International Journal of Theoretical Physics, Vol. 8, No. 1 (1973), pp. 65–83

Quantum Mechanics as Relativistic Statistics. I: The Two-Particle Case

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Received: 31 *May* 1972

Abstract

It is shown that non-relativistic quantum mechanics can be treated as a kind of relativistic statistical theory, which describes the indeterministic motion of classical particles. The theory is relativistic in the sense that the relativistic notion of the state and two-time equations of motion are used. The principles and relations of quantum mechanics are obtained from the principles of statistics and those of classical mechanics.

Ever since the inception of quantum mechanics, attempts were made to understand it from the point of view of classical mechanics.† These attempts were partly successful in the case of one-particle quantum mechanics, but difficulties were encountered in the field of quantum mechanics of many particles (Janossy, 1969).

This paper is a subsequent development (Rylov, 1971). According to present ideas, quantum mechanics is a non-relativistic approximation of relativistic statistics. This has been shown for the case of the one-particle system.

In the present paper two different notions of state are used: n-state and r-state. The n-state (non-relativistical state) of a system is used in the non-relativistic theory. The n-state of the physical system is a set of quantities, given at a certain moment of time. For instance, the particle n-state is determined at a certain moment by coordinates q and momenta p, i.e. by a point in the phase space of coordinates and momenta. The n-state obeys a motion equation which describes evolution of the n-state (a phase trajectory).

In the relativistic theory the simultaneity is relative, and it seems more consistent to use the notion of r-state (relativistical state). The r-state is given over all space-time. The particle r-state is equation of its world-line. For the deterministic particle the coordinates of its world-line obey some equations which are restrictions imposed upon possible r-states.

† See Kaliski (1970) for a comprehensive bibliography.

In that case, when the particle interacts with the medium (ether) by unpredictable manner, its behaviour becomes non-deterministic, i.e. one cannot calculate the particle n-state at moment t' if its n-state is given at moment $t(t < t')$. Thus the non-deterministic particle n-state does not obey any equation of motion.

For a description of the motion of non-deterministic particles, the notion of statistical ensemble is used. The statistical ensemble is a set of many independent identical systems which are in different states. The statistical ensemble is a deterministic dynamical system even if constituting systems are non-deterministic. This means that the ensemble n-state can be calculated at moment t' if its n-state at moment t $(t < t')$ is known.

For example, let the non-deterministic particle state be represented by a point in the phase space. The ensemble consists of N such independent particles $(N \to \infty)$. The state of every particle is represented by a point in the phase space. Let $d\Omega$ be an element of volume of phase space, and dN be the number of points in $d\Omega$. Then

$$dN = W d\Omega$$

where $W = W(q,p)$ is a state density.

In spite of the fact that any individual system of ensemble is indeterministic, it is found that evolution of the W can be calculated because the W obeys some equation, which form depends on the character of random forces acting upon particles of ensemble. Thus the W describes the n-state of the statistical ensemble as a dynamical system.

The equation which is obeyed by W is invariant with respect to transformation $W \to CW$, when $C =$ constant. This is so because the ensemble behaviour is not to depend on the number of systems constituting ensemble, if this number is large enough. The constant C can be chosen in such a way that $W(q,p)$ represents the probability of detecting the n-state of a particle in the volume $d\Omega$ of the phase space.

Thus the non-deterministic system corresponds to the deterministic system called statistical ensemble, the n-state W of the ensemble being a density of n-states (q,p) of systems constituting the ensemble. The n-state $W(q,p)$ of the ensemble is the probability density to detect the physical system in the state (q,p). For this reason, calculating $W(q,p)$ for any times t by means of the motion equation of the statistical ensemble one can calculate the evolution of the mean value $\langle F(q,p) \rangle$ of any function $F(q,p)$ of the n-state (q,p).

I have described in general the traditional scheme of the statistical ensemble application for a description of the behaviour of non-deterministic systems. One can emphasise three essential points in this scheme.

(1) Transition from the physical system to the ensemble, i.e. the method of construction of the ensemble state.
(2) Transition from the ensemble to the individual system, i.e. the method of calculation of statistical characteristics of non-deterministical system proceeding from the ensemble state.

(3) Determination of equations which are obeyed by the ensemble state, and the solution of these equations.

Let us consider now the relativistical case when the r-state is a basis. This means that one has to do with the world-line statistics but not with the statistics of points in the phase space. In this case the notion of statistical ensemble is changed slightly.

Let us consider a physical system consisting of a particle. The r-state of the particle is described by world-line $q^i = q^i(\tau)$ in the space-time (q^i are coordinates, $i = 0, 1, 2, 3$; and τ is a parameter along a world-line). Let there be an ensemble (i.e. a set) of such world-lines. Their density in the vicinity of the point q can be defined in the following way (Rylov, 1971). Let ds_i be an infinitesimal three-dimensional area at the point q^i and dN be the number of world-lines crossing ds_i. Then

$$dN = \sum_{i=0}^{3} j^i ds_i$$

where j^i is a factor which is, by definition, the density of r-states (world-lines) at the point q of the space-time.

Let us assume, as in the traditional approach, that the state density is the statistical ensemble state. Then vector j^i† is the ensemble state.

Let us consider a physical system consisting of two particles. The r-state of the first particle is world-line L_1: ${q_1}^i = {q_1}^i(\tau)$ ($i = 0,1,2,3$) in the space-time V_1 of the first particle. The r-state of the second particle is world-line L_2: ${q_2}^i = {q_2}^i(\tau)$ ($i = 0,1,2,3$) in the space-time V_2 of the second one. It is evident that any combinations of r-states of the first and second particle are possible. For this reason the r-state of the whole system is a two-dimensional surface $S = L_1 \otimes L_2$, in the eight-dimensional space $V_{12} = V_1 \otimes V_2$, where $\otimes$ denotes a tensor product. Let $x^a = \{{q_1}^i, {q_2}^i\}$ ($a = 1,2,\ldots,8$) be coordinates in V_{12}. Let there be an ensemble (a set) of surfaces S in space V_{12}. For the determination of the density of surfaces S in the vicinity of the point $x = \{q_1, q_2\}$ of the V_{12}, one chooses an infinitesimal six-dimensional area $dS_{ab} = -dS_{ba}$ ($a,b = 1,2,\ldots,8$) in the point x, and calculates the number dN of two-dimensional surfaces S crossing ds_{ab}. It is evident that

$$dN = \frac{1}{2} \sum_{a,b=1}^{8} j^{ab} ds_{ab}$$

where j^{ab} is a factor. The antisymmetric tensor j^{ab} is by definition the r-state density of two-particle systems in the point x of the space V_{12}. The j^{ab} is to be identified with the state of statistical ensemble of two-particle systems.

