on it. The T-geometry, equipped with additional structure, is a more pithy construction, than the T-geometry simply. But it is valid only in the case, when we consider the additional structure as an addition to the T-geometry. If we use an additional structure in construction of the geometry, we identify the additional structure with one of structures of the geometry. If we demand that the additional structure be a structure of T-geometry, we restrict an application of the deformation principle and reduce the list of possible generalized geometries, because coincidence of the additional structure with some structure of a geometry is possible not for all geometries, but only for some of them.

Let, for instance, we use concept of a curve $\mathcal{L}$ (1.7) for construction of a generalized geometry. The concept of curve $\mathcal{L}$, considered as a continuous mapping, is a topological structure, which cannot be expressed only via the distance or via the world function. A use of the mapping (1.7) needs an introduction of topological space and, in particular, the concept of continuity. If we identify the topological curve (1.7) with the "metrical" curve, defined as a broken line

$$\mathcal{T}_{\rm br}=\bigcup_i \mathcal{T}_{[P_iP_{i+1}]},\qquad \mathcal{T}_{[P_iP_{i+1}]}=\left\{R|\sqrt{2\sigma\left(P_i,P_{i+1}\right)}-\sqrt{2\sigma\left(P_i,R\right)}-\sqrt{2\sigma\left(R,P_{i+1}\right)}\right\} \tag{5.2}$$

consisting of the straight line segments $\mathcal{T}_{[P_iP_{i+1}]}$ between the points P_i, P_{i+1}, we truncate the list of possible geometries, because such an identification is possible only in some generalized geometries. Identifying (1.7) and (5.2), we eliminate all discrete geometries and those continuous geometries, where the segment $\mathcal{T}_{[P_iP_{i+1}]}$ of straight line is a surface, but not a one-dimensional set of points. Thus, additional structures may lead to (i) a fortified geometry, (ii) a restricted geometry and (iii) a restricted fortified geometry. The result depends on the method of the additional structure application.

Note that some constraints (continuity, convexity, lack of absolute parallelism), imposed on generalized geometries, are a result of a disagreement of the means of description, which are used at the geometry construction. In the T-geometry, which uses only the deformation principle, there is no such restrictions. Besides, the T-geometry has some new property of a geometry, which is not accepted by conventional versions of generalized geometry. This property, called the geometry nondegeneracy, follows directly from the application of arbitrary deformations to the proper Euclidean geometry.

Definition 2 *The geometry is degenerate at the point P_0 in the direction of the vector $\mathbf{Q}_0\mathbf{Q}$, $|\mathbf{Q}_0\mathbf{Q}|\neq 0$, if the relations*

$$\mathbf{Q}_0\mathbf{Q}\uparrow\uparrow\mathbf{P}_0\mathbf{R}:\qquad (\mathbf{Q}_0\mathbf{Q}.\mathbf{P}_0\mathbf{R})=\sqrt{|\mathbf{Q}_0\mathbf{Q}|\cdot|\mathbf{P}_0\mathbf{R}|},\qquad |\mathbf{P}_0\mathbf{R}|=a\neq 0 \tag{5.3}$$

considered as equations for determination of the point R, have not more, than one solution for any $a\neq 0$. Otherwise, the geometry is nondegenerate at the point P_0 in the direction of the vector $\mathbf{Q}_0\mathbf{Q}$.

Note that the first equation (5.3) is the condition of the parallelism of vectors $\mathbf{Q}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$.

The proper Euclidean geometry is degenerate, i.e. it is degenerate at all points in directions of all vectors. Considering the Minkowski geometry, one should distinguish between the Minkowski T-geometry and Minkowski geometry. The two geometries are described by the same world function and differ in the definition of the parallelism. In the Minkowski T-geometry the parallelism of two vectors $\mathbf{Q}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$ is defined by the first equation (5.3). This definition is based on the deformation principle. In the n-dimensional Minkowski geometry (n-dimensional pseudo-Euclidean geometry of index 1) the parallelism is defined by the relation of the type of (3.9)

