# Is the Dirac particle completely relativistic?

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**Abstract**

The Dirac particle, i.e. the dynamic system $\mathcal{S}_{\mathrm{D}}$, described by the free Dirac equation is investigated. Although the Dirac equation is written usually in the relativistically covariant form, the dynamic system $\mathcal{S}_{\mathrm{D}}$ is not completely relativistic, because its description contains such absolute objects as $\gamma$-matrices $\gamma^k$, forming a matrix vector. By means of the proper change of variables the $\gamma$-matrices are eliminated, but instead of them the constant timlike vector $f^k$ appears. The vector $f^k$ describes an absolute splitting of the space-time into space and time, which is characteristic for the nonrelativistic description. To investigate a degree of the violation of the $\mathcal{S}_{\mathrm{D}}$ relativistic description, we consider the classical Dirac particle $\mathcal{S}_{\mathrm{Dcl}}$, obtained from $\mathcal{S}_{\mathrm{D}}$ by means of the relativistic dynamic disquantization. The classical dynamic system $\mathcal{S}_{\mathrm{Dcl}}$ appears to be composite, because it has ten degrees of freedom. Six translational degrees of freedom are described relativistically (without a reference to $f^k$), whereas four internal degrees of freedom are described nonrelativistically, because their description refers to $f^k$. Coupling the absolute vector $f^k$ with the energy-momentum vector of $\mathcal{S}_{\mathrm{Dcl}}$, the classical Dirac particle $\mathcal{S}_{\mathrm{Dcl}}$ is modified minimally. The vector $f^k$ ceases to be absolute, and the modified classical Dirac particle $\mathcal{S}_{\mathrm{mDcl}}$ becomes to be completely relativistic. The dynamic equations for $\mathcal{S}_{\mathrm{mDcl}}$ are solved. Solutions for $\mathcal{S}_{\mathrm{Dcl}}$ and $\mathcal{S}_{\mathrm{mDcl}}$ are compared.

*Key words: disquantization, Dirac equation, relativistic invariance.*

# 1 Introduction

The question-mark in the title of the paper seems to be inadequate, because the problem of the relativistic invariance of the Dirac equation had been solved many years ago. The Dirac equation was invented by Dirac as a relativistic equation, and its relativistic invariance was proved by Dirac [1]. The Dirac equation was investigated by many authors [2] - [8], but this investigation was produced always in the framework of the quantum principles. The proof of the relativistic invariance of the Dirac equation can be found in any textbook on quantum mechanics. Why does this question arise?

The fact is that, in general, the relativistic covariance of dynamic equations is not sufficient for compatibility with the relativity principles. The Dirac equation contains such specific quantities as $\gamma$-matrices. If we make a change of variables, eliminating $\gamma$-matrices, we obtain the system of dynamic equations, which is not relativistically covariant. Such a transition to eight hydrodynamic variables: 4-vector $j^k = \bar{\psi}\gamma^k\psi$, $k = 0,1,2,3$ , 4-pseudovector $S^k = i\bar{\psi}\gamma_5\gamma^k\psi$, $k = 0,1,2,3$, scalar $\varphi$ and pseudoscalar $\kappa$ instead of eight real dependent variables $\psi$ is necessary, if we want to obtain classical analog $\mathcal{S}_{\rm Dcl}$ of the Dirac particle $\mathcal{S}_{\rm D}$ [9].

The Dirac particle $\mathcal{S}_{\rm D}$ is the dynamic system which is described by the action

$$\mathcal{S}_{\rm D}: \qquad \mathcal{A}_{\rm D}[\bar{\psi},\psi] = \int (-m\bar{\psi}\psi + \frac{i}{2}\hbar\bar{\psi}\gamma^l\partial_l\psi - \frac{i}{2}\hbar\partial_l\bar{\psi}\gamma^l\psi)d^4x \tag{1.1}$$

Here $\psi$ is four-component complex wave function, $\psi^*$ is the Hermitian conjugate wave function, and $\bar{\psi} = \psi^*\gamma^0$ is conjugate one. $\gamma^i$, $i = 0,1,2,3$ are $4\times 4$ complex constant matrices, satisfying the relation

$$\gamma^l\gamma^k + \gamma^k\gamma^l = 2g^{kl}I, \qquad k,l = 0,1,2,3. \tag{1.2}$$

where $I$ is the unit $4\times 4$ matrix, and $g^{kl} = \mathrm{diag}(c^{-2},-1,-1,-1)$ is the metric tensor. Considering dynamic system $\mathcal{S}_{\rm D}$, we choose for simplicity such units, where the speed of the light $c = 1$. The action (1.1) generates dynamic equation

$$i\hbar\gamma^l\partial_l\psi - m\psi = 0 \tag{1.3}$$

and expressions for physical quantities: the 4-current $j^k$ of particles and the energy-momentum tensor $T_l^k$

$$j^k = \bar{\psi}\gamma^k\psi, \qquad T_l^k = \frac{i}{2}\left(\bar{\psi}\gamma^k\partial_l\psi - \partial_l\bar{\psi}\cdot\gamma^k\psi\right) \tag{1.4}$$

The dynamic equation (1.3) is known as the Dirac equation.

We stress that the current $j^k$, as well as the energy-momentum tensor $T_l^k$ are attributes of the Dirac particle $\mathcal{S}_{\rm D}$. In particular, it means that, changing expression (1.4) for the current $j^k$, we change the dynamic system $\mathcal{S}_{\rm D}$, even if the dynamic equation (1.3) is not changed.

The classical Dirac particle $\mathcal{S}_{\rm Dcl}$ (classical analog of $\mathcal{S}_{\rm D}$) is a discrete dynamic system, i.e. the dynamic system, having finite number (ten) of the freedom degrees.

Action and dynamic equations for $\mathcal{S}_{\mathrm{Dcl}}$ are obtained as a result of dynamic disquantization of the dynamic system $\mathcal{S}_{\mathrm{D}}$ [9]. By definition the procedure of dynamic disquantization means the change

$$\partial^l \to \partial_{||}^l = \frac{j^l j^k}{j^s j_s}\partial_k, \qquad l = 0,1,2,3, \qquad \partial^k \equiv g^{kl}\partial_l \equiv g^{kl}\frac{\partial}{\partial x_l}, \qquad j_l \equiv g_{lk} j^k \quad (1.5)$$

in the action (1.1) and, hence, in the dynamic equation (1.3).

After such a change the system of partial differential equations (1.3) becomes to be equivalent to a system of ordinary differential equations, because all dynamic equations contain derivatives only in the direction of the vector $j^k$. The field of vector $j^k$ in the space-time determines a set of world lines $\mathcal{L}$ tangent to $j^k$. After dynamic disquantization (1.5) the dynamic equations contain only derivatives along $\mathcal{L}$. It means that giving initial values of dependent dynamic variables at some point of the world line $\mathcal{L}_1$, one can determine by means of dynamic equations the values of the dependent dynamic variables at all points of $\mathcal{L}_1$ independently of other world lines $\mathcal{L}$. It means that after dynamic disquantization the dynamic system $\mathcal{S}_{\mathrm{D}}$ turns into the dynamic system $\mathcal{S}_{\mathrm{Dqu}}$, describing the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{Dcl}}\right]$ of classical dynamic system $\mathcal{S}_{\mathrm{Dcl}}$.

Dynamic disquantization has the following properties:

1. Dynamic disquantization is determined completely by the dynamic system $\mathcal{S}_{\mathrm{D}}$.

2. Dynamic disquantization leads to the unique resulting dynamic system $\mathcal{S}_{\mathrm{Dcl}}$.

3. Dynamic disquantization is a relativistically covariant procedure.

It follows from the two last properties, that if the dynamic system $\mathcal{S}_{\mathrm{Dcl}}$ is nonrelativistic, the dynamic system $\mathcal{S}_{\mathrm{D}}$ cannot be relativistic.

In general, using the first equation (1.4), we can apply the procedure (1.5) directly to (1.3). But the obtained system of ordinary differential equations

$$i\hbar\left(\bar{\psi}\gamma_l\psi\right)\left(\bar{\psi}\gamma^k\psi\right)\gamma^l\partial_k\psi - m\left(\bar{\psi}\gamma_k\psi\right)\left(\bar{\psi}\gamma^k\psi\right)\psi = 0$$

is too complicated for investigation, because it is nonlinear. Description in terms of the wave function is effective only, if the dynamic equations are linear in terms of the wave function. In the case of nonlinear dynamic equations it is more effective to introduce dynamic variables, whose physical meaning is clear.

Investigation of the dynamic system $\mathcal{S}_{\mathrm{Dcl}}$ instead of $\mathcal{S}_{\mathrm{D}}$ is useful, because $\mathcal{S}_{\mathrm{Dcl}}$ has finite number of the freedom degrees. It may appear that some of degrees of freedom are described relativistically, but another ones are described nonrelativistically.

# 2 Transformation of variables

The state of dynamic system $\mathcal{S}_{\mathrm{D}}$ is described by eight real dependent variables (eight real components of four-component complex wave function $\psi$). Transforming the action (1.1), we use the mathematical technique [11, 12], where the wave function $\psi$ is considered to be a function of hypercomplex numbers $\gamma$ and coordinates $x$. In this case the dynamic quantities are obtained by means of a convolution of expressions $\psi^* O\psi$ with zero divisors. This technique allows one to work without fixing the $\gamma$-matrices representation.

Using designations

$$\gamma_5 = \gamma^{0123} \equiv \gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.1}$$

$$\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_3, \} = \{-i\gamma^2\gamma^3, -i\gamma^3\gamma^1, -i\gamma^1\gamma^2\} \tag{2.2}$$

we make the change of variables

$$\psi = Ae^{i\varphi+\frac{1}{2}\gamma_5\kappa} \exp\left(-\frac{i}{2}\gamma_5\boldsymbol{\sigma\eta}\right) \exp\left(\frac{i\pi}{2}\boldsymbol{\sigma}\mathbf{n}\right) \Pi \tag{2.3}$$

$$\bar{\psi} = \psi^*\gamma^0$$

$$\psi^* = A\Pi \exp\left(-\frac{i\pi}{2}\boldsymbol{\sigma}\mathbf{n}\right) \exp\left(-\frac{i}{2}\gamma_5\boldsymbol{\sigma\eta}\right) e^{-i\varphi-\frac{1}{2}\gamma_5\kappa} \tag{2.4}$$

where (*) means the Hermitian conjugation, and

$$\Pi = \frac{1}{4}(1+\gamma^0)(1+\mathbf{z}\boldsymbol{\sigma}), \qquad \mathbf{z} = \{z^\alpha\} = \text{const}, \qquad \alpha = 1,2,3; \qquad \mathbf{z}^2 = 1 \tag{2.5}$$

is a zero divisor. The quantities $A$, $\kappa$, $\varphi$, $\boldsymbol{\eta} = \{\eta^\alpha\}$, $\mathbf{n} = \{n^\alpha\}$, $\alpha = 1,2,3$, $\mathbf{n}^2 = 1$ are eight real parameters, determining the wave function $\psi$. These parameters may be considered as new dependent variables, describing the state of dynamic system $\mathcal{S}_{\mathrm{D}}$. The quantity $\varphi$ is a scalar, and $\kappa$ is a pseudoscalar. Six remaining variables $A$, $\boldsymbol{\eta} = \{\eta^\alpha\}$, $\mathbf{n} = \{n^\alpha\}$, $\alpha = 1,2,3$, $\mathbf{n}^2 = 1$ can be expressed through the flux 4-vector $j^l = \bar{\psi}\gamma^l\psi$ and the spin 4-pseudovector

$$S^l = i\bar{\psi}\gamma_5\gamma^l\psi, \qquad l = 0,1,2,3 \tag{2.6}$$

Because of two identities

$$S^l S_l \equiv -j^l j_l, \qquad j^l S_l \equiv 0. \tag{2.7}$$

there are only six independent components among eight components of quantities $j^l$, and $S^l$.

