Ptolemyness of conventional program for microcosm investigations and alternative research program

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Abstract

It is shown that the conventional approach to microcosm investigations uses an incorrect supposition (incorrect space-time model) whose incorrectness is compensated by means of additional hypotheses (quantum mechanics principles). Such a conception reminds the Ptolemaic doctrine of celestial mechanics. Alternative research program, which uses more correct space-time model and does not need additional hypotheses (quantum mechanics principles) for free explanation of quantum effects, is suggested. If the more correct space-time model were known in the beginning of XXth century, when research of microcosm started, quantum mechanics could develop in other way. The alternative research program appeared with a secular delay, because all this time the necessary mathematical technique was not available for researchers. Absence of necessary mathematical technique is connected with some prejudices which have been overcame at the construction of new conception of geometry and that of statistical description. Basic statements of the new mathematical technique and principles of its application in alternative research program are presented in the paper.

1 Introduction

When in the beginning of XX century one starts to investigate physical phenomena in microcosm, researchers met two serious problems, which could not be solved in the scope of the classical physics of that time. They demanded a new approaches. The first problem is the problem of microparticle motion with velocities close to the speed of the light. This problem had been solved by construction of special relativity theory. The concentrated expression of the relativity principles is the statement that the event space (space-time) is described by the Minkowski geometry, or what is the same by the world function [1]

$$\sigma_{\mathrm{M}}(x,x') = \sigma_{\mathrm{M}}(t,\mathbf{x},t',\mathbf{x}') = \frac{1}{2}\left(c^2(t-t')^2 - (\mathbf{x}-\mathbf{x}')^2\right) \tag{1.1}$$

where c is the speed of the light, $x = \{t,\mathbf{x}\}$ and $x' = \{t',\mathbf{x}'\}$ are coordinates of two arbitrary points in the event space.

The second problem is the problem of stochastic microparticles motion, which cannot be understood and explained in the scope of deterministic classical physics. To describe phenomena connected with the stochastic microparticle motion, one should modify the space-time geometry in addition. One should substitute the world function σ_{M} by σ

$$\sigma(x,x') = \sigma_{\mathrm{M}}(x,x') + D(\sigma_{\mathrm{M}}(x,x')), \tag{1.2}$$

where σ_{M} is the Minkowski world function (1.1), and

$$D = D(\sigma_{\mathrm{M}}) = \begin{cases} d & \sigma_{\mathrm{M}} > \sigma_0 \\ 0 & \sigma_{\mathrm{M}} \leq 0 \end{cases}, \tag{1.3}$$

$$d = \frac{\hbar}{2bc} = \text{const} \approx 10^{-21}\text{cm}^2, \qquad \sigma_0 = \text{const} \approx d \approx 10^{-21}\text{cm}^2$$

is a correction, called distortion. Here $\hbar$ is the quantum constant, $b \approx 10^{-17}$g/cm is a new universal constant. Values of distortion D within $[0,\sigma_0]$ are yet unknown. They are to be established as a result of further investigations.

Formally the modification of geometry is very slight, as far as the distortion D is a small correction to the Minkowski world function. Nevertheless at the transition from (1.1) to (1.2) the space-time model changes qualitatively no less, than at the transition from Newtonian model to the Minkowski one. The space-time geometry, generated by the world function (1.2), is not a Riemannian geometry. We shall refer to it as T-geometry. The T-geometry is nondegenerate geometry. It means that at any point of the space-time there exists many unit timelike vectors parallel to a given timelike vector, and motion of free particles is stochastic, although the T-geometry in itself is deterministic. It seems rather evident that the free particle motion in the space-time with stochastic geometry is stochastic[2, 3, 4, 5, 6, 7], but a stochastic motion of a free particle in the deterministic space-time looks rather unexpected and needs an explanation. It will be given in the second section.

The classical physics in the event space with T-geometry (1.2) explains phenomena, conditioned by the stochasticity of microparticle motion (known as quantum effects) freely and without any additional suppositions or hypotheses. The supposition (1.2) on the character of geometry is not an additional hypothesis. It is simply a correction of the Minkowski geometry, which is used instead of (1.1). In other words, the relation (1.2) is a hypothesis in the same degree, as the statement, that the space-time geometry is the Minkowski one, is a hypothesis. As one can see from (1.2), (1.3) the world function σ differs essentially from σ_{M} only for small space-time intervals of the order 10^{-10}cm, i.e. in the microcosm.

In the beginning of the XX century the T-geometry was not known for a number of reasons, which will be discussed in the second section, and the second problem was solved differently. The Newtonian space-time model was conserved, but a number of additional hypotheses on the microparticle motion laws was taken. These additional hypotheses are known as quantum mechanics principles. The conception of such a solution of the microparticle stochasticity problem is called the quantum mechanics (QM).

The quantum mechanics is a non-relativistic theory from outset, i.e. the first problem and the second one are solved separately. Thereafter the problem of unification of quantum mechanics (QM) with the relativity theory (RT) arises. The scheme of the conventional research program for microcosm investigation looks as follows

$$
\begin{array}{ccccccc}
 & \nearrow & \begin{array}{c}\text{large}\\ \text{velocity}\\ \text{problem}\end{array} & \to & \begin{array}{c}\text{Minkowski}\\ \text{geometry}\end{array} & \searrow & \\
\begin{array}{c}\text{classic}\\ \text{physics}\end{array} & & & & & & \begin{array}{c}\text{integration}\\ \text{problem of}\\ \text{QM and RT}\end{array}\\
 & \searrow & \begin{array}{c}\text{microparticle}\\ \text{stochasticity}\\ \text{problem}\end{array} & \to & \text{QM principles} & \nearrow &
\end{array} \tag{1.4}
$$

The quantum relativistic field theory and elementary particle theory are different sides of the problem of unification QM with RT.

Unprejudiced observer known another possible solution (1.2) of the problem is inclined to interpret this in the sense, that the research program (1.4) uses untrue assertion (inadequate space-time model) which manifests itself as contradictions appearing in different places of theory. One needs to introduce new hypotheses, compensating specific manifestations of the untrue assertion of the theory. At the same time researchers developing the theory are inclined to connect all arising problems with complexity of physical phenomena in microcosm. Such a situation took place in the science history. It is the Ptolemaic doctrine, using untrue assertion on the place of the Earth in the center of universe. Ptolemy and his successors succeeded to describe correctly heavenly bodies motion in spite of the untrue assertion on the place of the Earth in the center of universe. These results had come about through use of new additional hypotheses, compensating original untrue assertion on the place of the Earth. In spite of success in explanation of astronomical observations the Ptolemaic doctrine lead to blind alley finally, and the most reasonable way of overcoming all problems was a substitution of untrue assertion by the true one.

Something like that is observed in the solution of the integration problem of QM and RT. One uses inadequate space-time model, and this is an origin of many problems of contemporary quantum theory. A use of the adequate space-time model (1.2) removes the integration problem of QM and RT, because the conception (1.2) is relativistic and quantum originally (it is quantum in the sense that it includes the quantum constant $\hbar$). Besides it does not contain any additional hypotheses and principles. The integration problem and problem of concordance of different principles do not exist at all. Instead there exists the problem of statistical description of stochastic microparticles. It is a very serious problem, because the probabilistic statistical description, used in the nonrelativistic statistical physics, is ineligible for description of stochastic relativistic motion. (see detail in sec. 3)

As a whole the scheme of the alternative research program for the microcosm investigation looks as follows.

$$\begin{array}{ccccccc} & & \begin{array}{c}\text{large}\\ \text{velocity}\\ \text{problem}\end{array} & & & & \\ \begin{array}{c}\text{classic}\\ \text{physics}\end{array} & \begin{array}{c}\nearrow\\ \\ \searrow\end{array} & & \begin{array}{c}\searrow\\ \\ \nearrow\end{array} & \begin{array}{c}\text{nondegen.}\\ \text{geometry}\end{array} & \rightarrow & \begin{array}{c}\text{statistical}\\ \text{description}\end{array} \\ & & \begin{array}{c}\text{microparticle}\\ \text{stochasticity}\\ \text{problem}\end{array} & & & & \end{array} \tag{1.5}$$

The research program (1.4) is a compensating, or Ptolemaic conception, as far as it uses inadequate space-time model and quantum principles, compensating inadequacy of this model. One cannot say that this research program is untrue, because it explains the observed physical phenomena and yet this program is not a rigorous physical theory, because it uses inadequate space-time model. Now this program is developed in details, because several generations of researchers had been working with this program for the last century. Nevertheless this detailed development does not prevent from appearance of new problems, which need new hypotheses for their solution. In this relation the program (1.4) reminds the Ptolemaic doctrine. We mark this circumstance, referring to the program as "Ptolemy-2". Further such a defect of scientific conception will be marked by a special term "ptolemyness". Ptolemyness of the conventional research program (1.4) is not evident. It becomes clear only after appearance of alternative research program, which is not Ptolemaic.

Unlike the program (1.4), the research program (1.5) is a rigorous physical theory. It is not a theory of Ptolemaic type, because it does not use any additional (compensating) hypotheses besides those which were used in classical physics. This program is very simple and reasonable. Now it is very young and slightly developed. All this associates with the Copernican doctrine at the time of its appearance. We mark this circumstance referring to the program (1.5) as Copernicus-2. These names have been given to mark qualitative difference between the two conception Ptolemy-2 and Copernicus-2 and to underline that, analyzing and evaluating interplay between them, one cannot use the criteria, obtained at the work with Ptolemaic (compensating) conceptions.

The fact is that that on one hand the Ptolemaic conception, i.e. a theory containing untrue assertion and compensating it by means of additional hypotheses, is not a rigorous scientific theory. On the other hand, it is applied very wide, and researchers of microcosm were forced to work with different types of Ptolemaic conceptions. As a result one derived the rules of work and criteria of estimations of obtained results, which are suitable for work with Ptolemaic conceptions. But these rules and criteria are not effective with the work with rigorous scientific conceptions, which do not contains mistakes and inadequate assertions. For work with Ptolemaic conceptions one needs a "short logic", i.e. one uses hypotheses and tries to make such conclusions from them which could be quickly verified experimentally. The long chains of logical considerations which cannot be quickly verified by experiment seem to be doubtfull. Primacy of experiment over logic and principles is another peculiarity of a Ptolemaic conception, when any principles are suitable, provided one can explain experimental date by their use. In many cases a researcher, working only with Ptolemaic theories, have not enough experience of work with rigorous scientific theories and cannot evaluate them correctly.

As an example let us consider the following situation. There are two alternative Ptolemaic conceptions A and B. Let the conception A appear earlier and be accepted by the scientific community. After appearance of the conception B the proponents of the conception A evaluate the conception B as follows. One considers whether the conception B explains the observed phenomena, which can be explained by the conception A. If no, the conception B is worse. If the conception B explains all phenomena, which are explained by the conception A and besides it explains some phenomena which cannot be explained by the conception A, then one should prefer the conception B. Finally, if the conception B explains only those phenomena, which are explained by the conception A and nothing except for them, one should prefer the conception A, because it appeared earlier. The second conception, leading to the same results, is considered to be superfluous. Analysis of hypotheses, used in conceptions A and B, number of them and their quality is considered to be superfluous. All this is valid, provided both conceptions A and B are Ptolemaic. If the conception B pretends to being rigorous theory, but not Ptolemaic one, i.e. it contains essentially less additional hypotheses, than the conception B, one should prefer the conception B even in the case, when it does not explain nothing besides those phenomena, which are explained by the conception A. In this case one should use the conception B, because it is more promising, and after its development it will explain many phenomena, which it could not explain at the point in time of appearance.