$$\mathbf{Q}_0\mathbf{Q} \uparrow\uparrow \mathbf{P}_0\mathbf{R}: \qquad (\mathbf{P}_0\mathbf{P}_i.\mathbf{Q}_0\mathbf{Q}) = a\,(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{R}), \qquad i = 1,2,...n, \qquad a > 0 \tag{5.4}$$

where points $\mathcal{P}^n = \{P_0, P_1, ...P_n\}$ determine a rectilinear coordinate system with basic vectors $\mathbf{P}_0\mathbf{P}_i$, $i = 1,2,..n$ in the n-dimensional Minkowski space. Dependence of the definition (5.4) on the points $(P_0, P_1, ...P_n)$ is fictitious, but dependence on the number $n+1$ of points $\mathcal{P}^n$ is essential. Thus, definition (5.4) depends on the method of the geometry description.

The Minkowski T-geometry is degenerate at all points in direction of all timelike vectors, and it is nondegenerate at all points in direction of all spacelike vectors. The Minkowski geometry is degenerate at all points in direction of all vectors. Conventionally one uses the Minkowski geometry, ignoring the nondegeneracy in spacelike directions.

Considering the proper Riemannian geometry, one should distinguish between the Riemannian T-geometry and the Riemannian geometry. The two geometries are described by the same world function. They differ in the definition of the parallelism. In the Riemannian T-geometry the parallelism of two vectors $\mathbf{Q_0Q}$ and $\mathbf{P_0R}$ is defined by (5.3). In the Riemannian geometry the parallelism of two vectors $\mathbf{Q_0Q}$ and $\mathbf{P_0R}$ is defined only in the case, when the points P_0 and Q_0 coincide. Parallelism of remote vectors $\mathbf{Q_0Q}$ and $\mathbf{P_0R}$ is not defined, in general. This fact is known as absence of absolute parallelism.

The proper Riemannian T-geometry is locally degenerate, i.e. it is degenerate at all points P_0 in direction of all vectors $\mathbf{P}_0\mathbf{Q}$ with the origin at the point P_0. In the general case, when $P_0 \neq Q_0$, the proper Riemannian T-geometry is nondegenerate, in general. But it is degenerate locally as well as the proper Riemannian geometry. The proper Riemannian geometry is degenerate, because it is degenerate locally, whereas the illocal degeneracy is not defined in the Riemannian geometry, because of the lack of absolute parallelism. Conventionally one uses the Riemannian geometry (not Riemannian T-geometry) and ignores the property of the nondegeneracy completely.

From the viewpoint of the conventional approach to the physical geometry the nondegeneracy is an undesirable property of a physical geometry, although from the logical viewpoint and from viewpoint of the deformation principle the nondegeneracy is *an inherent property of a generalized geometry.* The illocal nondegeneracy is ejected from the proper Riemannian geometry by denial of existence of the remote vector parallelism. Nondegeneracy in the spacelike directions is ejected from the Minkowski geometry by means of the redefinition of the two vectors parallelism. But the nondegeneracy is an important property of the real space-time geometry. To appreciate this, let us consider an example.

6 Simple example of nondegenerate space-time geometry

Let the space-time geometry $\mathcal{G}_{\mathrm{d}}$ be described by the T-geometry, given on 4-dimensional manifold $\mathcal{M}_{1+3}$. The world function σ_{d} is described by the relation

$$\sigma_{\mathrm{d}}=\sigma_{\mathrm{M}}+D\left(\sigma_{\mathrm{M}}\right)=\begin{cases}\sigma_{\mathrm{M}}+d & \text{if } \sigma_0<\sigma_{\mathrm{M}}\\ \left(1+\frac{d}{\sigma_0}\right)\sigma_{\mathrm{M}} & \text{if } 0\leq\sigma_{\mathrm{M}}\leq\sigma_0\\ \sigma_{\mathrm{M}} & \text{if } \sigma_{\mathrm{M}}<0\end{cases} \tag{6.1}$$

where $d\geq 0$ and $\sigma_0>0$ are some constants. The quantity σ_{M} is the world function in the Minkowski space-time geometry $\mathcal{G}_{\mathrm{M}}$. In the orthogonal rectilinear (inertial) coordinate system $x=\{t,\mathbf{x}\}$ the world function σ_{M} has the form

$$\sigma_{\mathrm{M}}\left(x,x'\right)=\frac{1}{2}\left(c^2\left(t-t'\right)^2-\left(\mathbf{x}-\mathbf{x}'\right)^2\right) \tag{6.2}$$

where c is the speed of the light.