After transformation we obtain (see details in [9])

$$\mathcal{S}_{\mathrm{D}}: \qquad \mathcal{A}_{\mathrm{D}}[j,\varphi,\kappa,\boldsymbol{\xi}] = \int \mathcal{L} d^4x, \qquad \mathcal{L} = \mathcal{L}_{\mathrm{cl}} + \mathcal{L}_{\mathrm{q1}} + \mathcal{L}_{\mathrm{q2}} \tag{2.8}$$

$$\mathcal{L}_{\mathrm{cl}} = -m\rho - \hbar j^i\partial_i\varphi - \frac{\hbar j^l}{2(1+\boldsymbol{\xi}\mathbf{z})}\varepsilon_{\alpha\beta\gamma}\xi^\alpha\partial_l\xi^\beta z^\gamma, \qquad \rho \equiv \sqrt{j^l j_l} \tag{2.9}$$

$$\mathcal{L}_{\mathrm{q1}} = 2m\rho\sin^2(\frac{\kappa}{2}) - \frac{\hbar}{2}S^l\partial_l\kappa, \tag{2.10}$$

$$\mathcal{L}_{\mathrm{q2}}=\frac{\hbar(\rho+j_0)}{2}\varepsilon_{\alpha\beta\gamma}\partial^\alpha\frac{j^\beta}{(j^0+\rho)}\xi^\gamma-\frac{\hbar}{2(\rho+j_0)}\varepsilon_{\alpha\beta\gamma}\left(\partial^0 j^\beta\right)j^\alpha\xi^\gamma \tag{2.11}$$

Here and in what follows, a summation is produced over repeated Greek indices (1-3) and over repeated Latin indices (0-3). Lagrangian is a function of 4-vector $j^l$, scalar $\varphi$, pseudoscalar $\kappa$, and unit 3-pseudovector $\boldsymbol{\xi}$, which is connected with the spin 4-pseudovector $S^l$ by means of the relations

$$\xi^\alpha=\rho^{-1}\left(S^\alpha-\frac{j^\alpha S^0}{(j^0+\rho)}\right),\qquad \alpha=1,2,3;\qquad \rho\equiv\sqrt{j^l j_l} \tag{2.12}$$

$$S^0=\mathbf{j}\boldsymbol{\xi},\qquad S^\alpha=\rho\xi^\alpha+\frac{(\mathbf{j}\boldsymbol{\xi})j^\alpha}{\rho+j^0},\qquad \alpha=1,2,3 \tag{2.13}$$

The unit 3-pseudovector $\boldsymbol{\xi}=\{\xi_1,\xi_2,\xi_3\}$ is connected with the 3-vector $\mathbf{n}=\{n_1,n_2,n_3,\}$ in (2.3), (2.4) by means of the relation

$$\boldsymbol{\xi}=2\mathbf{n}(\mathbf{n}\mathbf{z})-\mathbf{z} \tag{2.14}$$

As one can see from (2.9) – (2.11), only two first terms in (2.9) and two term in (2.10) are written in the relativistically covariant form. They are invariants. Other terms are written in the non-covariant form. The non-covariant term in (2.9) contains three-dimensional Levi-Chivita pseudotensor $\varepsilon_{\alpha\beta\gamma}$. It can be considered as spatial components of the 4-dimensional Levi-Chivita pseudotensor $\varepsilon_{iklm}$ ($\varepsilon_{0123}=1$), convoluted with the constant timelike unit vector $f^l=\{1,0,0,0\}$. Then only spatial components of pseudotensor $\varepsilon_{iklm}f^m$ do not vanish

$$\varepsilon_{\alpha\beta\gamma}=-\varepsilon_{\alpha\beta\gamma m}f^m,\qquad \alpha,\beta,\gamma=1,2,3 \tag{2.15}$$

and one may substitute relation (2.15) in expression (2.9).

To write the expression (2.11) in the relativistically covariant form, we also introduce constant 4-vector $f^l$

$$f^i=\{1,0,0,0\} \tag{2.16}$$

and take into account that the three-dimensional Levi-Chivita pseudotensor $\varepsilon_{\alpha\beta\gamma}$ may be considered as the component $\varepsilon_{0\alpha\beta\gamma}$ of the 4-dimensional Levi-Chivita pseudotensor $\varepsilon_{iklm}$. It easy to verify that the expression (2.11) may be written in the form

$$\mathcal{L}_{\mathrm{q2}}=\frac{\hbar}{2}\varepsilon_{ikl\gamma}\left(j^i+f^i\rho\right)\partial^k\frac{j^l+f^l\rho}{(j^s f_s+\rho)}\xi^\gamma \tag{2.17}$$

where summation is produced over Latin indices $(0-3)$ and over Greek indices $(1-3)$.

Let us write relations (2.12), (2.13) in the covariant form. We shall consider 3-pseudovector $\boldsymbol{\xi}=\{\xi^1,\xi^2,\xi^3\}$ as spatial components of 4-pseudovector $\xi^k=\{\xi^0,\boldsymbol{\xi}\}=\{\xi^0,\xi^1,\xi^2,\xi^3\}$. Then relations (2.12), (2.13) take the form

$$\xi^k=\rho^{-1}\left(S^k-\frac{j^k S^l f_l}{(j^s f_s+\rho)}\right),\qquad k=0,1,2,3; \tag{2.18}$$

$$\xi^k \xi_k = \left( \frac{S^l f_l}{(j^s f_s + \rho)} \right)^2 - 1$$
$$S^k = \xi^k \rho + j^k (\xi^s f_s), \qquad k = 0,1,2,3 \tag{2.19}$$

We introduce the four-component quantity

$$\nu^k = \xi^k - f^k (\xi^s f_s), \qquad k = 0,1,2,3; \qquad \nu^l \nu_l = -1 \tag{2.20}$$

In the coordinate system, where the expression (2.17) is written and where the vector $f^k$ has the form (2.16), we obtain $\nu^0 = 0$, $\nu^\alpha = \xi^\alpha$, $\alpha = 1,2,3$. The relation (2.17) can be written in the covariant form

$$\mathcal{L}_{\mathrm{q}2} = \hbar \rho \varepsilon_{iklm} q^i \left( \partial^k q^l \right) \nu^m \tag{2.21}$$

where $q^k$ is the unit 4-vector

$$q^k = \frac{j^k + f^k \rho}{\sqrt{2\rho (j^s f_s + \rho)}}, \qquad q^k q_k = 1 \tag{2.22}$$

Analogously, one can write the expression (2.9) in the covariant form. Introducing the unit spacelike constant 4-pseudovector

$$z^k = \{0, \mathbf{z}\} = \left(0, z^1, z^2, z^3\right), \qquad z^k z_k = -1, \qquad \boldsymbol{\xi}\mathbf{z} = -\xi_l z^l \tag{2.23}$$

where the 3-pseudovector $\mathbf{z}$ is defined by the relation (2.5), we write the last term of the expression (2.9) in the form

$$-\frac{\hbar j^l}{2(1 + \boldsymbol{\xi}\mathbf{z})} \varepsilon_{\alpha\beta\gamma} \xi^\alpha \partial_l \xi^\beta z^\gamma = -\hbar j^s \varepsilon_{iklm} \frac{\xi^i}{\sqrt{2(1 - \xi_s z^s)}} \partial_s \frac{\xi^k}{\sqrt{2(1 - \xi_s z^s)}} z^l f^m \tag{2.24}$$

The factor $[2(1 - \xi^s z_s)]^{-1/2}$ is introduced under sign of derivative, because differentiation of it gives 0 in virtue of the vanishing factor $\varepsilon_{iklm} \xi^i \xi^k z^l f^m = 0$.

Finally, introducing 4-pseudovector

$$\mu^i \equiv \frac{\nu^i}{\sqrt{-(\nu^l + z^l)(\nu_l + z_l)}} = \frac{\nu^i}{\sqrt{2(1 - \nu^l z_l)}} = \frac{\nu^i}{\sqrt{2(1 + \boldsymbol{\xi}\mathbf{z})}}. \tag{2.25}$$

we can rewrite the last term of (2.9) in the form

$$-\frac{\hbar j^l}{2(1 + \boldsymbol{\xi}\mathbf{z})} \varepsilon_{\alpha\beta\gamma} \xi^\alpha \partial_l \xi^\beta z^\gamma = \hbar j^s \varepsilon_{iklm} \mu^i \partial_s \mu^k z^l f^m \tag{2.26}$$

Now we can write the action (2.8) in the covariant form

$$\mathcal{S}_{\mathrm{D}}: \qquad \mathcal{A}_{\mathrm{D}}[j, \varphi, \kappa, \boldsymbol{\xi}] = \int \mathcal{L} d^4 x, \qquad \mathcal{L} = \mathcal{L}_{\mathrm{cl}} + \mathcal{L}_{\mathrm{q}1} + \mathcal{L}_{\mathrm{q}2} \tag{2.27}$$

$$\mathcal{L}_{\mathrm{cl}} = -m\rho - \hbar j^i \partial_i \varphi + \hbar j^s \varepsilon_{iklm} \mu^i \partial_s \mu^k z^l f^m, \qquad \rho \equiv \sqrt{j^l j_l} \tag{2.28}$$

$$\mathcal{L}_{\mathrm{q1}} = 2m\rho \sin^2(\frac{\kappa}{2}) - \frac{\hbar}{2} S^l \partial_l \kappa, \tag{2.29}$$

$$\mathcal{L}_{\mathrm{q2}} = \hbar \rho \varepsilon_{iklm} q^i (\partial^k q^l) \nu^m \tag{2.30}$$

where $\varphi$ is a scalar, $\kappa$ is a pseudoscalar, the quantities $j^k$, $q^k$,$f^k$ are 4-vectors and the quantities $S^k$, $\mu^k$, $\nu^k$, $z^k$ are 4-pseudovectors. Vector $S^k$ is expressed via variables $j^k$ and $\xi^k$ by means of relations (2.19). 4-pseudovector $\xi^k$ contains only two independent components, because it satisfies the constraints

$$\xi^k \xi_k = (\xi^s f_s)^2 - 1, \qquad \left(j_k \xi^k\right) + (\xi^s f_s)\,\rho = 0 \tag{2.31}$$

which ensure fulfilment of relations (2.7).