It is this case that took place in the conflict between the Ptolemy's doctrine and Copernican one. In first time after appearance the Copernican doctrine explained nothing in the heavenly bodies motion that cannot be explained by the Ptolemy's doctrine. Further development of the Copernican doctrine leads to such results which cannot be imagined by proponents of Ptolemy. The Copernican doctrine was much simpler, because it did not use additional (compensating) assertions. It was the simplicity of the Copernican doctrine, conditioned by its rigor ("non-ptolemyness"), appears to be the main factor, providing its victory.

The research program Copernicus-2 pretends to the role of rigorous (non-Ptolemaic) conception, because it does not use the quantum principles for explanation of non-relativistic quantum effects. On the other hand, evaluation of the program Ptolemy-2 as a Ptolemaic conception is based on the fact that there exists the program Copernicus-2, which uses essentially less number of base assertions and which does not uses, in particular, QM principles. As for relativistic quantum effects, the program Copernicus-2 cannot say anything about them due to its insufficient development. It should keep in mind that the program Copernicus-2 is very young, whereas the program Ptolemy-2 has been developed by several generations of researchers in the course of several decades. But the program Copernicus-2 promises some progress in explanation of relativistic quantum effects as a result of further development, whereas the program Ptolemy-2 does not promise a progress. Experience of work with Ptolemy-2 shows that at its development the number of problems increases, and the conception becomes more and more complex and tangled.

The program Copernicus-2 cannot be considered to be a quite new conception. All its stages have been known since the beginning of XX century. The fact that quantum effects can be explained as a result of statistical description of microparticle stochastic behavior seems very reasonable for many researchers [8, 9]. Such an explanation seem to be very plausible in the light of the success of the statistical physics, which explains the nature of heat and thermal phenomena in such a way. The fact that the space-time geometry can be a reason of stochasticity is not new also [2, 3, 4, 5, 6, 7].

Difficulties of work with the program Copernicus-2 are connected with insufficient development of geometry and of the conception of the statistical description. In other words, there were no mathematical tools which should be sufficiently effective for description of microparticle stochastic motion. One was forced to construct these mathematical tools, developing a new conception of geometry and a new conception of statistical description. It is the development of these new conceptions, that allowed one to formulate and substantiate the research program Copernicus-2. Thus, at the development of the research program Coperncus-2 the technical and mathematical results are main and determining.

The newly developed conceptions of geometry and statistical description are more general, than existing before. From formal viewpoint the lager generality is achieved at the cost of reduction of the number of fundamental concepts, i.e. concepts used at the construction of the conception. In particular, in T-geometry the concept of a curve is not used, and in the dynamic conception of statistical description the concept of probability and that of probability density are not used.

In the second section the new conception of geometry is considered. The most attention is concentrated on conceptual problems, in particular, one investigates, how stochastic motion of free particles can appear in the space-time with deterministic geometry and what is the reason why such a simple and necessary construction as T-geometry has not been constructed earlier. In the third section one considers conceptual problems of statistical description – a new conception of statistical description restricted by no constraints of the probability theory. In the fourth section the dynamic conception of statistical description is applied to the description of quantum-stochastic particle.

2 Metric conception of geometry

Usually a geometry is constructed on the basis of linear space, where linear operations on vectors are defined. Vector (the main object of the linear space) is determined by two points: origin of the vector and its end. It is supposed that in the linear space the origins of all vectors coincide, and any vector is determined single-valuedly by the point which determines its end. After definition of the scalar product the linear space turns to vector Euclidean space. As far as there are one-to-one correspondence between the vectors of the linear space and points representative their end, the vector Euclidean space generates the point Euclidean space, where the main characteristic is the distance d between two points or the world function $\sigma = \frac{1}{2}d^2$. The scalar product in the vector Euclidean space is connected single-valuedly with the world function in the corresponding point Euclidean space. The scalar product in the vector Euclidean space determines the world function in the corresponding point Euclidean space, and vice versa.

Introduction of the linear space as a basis for construction of the Euclidean space is possible only in the continuous homogeneous space, where all points and all connections between them are similar. If the continuity of the space is violated, for instance, removing one point of it, the space stops to be linear space, because now linear operations are not defined properly. They lead to a definite result not always. In inhomogeneous space one has to introduce tangent linear space at any point, and this set of linear spaces forms a basis for construction of inhomogeneous (Riemannian) geometry.

The practical work with the event space, considered to be the Minkowski space, suggests that the geometry is determined by the world function (distance between any two points) of the event space and that the linear space is not a necessary attribute of geometry. It plays a role of some subsidiary construction, which is used for building of geometry and which can be removed after the geometry has been constructed. If it is really so, the geometry can be constructed without referring to a linear space. It may appear that some restrictions, imposed usually on geometry, are generated by the properties of the linear space, which is used at the construction of geometry, but not at the geometry itself.

Construction of a geometry, based only on information, contained in the world function, will be referred to as metric conception of geometry. This approach is well known as metric geometry [10, 11, 12], But one did not succeed to carry out it consequently (i.e. without invoking additional information) and to construct a geometry which should be as informative as the Euclidean one. One succeeded for the first time to make this in the papers [13, 14].

The idea of the geometry construction on the basis of only world function σ is very simple. All relations of Euclidean geometry are written in terms of the world function and declared to be valid for any world function, i.e. for any geometry. Practically it is important to represent in terms of world function only the scalar product, because all remaining relations are expressed finally through it. It is important also not to use the concept of a curve, defined as a continuous mapping of a segment of real axis on the space Ω

$$\mathcal{L}: \quad [0,1] \to \Omega. \tag{2.1}$$

Let Ω be a set of points with the world function σ, given on $\Omega \times \Omega$

$$\sigma: \qquad \Omega \times \Omega \to \mathbb{R} \tag{2.2}$$

$$\sigma(P,Q) = \sigma(Q,P), \qquad \sigma(P,P) = 0, \qquad \forall P,Q, \in \Omega \tag{2.3}$$

Let the totality $V = \{\sigma, \Omega\}$ be called σ-space. Vector $\overrightarrow{PQ} \equiv \mathbf{PQ}$ is an ordered set of two points $\{P,Q\}$ (point P is an origin of the vector and Q is its end). The length $|\overrightarrow{PQ}|$ of the vector is determined by the relation $|\overrightarrow{PQ}| = \sqrt{2\sigma(P,Q)}$. The scalar product of two vectors $\overrightarrow{P_0P_1}$, $\overrightarrow{P_0P_2}$, having a common origin, is given by the relation

$$\left(\overrightarrow{P_0P_1}.\overrightarrow{P_0P_2}\right) = \sigma(P_0,P_1) + \sigma(P_0,P_2) - \sigma(P_1,P_2), \tag{2.4}$$

It represents a formula of the cosine theorem for the triangle with vertices at points P_0, P_1, P_2, written in terms of the world function σ. The relation (2.4) may be interpreted as a definition of the scalar product, made without a reference to linear space. To stress independence on the linear space, the difinition (2.4) will be referred to as the scalar σ-product.

Note that the scalar σ-product can be determined for vectors $\overrightarrow{P_0P_1}$, $\overrightarrow{Q_0Q_1}$, having different origins. In this case the relation (2.4) takes the form

$$\left(\overrightarrow{P_0P_1}.\overrightarrow{Q_0Q_1}\right) = \sigma(P_0,Q_1) + \sigma(Q_0,P_1) - \sigma(P_0,Q_0) - \sigma(P_1,Q_1), \tag{2.5}$$

Dimension of the space is its another important property, determined by the maximal number of linearly independent vectors. For n vectors $\overrightarrow{P_0P_i}$ $\quad i = 1,2,...n$ of the Euclidean space were linearly independent, it is necessary and sufficient that the Gram's determinant vanishes

$$F_n(\mathcal{P}^n) = 0, \qquad \mathcal{P}^n \equiv \{P_0, P_1, \ldots P_n\} \subset \Omega, \tag{2.6}$$

where

$$F_n(\mathcal{P}^n) \equiv \det \left\| \left(\overrightarrow{P_0P_i}.\overrightarrow{P_0P_k}\right) \right\|, \qquad i,k = 1,2,\ldots n \tag{2.7}$$

It follows from (2.4) and (2.7), that linear independence of vectors can be defined in terms of the world function without a reference to linear space.

There exist necessary and sufficient conditions that the σ-space $V = \{\sigma, \Omega\}$ is n-dimensional Euclidean space. They state that there exists a set of $(n+3)$-point σ-subspaces $\{\sigma, \mathcal{P}^{n+2}\} \subset V$, whose world function satisfies some relations. σ-subspaces of this set have $n+1$ common fixed points $\mathcal{P}^n$. Other two points P_{n+1}, P_{n+2} are arbitrary points of V and running points of σ-subspaces $\{\sigma, \mathcal{P}^{n+2}\}$ of this set. Corresponding theorem was proved in [14].

It follows from the theorem that information, contained in the world function, is sufficient for construction of rigorous geometry which is as rich in content as the Euclidean geometry. Any choice of the world function, satisfying the condition (2.2), corresponds to some geometry. This choice is restricted neither continuity condition, nor condition of geometry degeneracy.

All known geometries (Riemannian, Euclidean) are degenerate geometries. Non-degenerate geometry is a new type of geometry, and the concept of degeneracy merits a special discussion. Two vectors $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0R}$, having common origin are called collinear $\overrightarrow{P_0P_1}||\overrightarrow{P_0R}$, if they are linearly dependent, i.e. if they satisfy the condition

$$F_2(P_0, P_1, R) = \begin{vmatrix} \left(\overrightarrow{P_0P_1}.\overrightarrow{P_0P_1}\right) & \left(\overrightarrow{P_0P_1}.\overrightarrow{P_0R}\right) \\ \left(\overrightarrow{P_0R}.\overrightarrow{P_0P_1}\right) & \left(\overrightarrow{P_0R}.\overrightarrow{P_0R}\right) \end{vmatrix} = 0, \tag{2.8}$$

which can be written in the form

$$\cos^2\vartheta = \frac{\left(\overrightarrow{P_0P_1}.\overrightarrow{P_0R}\right)^2}{\left|\overrightarrow{P_0P_1}\right|^2\left|\overrightarrow{P_0R}\right|^2} = 1 \tag{2.9}$$

The last relation means that the angle ϑ between vectors is equal to either 0, or π.

Let vector $\overrightarrow{P_0P_1}$ be given in n-dimensional Euclidean space, and $\overrightarrow{P_0R}$ is a vector collinear to $\overrightarrow{P_0P_1}$. Then the set $\mathcal{T}_{P_0P_1}$ of points R

$$\mathcal{T}_{P_0P_1} = \{R | F_2(P_0, P_1, R) = 0\} \tag{2.10}$$

is a straight line, passing through the points P_0, P_1, or, what is the same, it is a straight line, passing through the point P_0, parallel to vector $\overrightarrow{P_0P_1}$. On the other hand, at the arbitrary world function the set $\mathcal{T}_{P_0P_1}$, determined by one equation, describes, generally, $(n-1)$-dimensional surface. The fact, that in the case of Euclidean space this $(n-1)$-dimensional surface degenerates to one-dimensional line, is connected with the special form of the world function of the Euclidean space. Even small change of the world function either removes degeneration, and the one-dimensional line turns to hallow $(n-1)$-dimensional tube, enveloping the straight, or increases degeneration, and the one-dimensional line degenerates to two points P_0, P_1. Thus, in the non-degenerate geometry the straights are substituted by hallow tubes. This fact justifies the name of geometry – tubular geometry, or briefly T-geometry.