In the case of the three-particle system the r-state density is described by an antisymmetrical tensor j^{abc} in the twelve-dimensional space V_{123}, and so on.

† The j^i considered at a certain moment of time is the ensemble n-state. The same j^i considered in the whole space-time is the r-state. In this sense the r-state of the ensemble coincides with the n-state.

Thus the statistical ensemble density, which is based on the r-state, may be a vector j^i, a tensor j^{ab}, and so on, depending on the number of particles in the system.

The method of transition from the system r-state to the statistical ensemble state is defined. It coincides with the traditional way: the ensemble state is the state density of constituting ensemble systems.

The reverse transition from the statistical ensemble to the system properties cannot coincide with the traditional one, because the traditional way is based on the fact that the ensemble state $W(q,p)$ is the probability density to detect the system in the state (q,p). Strictly speaking, neither j^i nor j^{ab} can be treated in such a way. For this reason the transition from the statistical ensemble to the non-deterministic system is based on the use of additive quantities.

Definition. The quantity B is an additive one if the value of B for several independent systems is equal to the sum of values B for every system.

Energy, momentum, angular momentum and their densities are examples of additive quantities. The statistical ensemble is a set of independent systems. For this reason any additive quantity attributed to the statistical ensemble as a dynamical system is a sum of values of this quantity for all systems constituting an ensemble. As the ensemble behaviour does not depend on the number of systems in the ensemble, the equations for the ensemble state j are invariant with respect to transformation $j \to Cj$ ($C =$ constant). Hence j can be normed on one system. Here j denotes any ensemble state: $W, j^i, j^{ab}, \ldots$

In this case the value of any additive quantity of ensemble is equal to the mean (over ensemble) value of this quantity for systems constituting an ensemble.

In the non-relativistic approximation, when one of components j is non-negative and conserved (for instance, j^0 for a one-particle system), it is possible to treat this component as a corresponding probability density. In this case, it is possible to obtain additional information on motion of systems constituting an ensemble. But this takes place in non-relativistical approximation only.

Let us formulate the foregoing in the axiomatic form.

The statistical principle. A non-deterministic† dynamical system A, whose state is described by quantities ξ, corresponds to a deterministic dynamical system, which is called a statistical ensemble and has the following properties.

(1) A state j of the statistical ensemble is a state density of systems A.

† The statistical principle can be applied to the deterministic system, if its initial conditions are not determined exactly.

(2) The equations for the ensemble state j are invariant with respect to transformation $j \to Cj$ (C = constant).
(3) If the ensemble state j has a proper normalisation (on one system), every additive quantity B, attributed to the statistical ensemble as a dynamical system, is the mean value of quantity B for system A.

The statistical principle settles the question about determination of the ensemble state and about determination of the non-deterministic system properties, but it does not determine which equations are obeyed by ensemble state.

This problem needs special consideration. It was solved for the one-particle case (Rylov, 1971). On the basis of the statistical principle (although it has not been formulated in the definite form) it was shown that in the one-particle case it is possible to choose an equation (or Lagrangian) for the statistical ensemble so that conclusions, obtained on the base of statistical principle, are equivalent to those of one-particle quantum mechanics.

In this paper the analogous problem is put for the two-particle case. The problem is to define the statistical ensemble as a dynamical system in such a way that conclusions obtained on the basis of the statistical principle coincide with those of quantum mechanics. To define the statistical ensemble as a dynamical system means one has to write a motion equation (or Lagrangian) for the ensemble. It is shown in this paper that this is possible. The rule for writing a Lagrangian is given. For this purpose, the ensemble of two-particle deterministic systems is considered first. For such an ensemble, a Lagrangian is the sum of Lagrangians of individual systems. One takes into account indeterminism by means of an additional term containing Planck's constant. Then one gets conception which is equivalent to quantum mechanics. This is the purpose of the present paper. For simplicity, only the case of two independent particles in two-dimensional space-time is considered.

The two-particle system ensemble has two peculiarities that distinguish it from the one-particle system ensemble. Firstly, the space V_{12} contains two times and the ensemble state j^{ab} evolves in two times at once. For this reason the number of independent equations is twice that of independent components of j^{ab}. The compatibility conditions of the equations complicate attempts to conclude them from the variational principle.

Secondly, at every point of space V_{12} the direction of two-dimensional surface, which represents the r-state of two particles, can be described by means of a simple bivector† tangent to this surface. At every point the state surface density j^{ab} is proportional to a mean value of the tangent bivector in the vicinity of this point. In general, the mean tangent bivector is not a simple bivector and does not represent the direction of any two-dimensional surface, because the sum of simple bivectors is not a simple bivector in general.

† The simple bivector n^{ab} is defined as a bivector that can be represented in the form $n^{ab} = v_1{}^a v_2{}^b - v_1{}^b v_2{}^a$, where $v_1{}^a$ and $v_2{}^a$ are vectors.

1. *Ensemble of Two-Particle Systems*

Let us consider a system consisting of two non-interacting particles moving in a two-dimensional space-time. The r-state of the first particle is represented by world-line $L_1: q_1{}^i = q_1{}^i(\tau)$ $(i = 0, 1)$ in the two-dimensional space-time V_1. The r-state of the second particle is represented by world-line L_2: $q_2{}^i = q_2{}^i(\tau)$ $(i = 0, 1)$ in the two-dimensional space-time V_2. $t_1 = q_1{}^0$, $q_1 = q_1{}^1$ are coordinates on V_1; $t_2 = q_2{}^0$, $q_2 = q_2{}^1$ are coordinates in V_2. The r-state of the two-particle system is represented by the two-dimensional surface $S = L_1 \otimes L_2$ in the four-dimensional space $V_{12} = V_1 \otimes V_2$, where $\otimes$ stands for tensor product. Let x^a $(a = 1, 2, 3, 4)$ be coordinates in V_{12}.

$$x^1 = q_1{}^0, \qquad x^2 = q_1{}^1, \qquad x^3 = q_2{}^0, \qquad x^4 = q_2{}^1 \tag{1.1}$$

Tensor indices in V_{12} are numerated by the small letters of the Latin alphabet.