Let us compare the broken line (5.2) in Minkowski space-time geometry $\mathcal{G}_{\mathrm{M}}$ and in the distorted geometry $\mathcal{G}_{\mathrm{d}}$. We suppose that $\mathcal{T}_{\mathrm{br}}$ is timelike broken line, and all links $\mathcal{T}_{[P_iP_{i+1}]}$ of $\mathcal{T}_{\mathrm{br}}$ are timelike and have the same length

$$\left|\mathbf{P}_i\mathbf{P}_{i+1}\right|_{\mathrm{d}}=\sqrt{2\sigma_{\mathrm{d}}\left(P_i,P_{i+1}\right)}=\mu_{\mathrm{d}}>0,\qquad i=0,\pm1,\pm2,... \tag{6.3}$$

$$\left|\mathbf{P}_i\mathbf{P}_{i+1}\right|_{\mathrm{M}}=\sqrt{2\sigma_{\mathrm{M}}\left(P_i,P_{i+1}\right)}=\mu_{\mathrm{M}}>0,\qquad i=0,\pm1,\pm2,... \tag{6.4}$$

where indices "d" and "M" mean that the quantity is calculated by means of σ_{d} and σ_{M} respectively. Vector $\mathbf{P}_i\mathbf{P}_{i+1}$ is regarded as the momentum of the particle at the segment $\mathcal{T}_{[P_iP_{i+1}]}$, and the quantity $|\mathbf{P}_i\mathbf{P}_{i+1}|=\mu$ is interpreted as its (geometric) mass. It follows from definition (2.5) and relation (6.1), that for timelike vectors $\mathbf{P}_i\mathbf{P}_{i+1}$ with $\mu>\sqrt{2\sigma_0}$

$$\left|\mathbf{P}_i\mathbf{P}_{i+1}\right|_{\mathrm{d}}^2=\mu_{\mathrm{d}}^2=\mu_{\mathrm{M}}^2+2d,\qquad \mu_{\mathrm{M}}^2>2\sigma_0 \tag{6.5}$$

$$\left(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1}\right)_{\mathrm{d}}=\left(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1}\right)_{\mathrm{M}}+d \tag{6.6}$$

Calculation of the shape of the segment $\mathcal{T}_{[P_0P_1]}\left(\sigma_{\mathrm{d}}\right)$ in $\mathcal{G}_{\mathrm{d}}$ gives the relation

$$r^2(\tau)=\begin{cases}\tau^2\mu_{\mathrm{d}}^2\frac{\left(1-\frac{\tau d}{2(\sigma_0+d)}\right)^2}{\left(1-\frac{2d}{\mu_{\mathrm{d}}^2}\right)}-\frac{\tau^2\mu_{\mathrm{d}}^2\sigma_0}{(\sigma_0+d)}, & 0<\tau<\frac{\sqrt{2(\sigma_0+d)}}{\mu_{\mathrm{d}}}\\ \frac{3d}{2}+2d\left(\tau-1/2\right)^2\left(1-\frac{2d}{\mu_{\mathrm{d}}^2}\right)^{-1}, & \frac{\sqrt{2(\sigma_0+d)}}{\mu_{\mathrm{d}}}<\tau<1-\frac{\sqrt{2(\sigma_0+d)}}{\mu_{\mathrm{d}}}\\ \left(1-\tau\right)^2\mu_{\mathrm{d}}^2\left[\frac{\left(1-\frac{(1-\tau)d}{2(\sigma_0+d)}\right)^2}{\left(1-\frac{2d}{\mu_{\mathrm{d}}^2}\right)}-\frac{\sigma_0}{(\sigma_0+d)}\right], & 1-\frac{\sqrt{2(\sigma_0+d)}}{\mu_{\mathrm{d}}}<\tau<1\end{cases}, \tag{6.7}$$