If there is no continuous coordinate system on the set Ω, it is difficult to determine whether the set (2.10) is a one-dimensional line. In this case for estimation of the degeneracy degree one can consider intersection between the tube $\mathcal{T}_{P_0P_1}$ and the sphere of radius $r = \sqrt{2\sigma(P_0, Q)}$, which passes through the point Q and has its center at the point P_0

$$\mathcal{S}\left(P_0, Q\right) = \left\{R|\sigma\left(P_0, Q\right) = \sigma\left(P_0, R\right)\right\} \tag{2.11}$$

In the case of Euclidean space the intersection $\mathcal{T}_{P_0P_1} \cap \mathcal{S}\left(P_0, Q\right)$ consists of two points Q_1, Q_2. The vector $\overrightarrow{P_0Q_1}$ is parallel to the vector $\overrightarrow{P_0P_1}$, ($\overrightarrow{P_0Q_1} \uparrow\uparrow \overrightarrow{P_0P_1}$), and vector $\overrightarrow{P_0Q_2}$ is antiparallel to the vector $\overrightarrow{P_0P_1}$, ($\overrightarrow{P_0Q_2} \downarrow\uparrow \overrightarrow{P_0P_1}$).

In other words, at the degenerate geometry at any point P_0 there is only one vector of given length, which is parallel to the given vector $\overrightarrow{P_0P_1}$, and only one vector of given length, which is antiparallel $\overrightarrow{P_0P_1}$.

In the case of non-degenerate geometry the intersection $\mathcal{T}_{P_0P_1} \cap \mathcal{S}\left(P_0, Q\right) = \omega_+ \cup \omega_-$ is divided into two such subsets ω_+ и ω_-, that the points $Q_1 \in \omega_+$ determine vectors $\overrightarrow{P_0Q_1}$, $\overrightarrow{P_0Q_1} \uparrow\uparrow \overrightarrow{P_0P_1}$, and points $Q_2 \in \omega_-$ determine vectors $\overrightarrow{P_0Q_2}$, $\overrightarrow{P_0Q_2} \downarrow\uparrow \overrightarrow{P_0P_1}$. Each of subsets ω_+ and ω_- contains many points. This corresponds to the fact that in the non-degenerate geometry at any point there are many vectors of given length $r = \sqrt{2\sigma(P_0, Q)}$, which are parallel (antiparallel) to the given vector $\overrightarrow{P_0P_1}$.

Non-degeneracy of the space-time geometry, i.e. existence of many timelike vectors of fixed length parallel to a given timelike vector at any point, is a reason of the free particle stochastic motion. To show this, let us consider the event space, where at any point P_0 there are many timelike vectors $\mathbf{P}_0\mathbf{P}_1$ of the given length $|\mathbf{P}_0\mathbf{P}_1| = \mu$, parallel to the given timelike vector $\mathbf{P}_0\mathbf{Q}_1$. Note that in the Minkowski geometry there is only one timelike vector $\mathbf{P}_0\mathbf{P}_1$ of the given length, parallel to timelike vector $\mathbf{P}_0\mathbf{Q}_1$.

In the Minkowski space-time the particle world line can be approximated by a broken line, consisted of rectilinear links of the same length. Then the joining points $\ldots P_{i-1}, P_i, P_{i+1}$ are such, that the vector $\mathbf{P}_i\mathbf{P}_{i+1}$ is proportional to the particle momentum, and its length $|\mathbf{P}_i\mathbf{P}_{i+1}| = \mu$ is proportional to its mass $m = b\mu$, where $b \approx 10^{-17}$g/cm is some universal constant and μ is geometric particle mass. If the particle is free, according to the Galilean law of inertia the adjacent links are parallel, i.e. the vector $\mathbf{P}_i\mathbf{P}_{i+1}$ is parallel to the vector $\mathbf{P}_{i+1}\mathbf{P}_{i+2}$, $i = 0, \pm 1, \pm 2, \ldots$.

Let us define the world line of a free particle as a broken line with parallel links. Then in the Minkowski space-time position of all links is determined single-valuedly, provided one fixes position of one link. Determinism of the broken line means determinism of the particle world line, what conditions determinism of the free particle motion. In T-geometry, where there are many vectors, parallel to the given one, fixing of a position of one link does not lead to single-valued determination of the remaining links position. It means that in such a space-time the free particle motion is stochastic, although the geometry in itself is deterministic.

In general, the T-geometry is non-Riemannian geometry. In some cases, when the set of vectors of fixed length, parallel to a given vector, degenerates into one vector, T-geometry degenerates into a Riemannian geometry. For instance, being a pseudo-Riemannian geometry, the Minkowski geometry is a special case of T-geometry.

Thus, T-geometry is rather general construction, having such an important property as non-degeneracy. The non-degeneracy of a geometry is a new unknown earlier property of geometry. Importance of this property is comparable with such important properties of geometry as continuity and homogeneity. It seems rather enigmatic, why such a simple and general construction as T-geometry was not known earlier. Why was such a property of geometry as non-degeneracy not known before the end of XX century? Absence of T-geometry in the list of possible geometries does not allow to solve correctly the microparticle stochasticity problem.

Absence of T-geometry at the beginning of the XX century even in the form of a speculative construction is explained, apparently, by existence of a discriminator, used at the geometry construction. The point is that, constructing geometry in terms of some fundamental concepts (for instance, such as dimension, coordinate system, distance, curve, etc.), one discriminates automatically those geometries, which are incompatible with at least one of these fundamental concepts. For instance, the Cartesian coordinate system is a discriminator of inhomogeneous (Riemannian) geometry. That is the reason why a Riemannian geometry is constructed in arbitrary (not Cartesian) coordinate system with all its attributes in the form of Christoffel symbols and covariant derivatives. If one declares that a Riemannian geometry is described in the Cartesian coordinate system, where the metric tensor g_{ik} =const, the nonhomogeneity of geometry is discriminated, and only homogeneous (Euclidean) geometry remains. In XIX century the Cartesian coordinate system was considered as something immanent to geometry in itself, and apparently, this circumstance stipulates prejudice of many mathematicians of XIX century against the Riemannian geometry.

The concept of a curve is a discriminator of non-degenerate geometries. This fact was realized quite recently [15]. One attempted to generalize the Riemannian and metric geometries. One attempted to generalize the metric geometry, removing the triangle axiom. Such a geometry is referred to as distant geometry. K. Menger [16] and L. Blumenthal [17] attempted to construct distant geometry. But metric geometry, or distant geometry, constructed without a use of the concept of a curve appears to be very poor geometries, because they contained few geometrical objects. To obtain more rich in content geometry, one uses the concept of a curve. Essentially this discriminates any possibility of an effective application of the triangle axiom remove, and a non-degenerate geometry cannot appear.

Thus, on the one hand, at construction of a geometry a use of the concept of the curve discriminates its non-degeneracy automatically. On the other hand, the concept of the curve is necessary for constructing geometrical objects, and it is not clear, what can substitute this very important concept of Riemannian geometry. Now the most of mathematician consider the concept of the curve (2.1) as a necessary attribute of any geometry. This is an origin of their prejudice against the T-geometry, and reminds the situation of the end of XIX century, when, considering the Cartesian coordinate system to be an attribute of any geometry, the most of mathematician had prejudice against the Riemannian geometry.

In the Riemannian geometry the concept of a continuous curve has two base functions: (1) the curve is a fundamental concept, used at construction of a geometry, (2) the curve is a tool for construction of geometrical objects. Geometrical object is some set $\mathcal{O} \subset \Omega$ of points. Usually it is a continual set. In T-geometry all geometrical relations are expressed via the world function and the first function of the curve appears to be not claimed.

The second function of the curve is used in the Riemannian geometry, where a geometrical object is build usually as a trace of motion of a more simple geometrical object. For instance a one-dimensional curve $\mathcal{L}$ is considered to be a trace of a moving point. It is described by the continuous mapping (2.1). A two-dimensional surface $\mathcal{S}$ is considered to be trace of moving one-dimensional curve. It is described by a continuous mapping

$$\mathcal{S}: \qquad [0,1] \times [0,1] \to \Omega \tag{2.12}$$

etc. Such a construction of a geometrical object contains a continuous mapping of the type continuum $\to$ continuum, which is very difficult for investigations, because before investigations of such mappings one needs at least to label them. But even the problem of labelling of all possible mappings of the type continuum $\to$ continuum is very complicated because of large power of the set of such mappings.

To investigate mappings of such a kind and to use them in geometry, one needs to separate only small part of them, imposing constraints on properties of the space Ω (for instance such constraints as continuity and topological properties). These constraints reduce the geometry generality in incontrollable way.

In T-geometry a geometrical object $\mathcal{O}$ is described by means of the skeleton-envelope method. Any geometrical object $\mathcal{O}$ is considered to be a set of intersection and joins of elementary geometrical objects (EGO).

Elementary geometrical object $\mathcal{E}$ is described as a set of zeros of some function

$$f_{\mathcal{P}^n}: \qquad \Omega \to \mathbb{R}, \qquad \mathcal{P}^n \equiv \{P_0, P_1, ...P_n\} \subset \Omega \tag{2.13}$$

It is represented in the form

$$\mathcal{E} = \mathcal{E}_f\left(\mathcal{P}^n\right) = \left\{R | f_{\mathcal{P}^n}\left(R\right) = 0\right\} \tag{2.14}$$

The finite set $\mathcal{P}^n \subset \Omega$ will be referred to as the skeleton of elementary geometrical object $\mathcal{E} \subset \Omega$. The continual set $\mathcal{E} \subset \Omega$ is referred to as the envelope of the skeleton $\mathcal{P}^n$. The function $f_{\mathcal{P}^n}$, determining the elementary geometrical object (EGO) is a function of parameters $\mathcal{P}^n \subset \Omega$ and of the running point $R \in \Omega$. The function $f_{\mathcal{P}^n}$ is supposed to be algebraic function of several arguments $w = \{w_1, w_2, ...w_s\}$, $s = (n+2)(n+1)/2$. Each of arguments w_k is the world function $w_k = \sigma\left(Q_k, L_k\right)$ of two arguments $Q_k, L_k \in \{R, \mathcal{P}^n\}$, belonging either to the skeleton $\mathcal{P}^n$, or to the running point R.

For instance,

$$\mathcal{S}(P_0, P_1) = \left\{R | f_{P_0 P_1}\left(R\right) = 0\right\}, \qquad f_{P_0 P_1}\left(R\right) = \sqrt{2\sigma\left(P_0, P_1\right)} - \sqrt{2\sigma\left(P_0, R\right)} \tag{2.15}$$

is a sphere, passing through the point P_1 and having its center at the point P_0. Ellipsoid $\mathcal{EL}$, passing through the point P_2 and having the focuses at points P_0, P_1 ($P_0 \neq P_1$) is described by the relation

$$\mathcal{EL}(P_0, P_1, P_2) = \{R | f_{P_0 P_1 P_2}(R) = 0\}, \tag{2.16}$$

where

$$f_{P_0 P_1 P_2}(R) = \sqrt{2\sigma(P_0, P_2)} + \sqrt{2\sigma(P_1, P_2)} - \sqrt{2\sigma(P_0, R)} - \sqrt{2\sigma(P_1, R)} \tag{2.17}$$

If focuses P_0, P_1 coincide ($P_0 = P_1$), the ellipsoid $\mathcal{EL}(P_0, P_1, P_2)$ degenerates into a sphere $\mathcal{S}(P_0, P_2)$. If the points P_1, P_2 coincide ($P_1 = P_2$), the ellipsoid $\mathcal{EL}(P_0, P_1, P_2)$ degenerates into a segment of a straight line $\mathcal{T}_{[P_0 P_1]}$ between the points P_0, P_1.

$$\mathcal{T}_{[P_0 P_1]} = \mathcal{EL}(P_0, P_1, P_1) = \{R | f_{P_0 P_1 P_1}(R) = 0\}, \tag{2.18}$$

$$f_{P_0 P_1 P_1}(R) = S_2(P_0, R, P_1) \equiv \sqrt{2\sigma(P_0, P_1)} - \sqrt{2\sigma(P_0, R)} - \sqrt{2\sigma(P_1, R)} \tag{2.19}$$

Another functions f generate another envelopes of elementary geometrical objects for the given skeleton $\mathcal{P}^n$.