The two-dimensional surface S is described by the equation

$$q_A{}^i = q_A{}^i(\tau_A) \qquad (i = 0, 1;\ A = 1, 2) \tag{1.2}$$

where τ_A are parameters. Let us call S a state surface or the r-state of the two-particle system.

Motion of the two-particle system is assumed to be non-deterministic. This means that even if the tangential bivector of state surface S is known at the moment of time $t_1 = t_2 = 0$, it is impossible to determine the surface S (but it exists). Although the non-deterministic system behaviour cannot be described exactly, it can be described statistically. For this purpose, according to the statistical principle, a non-deterministic system corresponds to a deterministic system (statistical ensemble). According to definition, the ensemble state is the r-state density of the non-deterministic system (i.e., the density of state surface S). The r-state of the ensemble has the property that if it is given at moment $t_1 = t_2 = 0$, then it can be determined for any moments (t_1, t_2) by means of motion equations.

Using the fact that the ensemble state is the r-state density of a non-deterministic system, one can make some statistical conclusions about the evolution of the non-deterministic system.

For the determination of the density of surfaces S in the vicinity of a point x^a of space V_{12}, let us consider at the point x the infinitesimal two-dimensional area

$$ds_{ab} = \sqrt{|g|\, c^{-2}}\, \varepsilon_{abcd}\, d_1 x^c\, d_2 x^d \tag{1.3}$$

Here $d_1 x^c$ and $d_2 x^d$ are two infinitesimal linear independent vectors forming the two-dimensional area ds_{ab}, ε_{abcd} is antisymmetrical over all indices pseudotensor $(\varepsilon_{1234} = 1)$, g is determinant of the metric tensor in V_{12}, c is light speed.

The number dN of state surfaces crossing ds_{ab} is defined by the relation

$$dN = \tfrac{1}{2} j^{ab}\, ds_{ab} \qquad (a, b = 1, 2, 3, 4) \tag{1.4}$$

As usual, summation is made on like super- and subscripts. $j^{ab} = j^{ab}(x)$ is a factor. According to definition, j^{ab} is a state density at the point x.

j^{ab} is an antisymmetric tensor and describes the mean density and direction of state surfaces at the point x.

According to definition, j^{ab} is a state† of statistical ensemble.

The number of systems in ensemble is defined by

$$N = \frac{1}{2}\int\int_{\Sigma} j^{ab}\, ds_{ab}$$

where $\sum$ is an infinite spacelike surface in V_{12}. It means that the projection of any vector which is tangent to $\sum$ into space V_1 or V_2 is a spacelike vector.

Definition. Ensemble of surfaces S‡ is called a simple one in region Ω of space V_{12} if surfaces S of ensemble do not cross in Ω. Ensemble simple in the whole V_{12} is called simple one.

It is easy to see that any ensemble of surfaces S, which are derived as a tensor product of world-line L_1 and L_2, placed correspondingly in V_1 and V_2 has the properties

$$j^{12} = j^{21} = j^{34} = j^{43} = 0 \tag{1.5}$$

As a matter of fact, the bivector j^{ab} tangent to surface S has the form

$$j^{ab} = u_1{}^a u_2{}^b - u_1{}^b u_2{}^a \tag{1.6}$$

where

$$u_1{}^a = \{u_1{}^1, u_1{}^2, 0, 0\}, \qquad u_2{}^a = \{0, 0, u_2{}^3, u_2{}^4\}$$

(1.5) follows from (1.6) for simple ensemble. Equation (1.5) is valid for any ensemble because any ensemble can be represented as a sum of simple ensembles.

It follows from (1.6) that the simple ensemble has the property

$$j^{13} j^{24} - j^{14} j^{23} = 0 \tag{1.7}$$

2. *The Simple Ensemble*

At first, let us consider the case of a statistical ensemble of deterministic systems. This allows us to conclude two-time equations for the ensemble state j^{ab} from a variational principle. After this the variational principle needs only to be generalised in the case of an ensemble of non-deterministic systems adding some terms to the Lagrangian.

In this section the action and equations of motion for the statistical ensemble of the deterministic two-particle system will be obtained, proceeding from action for individual particles.

The motion law of every individual ensemble system is supposed to be known. For simplicity, systems consisting of two free particles with masses

† Tensor j^{ab} considered in the whole V_{12} is r-state, and considered at t_1, t_2 = constant it is n-state.

‡ The term 'ensemble of surfaces S' is used for short, instead of 'statistical ensemble of systems which states are represented by surfaces S'.

m_1 and m_2 are considered. Let each ensemble system be numerated by two numbers: ${\xi_1}^1$ and ${\xi_1}^2$. An action for a simple ensemble is the sum of actions for individual systems.

$$S = S_1 + S_2$$

$$S_A = S_A[{q_A}^i] = -m_A c \int \sqrt{\left(\frac{\partial {q_A}^i}{\partial \tau} g_{ik} \frac{\partial {q_A}^k}{\partial \tau}\right)} d{\xi_1}^1 d{\xi_1}^2 d\tau \qquad (i,k = 0,1; A = 1,2) \tag{2.1}$$

Summation is made on like tensor (Latin) super- and subscripts. Summation on capital indices which numerate particles is always designated by the sign of summation.

g_{ik} is the metric tensor

$$g_{ik} = \begin{Vmatrix} c^2 & 0 \\ 0 & -1 \end{Vmatrix} \tag{2.2}$$

$${q_A}^i = {q_A}^i(\tau, {\xi_1}^1, {\xi_1}^2) \qquad (i = 0,1; \qquad A = 1,2) \tag{2.3}$$

where c is light speed.

If ${\xi_1}^1$ and ${\xi_1}^2$ are fixed, then ${q_A}^i = {q_A}^i(\tau)$ describes the world-line of the Ath particle. τ is a parameter along a world-line of the first particle as well as the second one. Only a world-line has a physical sense but not a parameter along it. Formally, it follows from the invariance of actions S_1 and S_2 with respect to transformation that

$$\tau \to \tau' = \chi(\tau) \tag{2.4}$$

where $\chi(\tau)$ is an arbitrary monotone function of τ. The transformation may be different for S_1 and S_2.