where $r(\tau)$ is the spatial radius of the segment $\mathcal{T}_{[P_0P_1]}(\sigma_{\mathrm{d}})$ in the coordinate system, where points P_0 and P_1 have coordinates $P_0 = \{0,0,0,0\}$, $P_1 = \{\mu_{\mathrm{d}},0,0,0\}$ and τ is a parameter along the segment $\mathcal{T}_{[P_0P_1]}(\sigma_{\mathrm{d}})$, $(\tau(P_0) = 0,\ \tau(P_1) = 1)$. One can see from (6.7) that the characteristic value of the segment radius is equal to $\sqrt{d}$.

Let the broken tube $\mathcal{T}_{\mathrm{br}}$ describe the "world tube" of a free particle. It means by definition that any link $\mathbf{P}_{i-1}\mathbf{P}_i$ is parallel to the adjacent link $\mathbf{P}_i\mathbf{P}_{i+1}$

$$\mathbf{P}_{i-1}\mathbf{P}_i \uparrow\uparrow \mathbf{P}_i\mathbf{P}_{i+1}: \qquad (\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1}) - |\mathbf{P}_{i-1}\mathbf{P}_i| \cdot |\mathbf{P}_i\mathbf{P}_{i+1}| = 0 \tag{6.8}$$

Definition of parallelism is different in geometries $\mathcal{G}_{\mathrm{M}}$ and $\mathcal{G}_{\mathrm{d}}$. As a result links, which are parallel in the geometry $\mathcal{G}_{\mathrm{M}}$, are not parallel in $\mathcal{G}_{\mathrm{d}}$ and vice versa.

Let $\mathcal{T}_{\mathrm{br}}(\sigma_{\mathrm{M}})$ describe the world line of a free particle in the geometry $\mathcal{G}_{\mathrm{M}}$. The angle ϑ_{M} between the adjacent links in $\mathcal{G}_{\mathrm{M}}$ is defined by the relation

$$\cosh\vartheta_{\mathrm{M}} = \frac{(\mathbf{P}_{-1}\mathbf{P}_0.\mathbf{P}_0\mathbf{P}_1)_{\mathrm{M}}}{|\mathbf{P}_0\mathbf{P}_1|_{\mathrm{M}} \cdot |\mathbf{P}_{-1}\mathbf{P}_0|_{\mathrm{M}}} = 1 \tag{6.9}$$

The angle $\vartheta_{\mathrm{M}} = 0$, and the geometrical object $\mathcal{T}_{\mathrm{br}}(\sigma_{\mathrm{M}})$ is a timelike straight line on the manifold $\mathcal{M}_{1+3}$.

Let now $\mathcal{T}_{\mathrm{br}}(\sigma_{\mathrm{d}})$ describe the world tube of a free particle in the geometry $\mathcal{G}_{\mathrm{d}}$. The angle ϑ_{d} between the adjacent links in $\mathcal{G}_{\mathrm{d}}$ is defined by the relation

$$\cosh\vartheta_{\mathrm{d}} = \frac{(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1})_{\mathrm{d}}}{|\mathbf{P}_i\mathbf{P}_{i+1}|_{\mathrm{d}} \cdot |\mathbf{P}_{i-1}\mathbf{P}_i|_{\mathrm{d}}} = 1 \tag{6.10}$$

The angle $\vartheta_{\mathrm{d}} = 0$ also. If we draw the broken tube $\mathcal{T}_{\mathrm{br}}(\sigma_{\mathrm{d}})$ on the manifold $\mathcal{M}_{1+3}$, using coordinates of basic points P_i and measure the angle ϑ_{dM} between the adjacent links in the Minkowski geometry $\mathcal{G}_{\mathrm{M}}$, we obtain for the angle ϑ_{dM} the following relation

$$\cosh\vartheta_{\mathrm{dM}} = \frac{(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1})_{\mathrm{M}}}{|\mathbf{P}_i\mathbf{P}_{i+1}|_{\mathrm{M}} \cdot |\mathbf{P}_{i-1}\mathbf{P}_i|_{\mathrm{M}}} = \frac{(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1})_{\mathrm{d}} - d}{|\mathbf{P}_i\mathbf{P}_{i+1}|_{\mathrm{d}}^2 - 2d} \tag{6.11}$$