For instance, the set of two points $\{P_0, P_1\}$ forms a skeleton not only for the tube $\mathcal{T}_{P_0 P_1}$, but also for the segment $\mathcal{T}_{[P_0 P_1]}$ of the tube (straight) (2.18), and for the tube ray $\mathcal{T}_{[P_0 P_1}$, which is defined by the relation

$$\mathcal{T}_{[P_0 P_1} = \{R | S_2(P_0, P_1, R) = 0\} \tag{2.20}$$

where the function S_2 is defined by the relation (2.19).

Any mapping (2.13) of the type continuum$\rightarrow$continuum is given and fixed, because the function $f_{\mathcal{P}^n}$ is a known function of its argument and parameters $\mathcal{P}^n$. Any such function $f_{\mathcal{P}^n}$ determines some class of elementary geometrical objects (EGO). A set of such functions is n-parametric set of functions. To build and investigate this class of EGOs, one does not need to impose any constraints on the set Ω, or on the world function. Thus, the skeleton-envelope method of building of geometrical objects deals only with investigation of comparatively simple mappings of the form

$$m_n: \qquad I_n \rightarrow \mathbb{R}, \qquad I_n = \{0, 1, ..., n\} \tag{2.21}$$

and it does not need imposition of constraints on the set Ω. Such mappings are connected with construction and investigation of EGO skeletons. Investigating a skeleton, one investigates simultaneously corresponding classes of EGOs, because at the fixed function (2.13) any EGO is connected rigidly with its skeleton.

Sometimes, investigating a geometrical object, it is sufficient to investigate its skeleton, which a countable set of points and can be investigated easier, than the continual set of points, forming the geometrical object in itself. For instance, analyzing reasons of the free particle stochasticity, we have restricted ourselves to investigation of the skeleton $\ldots P_{i-1}, P_i, P_{i+1}, \ldots$ of the broken tube. It simplifies our analysis essentially.

The skeleton-envelope method simplifies essentially the problem of geometrical object building. It allows to separate the problem into informal problem of the skeleton construction and a formal procedure of the envelope construction, using its skeleton. Taking in to account that the problem of the envelope construction in accord with its skeleton is formalized, one can consider the envelope of the geometrical object to be an attribute of its skeleton.

3 Dynamic conception of statistical description

There are numerous attempts of considering the quantum description of microparticle motion as a result of statistical description of their stochastic motion [8, 9]. As a rule they are founded on the probability theory which is not suitable for description of relativistic stochastic motion. But stochastic motion, generated by the quantum stochasticity is relativistic. Inapplicability of the probability theory for description of relativistic stochastic processes is connected with the fact, that the concept of probability density supposes a possibility of the event space separation to sets of simultaneous independent events. It is impossible in the relativistic theory, where the absolute simultaneity is absent. Formally this is displayed in the fact, that at the description of stochastic relativistic particle the object of statistical description is such a lengthy physical object as world line in the space-time, whereas in the non-relativistic case the object of the statistical description is the pointlike particle in the three-dimensional space.

Numerous unsuccessful attempts of representing the quantum mechanics as a result of the probabilistic statistical description had discredited the idea in itself to reduce the quantum mechanical description to the statistical description of randomly moving particles. Now many serious researchers consider sceptically a possibility of the quantum mechanics reduction to the statistical description of stochastically moving particles, although the quantum mechanics is considered to be a statistical theory.

Strictly, the term "statistical description" means a description, containing many similar objects, a reference to a probability concept or probability density being unnecessary. Moreover, such a reference is undesirable, as far as the statistical description, founded on the concept of probability, is restricted by a possibility of the probability introduction. Dynamic conception of statistical description seems to be more effective, although it is less informative. Essence of the dynamic conception of statistical description is formulated as follows [18, 21].

Let $\mathcal{S}_{\rm st}$ be a stochastic system, i.e. dynamic system[1], experiments with which are irreproducible, and for which dynamic equations do not exist. For instance, let $\mathcal{S}_{\rm st}$ be an electron flying through a narrow slit in a diaphragm and hitting the screen

[1]Conventional terminology contains only terms "stochastic system" and "dynamic system". The concept collective with respect to the two terms is absent. For this reason the term "dynamic system" is used as a collective term with respect to terms "non-deterministic dynamic system" (instead of customary "stochastic system") and "deterministic dynamic system" (instead of customary "dynamic system").

at some point x_1. Another experiment, produced with an electron, prepared in the same way, leads to its hit at another point x_2, which does not coincide with x_1, generally. In other words the electron $\mathcal{S}_{\mathrm{st}}$ is a stochastic system, and experiments with it are irreproducible.

If one produces N, $(N \to \infty)$ experiments with $\mathcal{S}_{\mathrm{st}}$, the obtained distribution of electrons over the screen can be reproducible. It can be reproduced in other series of N_1, $(N_1 \to \infty)$ experiments. It means that the dynamic system $\mathcal{E}\left[N, \mathcal{S}_{\mathrm{st}}\right]$, consisting of many independent non-deterministic (stochastic) dynamic systems $\mathcal{S}_{\mathrm{st}}$, is a deterministic system, experiments with which are reproducible, and for which there are dynamic equations, although dynamic equations do not exist for $\mathcal{S}_{\mathrm{st}}$. The dynamic system $\mathcal{E}\left[\mathcal{S}\right] = \mathcal{E}\left[\infty, \mathcal{S}\right]$ is known as a statistical ensemble, and dynamic systems $\mathcal{S}$, constituting it are referred to as the statistical ensemble elements. Elements of the ensemble can be deterministic dynamic systems $\mathcal{S}_{\mathrm{d}}$, as well as stochastic ones $\mathcal{S}_{\mathrm{st}}$. Being a dynamic system, the statistical ensemble $\mathcal{E}$ may be an element of other statistical ensemble $\mathcal{E}'$, which in turn may be an element of the statistical ensemble $\mathcal{E}''$, etc.

Idea of the dynamic conception of the statistical description lies in the fact that it is impossible to investigate the stochastic system $\mathcal{S}_{\mathrm{st}}$, because of irreproducibility of experiments with it, but one can investigate the statistical ensemble $\mathcal{E}\left[N, \mathcal{S}_{\mathrm{st}}\right]$ as a deterministic dynamic system, and on the basis of these results one can make some conclusions on the properties of stochastic system $\mathcal{S}_{\mathrm{st}}$.

Why does the set $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$ of many independent stochastic systems $\mathcal{S}_{\mathrm{st}}$ turn to a deterministic dynamic system? Apparently, because that typical features are summed or averaged, but random ones compensate themselves. Is this so or not, but it is evident that $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$ is a deterministic dynamic system, and it is a basis of the statistical description. In any case one can consider this statement as a principle, which will be referred to as statistical principle [18, 21].

The statistical ensemble have several important properties. Using them, one can transform statistical description in such a way, that it loses its statistical features and will be perceived as purely dynamical. Such a description stops to resemble a statistical description, understood as a probabilistic statistical description. There are three basic properties of the statistical description.

(1) Properties of the statistical ensemble do not depend on the number N of its elements, if this number is enough large, i.e. $N \to \infty$.

(2) A statistical ensemble may be an element of other statistical ensemble.

(3) In the simplest case of pure ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$ of stochastic systems $\mathcal{S}_{\mathrm{st}}$ coincides with the dynamic system $\mathcal{E}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right] = \mathcal{S}\left[\mathcal{S}_{\mathrm{d}}\right]$, consisting of many interacting deterministic systems $\mathcal{S}_{\mathrm{d}}$. The form of interaction of deterministic systems $\mathcal{S}_{\mathrm{d}}$ is determined by the stochasticity character of stochastic systems $\mathcal{S}_{\mathrm{st}}$. This allows to label the stochasticity character by the form of interaction and to reduce description of stochasticity to interaction of deterministic systems.

Let us start from the first property, which admits to normalize the ensemble state. Let the stochastic system $\mathcal{S}_{\mathrm{st}}$ represent a microparticle, whose state is described by its position $\mathbf{x}$ and momentum $\mathbf{p}$. Then at large enough N the ensemble

$\mathcal{E}\left[N,\mathcal{S}_{\rm st}\right]$ represents a distributed fluidlike dynamic system. There are an action for such a system $\mathcal{A}\left[N,\varphi,\boldsymbol{\xi}\right]$, where φ=$\varphi\left(t,\mathbf{x}\right)$, and $\boldsymbol{\xi}=\boldsymbol{\xi}\left(t,\mathbf{x}\right)$ are dynamic variables, describing the fluid state. The action for the statistical ensemble has the property

$$\mathcal{A}\left[aN,\varphi,\boldsymbol{\xi}\right]=a\mathcal{A}\left[N,\varphi,\boldsymbol{\xi}\right],\qquad a=\mathrm{const},\qquad a>0. \tag{3.1}$$

It generates dynamic equations and the energy-momentum tensor T_{k}^{i}. Besides, for the dynamic system $\mathcal{E}\left[N,\mathcal{S}_{\rm st}\right]$ one can introduce the particle density j^{0} and the particle flux density j^{α}, $\alpha=1,2,3$. Due to relation (3.1) the ensemble properties do not depend on the number N of its elements. But one may consider that this property is fulfilled for any N and, setting formally $N=1$, consider an ensemble, consisting of one element. More exactly it means that, if $\mathcal{A}\left[N,\varphi,\boldsymbol{\xi}\right]$ is the action for the statistical ensemble $\mathcal{E}\left[N,\mathcal{S}_{\rm st}\right]$, the action

$$\mathcal{A}_{\rm av}\left[\varphi,\boldsymbol{\xi}\right]=\lim_{N\rightarrow\infty}\frac{\mathcal{A}\left[N,\varphi,\boldsymbol{\xi}\right]}{N}$$

is the action for $\left\langle \mathcal{S}_{\rm st}\right\rangle$. Such a statistical ensemble will be referred to as average dynamic system $\left\langle \mathcal{S}_{\rm st}\right\rangle$. Thus, $\left\langle \mathcal{S}_{\rm st}\right\rangle=\mathcal{E}\left[N,\mathcal{S}_{\rm st}\right]_{N=1}$. The average dynamic system $\left\langle \mathcal{S}_{\rm st}\right\rangle$, constructed on the basis of the stochastic system $\mathcal{S}_{\rm st}$, is a deterministic dynamic system, for which a value of any physical quantity q can be interpreted as the mean value $\left\langle q\right\rangle$ of the same quantity q for the stochastic system $\mathcal{S}_{\rm st}$. The average dynamic system $\left\langle \mathcal{S}_{\rm st}\right\rangle$ is a deterministic system, having dynamic equations. Using these equations, one can calculate evolution of the mean value $\left\langle q\right\rangle$ of any physical quantity q for the stochastic system $\mathcal{S}_{\rm st}$.