To obtain two-time equations from the variational principle, the action $S = S[\xi]$ is written in the form of an integral over a three-dimensional surface $\eta(x) = C = \text{constant}$ in space V_{12}, integrand containing only derivatives tangent to surface $\eta = C$. Equations obtained from the variational principle allow us to obtain a solution on the surface $\eta = C$. If initial data are given on two-dimensional surface σ (for instance, $t_1 = t_2 = 0$) then a solution can be calculated at any point x. To do this it is sufficient to choose the surface $\eta = C$ in such a manner that surface $\eta = C$ should pass through σ and x. If a solution does not depend on the choice of η, then two-time equations exist and are compatible. The solution can be calculated by means of these equations if initial data are given on the two-dimensional surface.

For realisation of this idea let us suppose

$${q_A}^i = {q_A}^i({\xi_0}^1, {\xi_0}^2, {\xi_1}^1, {\xi_1}^2) \qquad (i = 0,1; A = 1,2) \tag{2.5}$$

$${\xi_0}^1 = {\xi_0}^1(\tau), \qquad {\xi_0}^2 = {\xi_0}^2(\tau) \tag{2.6}$$

It is easy to see that (2.5) and (2.6) are equivalent to (2.3). After substitution of (2.5) and (2.6) into (2.1), action S takes the form of an integral over the surface (2.6). The equation of the surface can be written in the form

$$\eta({\xi_0}^1, {\xi_0}^2) = C = \text{constant} \tag{2.7}$$

Then (2.1) takes the form

$$S_A = S_A[q_A{}^i] = -m_A c \int \sqrt{\left(\frac{\partial(q_A{}^i,\eta)}{\partial(\xi_0{}^1,\xi_0{}^2)} g_{ik} \frac{\partial(q_A{}^k,\eta)}{\partial(\xi_0{}^1,\xi_0{}^2)}\right)} \delta(\eta - C)\, d^4\xi$$
$$(d^4\xi = d\xi_0{}^1\, d\xi_1{}^1\, d\xi_0{}^2\, d\xi_1{}^2) \quad (2.8)$$

Let us suppose now, that

$$\xi_i{}^A = \xi_i{}^A(x) \qquad (i = 0,1; A = 1,2) \quad (2.9)$$

$$\eta = \eta(\xi_0{}^1, \xi_0{}^2) = \eta(x) \quad (2.10)$$

Let us introduce designations

$$\xi_1 = \tau, \qquad \xi_2 = \eta, \qquad \xi_3 = \xi_1{}^1, \qquad \xi_4 = \xi_1{}^2 \quad (2.11)$$

$$J = \frac{\partial(\xi_1,\xi_2,\xi_3,\xi_4)}{\partial(x^1,x^2,x^3,x^4)}, \qquad \eta_a \equiv \frac{\partial\eta}{\partial x^a}, \qquad \tau_a \equiv \frac{\partial\tau}{\partial x^a} \quad (2.12)$$

(J is Jacobian).

$$g^1_{ab} = \begin{Vmatrix} c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}, \qquad g^2_{ab} = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix} \quad (2.13)$$

$$j^{ab}\eta_b = \frac{\partial J}{\partial \tau_a} = \frac{\partial^2 J}{\partial \tau_a \partial \eta_b}\eta_b \quad (2.14)$$

Action (2.8) can be written in the form

$$S = S[j^{ab}, \xi_a, p_b] = \int \left\{ -\sum_{A=1}^{2} \sqrt{(j^{ab}\eta_b g^A_{ac} j^{cd}\eta_d)} + p_a\eta_b \left(\frac{\partial^2 J}{\partial\tau_a\partial\eta_b} - j^{ab}\right)\right\}$$
$$\delta(\eta - C)\, d^4x \qquad (d^4x = dx^1\, dx^2\, dx^3\, dx^4) \quad (2.15)$$

Here p_a are Lagrangian multipliers, which introduce designation (2.14) into Lagrangian.

Let us suppose that one system of ensemble corresponds to a unit range of variables $\xi_1{}^1$ and $\xi_1{}^2$. It is easy to verify that j^{ab} defined by (2.14) is a density of state surfaces in the sense of (1.4).

Due to the identity

$$\frac{\partial^2 J}{\partial\tau_a\partial\eta_b} = \frac{1}{J}\left(\frac{\partial J}{\partial\tau_a}\frac{\partial J}{\partial\eta_b} - \frac{\partial J}{\partial\tau_b}\frac{\partial J}{\partial\eta_a}\right) \quad (2.16)$$

and the arbitrariness of η_a in (2.14) j^{ab} is a simple bivector.

Varying (2.15) with respect to j^{ab} and ξ_c one gets:

$$\frac{\delta S}{\delta j^{ab}} = \left\{ -\sum_{A=1}^{2} \frac{m_A c}{R_A} g^A_{ac}\eta_b j^{cd}\eta_d - p_a\eta_b \right\} \delta(\eta - C) = 0 \quad (2.17)$$

where

$$R_A = \sqrt{(j^{ab}\eta_b g^A_{ac} j^{cd}\eta_d)} \quad (2.18)$$

$$\frac{\partial S}{\delta \xi_c} = \frac{\partial}{\partial x^d}\left(p_a \eta_b \frac{\partial^3 J}{\partial \tau_a \partial \eta_b \partial \xi_{c,d}}\right) \delta(\eta - C) = 0 \qquad \left(\xi_{c,d} \equiv \frac{\partial \xi_c}{\partial x^d}\right) \quad (2.19)$$

Variation with respect to p_a leads to the equation (2.14), which is to be fulfilled on surface $\eta = C$.

According to (2.7) the surface $\eta = C$ is chosen arbitrarily, therefore only solutions which are defined in the whole V_{12} or in the four-dimensional region of V_{12} and do not depend on choice of η are of interest. Thus, one looks for solutions where p_a and j^{ab} do not depend on the choice $\eta = C$.

From (2.17) it follows that

$$p_a = -\sum_{A=1}^{2} \frac{m_A c g^A_{ac} j^{cd} \eta_d}{\sqrt{(j^{ef} \eta_f g^A_{eh} j^{hi} \eta_i)}} \quad (2.20)$$

According to (2.13) $g^A_{ac} j^{cd} \eta_d$ is a projection of vector $j^{cd}\eta_d$ into V_A. The two components do not vanish: $a = 1, 2$ for $A = 1$ and $a = 3, 4$ for $A = 2$. The vector $g^A_{ac} j^{cd} \eta_d / R_A$ is a unit vector with two non-vanishing components. The condition of independence of p_a on η is written in the form

$$\frac{j^{2c}\eta_c}{j^{1d}\eta_d} = v_1, \qquad \frac{j^{4c}\eta_c}{j^{3d}\eta_d} = v_2 \quad (2.21)$$

where v_1 and v_2 do not depend on η and are treated as mean velocities for the first and second particle.