Substituting the value of $(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1})_{\mathrm{d}}$, taken from (6.10), we obtain

$$\cosh\vartheta_{\mathrm{dM}} = \frac{\mu_{\mathrm{d}}^d - d}{\mu_{\mathrm{d}}^2 - 2d} \approx 1 + \frac{d}{\mu_{\mathrm{d}}^2}, \qquad d \ll \mu_{\mathrm{d}}^2 \tag{6.12}$$

Hence, $\vartheta_{\mathrm{dM}} \approx \sqrt{2d}/\mu_{\mathrm{d}}$. It means, that the adjacent link is located on the cone of angle $\sqrt{2d}/\mu_{\mathrm{d}}$, and the whole line $\mathcal{T}_{\mathrm{br}}(\sigma_{\mathrm{d}})$ has a random shape, because any link wobbles with the characteristic angle $\sqrt{2d}/\mu_{\mathrm{d}}$. The wobble angle depends on the space-time distortion d and on the particle mass μ_{d}. The wobble angle is small for the large mass of a particle. The random displacement of the segment end is of the order $\mu_{\mathrm{d}}\vartheta_{\mathrm{dM}} = \sqrt{2d}$, i.e. of the same order as the segment width. It is reasonable, because these two phenomena have the common source: the space-time distortion D.

One should note that the space-time geometry influences the stochasticity of particle motion illocally in the sense, that the form of the world function (6.1) for values of $\sigma_{\mathrm{M}} < \frac{1}{2}\mu_{\mathrm{d}}^2$ is unessential for the motion stochasticity of the particle of the mass μ_{d}.

Such a situation, when the world line of a free particle is stochastic in the deterministic geometry, and this stochasticity depends on the particle mass, seems to be rather exotic and incredible. But experiments show that the motion of real particles of small mass is stochastic indeed, and this stochasticity increases, when the particle mass decreases. From physical viewpoint a theoretical foundation of the stochasticity is desirable, and some researchers invent stochastic geometries, noncommutative geometries and other exotic geometrical constructions, to obtain the quantum stochasticity. But in the Riemannian space-time geometry the particle motion does not depend on the particle mass, and in the framework of the Riemannian space-time geometry it is difficult to explain the quantum stochasticity by the space-time geometry properties. The distorted geometry $\mathcal{G}_{\mathrm{d}}$ explains freely the stochasticity and its dependence on the particle mass. Besides, at proper choice of the distortion d the statistical description of stochastic $\mathcal{T}_{\mathrm{br}}$ leads to the quantum description (in terms of the Schrödinger equation) [6]. To do this, it is sufficient to set

$$d = \frac{\hbar}{2bc} \tag{6.13}$$

where $\hbar$ is the quantum constant, c is the speed of the light, and b is some universal constant, connecting the geometrical mass μ with the usual particle mass m by means of the relation $m = b\mu$. In other words, the distorted space-time geometry (6.1) is closer to the real space-time geometry, than the Minkowski geometry $\mathcal{G}_{\mathrm{M}}$.

Further development of the statistical description of geometrical stochasticity leads to a creation of the model conception of quantum phenomena (MCQP), which relates to the conventional quantum theory approximately in the same way as the statistical physics relates to the axiomatic thermodynamics. MCQP is the well defined relativistic conception with effective methods of investigation [10], whereas the conventional quantum theory is not well defined, because it uses incorrect space-time geometry, whose incorrectness is compensated by additional hypotheses (quantum principles). Besides, it has problems with application of the nonrelativistic quantum mechanical technique to the description of relativistic phenomena.

The geometry $\mathcal{G}_{\mathrm{d}}$, as well as the Minkowski geometry are homogeneous geometries, because the world function σ_{d} is invariant with respect to all coordinate transformations, with respect to which the world function σ_{M} is invariant.

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