# 3 Relativistic invariance

It is a common practice to think that if dynamic equations of a dynamic system can be written in the relativistically covariant form, such a possibility provides automatically relativistic character of considered dynamic system, described by these equations. In general, it is valid only in the case, when dynamic equations do not contain absolute objects, or these absolute objects has the Lorentz group as a group of their symmetry [10]. The absolute object is one or several quantities, which are the same for all states of the dynamic system [10]. A given external field, or metric tensor (when it is given, but not determined from the gravitational equations) are examples of absolute objects. In the case of dynamic system $\mathcal{S}_{\mathrm{D}}$, described by the action (1.1) the Dirac $\gamma$-matrices are absolute objects.

Anderson [10] investigated in details the role of absolute objects for symmetry of dynamic systems. His conclusion is as follows. If a dynamic system is described by dynamic equations, written in the covariant form, the symmetry group of the dynamic system is determined by the symmetry group of these absolute objects. Here we confirm this result in a simple example, when the dynamic equations of the certainly nonrelativistic dynamic system are written in a relativistically covariant form.

Let us consider a system of differential equations, consisting of the Maxwell equations for the electromagnetic tensor $F^{ik}$ in some inertial coordinate system

$$\partial_k F^{ik}(x) = 4\pi J^i, \qquad \varepsilon_{iklm} g^{jm} \partial_j F^{kl}(x) = 0, \qquad \partial_k \equiv \frac{\partial}{\partial x^k} \tag{3.1}$$

and equations

$$m\frac{d}{d\tau}\left[\left(l_k \dot{q}^k\right)^{-1} \dot{q}^i - \frac{1}{2} g^{ik} l_k \left(l_j \dot{q}^j\right)^{-2} \dot{q}^s g_{sl} \dot{q}^l\right] = eF^{il}(q)\, g_{lk} \dot{q}^k; \qquad i = 0,1,2,3 \tag{3.2}$$

$$\dot{q}^k \equiv \frac{dq^k}{d\tau}$$

where $q^i = q^i(\tau)$, $i = 0, 1, 2, 3$ describe coordinates of a pointlike charged particle as functions of a parameter $\tau$, the quantity $l_k$, $k = 0, 1, 2, 3$ is a constant timelike unit vector,

$$g^{ik} l_i l_k = 1; \tag{3.3}$$

and the speed of the light $c = 1$.

This system of equations is relativistically covariant with respect to quantities $q^i$, $F^{ik}$, $J^i$, $l_i$, $g_{ik}$, i.e. it does not change its form at any Lorentz transformation, which is accompanied by corresponding transformation of quantities $q^i$, $F^{ik}$, $J^i$, $l_i$, $g_{ik}$, where the quantities $q^i$, $J^i$, $l_i$ are transformed as 4-vectors and the quantities $F^{ik}$, $g_{ik}$ are transformed as 4-tensors

The reference to the quantities $q^i$, $F^{ik}$, $J^i$, $l_i$, $g_{ik}$ means that all these quantities are considered as formal dependent variables, when one compares the form of dynamic equations written in two different coordinate systems. For instance, if the reference to $J^i$ is omitted in the formulation of the relativistic covariance, it means that components of $J^i$ are considered as some functions of the coordinates $x$. Let $J^i \neq 0$, and $\tilde{J}^i$ be the quantity $J^i$, written in other coordinate system. Then, in general, $J^i$ and $\tilde{J}^i$ are different functions of the arguments $x$ and $\tilde{x}$ respectively, and the first equation (3.1) has different form in different coordinate systems. In other words, the dynamic equations (3.1)–(3.2) are not relativistically covariant, in general, with respect to quantities $q^i$, $F^{ik}$, $l_i$, $g_{ik}$, if $J^i \neq 0$. Thus, for the relativistic covariance it is important both the laws of transformation and how each of quantities is considered as a formal variable, or as some function of coordinates.

Following Anderson [10], we divide the quantities $q^i$, $F^{ik}$, $J^i$, $l_i$, $g_{ik}$ into two parts: dynamic objects (variables) $q^i$, $F^{ik}$ and absolute objects $J^i$, $l_i$, $g_{ik}$. According to definition of absolute objects they have the same value for all solutions of the dynamic equations, whereas dynamic variables are different, in general, for different solutions. If the dynamic equations are written in the relativistically covariant form, their symmetry group is determined by the symmetry group of the absolute objects $J^i$, $l_i$, $g_{ik}$. Dynamic equations are compatible with the principles of relativity if the Lorentz group is the symmetry group of the absolute objects.

Let for simplicity $J^i \equiv 0$. The symmetry group of the constant timelike vector $l_i$ is a group of rotations in the 3-plane orthogonal to the vector $l_i$. The Lorentz group is a symmetry group of the metric tensor $g_{ik} =$diag $\{1, -1, -1, -1\}$. Thus, the symmetry group of all absolute objects $l_i$, $g_{ik}$, $J^i \equiv 0$ is a subgroup of the Lorentz group (rotations in the 3-plane orthogonal to $l_i$). As far as the symmetry group is a subgroup of the Lorentz group and does not coincide with it, the system of equations (3.1)–(3.2) is nonrelativistic (incompatible with the relativity principles).

Of course, the compatibility with the relativity principles does not depend on the fact with respect to which quantities the relativistic covariance is considered. For instance, let us consider a covariance of equations (3.1), (3.2) with respect to the quantities $q^i$, $F^{ik}$, $J^i \equiv 0$. It means that now the quantities $l_i$ are to be considered to be functions of $x$ (in the given case these functions are constants), because a reference to $l_i$ as formal variables is absent. After the transformation to another coordinate system the equation (3.2) takes the form

$$m\frac{d}{d\tau}\left[\left(\tilde{l}_k\frac{d\tilde{q}^k}{d\tau}\right)^{-1}\frac{d\tilde{q}^i}{d\tau}-\frac{1}{2}g^{ik}\tilde{l}_k\left(\tilde{l}_j\frac{d\tilde{q}^j}{d\tau}\right)^{-2}\frac{d\tilde{q}^s}{d\tau}\frac{d\tilde{q}_s}{d\tau}\right]=e\tilde{F}^{il}\left(\tilde{q}\right)g_{lk}\frac{d\tilde{q}^k}{d\tau} \tag{3.4}$$

Here $\tilde{l}_i$ are considered as functions of $\tilde{x}$. But $\tilde{l}_i$ are other functions of $\tilde{x}$, than $l_i$ of $x$ (other numerical constants $\tilde{l}_k = l_j\partial x^j/\partial\tilde{x}^k$ instead of $l_k$), and equations (3.2) and (3.4) have different forms with respect to quantities $q^i$, $F^{ik}$, $J^i \equiv 0$. It means that (3.2) is not relativistically covariant with respect to $q^i$, $F^{ik}$, $J^i \equiv 0$, although it is relativistically covariant with respect to $q^i$, $F^{ik}$, $l_i$, $J^i \equiv 0$.

Setting $l_i = \{1, 0, 0, 0\}$, $t = q^0(\tau)$ in (3.2), we obtain

$$m\frac{d^2q^\alpha}{dt^2} = eF^\alpha_{.0} + eF^\alpha_{.\beta}\frac{dq^\beta}{dt}, \qquad i = \alpha = 1, 2, 3; \tag{3.5}$$

$$\frac{m}{2}\frac{d}{dt}\left(\frac{dq^\alpha}{dt}\frac{dq^\alpha}{dt}\right) = eF^\alpha_{.0}\frac{dq^\alpha}{dt}, \qquad i = 0. \tag{3.6}$$

Now the equations (3.5), (3.6) do not contain the absolute object $l_k$. It is easy to see that these equations describe a nonrelativistic motion of a charged particle in a given electromagnetic field $F^{ik}$. The fact that the equations (3.2) or (3.5) are nonrelativistic is connected with the space-time splitting into space and time that is characteristic for Newtonian mechanics. Any 3-plane, orthogonal to the vector $l^k$, consists of simultaneous events. This space-time splitting is described in different ways in equations (3.2) and (3.5). It is described by the constant timelike vector $l_k$ in (3.2). In the equation (3.5) the space-time splitting is described by a special choice of the coordinate system, whose time axis is directed along the vector $l^k$, and all events having the same coordinate $t$ are simultaneous absolutely.

From physical viewpoint the Newtonian conception of space-time and that of the special relativity distinguish in the number of invariants, describing the space-time. In the Newtonian space-time conception there are two independent invariant quantities: time $t$ and distance $r = |\mathbf{x}|$. This conception is associated with the description in terms of the time $t$ and spatial coordinates $\mathbf{x}$. This description is known as noncovariant description, and (3.5), (3.6) is an example of such a description. In the relativistic conception of the space-time there is only one invariant quantity: space-time interval $s = \sqrt{c^2t^2 - r^2}$. The relativistic conception of the space-time associated with the description in terms of the 4-vector $x^k = \{t, \mathbf{x}\}$. This description is known as (relativistically) covariant description. Can we describe a relativistic phenomenon, using noncovariant description, associated with the Newtonian conception? Yes, it is possible, and noncovariant description of relativistic phenomena is used frequently. Vice versa, can we describe a nonrelativistic phenomenon, using relativistically covariant description, associated with the relativistic conception of the space-time? Yes, it is possible. Relativistically covariant description of the nonrelativistic phenomenon is possible, provided this description contains two independent space-time invariants: the time $t$ and the distance $r$. To obtain two invariants: $t$ and $r$ from the space-time interval $s$, one needs to introduce an absolute constant timelike 4-vector $l^k$ ( $l^k l_k = 1$). This vector is associated with the space-time and admits one to construct two space-time invariants $t$ and $r$ from the space-time interval $s$

$$t = x^k l_k, \qquad r = \sqrt{(x^k l_k)^2 - x^k x_k}, \qquad c = 1$$

Thus, any appearance of the absolute constant timelike 4-vector $l^k$ in the relativistically covariant description of a physical phenomenon is open to suspicion that the phenomenon is nonrelativistic (incompatible with the relativity theory). The physical phenomenon is relativistic, if the vector $l^k$ is fictitious in the given description. On the contrary, the physical phenomenon is nonrelativistic, because its description contains two space-time invariants $t$ and $r$. The relativistically covariant description is not used practically for description of nonrelativistic phenomena, and most researchers are convinced, that any relativistically covariant description is a description of a relativistic phenomenon.

The considered example shows that nonrelativistic equation (3.5) can be written in a relativistically covariant form (3.2), provided one introduces an absolute object $l_i$, describing space-time splitting.

Dirac matrices $\gamma^k$ are absolute objects, as well as the metric tensor $g^{kl}$, which may be considered as a derivative absolute object determined by the relation (1.2). It follows from the relation (1.2) that the $\gamma$-matrices are to be transformed as 4-vectors at the linear transformations of the space-time coordinates $x$, because the rhs of (1.2) is a tensor at the linear transformations of coordinates $x$. Although at the Lorentz transformation rhs of (1.2) is an invariant, we cannot suppose that $\gamma^i$ may be invariants, because $\gamma^i$ cannot be invariants under linear transformations of coordinates.