As a result of such approach the statistical description of stochastic system $\mathcal{S}_{\rm st}$ reduces to consideration of a deterministic system $\left\langle \mathcal{S}_{\rm st}\right\rangle$, but the circumstance that $\left\langle \mathcal{S}_{\rm st}\right\rangle$ remains to be a statistical ensemble may drop out of consideration.

Thus, one can consider simultaneously two dynamic systems $\mathcal{S}_{\rm st}$ and $\left\langle \mathcal{S}_{\rm st}\right\rangle$. The system $\mathcal{S}_{\rm st}$ is concentrated, but stochastic. The system $\left\langle \mathcal{S}_{\rm st}\right\rangle$ is distributed, but deterministic. They cannot be confused, and one should use different terms and designations for them. The state of the distributed system $\left\langle \mathcal{S}_{\rm st}\right\rangle$ can be described by the wave function ψ (it will be shown below). It is this system, that is considered usually in quantum mechanics. It is considered as a dynamic system, describing a real physical particle. As for the stochastic system $\mathcal{S}_{\rm st}$, it does not appear in the quantum mechanics technique. It may be disregarded, until one deals only with dynamics, where only the average dynamic system $\left\langle \mathcal{S}_{\rm st}\right\rangle$ appears. But discussing the measurement processes, such a disregard of the stochastic system $\mathcal{S}_{\rm st}$ is inadmissible, because there are several different measurement procedures, where the systems $\mathcal{S}_{\rm st}$ and $\left\langle \mathcal{S}_{\rm st}\right\rangle$ play different roles.

Unfortunately, in quantum mechanics almost never one differs systems $\mathcal{S}_{\rm st}$ and $\left\langle \mathcal{S}_{\rm st}\right\rangle$. Furthermore, considering the measurement process, one uses the same term for them, what is inadmissible even from viewpoint of usual logic. Besides, at such an "generalized terminology" different measurement procedures merge into one procedure, which is interpreted by different researchers in different ways, depending on, which of two systems $\mathcal{S}_{\rm st}$ or $\langle\mathcal{S}_{\rm st}\rangle$ is taken into account at this time. Numerous paradoxes (wave function collapse, Schrödinger cat paradox, Einstein – Podolski – Rosen paradox [22], etc.) are corollaries of such a consideration, although in reality there are no paradoxes. There is only confusion, when the same term is used for two different objects. Note, that paradoxes arise only at the discussion of the measurement process, where both systems $\mathcal{S}_{\rm st}$ and $\langle\mathcal{S}_{\rm st}\rangle$ appear. At the discussion of dynamics, where only the system $\langle\mathcal{S}_{\rm st}\rangle$ appears, there are no paradoxes.

The second property of the statistical ensemble means that one statistical ensemble may be an element of the other one. Such an organization of a statistical description is useful in the following relation. If elements of a statistical ensemble are deterministic dynamic systems $\mathcal{S}_{\rm d}$, i.e. such dynamical systems, for which there are dynamic equations, a construction of dynamic equations for the statistical ensemble $\mathcal{E}\left[\infty,\mathcal{S}_{\rm d}\right]$ is a formal procedure, which can be carried out easily, provided dynamic equations for $\mathcal{S}_{\rm d}$ are known. If elements of the statistical ensemble are nondeterministic dynamic systems $\mathcal{S}_{\rm st}$, i.e. such dynamic systems, for which there are no dynamic equations, construction of dynamic equations for the statistical ensemble $\mathcal{E}\left[\infty,\mathcal{S}_{\rm st}\right]$ is a complicated informal procedure.

Let us explain this in an example of a description of deterministic particle $\mathcal{S}_{\rm d}$, whose motion is described by the Hamilton function $H(t,\mathbf{x},\mathbf{p},)$, where $\mathbf{x}=\{x^\alpha\}\quad \alpha=1,2,...n$, are generalized coordinates and $\mathbf{p}=\{p_\alpha\}\quad \alpha=1,2,...n$ is a generalized momentum. The most general statistical ensemble $\mathcal{E}_{\rm gen}\left[\mathcal{S}_{\rm d}\right]$ is described usually by the distribution function $F(t,\mathbf{x},\mathbf{p})$, satisfying the Liouville equation. $\mathcal{E}_{\rm gen}\left[\mathcal{S}_{\rm d}\right]$ may be considered to be a statistical ensemble $\mathcal{E}_{\rm gen}\left[\mathcal{E}_{\rm p}\right]$, whose elements are statistical ensembles $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm d}\right]$ of special type, whose elements are dynamic systems $\mathcal{S}_{\rm d}$.

Following von Neumann [23], we shall refer to the statistical ensemble of special type $\mathcal{E}_{\rm p}\left[\mathcal{S}\right]$ as a pure ensemble, because it admits a description in terms of the wave function. (It will be shown below). By definition the pure statistical ensemble is such a statistical ensemble $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm d}\right]$, which is described by the distribution function

$$F_{\rm p}(t,\mathbf{x},\mathbf{p})=\rho(t,\mathbf{x})\delta(\mathbf{p}-\mathbf{P}(t,\mathbf{x})) \tag{3.2}$$

It satisfies a system of dynamic equations written for independent variables $\{t,\mathbf{x}\}$, i.e. in the configuration space of coordinates $\mathbf{x}$. In other words, the pure statistical ensemble is described in terms of several functions $\rho(t,\mathbf{x})$ and $\mathbf{P}(t,\mathbf{x})$ of only argument $\mathbf{x}$ instead of a description in terms of one function of arguments $\mathbf{x},\mathbf{p}$. The system of dynamic equations for these functions is derived as a result of the substitution (3.2) into the Liouville equation for the distribution function $F(t,\mathbf{x},\mathbf{p})$ and subsequent integration with respect to variable $\mathbf{p}$ with the weight multipliers 1 and $\mathbf{p}$.

Now if the particle is a stochastic one $\mathcal{S}_{\rm st}$, an informal procedure is only construction of the statistical ensemble $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm st}\right]$ with nondeterministic elements $\mathcal{S}_{\rm st}$, (i.e. the transition $\mathcal{S}_{\rm st}\rightarrow\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm st}\right]$). As far as $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm st}\right]$ is a deterministic dynamic system, a construction of the statistical ensemble $\mathcal{E}_{\rm gen}\left[\mathcal{E}_{\rm p}\right]$, whose elements are the statistical ensembles $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm st}\right]$ (i.e. the transition $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm st}\right]\rightarrow\mathcal{E}_{\rm gen}\left[\mathcal{E}_{\rm p}\right]$), is a comparatively simple formal procedure. Thus, only the transition $\mathcal{S}_{\mathrm{st}} \rightarrow \mathcal{E}_{\mathrm{p}}\left[\mathcal{S}_{\mathrm{st}}\right]$ is conceptual. The most attention will be concentrated on this procedure.

The state $F(t, \mathbf{x}, \mathbf{p})$ of an ensemble of general form $\mathcal{E}_{\text{gen}}\left[\mathcal{S}_{\mathrm{d}}\right]$ evolves according to the Liouville equation

$$\mathcal{E}_{\text{gen}}\left[\mathcal{S}_{\mathrm{d}}\right]: \qquad \frac{\partial F}{\partial t}+\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial H}{\partial p_{\alpha}} F\right)-\frac{\partial}{\partial p_{\alpha}}\left(\frac{\partial H}{\partial x^{\alpha}} F\right)=0 \tag{3.3}$$

where $H=H(t, \mathbf{x}, \mathbf{p})$ is the Hamilton function for the dynamic system $\mathcal{S}_{\mathrm{d}}$. A summation is made over repeated Greek indices from 1 to n.

Dynamic equations for the statistical ensemble of a special form $\mathcal{E}_{\mathrm{p}}\left[\mathcal{S}_{\mathrm{d}}\right]$ have the form

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x^{\alpha}}\left[\rho \frac{\partial H}{\partial p_{\alpha}}(t, \mathbf{x}, \mathbf{p})\right]_{\mathbf{p}=\mathbf{P}}=0 \tag{3.4}$$

$$\frac{\partial}{\partial t}\left(\rho P_{\beta}\right)+\frac{\partial}{\partial x^{\alpha}}\left(\rho P_{\beta}\left[\frac{\partial H}{\partial p_{\alpha}}(t, \mathbf{x}, \mathbf{p})\right]_{\mathbf{p}=\mathbf{P}}\right)+\rho \frac{\partial H}{\partial x^{\beta}}(t, \mathbf{x}, \mathbf{P})=0, \qquad \beta=1,2, \ldots n \tag{3.5}$$

Let us interpret ρ as a particle density, and $\mathbf{v}=\partial H / \partial \mathbf{p}$ as a generalized velocity. Then the equation (3.4) is regarded as a continuity equation, and equations (3.5) may be interpret as generalized Euler equations for some fluid without pressure.

The system of equations (3.4), (3.5) is closed, but it is not complete, and it cannot be obtained from the variational principle. Let us add to the generalized Euler equations the equations

$$\frac{d x^{\beta}}{d t}=\frac{\partial H}{\partial P_{\beta}}(t, \mathbf{x}, \mathbf{P}), \qquad \beta=1,2, \ldots n \tag{3.6}$$

describing a particle motion in a given velocity field $\mathbf{v}=\partial H / \partial \mathbf{P}$. These equations can be rewritten in the form, known in hydrodynamics as Lin constraints [24]

$$\frac{\partial \xi_{\beta}}{\partial t}+\frac{\partial H}{\partial P_{\alpha}}(t, \mathbf{x}, \mathbf{P}) \partial_{\alpha} \xi_{\beta}=0, \qquad \beta=1,2, \ldots n, \qquad \partial_{k} \equiv \frac{\partial}{\partial x^{k}}, \qquad k=0,1, \ldots n \tag{3.7}$$

Here $\boldsymbol{\xi}(t, \mathbf{x})=\left\{\xi_{\alpha}(t, \mathbf{x})\right\}, \quad \alpha=1,2, \ldots n$ are n independent integrals of equations (3.6).

The system of $2n+1$ equations (3.4), (3.5), (3.7) forms a complete system of dynamic equations, describing evolution of the pure statistical ensemble $\mathcal{E}_{\mathrm{p}}\left[\mathcal{S}_{\mathrm{d}}\right]$. It can be integrated and reduced to a system of $n+2$ equations for $n+2$ variables $\rho, \varphi, \boldsymbol{\xi}$

$$b_{0}\left[\partial_{0} \varphi+g^{\alpha}(\boldsymbol{\xi}) \partial_{0} \xi_{\alpha}\right]+H(\mathbf{x}, \mathbf{P})=0 \tag{3.8}$$

$$\partial_{0} \rho+\partial_{\alpha}\left(\rho \frac{\partial H}{\partial P_{\alpha}}(t, \mathbf{x}, \mathbf{P})\right)=0 \tag{3.9}$$

$$\frac{\partial \xi_{\beta}}{\partial t}+\frac{\partial H}{\partial P_{\alpha}}(t, \mathbf{x}, \mathbf{P}) \partial_{\alpha} \xi_{\beta}=0, \qquad \beta=1,2, \ldots n \tag{3.10}$$

where φ is a new variable, and $\mathbf{P}$ is expressed via n arbitrary functions $\mathbf{g}(\boldsymbol{\xi}) = \{g^{\alpha}(\boldsymbol{\xi})\}, \quad \alpha = 1, 2, ...n$ of argument $\boldsymbol{\xi}$.

$$P_{\beta} = b_0 \left(\partial_{\beta}\varphi + g^{\alpha}(\boldsymbol{\xi})\right) \partial_{\beta}\xi_{\alpha}, \qquad \beta = 1, 2, ...n \tag{3.11}$$

The validity of the statement on integration of the system (3.4) (3.5), (3.7) can be verified either by means of a direct substitution of (3.11) into (3.5), or by use of technique, developed in [25]. b_0 is an arbitrary constant, which may be incorporated in the variable φ and arbitrary functions $\mathbf{g}(\boldsymbol{\xi})$.