Let us introduce the designation

$$j^{13} = \rho \quad (2.22)$$

Then from (2.21) it follows due to arbitrariness of η

$$j^{23} = \rho v_1, \qquad j^{14} = \rho v_2 \quad (2.23)$$

$$j^{24} = \rho v_1 v_2 \quad (2.24)$$

It is easy to see that (1.7) follows from (2.22), (2.23) and (2.24). Substitution of (2.22)–(2.24) into (2.20) leads to

$$p_{\binom{0}{A}} = -\frac{m_A c^2}{\sqrt{(1 - v_A^2/c^2)}}, \qquad p_{\binom{1}{A}} = \frac{m_A v_A}{\sqrt{(1 - v_A^2/c^2)}} \qquad (A = 1, 2) \quad (2.25)$$

Here, another mode of designation is used. The index $a = 1, 2, 3, 4$ is replaced by the double index $\binom{i}{\tilde{A}}$ $(i = 0, 1; A = 1, 2)$

$$1 \to \binom{0}{1}, \qquad 2 \to \binom{1}{1}, \qquad 3 \to \binom{0}{2}, \qquad 4 \to \binom{1}{2} \quad (2.26)$$

where A is the number of the particle and i is the tensor index in space V_A.

Due to the arbitrariness of η, from (2.14) we have

$$j^{ab} = \frac{\partial^2 J}{\partial \tau_a \partial \eta_b} \quad (2.27)$$

After transformations, the equations (2.17) are reduced to the form

$$\frac{\partial v_A}{\partial t_1} + v_1 \frac{\partial v_A}{\partial q_1} = 0 \qquad (t_1 = q_1{}^0; q_1 = q_1{}^1; A = 1,2)$$

$$\frac{\partial v_A}{\partial t_2} + v_A \frac{\partial v_A}{\partial q_2} = 0 \qquad (t_2 = q_2{}^0; q_2 = q_2{}^1; A = 1,2) \tag{2.28}$$

The j^{ab} obeys a conservation law

$$\partial_a j^{ab} = 0 \tag{2.29}$$

This is verified easily by substituting (2.27) into (2.29). The identity appears.

Substituting (2.22)–(2.24) into equation (2.29), one obtains the following for $b = 1$ and $b = 3$:

$$\frac{\partial \rho}{\partial t_1} + \frac{\partial(\rho v_1)}{\partial q_1} = 0$$

$$\frac{\partial \rho}{\partial t_2} + \frac{\partial(\rho v_2)}{\partial q_2} = 0 \tag{2.30}$$

It is easy to verify that equation (2.29) for $b = 2, 4$ is a corollary of (2.28) and (2.29).

Thus, j^{ab} is determined by three independent quantities ρ, v_1, v_2, which obey six equations (2.28) and (2.30). The equations determine derivatives of ρ, v_1, v_2 with respect to t_1 and t_2 if ρ, v_1, v_2 are given as a function of q_1, q_2. The two-time system of equations is obtained. One can verify that all equations are compatible, and the compatibility conditions do not add any new restrictions.

3. *Quantum Ensemble*

In the foregoing section the equations for a simple ensemble were obtained from the variational principle. It is necessary to generalise the method for a non-simple ensemble where the condition (1.7) is not fulfilled.

It will be done in this section. The expression for action of the statistical ensemble of the non-deterministic system will be obtained. It will be shown that the description of the non-deterministic system by means of this ensemble is equivalent to the quantum mechanical description of two independent particles.

(1.7) follows from relation (2.27), identity (2.16) and relation (1.5). The equation (2.16) does not occur if in (2.15) and (2.16) J is replaced by

$$J' = \sum_{i=1}^{s-1} \frac{\partial(\tau, \eta, \xi_{2i+1}, \xi_{2i+2})}{\partial(x^1, x^2, x^3, x^4)} \qquad (s > 2) \tag{3.1}$$

where ξ are variables.

Besides (2.16), the relation (1.7) follows from (2.21) if it is valid for arbitrary function η. To avoid this the arbitrariness of choice of the surface $\eta =$ constant has to be restricted. Let us suppose that

$$\eta = \eta(t_1, t_2) \tag{3.2}$$

This is a non-relativistical condition, and it is no use considering relativistic action. Expand the Lagrangian (2.15) in a series over powers of c^{-1}. A non-relativistic case is supposed to occur, i.e. inequalities are fulfilled

$$\frac{1}{c}|j^{14}| \ll |j^{13}|, \qquad \frac{1}{c}|j^{23}| \ll |j^{13}|$$

$$\frac{1}{c^2}|j^{24}| \ll |j^{13}| \tag{3.3}$$

Giving to action a non-relativistic form, the relativistic content is conserved; namely, the two-time and tensor character of the ensemble state j^{ab}.

Let us consider now an ensemble of non-deterministic systems. An action is chosen in the form

$$S = S[j^{ab}, p_a, \xi_i] = S_{cl} + S_q \tag{3.4}$$

$$S_{cl} = \int d^4x\, \delta(\eta - C)\left\{\sum_{A=1}^{2} m_A \frac{(j^{(1)_A{}_b}\eta_b)^2}{2j^{(0)_A{}_c}\eta_c} - p_a\eta_b\left(j^{ab} - \frac{\partial^2 J}{\partial\tau_a\,\partial\eta_b}\right)\right\} \tag{3.5}$$

$$S_q = -\int d^4x\, \delta(\eta - C)\sum_{A=1}^{2}\frac{\hbar^2}{2m_A}\frac{[(\partial j^{(0)_A{}_a}/\partial q_A{}^1)\eta_a]^2}{j^{(0)_A{}_c}\eta_c} \tag{3.6}$$

Here, both manners of designation (2.26) are used. S_q is an additional term, which takes into account non-determinism of motion of individual systems, i.e. the influence of deflection of individual system motion from mean motion upon mean motion. The more the gradient of state density j^{ab} is, the more the influence. In the non-relativistic case there is only one large component $j^{13} = \rho$. This is taken into account in (3.6). Planck's constant $\hbar$ is a factor of proportionality and describes the influence of the chaotic motion upon the mean one.