There are two approaches to the Dirac equation. In the first approach [11, 12] the wave function $\psi$ is considered to be a scalar function defined on the field of Clifford numbers $\gamma^l$,

$$\psi = \psi(x,\gamma)\Gamma, \qquad \bar{\psi} = \Gamma\bar{\psi}(x,\gamma), \tag{3.7}$$

where $\Gamma$ is a constant nilpotent factor which has the property $\Gamma F(\gamma)\Gamma = a\Gamma$. Here $F(\gamma)$ is arbitrary function of $\gamma^l$ and $a$ is a complex number, depending on the form of the function $F$. Within such an approach $\psi$, $\bar{\psi}$ are transformed as scalars under the Lorentz transformations, whereas $\gamma^l$ are transformed as components of a 4-vector. In this case the symmetry group of $\gamma^l$ is a subgroup of the Lorentz group, and $\mathcal{S}_{\rm D}$ is a nonrelativistic dynamic system. Then the matrix vector $\gamma^l$ describes some preferred direction in the space-time.

In the second (conventional) approach $\psi$ is considered to be a spinor, and $\gamma^l$, $l = 0, 1, 2, 3$ are scalars with respect to the transformations of the Lorentz group. In this case the symmetry group of the absolute objects $\gamma^l$ is the Lorentz group, and dynamic system $\mathcal{S}_{\rm D}$ is considered to be a relativistic dynamic system.

Of course, the approaches leading to incompatible conclusions cannot be both valid. At least, one of them is wrong. Analyzing the two approaches, Sommerfeld

[12] considered the first approach to be more reasonable. In the second case the analysis is rather difficult due to non-standard transformations of $\gamma^l$ and $\psi$ under linear coordinate transformations $T$. Indeed, the transformation $T$ for the vector $j^l = \bar{\psi}\gamma^l\psi$ has the form

$$\tilde{\bar{\psi}}\tilde{\gamma}^l\tilde{\psi} = \frac{\partial \tilde{x}^l}{\partial x^s}\bar{\psi}\gamma^s\psi, \tag{3.8}$$

where the quantities marked by tilde mean the quantities in the transformed coordinate system. This transformation can be carried out by two different ways

$$1: \quad \tilde{\psi} = \psi, \qquad \tilde{\overline{\psi}} = \overline{\psi}, \qquad \tilde{\gamma}^l = \frac{\partial \tilde{x}^l}{\partial x^s}\gamma^s, \qquad l = 0,1,2,3 \tag{3.9}$$

$$2: \quad \tilde{\gamma}^l = \gamma^l, \qquad l = 0,1,2,3, \qquad \tilde{\psi} = S(\gamma,T)\psi, \qquad \tilde{\overline{\psi}} = \overline{\psi}S^{-1}(\gamma,T), \tag{3.10}$$

$$S^*(\gamma,T)\gamma^0 = \gamma^0 S^{-1}(\gamma,T) \tag{3.11}$$

The relations (3.9) correspond to the first approach and the relations (3.10) correspond to the second one. Both ways (3.9) and (3.10) lead to the same result, provided

$$S^{-1}(\gamma,T)\gamma^l S(\gamma,T) = \frac{\partial \tilde{x}^l}{\partial x^s}\gamma^s \tag{3.12}$$

In particular, for the infinitesimal Lorentz transformation $x^i \to x^i + \delta\omega^i_{.k}x^k$ the matrix $S(\gamma,T)$ has the form [13]

$$S(\gamma,T) = \exp\left(\frac{\delta\omega_{ik}}{8}\left(\gamma^i\gamma^k - \gamma^k\gamma^i\right)\right) \tag{3.13}$$

The second way (3.10) has two defects. First, the transformation law of $\psi$ depends on $\gamma$, i.e. under linear transformation $T$ of coordinates the components of $\psi$ are transformed through $\psi$ and $\gamma^l$, but not only through $\psi$. Note that the tensor components in a coordinate system are transformed only through the tensor components in other coordinate system, and this transformation does not contain any absolute objects. (for instance, the relation (3.8)). Second, the relation (3.12) is compatible with (1.2) only under transformations $T$ between orthogonal coordinate systems, when components $g^{lk} = \{1,-1,-1,-1\}$ of the metric tensor are invariant. Indeed, lhs of the relation (1.2) is a scalar under any linear coordinate transformations, whereas rhs of (1.2) is invariant only under orthogonal (Lorentz) transformations. Rhs of the relation (1.2) is transformed as a tensor under linear coordinate transformations. In other words, at the second approach the relation (1.2) is not covariant, in general, with respect to arbitrary linear transformations of coordinates. In this case one cannot be sure that the symmetry group of the dynamic system coincides with the symmetry group of absolute objects.

The fact that the symmetry group of the dynamic system coincides with the symmetry group of absolute objects was derived with the supposition, that under the coordinate transformation any object is transformed only via its components.

This condition is violated in the second case, and one cannot be sure that the symmetry group of dynamic system coincides with that of absolute objects.

After change of variables the action (1.1) is transformed to the form (2.8) – (2.11), the $\gamma$-matrices being eliminated. But after reduction of the action to the relativistically covariant form two new absolute objects appear: constant 4-vector $f^k$ and constant 4-pseudovector $z^i$. The 4-pseudovector $z^i$ appears to be fictitious. But the action depends really on 4-vector $f^k$, which resembles the vector $l_k$ in the considered example (3.2). It means that the dynamic system $\mathcal{S}_\mathrm{D}$ is nonrelativistic, because it supposes an absolute separation of the space-time into the space and the time.

# 4 Modification of the classical Dirac particle

Not all terms in the action (2.27) – (2.30) contain absolute objects $f^k$ and $z^k$. Two first terms of (2.28) and the term (2.29) do not contain absolute objects. It means that the Dirac particle $\mathcal{S}_\mathrm{D}$ is described partly relativistically and partly nonrelativistically. To determine what is described relativistically, it is useful to investigate the classical Dirac particle $\mathcal{S}_\mathrm{Dcl}$, which is a discrete dynamic system, i.e. the dynamic system having a finite number of the freedom degrees. The classical Dirac particle $\mathcal{S}_\mathrm{Dcl}$ is determined uniquely by the Dirac particle $\mathcal{S}_\mathrm{D}$ by means of the relativistic procedure (1.5). Investigating the classical Dirac particle $\mathcal{S}_\mathrm{Dcl}$, we can determine which degrees of freedom are described nonrelativistically and correct the nonrelativistic description.

After dynamic disquantization (1.5) of the action (2.8) we obtain the action for the classical Dirac particle $\mathcal{S}_\mathrm{Dcl}$. According to calculations of [9] we obtain

$$\mathcal{S}_\mathrm{Dcl}: \qquad \mathcal{A}_\mathrm{Dcl}[x,\boldsymbol{\xi}]=\int\left\{-\kappa_0 m\sqrt{\dot{x}^i\dot{x}_i}+\hbar\frac{(\dot{\boldsymbol{\xi}}\times\boldsymbol{\xi})\mathbf{z}}{2(1+\boldsymbol{\xi}\mathbf{z})}+\hbar\frac{(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\boldsymbol{\xi}}{2\sqrt{\dot{x}^s\dot{x}_s}(\sqrt{\dot{x}^s\dot{x}_s}+\dot{x}^0)}\right\}d\tau_0 \tag{4.1}$$

where coordinates $x=x(\tau_0)=\{x^0,\mathbf{x}\}=\{x^0,x^2,x^2,x^3\}$ of the classical Dirac particle $\mathcal{S}_\mathrm{Dcl}$ and its internal variables $\boldsymbol{\xi}=\boldsymbol{\xi}(\tau_0)=\{\xi_1,\xi_2,\xi_3\}$, $\boldsymbol{\xi}^2=1$ are considered to be functions of $\tau_0$. The quantity $\kappa_0=\pm 1$ is obtained as a result of solution [9] of dynamic equation for the dynamic variable $\kappa$.

The action (4.1) can be written in the relativistically covariant form

$$\mathcal{S}_\mathrm{Dcl}: \qquad \mathcal{A}_\mathrm{Dcl}[x,\xi]=\int\{-\kappa_0 m\sqrt{\dot{x}^i\dot{x}_i}-\hbar\frac{\varepsilon_{iklm}\xi^i\dot{\xi}^k f^l z^m}{2(1-\xi^s z_s)}+\frac{\hbar}{2}Q\varepsilon_{iklm}\dot{x}^i\ddot{x}^k f^l\xi^m\}d\tau_0 \tag{4.2}$$

$$Q=Q(\dot{x},f)=\frac{1}{\sqrt{\dot{x}^s\dot{x}_s}(\dot{x}^l f_l+\sqrt{\dot{x}^l\dot{x}_l})},\qquad \dot{x}^l\equiv\frac{dx^l}{d\tau_0} \tag{4.3}$$

where 4-vectors $f^k$, $z^k$ are defined respectively by relations (2.16), (2.23) and

$$\xi^k=\{\xi_0,\boldsymbol{\xi}\},\qquad \xi^l f_l=0 \tag{4.4}$$

The 4-vector $z^k$ appears to be fictitious (see Appendix B of this paper or of the paper [9]). But the 4-vector $f^k$ is not fictitious.

The action (4.2) is invariant with respect to a change of independent variable $\tau_0$

$$\tau_0 \to \tilde{\tau}_0 = F(\tau_0) \tag{4.5}$$

where $F$ is an arbitrary monotone function. The variable $\tau_0$ may be chosen in such a way that

$$\sqrt{\dot{x}^s \dot{x}_s} = 1 \tag{4.6}$$

for all values of the independent variable $\tau_0$. Formally the expression (4.6) appears to be an integral of dynamic equations generated by the action (4.2).

The first term in (4.2) is described relativistically, because it does not contain the absolute object $f^k$. This term describes the translation degrees of freedom. Two last terms contain the absolute object $f^k$. They describe internal degrees of freedom. Description of internal degrees of freedom appears to be nonrelativistic.

If we set $\hbar = 0$ in the action (4.2), we suppress internal degrees of freedom of the classical Dirac particle, and description of $\mathcal{S}_{\mathrm{Dcl}}$ becomes relativistical. The classical Dirac particle becomes relativistic, provided the absolute object $f^k$ is changed by the dynamical variables. For instance, we may identify the unit timelike 4-vector $f^k$ with the constant energy-momentum vector $p_k$, normalized in a proper way. At such a change the classical Dirac particle $\mathcal{S}_{\mathrm{Dcl}}$ turns into the modified classical Dirac particle $\mathcal{S}_{\mathrm{mDcl}}$.