The system of $n+2$ equations (3.8), (3.9), (3.10) is complete. It is remarkable in the relation, that it can be described in terms of many-component complex function ψ (wave function). This transformation can be carried out, using the Hamilton variational principle.

One can show, that dynamic equations (3.8), (3.9), (3.10) for the pure statistical ensemble $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{d}}]$ of deterministic dynamic systems $\mathcal{S}_{\mathrm{d}}$ are derived from the variational principle with the action

$$\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{d}}]: \qquad \mathcal{A}[\rho, \varphi, \boldsymbol{\xi}] = -\int \rho\{H(t, \mathbf{x}, \mathbf{p}) + p_0\} d^{n+1}x, \tag{3.12}$$

$$p_k = b_0 \left[\partial_k\varphi + g^{\alpha}(\boldsymbol{\xi})\partial_k\xi_{\alpha}\right], \qquad \partial_k \equiv \frac{\partial}{\partial x^k}, \qquad k = 0, 1, ..., n \tag{3.13}$$

where $\rho, \varphi, \boldsymbol{\xi}$ are dependent variables, considered to be functions of argument $x = \{x^0, \mathbf{x}\} = \{t, \mathbf{x}\}$. $H(t, \mathbf{x}, \mathbf{p})$ is the Hamilton function for $\mathcal{S}_{\mathrm{d}}$. b_0 is an arbitrary constant, and $g^{\alpha}(\boldsymbol{\xi}), \quad \alpha = 1, 2, ..., n$ are arbitrary functions of argument $\boldsymbol{\xi}$. Dynamic variables $\varphi, \boldsymbol{\xi}$ are hydrodynamic potentials (Clebsch potentials). Clebsch [26, 27] had introduced them for description of incompressible fluid. The variables $\varphi, \boldsymbol{\xi}$ are referred to as potentials, because the momentum $\mathbf{p} = \mathbf{P}(t, \mathbf{x})$ is expressed via derivatives of the potentials $\varphi, \boldsymbol{\xi}$, as one can see this from relations (3.13). The Hamilton function $H(t, \mathbf{x}, \mathbf{p})$ is a function, which determines the form of the action (3.12), and the variational principle, based on (3.12), may be referred to as the Hamilton variational principle.

Let us introduce a k-component complex function $\psi = \{\psi_{\alpha}\}, \quad \alpha = 1, 2, \ldots k$, defining it by the relations

$$\psi_{\alpha} = \sqrt{\rho} e^{i\varphi} u_{\alpha}(\boldsymbol{\xi}), \qquad \psi_{\alpha}^* = \sqrt{\rho} e^{-i\varphi} u_{\alpha}^*(\boldsymbol{\xi}), \qquad \alpha = 1, 2, \ldots k \tag{3.14}$$

$$\psi^*\psi \equiv \sum_{\alpha=1}^{k} \psi_{\alpha}^*\psi_{\alpha}$$

where $(^*)$ means a complex conjugate, $u_{\alpha}(\boldsymbol{\xi}), \quad \alpha = 1, 2, \ldots k$ are functions of only variables $\boldsymbol{\xi}$. They satisfy the relations

$$-\frac{i}{2}\sum_{\alpha=1}^{k}\left(u_{\alpha}^*\frac{\partial u_{\alpha}}{\partial \xi_{\beta}} - \frac{\partial u_{\alpha}^*}{\partial \xi_{\beta}} u_{\alpha}\right) = g^{\beta}(\boldsymbol{\xi}), \qquad \beta = 1, 2, ...n, \qquad \sum_{\alpha=1}^{k} u_{\alpha}^* u_{\alpha} = 1 \tag{3.15}$$

k is such a natural number that equations (3.15) admit a solution. In general, k depends on arbitrary integration functions $\mathbf{g} = \{g^\beta(\boldsymbol{\xi})\}$, $\beta = 1, 2, ...n$.

It is easy to verify that

$$\rho = \psi^*\psi, \qquad p_l(\varphi, \xi) = -\frac{ib_0}{2\psi^*\psi}(\psi^*\partial_l\psi - \partial_l\psi^* \cdot \psi), \qquad l = 0, 1, ...n \tag{3.16}$$

The variational problem with the action (3.12) appears to be equivalent to the variational problem with the action functional

$$A[\psi, \psi^*] = \int \left\{ \frac{ib_0}{2}(\psi^*\partial_0\psi - \partial_0\psi^* \cdot \psi) \right.$$

$$\left. -H\left(x, -\frac{ib_0}{2\psi^*\psi}(\psi^*\boldsymbol{\nabla}\psi - \boldsymbol{\nabla}\psi^* \cdot \psi)\right)\psi^*\psi \right\} \mathrm{d}^{n+1}x \tag{3.17}$$

where $\boldsymbol{\nabla} = \{\nabla_\alpha\} = \{\partial_\alpha\}$, $\alpha = 1, 2, ...n$.

Let us note, that the function ψ, considered to be a function of independent variables $x = \{t, \mathbf{x}\}$ is very indefinite in the sense, that the same state $\{\rho(t, \mathbf{x}), \mathbf{P}(t, \mathbf{x})\}$ of the statistical ensemble $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]$ can be described by various ψ-functions. There are two reasons for such an indefiniteness. First, the functions $u_\alpha(\boldsymbol{\xi})$ are not determined single-valuedly by the equations (3.15). Second, their arguments $\boldsymbol{\xi}$ as functions of x are determined within the relabelling transformation

$$\xi_\alpha \to \tilde{\xi}_\alpha = \tilde{\xi}_\alpha(\boldsymbol{\xi}), \qquad \det \parallel \partial\tilde{\xi}_\alpha/\partial\xi_\beta \parallel = 1, \qquad \alpha, \beta = 1, 2, ...n \tag{3.18}$$

Description of the statistical ensemble $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]$ in terms of the function ψ is more indefinite, than a description in terms of hydrodynamic potentials $\boldsymbol{\xi}$. Information on initial and boundary conditions, contained in functions $\mathbf{g}(\boldsymbol{\xi})$, is lost at the description in terms of ψ-function.

The dynamic equations have the form

$$\delta\psi_\beta^*: \qquad \left[ib_0\partial_0 - H + \frac{\partial H}{\partial p_\alpha}p_\alpha + \frac{ib_0}{2}\left(\frac{\partial H}{\partial p_\alpha}\nabla_\alpha + \nabla_\alpha\frac{\partial H}{\partial p_\alpha}\right)\right]\psi_\beta = 0, \qquad \beta = 1, 2, ...k \tag{3.19}$$

$$\delta\psi_\beta: \qquad \left[-ib_0\partial_0 - H + \frac{\partial H}{\partial p_\alpha}p_\alpha - \frac{ib_0}{2}\left(\frac{\partial H}{\partial p_\alpha}\nabla_\alpha + \nabla_\alpha\frac{\partial H}{\partial p_\alpha}\right)\right]\psi_\beta^* = 0, \qquad \beta = 1, 2, ...k \tag{3.20}$$

where $H = H(x, \mathbf{p})$ and $\frac{\partial H}{\partial p_\alpha}(x, \mathbf{p})$ are considered to be multiplication operators by these quantities, the expression (3.16) has to substituted instead of $\mathbf{p}$, and thereafter the operator $\boldsymbol{\nabla}$ has to act. In general, dynamic equations (3.19), (3.20) are nonlinear with respect to ψ-function, although they appear to be linear in some cases. In these cases the dynamic equations can be solved easily.

The number k of the ψ-function components in the action (3.17) is arbitrary. A formal variation of the action with respect to ψ_α and ψ_α^*, $\alpha = 1, 2, \dots k$ leads to $2k$ real dynamic equations, but not all of them are independent. There are such combinations of variations $\delta\psi_\alpha$, $\delta\psi_\alpha^*$, $\alpha = 1, 2, \dots k$, do not change expressions

(3.16). Such combinations of variations $\delta\psi_\alpha$, $\delta\psi_\alpha^*$, $\alpha = 1,2,\dots k$ do not change the action (3.17), and corresponding combinations of dynamic equations $\delta\mathcal{A}/\delta\psi_\alpha = 0$, $\delta\mathcal{A}/\delta\psi_\alpha^* = 0$ are identities. It associates with a connection between dynamic equations.

Thus, the number of equations increases at increase of the number k, but the number of independent dynamic equations remains the same. The number k is restricted from below by the constraint, that the equations (3.15) have a solution. in other words, the minimal number k_m of the ψ-function components depends on the form of functions $\mathbf{g}(\boldsymbol{\xi})$, i.e. on the initial conditions. This number k_m associates with a kinematic spin (k-spin) $s = 2k_m + 1$ of the ensemble state [25].

ψ-function and k-spin remind respectively wave function and spin. ψ-function coincides with the wave function, provided dynamic equations (3.19), (3.20) becomes linear. It appears to be possible for a pure statistical ensemble $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{st}]$ of stochastic systems $\mathcal{S}_\mathrm{st}$. In this case the k-spin associates with the spin of a particle, but the k-spin remains to be a property of the statistical ensemble $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{st}]$ (i.e. a collective property), whereas in quantum mechanics the spin is considered to be a property of a single particle.

For this reason one should note that in quantum mechanics the spin is a property of a single particle not always. In the paper [28] the properties of two dynamic systems $\mathcal{S}_\mathrm{S}$ and $\mathcal{S}_\mathrm{P}$, described respectively by the Schrödinger equation and by the Pauli one, were analyzed. It appears that in the classical approximation both dynamic systems can be interpreted as pure statistical ensembles respectively $\mathcal{E}_\mathrm{S}[\mathcal{S}_\mathrm{d}]$ and $\mathcal{E}_\mathrm{P}[\mathcal{S}_\mathrm{d}]$, whose elements appear to be the same dynamic system $\mathcal{S}_\mathrm{d}$. The statistical ensembles $\mathcal{E}_\mathrm{S}[\mathcal{S}_\mathrm{d}]$, $\mathcal{E}_\mathrm{P}[\mathcal{S}_\mathrm{d}]$ differ only in their structure, i.e. in a choice of functions $\mathbf{g}(\boldsymbol{\xi})$.

Thus, analysis of the description methods of the pure statistical ensemble description shows that the wave function and spin are not specific quantum objects. The wave function is simply a set of complex potentials, and it contains not more mysticism, than electromagnetic potentials.

4 Pure statistical ensemble of stochastic systems

Let us consider a statistical ensemble$_\mathrm{p}[\mathcal{S}_\mathrm{st}]$ of stochastic systems $\mathcal{S}_\mathrm{st}$. There are no dynamic equations for $\mathcal{S}_\mathrm{st}$, and dynamic equations for $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{st}]$ cannot be derived from dynamic equations for $\mathcal{S}_\mathrm{st}$. But we believe that dynamic equations for $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{st}]$ do exist, as far as experiments with statistical ensembles of stochastic particles $\mathcal{S}_\mathrm{st}$ are reproducible.

Let us consider a motion of stochastic particle $\mathcal{S}_\mathrm{st}$ as a result of interaction between a deterministic particle $\mathcal{S}_\mathrm{d}$ and some stochastic agent, which perturbs motion of $\mathcal{S}_\mathrm{d}$ and make it to be stochastic. To derive dynamic equations for $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{st}]$, some suppositions on properties of this agent are to be made, because it is impossible to derive dynamic equations for $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{st}]$ from nothing. If $\mathcal{S}_\mathrm{st}$ is a Brownian particle, moving in a gas, one supposes that the Brownian particle collides with gas molecules, and these collisions make the Brownian particle motion to be stochastic. These collisions are supposed to be independent and random. The Brownian particle motion appears to be a Markovian process. The dynamic system $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{st}}]$ appears to be dissipative, and there is no variational principle for it.