I shall call statistical ensemble (3.4)–(3.6) the quantum ensemble for two particles. The prime by J' will be omitted.

Let (1.5) be fulfilled. j^{ab} and p_a do not depend on η. Then, using (2.22) one gets relations (2.23), and instead of (2.25) one gets relations

$$\pi_A = p_{(1)_A} = m_A v_A \qquad (A = 1,2)$$

$$\varepsilon_A = p_{(0)_A} = -\frac{\pi_A{}^2}{2m_A} + \frac{\hbar^2}{2m_A}\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial q_A{}^1\,\partial q_A{}^1} \qquad (A = 1,2) \tag{3.7}$$

Variation with respect to p_A leads to relation (2.14). For all cases except $a = 2$, $b = 4$ and $a = 4$, $b = 2$, (2.27) follows from (2.14) due to (3.2). The quantity j^{24} is absent in equations obtained as a result of variation of (3.4) therefore j^{24} can be chosen in such a way that equation (2.29) should always be fulfilled.

Variation with respect to ξ_c ($c = 3,4,\ldots,2S$) leads to equation (2.19), where $c = 1, 2, \ldots 2s$.† Using (2.27) and identities

$$\frac{\partial}{\partial x^a}\frac{\partial^3 J}{\partial\tau_a\,\partial\eta_b\,\partial\xi_{i,d}} = 0 \qquad \left(\xi_{i,d} \equiv \frac{\partial\xi_i}{\partial x^d}\right) \tag{3.8}$$

$$\sum_{i=1}^{2S}\frac{\partial^3 J}{\partial\tau_a\,\partial\eta_b\,\partial\xi_{i,d}}\xi_{i,c} = \frac{\partial^2 J}{\partial\tau_a\,\partial\eta_b}\delta_c{}^d + \frac{\partial^2 J}{\partial\tau_a\,\partial\eta_b}\delta_c{}^a + \frac{\partial^2 J}{\partial\tau_a\,\partial\eta_b}\delta_c{}^b \tag{3.9}$$

one can get from (2.19)

$$R_{ab}z^b = 0 \tag{3.10}$$

where

$$R_{ab} = \tfrac{1}{2}\varepsilon_{abcd}\frac{\partial^2 J}{\partial\tau_c\,\partial\eta_d}$$

$$f_{ab} = \partial_a p_b - \partial_b p_a, \qquad z^d = \tfrac{1}{2}\varepsilon^{abcd}f_{bc}\eta_d \tag{3.11}$$

and ε_{abcd} is antisymmetrical over all indices unit pseudo-tensor.

Let

$$\det\|R_{ab}\| = \left(\frac{\partial^2 J}{\partial\tau_2\,\partial\eta_4}j^{13} - j^{14}j^{23}\right)^2 \neq 0 \tag{3.12}$$

Then (3.10) has only trivial solution $z^b = 0$, and therefore

$$f_{ab}\eta_c + f_{ca}\eta_b + f_{bc}\eta_a = \varepsilon_{abcd}z^d = 0. \tag{3.13}$$

Being equal to zero coefficients before η_1 and η_2, one gets

$$f_{ab} = \partial_a p_b - \partial_b p_a = 0 \tag{3.14}$$

for any combinations of a and b except $a = 1$, $b = 3$ and $a = 3$, $b = 1$.

In the case, when $\det\|R_{ab}\| = 0$, it follows from (3.10)

$$f_{12} = 0, \qquad f_{34} = 0$$

$$f_{23} = -v_2 f_{24}, \qquad f_{14} = -v_1 f_{24} \tag{3.15}$$

Consider the general case (3.12). The general solution of (3.14) has the form

$$\pi_A = \frac{\partial\phi}{\partial q_A}, \qquad \varepsilon_A = \frac{\partial\phi}{\partial t_A} + f_A(t_1,t_2) \qquad (A = 1,2) \tag{3.16}$$

where

$$\phi = \phi(x), \qquad x = \{x^1,x^2,x^3,x^4\}, \qquad f_A = f_A(t_1,t_2)$$

are arbitrary functions of their arguments. Substituting (3.16) into (3.7) one obtains

$$\frac{\partial\phi}{\partial t_A} + \frac{1}{2m_A}\left(\frac{\partial\phi}{\partial q_A}\right)^2 - \frac{\hbar^2}{2m_A}\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial q_A\,\partial q_A} = f_A(t_1,t_2) \tag{3.17}$$

The condition of compatibility of two equations (3.17) has the form

$$f_A = \frac{\partial f}{\partial t_A} \qquad [A = 1,2; f = f(t_1,t_2)] \tag{3.18}$$

† Equation (2.19) is the identity for $c = 1, 2$ ($\tau = \xi_1, \eta = \xi_2$).

Hence the f can be included in the ϕ and one can suppose in (3.17) without loss of generality $f_1 = f_2 = 0$.

By means of (3.16) and (3.7) the conservation laws (2.30) can be written in the form

$$\frac{\partial \rho}{\partial t_A} + \frac{\partial}{\partial q_A}\left(\frac{\rho}{m_A}\frac{\partial \phi}{\partial q_A}\right) = 0 \qquad (A = 1,2) \tag{3.19}$$

Equations (3.17), (3.19) describe the evolution of ρ and ϕ in times t_1 and t_2. The compatibility of these equations can be proved by a direct verification. In order that the conservation laws (2.23) are valid in the cases $b = 2$ and $b = 4$, it is necessary to suppose instead of (2.24)

$$j^{24} = \rho v_1 v_2 + \Phi,$$

$$\Phi = -\frac{\hbar^2}{4m_1 m_2}\left(\frac{\partial^2 \rho}{\partial q_1 \partial q_2} - \frac{1}{\rho}\frac{\partial \rho}{\partial q_1}\frac{\partial \rho}{\partial q_2}\right) + \alpha \tag{3.20}$$

where α does not depend on q_1 and q_2.

Let $\alpha = 0$. Now the ensemble state j^{ab} is determined by two quantities ρ and ϕ. Neglecting quantum effects ($\hbar = 0$), or in the case when $\rho = \rho_1(q_1)\rho_2(q_2)$, the ensemble satisfies the condition (1.6), and it may be corresponded by a simple ensemble.