To make this change we introduce designations

$$y^l = \dot{x}^l, \qquad l = 0, 1, 2, 3 \tag{4.7}$$

and rewrite the action (4.2) in the form

$$\mathcal{S}_{\mathrm{Dcl}}: \quad \mathcal{A}_{\mathrm{Dcl}}[x, y, \xi, p] = \int \left\{ L\left(y, \dot{y}, \xi, \dot{\xi}, f\right) - p_l\left(y^l - \dot{x}^l\right) \right\} d\tau_0 \tag{4.8}$$

where

$$L\left(y, \dot{y}, \xi, \dot{\xi}, f\right) = -\kappa_0 \sqrt{y_l y^l} - \hbar \frac{\varepsilon_{iklm} \xi^i \dot{\xi}^k f^l z^m}{2(1 - \xi^s z_s)} + \frac{\hbar}{2} Q \varepsilon_{iklm} y^i \dot{y}^k f^l \xi^m \tag{4.9}$$

$$Q = Q(y, f) = \frac{1}{\sqrt{y^s y_s}(y^l f_l + \sqrt{y^l y_l})}, \tag{4.10}$$

Variables $p_l = p_l(\tau_0)$, $l = 0, 1, 2, 3$ are the Lagrange multipliers, introducing designations (4.7). The quantities $f^k$ and $z^k$ are not dynamical variables and they are not to be varied.

Dynamic equations generated by the action (4.8) have the form

$$\frac{\delta \mathcal{A}_{\mathrm{Dcl}}}{\delta x^l} = -\dot{p}_l = 0, \qquad p_l = \mathrm{const} \tag{4.11}$$

$$\frac{\delta \mathcal{A}_{\mathrm{Dcl}}}{\delta y^l} = \frac{\partial L}{\partial y^l} - \frac{d}{d\tau_0} \frac{\partial L}{\partial \dot{y}^l} - p_l = 0, \tag{4.12}$$

$$\begin{aligned} \frac{\delta \mathcal{A}_{\mathrm{Dcl}}}{\delta \xi^l} &= \left(\frac{\partial L}{\partial \xi^s}-\frac{d}{d\tau_0}\frac{\partial L}{\partial \dot{\xi}^s}\right)\left(\delta_l^s - f_l f^s + \xi_l \xi^s\right) = 0 &(4.13) \\ \frac{\delta \mathcal{A}_{\mathrm{Dcl}}}{\delta p_l} &= \dot{x}^l - y^l = 0 &(4.14)\end{aligned}$$

The second multiplier in (4.13) takes into account that the pseudovector $\xi^k$ restricted by the constraints $\xi_k \xi^k = -1$, $\xi_k f^k = 0$.

As far as $p_l$ =const, $l = 0,1,2,3$ in force of dynamic equations, we can express the constant unit 4-vector $f_l$ in the form

$$f_l = \frac{\varepsilon p_l}{\sqrt{p_s p^s}} \qquad l = 0,1,2,3, \qquad \varepsilon = \mathrm{sgn} p_0 = \pm 1 \tag{4.15}$$

and substitute $f_l$ from (4.15) in the action (4.8). We obtain

$$\mathcal{S}_{\mathrm{mDcl}}: \qquad \mathcal{A}_{\mathrm{mDcl}}[x,y,\xi,p] = \int \left\{ L\left(y,\dot{y},\xi,\dot{\xi},\frac{\varepsilon p_l}{\sqrt{p_s p^s}}\right) + p_l\left(\dot{x}^l - y^l\right)\right\} d\tau_0 \tag{4.16}$$

The action (4.16) describes dynamic system $\mathcal{S}_{\mathrm{mDcl}}$, which distinguishes, in general, from the dynamic system $\mathcal{S}_{\mathrm{Dcl}}$. The dynamic system $\mathcal{S}_{\mathrm{mDcl}}$ is compatible with the relativity principles, whereas the dynamic system $\mathcal{S}_{\mathrm{Dcl}}$ is not. In particular, it means as follows. Let $\{x,y,\xi,p\}$ be a solution of dynamic equations, generated by the action (4.16) and $\left\{\tilde{x},\tilde{y},\tilde{\xi},\tilde{p}\right\}$ have been obtained from $\{x,y,\xi,p\}$ by means of some transformation of the Lorentz group. Then $\left\{\tilde{x},\tilde{y},\tilde{\xi},\tilde{p}\right\}$ is a solution of the same dynamic equations. Solutions of dynamic equations (4.11) – (4.14) have not these property, in general.

All dynamic equations, generated by the action (4.16) coincide with the dynamic equations (4.11) – (4.14) except for the dynamic equation (4.14), which has now the form

$$\frac{\delta \mathcal{A}_{\mathrm{mDcl}}}{\delta p_l} = -y^l + \dot{x}^l + \frac{\varepsilon}{\sqrt{p_s p^s}}\left[\frac{\partial L\left(y,\dot{y},\xi,\dot{\xi},f\right)}{\partial f_k}\right]_{f_i = \frac{\varepsilon p_i}{\sqrt{p_s p^s}}}\left(\delta_k^l - \frac{p_k p^l}{p_s p^s}\right) = 0 \tag{4.17}$$

We write dynamic equations for the action (4.16) in the coordinate system, where the 4-vector $f_l$ and the energy-momentum 4-vector $p_l$ have the form

$$p_l = \{p_0,0,0,0\}, \qquad f_l = \{1,0,0,0\} \tag{4.18}$$

Solutions for other values of the canonical momentum $p_l = \tilde{p}_l$ may be obtained from solutions for $p_l = \{p_0,0,0,0\}$ by means of the coordinate transformation of the Lorentz group, which transforms $p_l = \{p_0,0,0,0\}$ into $p_l = \tilde{p}_l$, $l = 0,1,2,3$.

Dynamic equations (4.11) – (4.13) generated by the action (4.8) as well as by the action (4.16) do not depend on the variables $x^l$. They can be solved independently of the solution of the equation (4.14), or (4.17). In the paper [9] these equations have been solved in the coordinate system, where the Dirac particle is at rest and the energy-momentum vector $p_l$ has the form (4.18). For the sake of convenience this solution is presented in the Appendix A.

The solution (A.33) has the form

$$\begin{aligned} p_k &= \left\{-\frac{\kappa_0 m}{\gamma}, 0, 0, 0\right\} \qquad (4.19) \\ \zeta^l &= \left\{0, z^1, z^2, z^3\right\}, \qquad \xi^l = \{0, 0, 0, \varepsilon_0\}, \qquad \varepsilon_0 = \pm 1 \qquad (4.20) \end{aligned}$$

$$\begin{aligned} y^k &= y^k(\tau_0) = \left\{\gamma, \sqrt{\gamma^2 - 1}\cos\Phi, \sqrt{\gamma^2 - 1}\sin\Phi, 0\right\} \qquad (4.21) \\ \dot{y}^k &= \left\{0, -\omega\sqrt{\gamma^2 - 1}\sin\Phi, \omega\sqrt{\gamma^2 - 1}\cos\Phi, 0\right\} \qquad (4.22) \end{aligned}$$

$$\Phi = -\frac{2\varepsilon_0\kappa_0 m}{\hbar\gamma}\tau_0 + \phi, \qquad \omega = \frac{d\Phi}{d\tau_0} = -\frac{2\varepsilon_0\kappa_0 m}{\hbar\gamma} \qquad (4.23)$$

where $\phi$ and $\gamma$ ($\gamma^2 \geq 1$) are arbitrary constants.

Substituting relations (4.19) - (4.22) in the equation (4.17), we can express the variables $\dot{x}^k$ as functions of $\tau_0$. These equations can be integrated easily.

From the relations (4.9), (4.10) we obtain the following expression

$$\frac{\partial L}{\partial f_l} = -\hbar g^{lp}\frac{\varepsilon_{pikm}\xi^i\dot{\xi}^k z^m}{2(1-\xi^s z_s)} + \frac{\hbar}{2}g^{lp}\frac{\varepsilon_{pikm}y^i\dot{y}^k\xi^m}{\sqrt{y^s y_s}(y^s f_s + \sqrt{y^l y_l})} - \frac{\hbar}{2}\frac{y^l\varepsilon_{iksm}y^i\dot{y}^k f^s\xi^m}{\sqrt{y^s y_s}(y^s f_s + \sqrt{y^l y_l})^2} \qquad (4.24)$$

The action (4.16) as well as the original action (4.8) are invariant with respect to transformation of the independent variable $\tau_0$

$$\tau_0 \to \tilde{\tau}_0 = F(\tau_0) \qquad y^l \to \tilde{y}^l = y^l\left(\frac{dF(\tau_0)}{d\tau_0}\right)^{-1}, \qquad \xi^l \to \xi^l, \qquad p_l \to p_l, \qquad x^l \to x^l$$

and we choose the independent variable $\tau_0$ in such a way, that

$$y_s y^s = 1 \qquad (4.25)$$

For the dynamic system $\mathcal{S}_{\mathrm{Dcl}}$ the condition (4.25) is equivalent to the condition

$$\dot{x}_s\dot{x}^s = 1, \qquad (4.26)$$

because in this case $\dot{x}^l = y^l$. But for the dynamic system $\mathcal{S}_{\mathrm{mDcl}}$ the conditions (4.25) and (4.26) are not equivalent, because in this case $\dot{x}^l \neq y^l$, in general, as it follows from (4.17).

Under constraints (4.19), (4.25) the additional term of the modified equation (4.17) has the form

$$\left[\frac{\partial L\left(y, \dot{y}, \xi, \dot{\xi}, f\right)}{\partial f_k}\right]_{f_l = \frac{\varepsilon p_l}{\sqrt{p_i p^i}}}\left(\delta_k^l - \frac{p_k p^l}{p_s p^s}\right) = \begin{cases} 0, & \text{if} \quad l = 0 \\ \frac{\varepsilon}{|p_0|}\frac{\partial L}{\partial f_\mu}, & \text{if} \quad l = \mu = 1, 2, 3 \end{cases} \qquad (4.27)$$

$$\begin{aligned}
\frac{\partial L}{\partial f_\mu} &= -\frac{\hbar}{2}\frac{\varepsilon_{\mu 0\alpha\beta}y^0\dot{y}^\alpha\xi^\beta}{(y^0+1)}+\frac{\hbar}{2}\frac{\varepsilon_{\mu 0\alpha\beta}\dot{y}^0y^\alpha\xi^\beta}{(y^0+1)}-\frac{\hbar}{2}\frac{y^\mu\varepsilon_{\alpha\beta\gamma}y^\alpha\dot{y}^\beta\xi^\gamma}{(y^0+1))^2}\\
&= \frac{\hbar}{2(y^0+1)}\left(-\varepsilon_{\mu\alpha\beta}y^0\dot{y}^\alpha\xi^\beta+\varepsilon_{\mu\alpha\beta}\dot{y}^0y^\alpha\xi^\beta+\frac{y^\mu\varepsilon_{\alpha\beta\gamma}y^\alpha\dot{y}^\beta\xi^\gamma}{(y^0+1)}\right), \qquad (4.28)\\
\mu &= 1,2,3
\end{aligned}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Chivita pseudotensor in the three-dimensional space. Let us substitute (4.27), (4.28) in the dynamic equation (4.17). We obtain after calculation

$$\dot{x}^0=\gamma,\qquad \dot{x}^1=\gamma\sqrt{\frac{\gamma-1}{\gamma+1}}\cos\Phi,\qquad \dot{x}^2=\gamma\sqrt{\frac{\gamma-1}{\gamma+1}}\sin\Phi,\qquad \dot{x}^3=0 \tag{4.29}$$

where $\Phi$ is determined by the relation (4.23).