Such a way of description is not suit for description of the geometric stochasticity influence, because the random component of the particle motion is relativistic, the probabilistic statistical description cannot be used. It is supposed that the stochastic agent influence manifests in the averaging of parameters of the Hamilton function H, describing motion of $\mathcal{S}_{\mathrm{d}}$. These parameters start to depend on the state of the statistical ensemble $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{d}}]$, i.e. on the collective variable ρ. Elements of the statistical ensemble start to interact between themselves and stop to be independent. The dynamic system $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{d}}]$ stops to be a statistical ensemble and turns to a dynamic system $\mathcal{E}_{\mathrm{red}}[\mathcal{S}_{\mathrm{d}}]$, which will be referred to as reduced ensemble (the word "statistical" is omitted).

For a free relativistic deterministic particle the Hamilton function has the form

$$H(x,\mathbf{p}) = \sqrt{m^2c^4 + \mathbf{p}^2c^2} \tag{4.1}$$

where the mass m is the only parameter of Hamiltonian of the system $\mathcal{S}_{\mathrm{d}}$. The variational principle (3.12) for dynamic system $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{d}}]$ has the form

$$\mathcal{A}[\rho,\varphi,\boldsymbol{\xi}] = \int \rho\{-\sqrt{m^2c^4+\mathbf{p}^2c^2} - b_0[\partial_0\varphi + g^\alpha(\boldsymbol{\xi})\partial_0\xi_\alpha]\}\mathrm{d}^4x, \tag{4.2}$$

where $\mathbf{p}$ is given by the relation (3.13) with $n=3$. After averaging [19, 20], which is produced with taking into account the world function (1.2), (1.3), the effective mass m of the particle changes

$$m^2 \to m_{\mathrm{q}}^2 = m^2 + \frac{\hbar^2}{c^2}(\boldsymbol{\nabla}\ln\rho)^2 \tag{4.3}$$

After substitution $m^2 \to m_{\mathrm{q}}^2$ the action takes the form

$$\mathcal{A}[\rho,\varphi,\boldsymbol{\xi}] = \int \rho\{-\sqrt{m^2c^4+\mathbf{p}^2c^2+\hbar^2c^2(\boldsymbol{\nabla}\ln\rho)^2} - b_0[\partial_0\varphi + g^\alpha(\boldsymbol{\xi})\partial_0\xi_\alpha]\}\mathrm{d}^4x, \tag{4.4}$$

The Hamilton function

$$H_{\mathrm{eff}}(x,\mathbf{p}) = \sqrt{m^2c^4+\hbar^2c^2(\boldsymbol{\nabla}\ln\rho)^2+\mathbf{p}^2c^2} \tag{4.5}$$

appears to be invariant with respect to transformation $\rho\to a\rho$, $a=$const.

The action (4.4) is an action for some statistical ensemble, because for the action (4.4) the condition (3.1) of independence on the number of elements takes the form

$$\mathcal{A}[a\rho,\varphi,\boldsymbol{\xi}] = a\mathcal{A}[\rho,\varphi,\boldsymbol{\xi}], \qquad a=\mathrm{const}, \qquad a>0. \tag{4.6}$$

This condition is satisfied, but now the action (4.4) cannot be interpreted as an action for a pure statistical ensemble, whose elements are some deterministic systems

$\mathcal{S}_{\rm d}$, because these dynamic systems $\mathcal{S}_{\rm d}$ interact between themselves and are not independent. It means that the action (4.4) can be and must be interpreted as an action for a pure statistical ensemble $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm st}\right]$, whose elements are stochastic systems $\mathcal{S}_{\rm st}$.

In the nonrelativistic approximation the action (4.4) has the form

$$\mathcal{A}[\rho,\varphi,\boldsymbol{\xi}]=\int\rho\{-mc^2-\frac{\mathbf{p}^2}{2m}-\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla}\ln\rho\right)^2-b_0[\partial_0\varphi+g^\alpha(\boldsymbol{\xi})\partial_0\xi_\alpha]\}\mathrm{d}^4x, \qquad (4.7)$$

where $\mathbf{p}$ is determined by the relation (3.13). The action (4.7) cannot be interpreted as an action for a statistical ensemble $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm d}\right]$ of deterministic systems $\mathcal{S}_{\rm d}$, but it can be regarded as an action for the set $\mathcal{E}_{\rm red}\left[\mathcal{S}_{\rm d}\right]$ of deterministic systems $\mathcal{S}_{\rm d}$, interacting between themselves by means of the potential energy

$$E_{\rm pot}=\frac{\mathbf{p}_{\rm st}^2}{2m}=\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla}\ln\rho\right)^2. \qquad (4.8)$$

where $\mathbf{p}_{\rm st}=-\hbar\boldsymbol{\nabla}\ln\rho$ is the mean momentum of the stochastic component of the particle motion. Thus, on the one hand, (4.7) is an action for the statistical ensemble $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm st}\right]$ of stochastic systems $\mathcal{S}_{\rm st}$, but on the other hand, (4.7) is an action for the set $\mathcal{E}_{\rm red}\left[\mathcal{S}_{\rm d}\right]$ of interacting deterministic systems $\mathcal{S}_{\rm d}$. It means that one can set up a correspondence between the stochasticity character and the form of deterministic systems $\mathcal{S}_{\rm d}$ interaction. Then one can label the stochasticity character by the form of this interaction. Essentially, such a reduction of a stochasticity to an interaction is the only possible way of a mathematical description of a stochasticity. It is described by the relation

$$\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm st}\right]=\mathcal{E}_{\rm red}\left[\mathcal{S}_{\rm d}\right] \qquad (4.9)$$

In terms of ψ-function (3.14) the action (4.7) is written in the form

$$\begin{aligned}A[\psi,\psi^*] &= \int\{\frac{ib_0}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-mc^2\rho-\frac{\hbar^2\left(\boldsymbol{\nabla}\rho\right)^2}{2m\rho}\\ &\quad+\frac{b_0^2}{8\rho m}(\psi^*\boldsymbol{\nabla}\psi-\boldsymbol{\nabla}\psi^*\cdot\psi)^2\}\mathrm{d}^4x, \end{aligned}\qquad (4.10)$$

where $\rho\equiv\psi^*\psi$.

Let the function ψ have k components. Regrouping components of the function ψ of the action (4.10), one obtains it in the form

$$\mathcal{A}[\psi,\psi^*]=\int\{\frac{ib_0}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-\frac{b_0^2}{2m}\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi$$

$$+\frac{b_0^2}{4}\sum_{\alpha,\beta=1}^{k}Q_{\alpha\beta,\gamma}^*Q_{\alpha\beta,\gamma}\rho+\frac{b_0^2-\hbar^2}{8\rho m}\left(\nabla\rho\right)^2-mc^2\rho\}\mathrm{d}^4x, \qquad \rho\equiv\psi^*\psi \qquad (4.11)$$

where a summation over γ is supposed from 1 to 3,

$$Q_{\alpha\beta,\gamma}=\frac{1}{\psi^*\psi}\left|\begin{array}{cc}\psi_\alpha & \psi_\beta\\ \partial_\gamma\psi_\alpha & \partial_\gamma\psi_\beta\end{array}\right|, \qquad \alpha,\beta=1,2,\ldots k \qquad \gamma=1,2,3 \qquad (4.12)$$

and $Q^*_{\alpha\beta,\gamma}$ is the complex conjugate to the quantity $Q_{\alpha\beta,\gamma}$.

In the simplest case, when the ψ-function has only one component, all quantities $Q_{11,\gamma}=0, \quad \gamma=1,2,3$, and the ensemble particle motion is irrotational. Then the action (4.11) reduces to the form

$$\mathcal{A}[\psi,\psi^*]=\int\{\frac{ib_0}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-\frac{b_0^2}{2m}\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi$$

$$-mc^2\rho+\frac{b_0^2-\hbar^2}{8\rho m}(\nabla\rho)^2\}\mathrm{d}^4x, \qquad \rho\equiv\psi^*\psi \tag{4.13}$$

Due to the last term in the action (4.13) the dynamic equation, generated by the action (4.13) is nonlinear, except for the case, when $b_0^2=\hbar^2$, although b_0 is an integration constant, and the action (4.13) describes the same dynamic system for any value of b_0. Equating the arbitrary constant b_0 to $\hbar$, $(b_0=\hbar)$, one obtains instead of (4.13)

$$\mathcal{A}[\psi,\psi^*]=\int\{\frac{i\hbar}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-\frac{\hbar^2}{2m}\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi-mc^2\psi^*\cdot\psi\}\mathrm{d}^4x \tag{4.14}$$

It is easy to see that the dynamic equation, generated by the action (4.14) is linear. After the substitution $\psi\rightarrow\exp\left(-imc^2t/\hbar\right)\psi$, removing the rest mass, the equation turns to the Schrödinger equation in its conventional form

$$i\hbar\partial_0\psi+\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi=0. \tag{4.15}$$

The constant b_0 describes the phase scale of the ψ-function, and the transformation of the ψ-function phase

$$\psi\rightarrow\tilde{\psi}=|\psi|\exp\left(\frac{\tilde{b}_0}{b_0}\log\frac{\psi}{|\psi|}\right), \tag{4.16}$$

changes the constant b_0 to the constant $\tilde{b}_0$ in the action (4.13). The actions (4.13) and (4.14) distinguish very strongly between themselves, although both describe the same dynamic system. The action (4.13) contains only one quantum term, i.e. the term, containing $\hbar$, and setting $\hbar=0$, one passes automatically from quantum description to classical one. Vice versa, in the action almost all terms are quantum, and one cannot set $\hbar=0$, because then any dynamic system description disappears. For derivation of classical description from the action (4.14) it is to use subtle methods of quasi-classical description. Linearity of dynamic equation, arising at the transition from the action (4.13) to the action (4.14), looks rather as a happy chance, than a manifestation of quantum-mechanical principle of dynamic equation linearity.

Describing stochastic systems $\mathcal{S}_{\mathrm{st}}$ by means of the action (4.14), one can interpret the quantity $\psi^*(\mathbf{x})\psi(\mathbf{x})$ as the probability density to discover a particle at the point

$\mathbf{x}$. It is connected with the fact that the quantity $\psi^*(\mathbf{x})\psi(\mathbf{x})$ is non-negative, and integral from it is conserved due to dynamic equation. The probability density, introduced in such a way, is very convenient, but it has not a direct relation to the statistical description. In general, consideration of the action (4.14) for the dynamic system $\mathcal{E}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right]$ does not associate with conventional conception of the statistical description.

One can show [29], that setting of a dynamic system, i.e. setting of the action (4.14), is enough for a description of all quantum effects (diffraction, interference, tunneling, uncertainty relation, determination of eigenvalues for stationary states, etc.). In other words, if the dynamic system (4.14) is given, one can describe all quantum effects without a reference to quantum mechanics principles. This statement is valid not only in the special case of the action (4.14), but in the general case of the action, appeared as a corollary of statistical description. This statement finishes the logical scheme of the research program Copernicus-2.