Let us multiply (3.17) by $\sqrt{\rho}\exp(i\phi/\hbar)$ and (3.19) by $i\hbar\exp(i\phi/\hbar)/(2\sqrt{\rho})$ and add them. As a result one has

$$i\hbar\frac{\partial \psi}{\partial t_A} + \frac{\hbar^2}{2m_A}\frac{\partial^2 \psi}{\partial q_A \partial q_A} = 0 \qquad (A = 1,2) \tag{3.21}$$

where

$$\psi = \sqrt{\rho}\exp(i\phi/\hbar) \tag{3.22}$$

Compatibility of two equations (3.21) is evident. Equation (3.21) is equivalent to the equations (3.17)–(3.19) and equations (3.17), (3.19) are compatible too. Ensemble evolution is completely described by equations.

Let us take the non-relativistical point of view, i.e. renounce the two-time description. This means that only ensemble behaviour at like times $t_1 = t_2$ is of interest. Let the plane $t_1 = t_2$ of space V_{12} be called V_3'. Let us realize a transformation of variables in (3.21)

$$t = \frac{t_1 + t_2}{2}, \qquad T = \frac{t_1 - t_2}{2} \tag{3.23}$$

As a result one has instead of (3.21)

$$\left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial q_1 \partial q_1} + \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial q_2 \partial q_2}\right)\psi = 0 \tag{3.24}$$

$$\left(i\hbar\frac{\partial}{\partial T} + \frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial q_1 \partial q_1} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial q_2 \partial q_2}\right)\psi = 0 \tag{3.25}$$

It is remarkable that equation (3.24) does not contain a derivative with respect to T, instead it contains T as a parameter. This means that if function ψ is given at a certain t and $T=0$ (i.e. $t=t_1=t_2$), then ψ can be calculated by means of (3.24) for any t and $T=0$, equation (3.25) being not used.

Thus in the space V_3' the ensemble evolution is described by (3.24), which is Schrödinger's equation for two particles.

Let us represent now the state of ensemble in the space V_3'. Let us introduce coordinates y^a, $a=-1, 0, 1, 2$ in the space V_{12} and coordinates y^i $i=0, 1, 2$ in the space V_3'

$$y^{-1}=T, \qquad y^0=t, \qquad y^1=q_1, \qquad y^2=q_2$$

In the space V_3' the system r-state is represented by a one-dimensional line, and therefore a number of r-states dN in a two-dimensional area $ds_i=c^{-1}\sqrt{|g_3|}\varepsilon_{ikl}d_1y^k d_2y^l$ is

$$dN=\sum_{i,k,l=0}^{2} c^{-1}\sqrt{|g_3|}\varepsilon_{ikl}j^i d_1y^k d_2y^l = j^i ds_i \tag{3.26}$$

where g_3 is determinant of the metric tensor in the space V_3', j^i is the state density in V_3', ε_{ikl} is a unit antisymmetrical over all indices pseudotensor in space V_3', d_1y^k, and d_1y^l are two infinitesimal vectors forming area ds_i. On the other hand, according to (1.3) and (1.4)

$$dN=(2c^2)^{-1}\sqrt{|g_4|}\sum_{a,b=-1}^{2}\sum_{k,l=0}^{2}\varepsilon_{abkl}j^{ab}d_1y^k d_2y^l \tag{3.27}$$

Where g_4 is determinant of the metric tensor in V_{12}. Comparison of (3.26) with (3.27) leads to the relation

$$V|g_4/g_3|c^{-1}j^{-1i}, \quad (i=0,1,2) \tag{3.28}$$

By means of (2.22), (2.23) and (3.20) one calculates in the coordinate system y^a, that j^{ab} has the form

$$2j^{ab}=\begin{array}{c|cccc} & T & t & q_1 & q_2 \\ \hline & 0 & \rho & \rho v_1 & \rho v_2 \\ & -\rho & 0 & -\rho v_1 & \rho v_2 \\ & -\rho v_1 & \rho v_1 & 0 & \rho v_1\sigma_2+\Phi \\ & -\rho v_2 & -\rho v_2 & -\rho v_1 v_2-\Phi & 0 \end{array} \tag{3.29}$$

Using (3.28) one gets for j^i

$$j^0=V_2\,\rho, \qquad j^1=V_2\,\rho v_1, \qquad j^2=V_2\,\rho v_2. \tag{3.30}$$

The conservation law for j^i follows from (2.30)

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho v_1)}{\partial q_1}+\frac{\partial(\rho v_2)}{\partial q_2}=0 \tag{3.31}$$

If $\rho(q_1,q_2)$ is normed properly, then it can be treated as a probability density to detect the first particle at the point q_1 and the second one at the point q_2. j^1 and j^2 can be treated as spatial components of probability flux. j^i is expressed through ψ in the usual way.

4. *Energy and Momentum of Ensemble*

In this section it will be shown that expressions for energy, momentum and angular momentum of statistical ensemble coincide with expressions for these quantities defined according to the rules of quantum mechanics.

Energy, momentum and angular momentum can be attributed to any dynamical system, including quantum statistical ensemble. These quantities can be obtained in the canonical way from the variational principle. Let us present the equation of the integration surface $\eta = C$ in the form

$$x^a = x^a(\alpha) \quad (a = 1, 2, 3, 4; \alpha = \{\alpha_1, \alpha_2, \alpha_3\}) \tag{4.1}$$

Then expression (2.15) can be represented in the form of an integral over a region Ω of variables $\alpha_1, \alpha_2, \alpha_3$. Varying functions (4.1), the variation δS of action (2.15) can be expressed in the form of integral over boundary $\sum$ of region Ω, becuase coefficients before variations δx^a vanish inside Ω due to equations of motion.

Let us designate the two-dimensional region of variables x^a corresponding to region $\sum$ of variables $\alpha_1, \alpha_2, \alpha_3$ through σ. The variation δS can be expressed in the form of an integral over region σ of variables x^a

$$\delta S = \tfrac{1}{2} \int_\sigma T_b^{ac}\, \delta x^b\, ds_{ac} \qquad (a,b,c = 1,2,3,4) \tag{4.2}$$

where ds_{ab} is given by the expression (1.3).