Instead of (4.26) we obtain

$$\dot{x}^l\dot{x}_l=\gamma^2-\left(\gamma^2-1\right)\left(\frac{\gamma}{\gamma+1}\right)^2=2\frac{\gamma^2}{\gamma+1} \tag{4.30}$$

Let now $t=x^0$ be the independent variable instead of $\tau_0$. Then equations (4.29) are transformed into

$$\frac{dx^k}{dt}=\left\{1,\sqrt{\frac{\gamma-1}{\gamma+1}}\cos\Phi,-\varepsilon_0\kappa_0\sqrt{\frac{\gamma-1}{\gamma+1}}\sin\Phi,0\right\} \tag{4.31}$$

where

$$\Phi=\Omega t-\varepsilon_0\kappa_0\phi,\qquad \Omega=\frac{2m}{\hbar\gamma^2} \tag{4.32}$$

Integration of (4.31) leads to the relations

$$x^k=\left\{t,\frac{\hbar\gamma^2}{2m}\sqrt{\frac{\gamma-1}{\gamma+1}}\sin\Phi,\frac{\hbar\varepsilon_0\kappa_0\gamma^2}{2m}\sqrt{\frac{\gamma-1}{\gamma+1}}\cos\Phi,0\right\} \tag{4.33}$$

which describe the world line of the modified classical Dirac particle $\mathcal{S}_{\mathrm{mDcl}}$. The total mass $M_{\mathrm{mDcl}}$ of $\mathcal{S}_{\mathrm{mDcl}}$ is described by the relation

$$M_{\mathrm{mDcl}}=\sqrt{p_0^2-\mathbf{p}^2}=|p_0|=\frac{m}{\gamma},\qquad \gamma\geq 1 \tag{4.34}$$

World line of $\mathcal{S}_{\mathrm{mDcl}}$ is a helix with the radius

$$a_{\mathrm{mDcl}}=\frac{\hbar\gamma^2}{2m}\sqrt{\frac{\gamma-1}{\gamma+1}} \tag{4.35}$$

The constant $\gamma\geq 1$ describes the intensity of excitation of the internal degrees of freedom.

For the classical Dirac particle $\mathcal{S}_{\mathrm{Dcl}}$ with nonrelativistic description of the internal degrees of freedom the relations (4.33) - (4.35) have another form [9]

$$x_{\mathrm{Dcl}}^k=\left\{t,\frac{\hbar\gamma}{2m}\sqrt{\gamma^2-1}\sin\left(\Omega_{\mathrm{Dcl}}t\right),\frac{\hbar\gamma}{2m}\sqrt{\gamma^2-1}\cos\left(\Omega_{\mathrm{Dcl}}t\right),0\right\} \tag{4.36}$$

$$M=M_{\rm Dcl}=M_{\rm mDcl}=\frac{m}{\gamma},\qquad \Omega=\Omega_{\rm Dcl}=\Omega_{\rm mDcl}=\frac{2m}{\hbar\gamma^2},\qquad a_{\rm Dcl}=\frac{\hbar\gamma}{2m}\sqrt{\gamma^2-1} \tag{4.37}$$

The dynamical systems $\mathcal{S}_{\rm mDcl}$ and $\mathcal{S}_{\rm Dcl}$ distinguish only in the radius $a$ of the helix

$$a_{\rm mDcl}=\frac{\gamma}{\gamma+1}a_{\rm Dcl}=\frac{\hbar\gamma^2}{2m}\sqrt{\frac{\gamma-1}{\gamma+1}} \tag{4.38}$$

The ratio between $a_{\rm mDcl}$ and $a_{\rm Dcl}$ is maximal in the case of the slight excitation, when $\gamma=1$, whereas the difference

$$a_{\rm Dcl}-a_{\rm mDcl}=\frac{\hbar}{2m}\gamma\sqrt{\frac{\gamma-1}{\gamma+1}} \tag{4.39}$$

has minimum at $\gamma=1$ and no maximum.

Interpretation of the modified classical Dirac particle $\mathcal{S}_{\rm mDcl}$ is the same as the interpretation of $\mathcal{S}_{\rm Dcl}$ [9]. The dynamic system $\mathcal{S}_{\rm mDcl}$ is interpreted as a relativistic rotator (two coupled particles of the mass $m_0$, rotating around their mass center). Coordinates $x_{(1)}$, $x_{(2)}$ of the rotator constituents in the center mass coordinate system are described by the relations

$$x_{(1)}^k = \{t, a\sin(\Omega t+\phi), a\sin(\Omega t+\phi), 0\} \tag{4.40}$$

$$x_{(2)}^k = \{t, -a\sin(\Omega t+\phi), -a\sin(\Omega t+\phi), 0\} \tag{4.41}$$

where $a$ , $M$ and $\Omega$ are respectively the radius of the rotator helix, the total mass and the frequency determined by the relations (4.38), (4.37). The quantities $a$, $M$, $\Omega$ are connected by the relation

$$M=\frac{2m_0}{\sqrt{1-a^2\Omega^2}} \tag{4.42}$$

The main characteristic of the relativistic rotator is the rigidity function $f_{\rm r}$, describing rigidity of coupling between the rotator constituents. It is defined by the relation

$$f_{\rm r}=f_{\rm r}(a)=\frac{M-2m_0}{2m_0}=\frac{1}{\sqrt{1-a^2\Omega^2}}-1 \tag{4.43}$$

where $f_{\rm r}$ is considered as a function of the radius $a$. Giving the rigidity function as a function of $a$ , we determine by means of (4.43) the relation between $\Omega$ and $a$. As a result we obtain that the quantities $a$, $M$, $\Omega$ are functions of the mass $m_0$, and of some parameter $\gamma$, describing the state of the rotator. Relations (4.37), (4.38) determine the quantities $a$, $M$, $\Omega$ of the modified Dirac particle $\mathcal{S}_{\rm mDcl}$ as functions of the parameter $\gamma$, describing the state of $\mathcal{S}_{\rm mDcl}$, and of the mass $m$, which is a parameter of $\mathcal{S}_{\rm mDcl}$. Identification of $\mathcal{S}_{\rm mDcl}$ with the relativistic rotator is possible, provided we assume that the relation between parameter $m$ of $\mathcal{S}_{\rm mDcl}$ and the rotator parameter $m_0$ depends on the state $\gamma$ of dynamic system $\mathcal{S}_{\rm mDcl}$.

Identifying the rotator with $\mathcal{S}_{\rm mDcl}$, we are forced to choose, what of quantities $m$ or $m_0$ is a parameter of $\mathcal{S}_{\rm mDcl}$. From physical viewpoint it seems that $m_0$ is to be the real parameter of $\mathcal{S}_{\mathrm{mDcl}}$. At such a choice the Dirac mass $m$, as well as the total mass $M$ are functions of the rotation state, described by the parameter $\gamma$, which is an integral of motion of $\mathcal{S}_{\mathrm{mDcl}}$. It means that the Dirac mass $m$ as well as the quantities $m$, $\Omega$, $a$, $M$ are integrals of motion of the rotator, whereas $m_0$ is a parameter of the dynamic system. Unfortunately, such an identification disagrees with the definition of $\mathcal{S}_{\mathrm{mDcl}}$ as a dynamic system described by the action (4.16). On the other hand, if we consider the Dirac mass $m$ as a parameter of $\mathcal{S}_{\mathrm{mDcl}}$ (but not as an integral of motion), the situation, described by the relations (4.37), is such, that the total mass $M = m/\gamma$ decreases, whereas the internal velocity $(\gamma-1)^{1/2}(\gamma+1)^{-1/2}$ increases with increasing $\gamma$. It means that the mass $m_0$ of the kinetic energy bearer decreases with increasing $\gamma$.

Eliminating $M, \Omega$ and $a$ from relations (4.42), (4.38) and (4.37), we obtain the relation between $m$ and $m_0$ for $\mathcal{S}_{\mathrm{mDcl}}$ and for $\mathcal{S}_{\mathrm{Dcl}}$

$$\mathcal{S}_{\mathrm{mDcl}}: \qquad m = \sqrt{2\gamma(\gamma+1)}m_0 \tag{4.44}$$

$$\mathcal{S}_{\mathrm{Dcl}}: \qquad m = 2m_0\gamma^2 \tag{4.45}$$

The rigidity function for $\mathcal{S}_{\mathrm{Dcl}}$ has the form

$$\mathcal{S}_{\mathrm{Dcl}}: \qquad f_{\mathrm{r}}(a) = \frac{1}{\sqrt{1-\left(\frac{4am_0}{\hbar}\right)^2}} - 1 \tag{4.46}$$

The rigidity function $f_{\mathrm{r}}(a)$ for $\mathcal{S}_{\mathrm{mDcl}}$ is given implicitly by the relation

$$\mathcal{S}_{\mathrm{mDcl}}: \qquad a = \frac{\hbar}{4m_0}\frac{\left(2(f_{\mathrm{r}}+1)^2-1\right)^{3/2}}{\sqrt{2}(f_{\mathrm{r}}+1)^3}\sqrt{(f_{\mathrm{r}}+1)^2-1} \tag{4.47}$$

To compare the rigidity functions for $\mathcal{S}_{\mathrm{Dcl}}$ and $\mathcal{S}_{\mathrm{mDcl}}$ we write the relation (4.46) in the form resolved with respect to $a$. We obtain

$$\mathcal{S}_{\mathrm{Dcl}}: \qquad a = \frac{\hbar}{4m_0}\sqrt{1-\frac{1}{(f_{\mathrm{r}}+1)^2}} \tag{4.48}$$

Comparison of (4.47) and (4.48) shows, that both rigidity functions are close at small $a$ $\quad (0 < 4m_0a/\hbar < 0.4)$. For large $a$ the coupling, described by the rigidity function $f_{\mathrm{r}}$ is more rigid for $\mathcal{S}_{\mathrm{Dcl}}$, than for $\mathcal{S}_{\mathrm{mDcl}}$.

# 5 Discussion

Almost eighty years since its appearance the Dirac equation exemplified the most useful relativistic dynamic equation. Hyperfine structure of the hydrogen spectrum has been explained by means of the Dirac equation. The quantum electrodynamics is based on the Dirac equation. How can the newly discovered internal degrees of freedom of the Dirac particle influence on these well established results?

Only internal degrees of freedom of the Dirac particle are described incorrectly (nonrelativistically), whereas the translation degrees of freedom are described correctly (relativistically). Energetic levels, connected with the internal degrees of freedom are very high. They are not excited at the processes, which are considered in the quantum electrodynamics and in the theory of atomic spectra. In these physical processes one may ignore the internal degrees of freedom of the Dirac particle. Thus, the correct description of the internal degrees of freedom does not change anything in the quantum electrodynamics and in the theory of atomic spectra. However, in the theory of the elementary particles, where there are high energetic processes, we should take into account the internal degrees of freedom of the Dirac particle. We are to describe them relativistically.