Thus, in the non-relativistic approximation the program Copernicus-2 gives the quantum mechanical description, basing only on the space-time geometry without QM principles. The general relativistic case has been developed insufficiently, but the statistical description, which leads to the dynamic system $\mathcal{S}_{\mathrm{KG}}$, described by the Klein-Gordon equation has been obtained in [29]. For its derivation one needs to use a relativistic version, where nonrelativistic effective mass $m_{\mathbf{q}}$, given by the relation (4.3) is substituted by its relativistic version, and the temporal component $j^0 = \rho$ is substituted by corresponding relativistic invariant j^0/H. But the nonrelativistic Hamilton variational principle is slightly suit for dealing with relativistic quantities. It is more convenient to use the Lagrange variational principle equivalent to (3.12)

$$\mathcal{E}\left[\mathcal{S}_{\mathrm{d}}\right]: \qquad \mathcal{A}[j,\varphi,\boldsymbol{\xi}] = \int \{L(x, \frac{\mathbf{j}}{j^0})j^0 - b_0 j^i[\partial_i\varphi + g^\alpha(\boldsymbol{\xi})\partial_i\xi_\alpha]\}\mathrm{d}^{n+1}x, \qquad (4.17)$$

where $L(x, \frac{d\mathbf{x}}{dt})$ is the Lagrangian of the system $\mathcal{S}_{\mathrm{d}}$ and $\{j^0, \mathbf{j}\} = \{j^i\}, \quad i = 0, 1, ...n$ is the flux of particle $\mathcal{S}_{\mathrm{d}}$ in the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{d}}\right]$. Then the variational principle for the statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right] = \mathcal{E}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right]$ of stochastic systems $\mathcal{S}_{\mathrm{st}}$ takes the form

$$\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]: \qquad \mathcal{A}[j,\varphi,\boldsymbol{\xi},\kappa] = \int \{-mcK\sqrt{j^i g_{ik} j^k} - b_0 j^i[\partial_i\varphi + g^\alpha(\boldsymbol{\xi})\partial_i\xi_\alpha]\}\mathrm{d}^3x, \quad (4.18)$$

$$m_{\mathrm{q}} = mK, \qquad K \equiv \sqrt{1 + \lambda^2(\partial_l\kappa^l + \kappa^l\kappa_l)}, \qquad \partial_k \equiv \partial/\partial x^k, \qquad (4.19)$$

where $g_{ik} = \mathrm{diag}\{c^2, -1, -1, -1\}$ is the metric tensor, m is the particle mass and $\lambda \equiv \hbar/mc$ is its Compton wavelength. $\boldsymbol{\xi} = \{\xi_\alpha\}, \quad \alpha = 1, 2, 3$; and $\kappa^l = \kappa^l(x), \quad x = \{x^l\}, \quad l = 0, 1, 2, 3$. A summation is made over repeating indices, for Latin indices from 0 to 3, and for Greek ones from 1 to 3. Here the effective mass $m_{\mathrm{q}} = mK$ is expressed via some κ-field, describing interaction of particles $\mathcal{S}_{\mathrm{d}}$ in the dynamic system $\mathcal{E}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right]$. At the same time the κ-field describes stochasticity of the systems $\mathcal{S}_{\mathrm{st}}$.

It follows [29] from dynamic equations, that the κ-field has a potential, designed by means of $\frac{1}{2}\ln\rho$, i.e. $\kappa_l = \frac{1}{2}\partial_l \ln\rho$. Then one can introduce the ψ-function by means of relations (3.14). In the simplest case, when ψ-function has only one component, the dynamic equation for it coincides with the Klein-Gordon equation and with the Schrödinger one in the nonrelativistic approximation.

The κ-field has all characteristic properties of a field, i.e. it has a proper energy, it can exist in the absence of a matter, i.e. at $j^k = 0, \quad k = 0,1,2,3$. Besides, it enables to produce pairs particle–antiparticle and is responsible for quantum effects. It means, that at $\kappa^i \equiv 0, \quad i = 0,1,2,3$ the statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ turns to the statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{d}}]$.

5 Concluding remarks

The research program Copernicus-2 is more perfect logically, than Ptolemy-2, because it was founded on the basis of more general and perfect conceptions of geometry and statistical description. It is important to understand, that these more general conceptions are not a result of some successful hypotheses, or restrictions. On the contrary, the larger generality and efficiency of the new conceptions of geometry and statistical description appear as a corollary of a removal of unfounded constraints, used earlier. The new conception of geometry does not use the concept of a curve, because it is too restrictive. The new conception of the statistical description does not use concept of probability and that of probability density, because they are also too restrictive. It is the point, that *simultaneous application* of both T-geometry and dynamical conception of statistical description is very important also. A use of only T-geometry explains the origin of quantum stochasticity, but it does not admit to reconstruct the mathematical technique of quantum mechanics. A use of only dynamical conception of statistical descriptions admits one to derive the mathematical technique of quantum mechanics, but it does not explains the origin of quantum stochasticity, and does not permit to develop this technique in the "geometrical direction", that is characteristic for the whole development of physics in the last century

From the fact, that the research program Copernicus-2 explains quantum effects without a reference to additional hypotheses (QM principles), it follows that Copernicus-2 is more logically consistent, than Ptolemy-2. The last program uses inadequate space-time geometry, which is to be corrected. But the program Ptolemy-2 works almost hundred years. All descriptions of quantum phenomena and corresponding calculations are produced in terms of quantum mechanics. Vast factual data were collected, and revision of these data is difficult and undesirable. In this connection it is very important to know, to what degree a transition from the program Ptolemy-2 to the program Copernicus-2 concerns existing results, obtained on the basis of quantum mechanics.

To estimate this, it is useful to turn to an experience of interplay between the axiomatic thermodynamics and statistical physics, which founded thermodynamics and determined limits of applicability of its relations. This experience shows that restrictions imposed by the statistical physics, concern only a small part of thermodynamics results. Nothing changed in the field, where thermodynamics was used for practical goals. One should expect that a transition to the program Copernicus-2 will change nothing in nonrelativistic quantum mechanics, which has been developed mostly and has practical applications. In the relativistic quantum mechanics, especially in the elementary particle theory the changes may be essential.

Let us note an important problem, connected with the dynamic system $\mathcal{S}_{\rm D}$, described by the Dirac equation, or by the action

$$\mathcal{S}_{\rm D}: \qquad \mathcal{A}_{\rm D}[\bar{\psi},\psi] = \int (-m\bar{\psi}\psi + \frac{i}{2}\hbar\bar{\psi}\gamma^l\partial_l\psi - \frac{i}{2}\hbar\partial_l\bar{\psi}\gamma^l\psi)d^4x \tag{5.1}$$

where ψ is a four-component complex wave function. It is known that the Dirac equation is a relativistic equation, but the dynamic system $\mathcal{S}_{\rm D}$ is not relativistic, and it is very unexpected. This fact was discovered at the analysis of dynamic system $\mathcal{S}_{\rm D}$ [30], undertaken for investigation of what a geometrical object is associated with $\mathcal{S}_{\rm D}$. The meaning of Dirac matrices γ^i in the action (5.1) is obscure. They were eliminated, and the system $\mathcal{S}_{\rm D}$ was investigated in tensor variables j^l, S^l, $(l = 0,1,2,3)$, φ, κ, determined by the relations

$$j^l = \bar{\psi}\gamma^l\psi, \qquad l = 0,1,2,3, \qquad \bar{\psi} = \psi^*\gamma^0, \tag{5.2}$$

$$S^l = i\bar{\psi}\gamma_5\gamma^l\psi, \qquad l = 0,1,2,3, \qquad \gamma_5 = \gamma^{0123} \equiv \gamma^0\gamma^1\gamma^2\gamma^3, \tag{5.3}$$

Here γ^l, $l = 0,1,2,3$ are Dirac γ-matrices, satisfying the commutation relations

$$\gamma^i\gamma^k + \gamma^k\gamma^i = 2g^{ik}, \qquad i,k = 0,1,2,3, \tag{5.4}$$

where g^{ik} =diag$(1,-1,-1,-1)$ is the metric tensor. Only two of components of the pseudovector S^l are independent, because there are two identities

$$S^lS_l \equiv -j^lj_l, \qquad j^lS_l \equiv 0. \tag{5.5}$$

To describe $\mathcal{S}_{\rm D}$ in tensor variables, the change of variables is made

$$\psi = Ae^{i\varphi+\frac{1}{2}\gamma_5\kappa}\exp(-\frac{i}{2}\gamma_5\boldsymbol{\sigma\eta})(\boldsymbol{\sigma}\mathbf{n})\Pi, \qquad \bar{\psi} = A\Pi(\boldsymbol{\sigma}\mathbf{n})\exp(-\frac{i}{2}\gamma_5\boldsymbol{\sigma\eta})e^{-i\varphi+\frac{1}{2}\gamma_5\kappa} \tag{5.6}$$

where Π is the zero devisor

$$\Pi = \frac{1}{4}\left(1+\gamma^0\right)\left(1+\mathbf{z}\boldsymbol{\sigma}\right), \qquad \mathbf{z} = \left\{z^1,z^2,z^3\right\}, \qquad \mathbf{z}^2 = 1 \tag{5.7}$$

$$\boldsymbol{\sigma} = \{\sigma_1,\sigma_2,\sigma_3,\}, \qquad \sigma_1 = -i\gamma^2\gamma^3, \qquad \sigma_2 = -i\gamma^3\gamma^1, \qquad \sigma_3 = -i\gamma^1\gamma^2 \tag{5.8}$$

The variables $A, \boldsymbol{\eta} = \{\eta^1,\eta^2,\eta^3\}$, $\mathbf{n} = \{n^1,n^2,n^3\}$, $(\mathbf{n}^2 = 1)$ are six intermediate variables, and $\mathbf{z}$ is a constant unite 3-vector. Substituting (5.6) in (5.1) and using (5.2), (5.3), one can express the action (5.1) in terms of tensor variables $j^l, S^l, \kappa, \varphi$ with eight independent real components.

One expected that after transformation to tensor variables j^l, S^l, φ, κ, one succeeded to write the action (5.1) in the relativistically covariant form. But it failed. The action and dynamic equations are written in the relativistically covariant form only after introduction of constant unit timelike 4-vector f^i. This 4-vector is an absolute object in the sense of Anderson [31]. (Note that the constant vector $\mathbf{z}$ is another absolute object, but it appears to be fictitious.) The 4-vector f^i describes separation of the space-time into space and time. In other words, the dynamic system $\mathcal{S}_{\rm D}$ appears to be nonrelativistic. Of course, it is nonrelativistic at the description in terms of the wave function ψ also, but in this case the 4-vector f^i is absorbed by other absolute objects (γ-matrices), and one cannot discover it at once (see discussion in [30]). One may think that appearance of f^i is a result of a calculation mistake (transformation of the action (5.1) to tensor variables is rather bulky). But the same timelike vector f^i appears in a more simple case of two-dimensional space-time, when a transformation of the system $\mathcal{S}_{\rm D}$ to the dynamic system $\mathcal{S}_{\rm KG}$, described by the Klein-Gordon equation, is possible [32]. Unfortunately, this circumstance forces one to think that the conclusion on nonrelativistic character of dynamic system $\mathcal{S}_{\rm D}$ is valid.

Thus, the dynamic system $\mathcal{S}_{\rm D}$ is nonrelativistic, and it is a serious test for both research programs Ptolemy-2 and Copernicus-2. Establishing of reasons of this circumstance could advance us in explanation of microcosm phenomena.

The research programs Ptolemy-2 and Copernicus-2 have guided the different development of further fundamental investigations, and therein lies the main difference between them. The key word for further investigation under program Ptolemy-2 is *linearity*, whereas for the program Copernicus-2 the key word is *geometrization*.

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