We have produced the minimal modification of the classical Dirac particle $\mathcal{S}_{\mathrm{Dcl}}$, but the description of the Dirac particle $\mathcal{S}_{\mathrm{D}}$ remains to be not completely relativistic. This defect should be eliminated also. We can make this, adding the eliminated "quantum" terms with transversal derivatives to the action for the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{mDcl}}\right]$ of the modified classical Dirac particles $\mathcal{S}_{\mathrm{mDcl}}$. This procedure is not unique. Besides, we cannot be sure, that the dynamic equations in terms of the wave function appear to be linear.

Since 1925, when the Dirac equation has been invented, it was considered to be the completely relativistic equation, describing the particles of spin 1/2 in the best way. Why was not the incompleteness of its relativity discovered earlier? This question is very important for the correct evaluation of the microcosm investigation strategy. The answer is rather unexpected. The nonrelativistic features of the Dirac equation was discovered in 1995 [14, 15], as soon as the Dirac particle became to be investigated simply as a dynamic system (but not as the quantum dynamic system). The quantum dynamic system distinguishes from the usual dynamic system in its compatibility with the quantum principles. The quantum principles restrict the way of description of the dynamic system. They demand that the quantum dynamic system be described in terms of the world function and dynamic equations be linear in these terms. The quantum principles admit only linear transformation of dependent variables (wave function).

The quantum principles impose the constraints on the description of a quantum dynamic system and cumber its investigation. They do not admit to eliminate $\gamma$-matrices, what is necessary for discovery of the dynamic equation incompatibility with the principles of relativity.

# Mathematical Appendices

# Appendix A Solution of dynamic equations common to $\mathcal{S}_{\rm Dcl}$ and $\mathcal{S}_{\rm mDcl}$

Dynamic equations generated by the action (4.8) have the form

$$\frac{\delta\mathcal{A}_{\rm Dcl}}{\delta x^l} = -\dot{p}_l = 0, \qquad p_l = \text{const} \tag{A.1}$$

$$\frac{\delta\mathcal{A}_{\rm Dcl}}{\delta y^l} = \frac{\partial L}{\partial y^l} - \frac{d}{d\tau_0}\frac{\partial L}{\partial \dot{y}^l} - p_l = 0, \tag{A.2}$$

$$\frac{\delta\mathcal{A}_{\rm Dcl}}{\delta \xi^l} = \left(\frac{\partial L}{\partial \xi^s} - \frac{d}{d\tau_0}\frac{\partial L}{\partial \dot{\xi}^s}\right)\left(\delta_l^s + \frac{\xi_l\xi^s}{\xi_k\xi^k}\right) = 0 \tag{A.3}$$

$$\frac{\delta\mathcal{A}_{\rm Dcl}}{\delta p_l} = \dot{x}^l - y^l = 0 \tag{A.4}$$

Let

$$y^k = \left\{y^0, \mathbf{y}\right\}, \qquad p_k = \{p_0, \mathbf{p}\} = \{p_0, 0\}, \qquad \xi^k = \{0, \boldsymbol{\xi}\}, \qquad \boldsymbol{\xi}^2 = 1$$

Then for $l = 1, 2, 3$ the dynamic equations (A.2) can be written in the form

$$-\kappa_0 m\mathbf{y} + \frac{\hbar Q}{2}(\boldsymbol{\xi}\times\dot{\mathbf{y}}) - \frac{\hbar}{2}\frac{\partial Q}{\partial \mathbf{y}}(\mathbf{y}\times\dot{\mathbf{y}})\boldsymbol{\xi} + \frac{\hbar}{2}\frac{d}{d\tau_0}\left(Q(\boldsymbol{\xi}\times\mathbf{y})\right) = \mathbf{p} = 0 \tag{A.5}$$

where

$$Q = Q(y) = \left(\sqrt{y^s y_s}(\sqrt{y^s y_s} + y^0)\right)^{-1} \tag{A.6}$$

For $l = 0$ the dynamic equations (A.2) can be written in the form

$$-\kappa_0 m\frac{y^0}{\sqrt{y^s y_s}} + \frac{\hbar}{2}\frac{\partial Q}{\partial y^0}(\mathbf{y}\times\dot{\mathbf{y}})\boldsymbol{\xi} = p_0 \tag{A.7}$$

The equation (A.3) has the form

$$-\hbar\frac{\left(\dot{\boldsymbol{\xi}}\times\mathbf{z}\right)\times\boldsymbol{\xi}}{2(1+\mathbf{z}\boldsymbol{\xi})} + \hbar\left(-\frac{d}{d\tau_0}\frac{(\boldsymbol{\xi}\times\mathbf{z})}{2(1+\mathbf{z}\boldsymbol{\xi})} - \frac{(\dot{\boldsymbol{\xi}}\times\boldsymbol{\xi})\mathbf{z}}{2(1+\mathbf{z}\boldsymbol{\xi})^2}\mathbf{z}\right)\times\boldsymbol{\xi} + \hbar\frac{(\mathbf{y}\times\dot{\mathbf{y}})\times\boldsymbol{\xi}}{2}Q = 0 \tag{A.8}$$

After transformations this equation is reduced to the equation (see Appendix B)

$$\dot{\boldsymbol{\xi}} = -(\mathbf{y}\times\dot{\mathbf{y}})\times\boldsymbol{\xi}Q \tag{A.9}$$

which does not contain the vector $\mathbf{z}$. It means that $\mathbf{z}$ determines a fictitious direction in the space-time. Note that $\mathbf{z}$ in the action (2.8) for the system $\mathcal{S}_{\rm D}$ is fictitious also, because there is only longitudinal derivative $j^l\partial_l$ in the term of the action (2.8) for $\mathcal{S}_{\rm D}$, which contains $\mathbf{z}$. This term is not changed at the dynamical disquantization.

Constraint (4.6) on the variables $\dot{x}^k$ remains to be valid in the considered case. It takes the form

$$\sqrt{y_s y^s} = 1, \qquad y^0 = \sqrt{1+\mathbf{y}^2} \tag{A.10}$$

Taking into account the condition (A.10) we obtain from (A.6) for quantities $Q$, $\partial Q/\partial y_0$, $\partial Q/\partial \mathbf{y}$

$$Q=\frac{1}{1+y_0},\qquad \frac{\partial Q}{\partial y_0}=-1,\qquad \frac{\partial Q}{\partial \mathbf{y}}=\frac{\mathbf{y}\,(2+y_0)}{(1+y_0)^2} \qquad \text{(A.11)}$$

The dynamic equations (A.9), (A.7) and (A.5) can be rewritten in the form

$$\dot{\boldsymbol{\xi}}=-\frac{(\mathbf{y}\times\dot{\mathbf{y}})\times\boldsymbol{\xi}}{1+y_0} \qquad \text{(A.12)}$$

$$\kappa_0 m y_0+\frac{\hbar}{2}\,(\mathbf{y}\times\dot{\mathbf{y}})\,\boldsymbol{\xi}=-p_0 \qquad \text{(A.13)}$$

$$-\kappa_0 m\mathbf{y}+\frac{\hbar(\boldsymbol{\xi}\times\dot{\mathbf{y}})}{2\,(1+y_0)}-\frac{\hbar}{2}\frac{\mathbf{y}\,(2+y_0)}{(1+y_0)^2}(\mathbf{y}\times\dot{\mathbf{y}})\boldsymbol{\xi}+\frac{\hbar}{2}\frac{d}{d\tau_0}\left(\frac{(\boldsymbol{\xi}\times\mathbf{y})}{1+y_0}\right)=0 \qquad \text{(A.14)}$$

Equations (A.12), (A.13), (A.14) admit the trivial solution

$$\mathbf{y}=0,\qquad y^0=-\frac{\kappa_0 p_0}{m}=\pm 1,\qquad \boldsymbol{\xi}=\boldsymbol{\xi}_0=\text{const},\qquad \boldsymbol{\xi}^2=1 \qquad \text{(A.15)}$$

$$x^k=\left\{-\varepsilon\kappa_0\tau_0,X^1,X^2,X^3\right\},\qquad X^\alpha=\text{const},\quad \alpha=1,2,3,\qquad \varepsilon=\operatorname{sgn}p_0 \qquad \text{(A.16)}$$

In this case the internal degrees of freedom are not excited, and the world line of the classical Dirac particle $\mathcal{S}_{\text{Dcl}}$ is a timelike straight in the space-time.

For solution of dynamic equations it is important to take into account that three 3-vectors $\boldsymbol{\xi}$,$\mathbf{y}$, $\dot{\mathbf{y}}$ are orthogonal between themselves

$$\boldsymbol{\xi}\mathbf{y}=0,\qquad \boldsymbol{\xi}\dot{\mathbf{y}}=0,\qquad \mathbf{y}\dot{\mathbf{y}}=0,\qquad \mathbf{y}^2=\gamma^2-1=\text{const}, \qquad \text{(A.17)}$$

To prove relations (A.17) we transform the relation (A.14), calculating the last term.

$$-\kappa_0 m\mathbf{y}+\frac{\hbar(\boldsymbol{\xi}\times\dot{\mathbf{y}})}{(1+y_0)}-\frac{\hbar}{2}\frac{\mathbf{y}\,(2+y_0)}{(1+y_0)^2}(\mathbf{y}\times\dot{\mathbf{y}})\boldsymbol{\xi}+\frac{\hbar}{2}\frac{(\dot{\boldsymbol{\xi}}\times\mathbf{y})}{1+y_0}=0 \qquad \text{(A.18)}$$

Using (A.12), we eliminate $\dot{\boldsymbol{\xi}}$ and obtain

$$-\kappa_0 m\mathbf{y}+\frac{\hbar(\boldsymbol{\xi}\times\dot{\mathbf{y}})}{(1+y_0)}-\frac{\hbar}{2}\frac{\mathbf{y}\,(2+y_0)}{(1+y_0)^2}(\mathbf{y}\times\dot{\mathbf{y}})\boldsymbol{\xi}-\frac{\hbar}{2}\frac{((\mathbf{y}\times\dot{\mathbf{y}})\times\boldsymbol{\xi})\times\mathbf{y}}{(1+y_0)^2}=0 \qquad \text{(A.19)}$$

Calculating the double vector product in the last term of (A.19) and combining it with the second term, we obtain

$$\frac{\hbar}{(1+y_0)}\left(\boldsymbol{\xi}+\frac{1}{2}\frac{(\boldsymbol{\xi}\mathbf{y})\,\mathbf{y}}{(1+y_0)}\right)\times\dot{\mathbf{y}}-\left(\frac{\hbar}{2}\frac{(2+y_0)}{(1+y_0)^2}(\mathbf{y}\times\dot{\mathbf{y}})\boldsymbol{\xi}+\kappa_0 m\right)\mathbf{y}=0 \qquad \text{(A.20)}$$

It follows from (A.20) that the vector $\mathbf{y}$ is orthogonal to vectors $\dot{\mathbf{y}}$ and $\boldsymbol{\xi}+\frac{1}{2}\,(1+y_0)^{-1}\,(\boldsymbol{\xi}\mathbf{y})\,\mathbf{y}$, provided

$$\frac{\hbar}{2}\frac{(2+y_0)}{(1+y_0)^2}(\mathbf{y}\times\dot{\mathbf{y}})\boldsymbol{\xi}+\kappa_0 m\neq 0 \qquad \text{(A.21)}$$

It means that the last two relations (A.17) are fulfilled. Besides

$$\left(1+\frac{\mathbf{y}^2}{2\left(1+y_0\right)}\right)(\boldsymbol{\xi}\mathbf{y})=0 \tag{A.22}$$

and the first relation (A.17) takes place, as far as according to the last relation (A.17) $\gamma^2>1$, and

$$1+\frac{\mathbf{y}^2}{2\left(1+y_0\right)}=\frac{\gamma+1}{2}\neq 0 \tag{A.23}$$

(In the case $\gamma^2=1$ we have the trivial solution $\mathbf{u}^2=0$). Differentiating of the first relation (A.17) and using the relation (A.12) for elimination of $\dot{\boldsymbol{\xi}}$, we obtain

$$(\boldsymbol{\xi}\dot{\mathbf{y}})=-\left(\dot{\boldsymbol{\xi}}\mathbf{y}\right)=\mathbf{y}\frac{(\mathbf{y}\times\dot{\mathbf{y}})\times\boldsymbol{\xi}}{1+y_0}=-\frac{\mathbf{y}^2\left(\boldsymbol{\xi}\dot{\mathbf{y}}\right)}{1+y_0}=-\left(\boldsymbol{\xi}\dot{\mathbf{y}}\right)(\gamma-1) \tag{A.24}$$

As far as $\gamma^2\geq 1$, the second relation (A.17) follows from (A.24). Thus, if the condition (A.21) is fulfilled the relations (A.17) are also fulfilled.

Let now the condition (A.21) be violated. Eliminating $(\mathbf{y}\times\dot{\mathbf{y}})\boldsymbol{\xi}$ from the relation

$$\frac{\hbar}{2}\frac{(2+y_0)}{\left(1+y_0\right)^2}(\mathbf{y}\times\dot{\mathbf{y}})\boldsymbol{\xi}+\kappa_0 m=0 \tag{A.25}$$

by means of the dynamic equation (A.13), we obtain

$$(2+y_0)\left(-p_0-\kappa_0 m y_0\right)+\kappa_0 m\left(1+y_0\right)^2=0 \tag{A.26}$$

Resolving (A.26) with respect to $y_0$, we obtain

$$y_0=\frac{\kappa_0 m}{p_0}-2=\text{const} \tag{A.27}$$

In this case it follows from (A.20)

$$\dot{\mathbf{y}}=\alpha\left(\boldsymbol{\xi}+\frac{1}{2}\frac{(\boldsymbol{\xi}\mathbf{y})\,\mathbf{y}}{\left(1+y_0\right)}\right) \tag{A.28}$$

where $\alpha$ is some real number. Besides, it follows from (A.10) that $\mathbf{y}^2=$const and hence the last two relations (A.17) are fulfilled. Multiplying (A.28) by $\mathbf{y}$, we obtain

$$\dot{\mathbf{y}}\mathbf{y}=\alpha\left(1+\frac{\mathbf{y}^2}{2\left(1+y_0\right)}\right)(\boldsymbol{\xi}\mathbf{y})=0 \tag{A.29}$$

Hence, either $\alpha=0$, or $(\boldsymbol{\xi}\mathbf{y})=0$. If $\alpha=0$, then it follows from (A.28), that $\dot{\mathbf{y}}=0$. But the condition $\dot{\mathbf{y}}=0$ is compatible with the relation (A.25) only if $m=0$. Hence, $\alpha\neq 0$, and it follows from (A.29), that $(\boldsymbol{\xi}\mathbf{y})=0$, and we obtain from (A.28)

$$\dot{\mathbf{y}}=\alpha\boldsymbol{\xi} \tag{A.30}$$

In the case (A.30) $(\mathbf{y}\times\dot{\mathbf{y}})\boldsymbol{\xi}=0$, and the relation (A.30) is compatible with the relation (A.25) only if $m=0$.

We suppose that $m\neq 0$, and the relation (A.25) is fulfilled never. The opposite relation (A.21) is fulfilled always, as well as the relations (A.17).

According to constraints (A.17) the dynamic equation (A.12) turns into

$$\dot{\boldsymbol{\xi}}=0,\qquad \boldsymbol{\xi}=\text{const},\qquad \boldsymbol{\xi}^2=1 \tag{A.31}$$

We choose the coordinate system in such a way that

$$\boldsymbol{\xi}=\{0,0,\varepsilon_0\},\qquad \varepsilon_0=\pm 1 \tag{A.32}$$

According to (A.17) vectors $\mathbf{y}$ and $\dot{\mathbf{y}}$ in this coordinate system can be written as follows

$$\mathbf{y} = \left(y^1,y^2,0\right)=\left\{\sqrt{\gamma^2-1}\cos\Phi,\sqrt{\gamma^2-1}\sin\Phi,0\right\},\qquad \gamma=\text{const}, \tag{A.33}$$

$$\dot{\mathbf{y}} = \left(\dot{y}^1,\dot{y}^2,0\right)=\left\{-\sqrt{\gamma^2-1}\omega\sin\Phi,\sqrt{\gamma^2-1}\omega\cos\Phi,0\right\},\qquad \omega=\frac{d\Phi}{d\tau_0} \tag{A.34}$$

It follows from the last relation (A.17) and (A.10), that

$$y^0=\gamma=\text{const} \tag{A.35}$$

Substituting (A.32), (A.33), and (A.34) in the dynamic equation (A.13), we obtain

$$(\mathbf{y}\times\dot{\mathbf{y}})\,\boldsymbol{\xi}=\varepsilon_0\left(\gamma^2-1\right)\omega\qquad \gamma=-\frac{\kappa_0 p_0}{m}-\varepsilon_0\kappa_0\frac{\hbar}{2m}\left(\gamma^2-1\right)\omega \tag{A.36}$$

Then the second relation (A.36) leads to the conclusion that $\omega$ =const.

Substituting (A.32), and (A.36) in the dynamic equation (A.20), we obtain

$$\frac{\hbar}{(1+y_0)}\boldsymbol{\xi}\times\dot{\mathbf{y}}-\left(\frac{\hbar}{2}\frac{(2+y_0)}{(1+y_0)^2}\varepsilon_0\left(\gamma^2-1\right)\omega+\kappa_0 m\right)\mathbf{y}=0 \tag{A.37}$$

It follows from (A.33), (A.34) and (A.32), that

$$\boldsymbol{\xi}\times\dot{\mathbf{y}}=-\varepsilon_0\omega\mathbf{y} \tag{A.38}$$

Substituting (A.32) and (A.35) in (A.37), we obtain

$$-\varepsilon_0\omega\mathbf{y}-\left(\frac{(2+\gamma)}{2(1+\gamma)}\varepsilon_0\left(\gamma^2-1\right)\omega+\frac{\kappa_0 m}{\hbar}(1+\gamma)\right)\mathbf{y}=0 \tag{A.39}$$

Resolving (A.39) with respect to $\omega$, we obtain

$$\omega=-2\kappa_0\varepsilon_0\frac{m}{\gamma\hbar} \tag{A.40}$$

It follows from the second equation (A.36) and (A.40), that

$$p_0 = -\frac{\kappa_0 m}{\gamma} \tag{A.41}$$

The solution (A.33) has the form

$$y^k = \left\{\gamma, \sqrt{\gamma^2-1}\cos\left(\omega\tau_0+\phi\right), \sqrt{\gamma^2-1}\sin\left(\omega\tau_0+\phi\right), 0\right\}, \tag{A.42}$$

where $\omega$ is determined by the relation (A.40), and $\gamma^2 \geq 1$, $\phi$, are constants.

If $\gamma = 1$ we obtain

$$y^k = \{1, 0, 0, 0\}, \tag{A.43}$$

This solution coincides with the trivial solution (A.15), (A.16).

# Appendix B Transformation of equation for variable $\boldsymbol{\xi}$

Multiplying equation (A.8) by $(1+\mathbf{z}\boldsymbol{\xi})$ and keeping in mind $\boldsymbol{\xi}^2 = 1$ and $\mathbf{z}^2 = 1$, we obtain

$$\boldsymbol{\xi}\times\left(-\dot{\boldsymbol{\xi}}\times\mathbf{z}+\frac{(\mathbf{z}\dot{\boldsymbol{\xi}})}{2(1+\mathbf{z}\,\boldsymbol{\xi})}\boldsymbol{\xi}\times\mathbf{z}-\frac{\dot{\boldsymbol{\xi}}(\boldsymbol{\xi}\times\mathbf{z})}{2(1+\mathbf{z}\,\boldsymbol{\xi})}\mathbf{z}-\frac{(1+\mathbf{z}\,\boldsymbol{\xi})}{2}\mathbf{b}\right)=0, \qquad \mathbf{b}=-(\mathbf{y}\times\dot{\mathbf{y}})Q \tag{B.1}$$

Two middle terms could be represented as the double vector product

$$\boldsymbol{\xi}\times\left(-\dot{\boldsymbol{\xi}}\times\mathbf{z}+\frac{1}{2(1+\mathbf{z}\,\boldsymbol{\xi})}\left(\dot{\boldsymbol{\xi}}\times((\boldsymbol{\xi}\times\mathbf{z})\times\mathbf{z})\right)-\frac{(1+\mathbf{z}\,\boldsymbol{\xi})}{2}\mathbf{b}\right)=0 \tag{B.2}$$

This equation can be rewritten in the form

$$\boldsymbol{\xi}\times\left(\dot{\boldsymbol{\xi}}\times\left(-\mathbf{z}+\frac{(\mathbf{z}\boldsymbol{\xi})\,\mathbf{z}-\boldsymbol{\xi}}{2(1+\mathbf{z}\,\boldsymbol{\xi})}\right)-\frac{(1+\mathbf{z}\,\boldsymbol{\xi})}{2}\mathbf{b}\right)=0 \tag{B.3}$$

Now calculating the double vector products and taking into account that $\boldsymbol{\xi}\dot{\boldsymbol{\xi}} = 0$, we obtain

$$\left(\dot{\boldsymbol{\xi}}\left(-\mathbf{z}\boldsymbol{\xi}+\frac{(\mathbf{z}\boldsymbol{\xi})^2-1}{2(1+\mathbf{z}\,\boldsymbol{\xi})}\right)\right)-\frac{(1+\mathbf{z}\,\boldsymbol{\xi})}{2}(\boldsymbol{\xi}\times\mathbf{b})=0$$

$$\left(\dot{\boldsymbol{\xi}}\left(-\mathbf{z}\boldsymbol{\xi}+\frac{(\mathbf{z}\boldsymbol{\xi})-1}{2}\right)\right)-\frac{(1+\mathbf{z}\,\boldsymbol{\xi})}{2}(\boldsymbol{\xi}\times\mathbf{b})=0$$

$$-\dot{\boldsymbol{\xi}}-(\boldsymbol{\xi}\times\mathbf{b})=0 \tag{B.4}$$

or using designation (B.1)

$$\dot{\boldsymbol{\xi}}=(\boldsymbol{\xi}\times(\mathbf{y}\times\dot{\mathbf{y}}))\,Q \tag{B.5}$$

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