It is remarkable that in the case when condition (3.2) is fulfilled, T_b^{ac} does not depend on the manner of parametrisation (4.1) and the choice of surface $\eta = C$, if its boundary σ is fixed. T_b^{ac} is energy-momentum tensor. T_b^{ac} is a tensor of rank 3, not of 2 as is usually the case. It is connected with the fact that the action variation is expressed in the form of an integral over a two-dimensional surface in a four-dimensional space. T_b^{ac} satisfies a conservation law

$$\eta_c \frac{\partial}{\partial x^a} T_b^{ac} = 0 \qquad (b = 1,2,3,4)$$

As $\eta_2 = \eta_4 = 0$ and η_1, η_3 are arbitrary

$$\frac{\partial}{\partial x^a} T_b^{ac} = 0 \qquad \text{for } c = 1,3,\ b = 1,2,3,4 \tag{4.3}$$

The conservation law allows us to introduce a conservative vector in space V_{12}

$$P_b = -\tfrac{1}{2} \int_{\sigma'} T_b^{ac}\, ds_{ac} = \int_{-\infty}^{\infty}\int T_b^{13}\, dq_1\, dq_2 \tag{4.4}$$

where σ' is chosen in the form $t_1 = \text{constant}$, $t_2 = \text{constant}$.

According to definition, P_b is an additive quantity. Besides, components $P_1 = P_{(1)}^{(2)}$ and $P_3 = P_{(2)}^{(2)}$ are connected with t_1 and t_2 time translations respectively. For this reason, according to statistical principle they can be identified with the mean energy of the first and second particles respectively.

$P_2 = P_{\binom{1}{1}}$ and $P_4 = P_{\binom{2}{2}}$ are connected with spatial translations in the directions of axes q_1 and q_2 and can be identified with the mean momentum of the first and second particles respectively.

Thus the variational principle allows us to obtain not only energy and momentum but two energies and two momenta according to a number of particles in the system.

The calculation gives for T_b^{ac}

$$T_b^{ac} = -\tfrac{1}{2}\Bigg\{p_b \frac{\partial^2 J}{\partial\tau_a \partial\eta_c} + \sum_{A=1}^{2} \frac{\hbar^2}{2m_A}(-1)^{1-A} \rho_{AA} \delta^{a,\binom{0}{3-A}} \delta_b{}^c$$

$$+ \sum_{A=1}^{2} \frac{\hbar^2}{2m_A}\left[\rho_{AA} v_{3-A} + \frac{\rho_A{}^2}{\rho} v_{3-A} + \frac{\rho_A}{\rho}\frac{\partial}{\partial q_A}(\rho v_{3-A})\right](-1)^{1-A} \delta^{a,\binom{1}{3-A}} \delta_b{}^c$$

$$+ \sum_{A=1}^{2} \frac{\hbar^2}{2m_A}\frac{\rho_A}{\rho} \rho_b j^{\binom{0}{A}c} \delta^{a,\binom{1}{A}}\Bigg\} - (c \leftrightarrow a) \tag{4.5}$$

where

$$\rho_A = \frac{\partial \rho}{\partial q_A}, \qquad \rho_{AA} = \frac{\partial^2 \rho}{\partial q_A \partial q_A}, \qquad \rho_b = \frac{\partial \rho}{\partial x^b}$$

$-(c \leftrightarrow a)$ means that it is necessary to subtract the term obtained from the foregoing term by transposition indices c and a.

For energy-momentum density in particular, one gets

$$T_b^{13} = -p_b \rho + \frac{\hbar^2}{2m_1}\delta_b{}^1 \frac{\partial^2 \rho}{\partial q_1 \partial q_1} + \frac{\hbar^2}{2m_2}\delta_b{}^3 \frac{\partial^2 \rho}{\partial q_2 \partial q_2} \tag{4.6}$$

Let us introduce an operator

$$\hat{p}_A = -i\hbar \frac{\partial}{\partial q_A}, \qquad A = 1, 2 \tag{4.7}$$

and suppose that there is such an ensemble that the wave function (3.22) vanishes rapidly enough if $q_1, q_2 \to \infty$ and it can be normed by

$$\int_{-\infty}^{\infty}\int \psi^* \psi \, dq_1 \, dq_2 = 1 \tag{4.8}$$

where ψ^* is the complex conjugate of ψ.

Substituting (4.6) into (4.4) and using (3.22) (3.7), one gets for mean energy $\langle E_A \rangle$ and mean momentum $\langle P_A \rangle$ of the particle

$$\langle E_A \rangle = P_{\binom{0}{A}} = \int_{-\infty}^{\infty}\int \psi^* \frac{\hat{p}_A \hat{p}_A}{2m_A} \psi \, dq_1 \, dq_2 \qquad (A = 1, 2) \tag{4.9}$$

$$\langle P_A \rangle = P_{\binom{1}{A}} = -\int_{-\infty}^{\infty}\int \psi^* \hat{p}_A \psi \, dq_1 \, dq_2 \qquad (A = 1, 2) \tag{4.10}$$

Expressions (4.9), (4.10) coincide with the expressions for energy and momentum which are given by quantum mechanics. The mean value of any function $F(q_1,q_2)$ is calculated by the formula

$$\langle F\rangle = \int\int_{-\infty}^{\infty} \psi^* F(q_1,q_2)\psi \, dq_1 \, dq_2 \tag{4.11}$$

because $\psi^*\psi$ is the probability density for the first particle at point q_1 and the second one at point q_2.

Thus, formulae (4.9)–(4.11) allow us to calculate the mean value of the arbitrary function of spatial coordinates, and the mean energy and mean momentum of each particle (angular momentum is equal to zero in one-dimensional configuration space).

Traditional quantum mechanics (Neumann, 1932) uses the following rule for calculation of the mean value of an arbitrary function of coordinates and momenta $F(q_1,q_2,p_1,p_2)$

$$\langle F\rangle = \int\int_{-\infty}^{\infty} \psi^* F(q_1,q_2,\hat{p}_1,\hat{p}_2)\psi \, dq_1 \, dq_2 \tag{4.12}$$

From my point of view the rule (4.12) is a formal extrapolation of relations (4.9)–(4.11). Correctness of this extrapolation can be verified in experiment for arbitrary function of coordinates and for conservative quantities—energy, momentum, angular momentum—only. In this case formulae obtained from statistical principle coincide with traditional formulae of quantum mechanics (Rylov 1971).

5. *The Case of Identical Particles*

If particles are identical and indistinguishable then $j^{(0)(0)}_{(1)(2)}(t_1,q_1,t_2,q_2)$ $dq_1\,dq_2$ is to be treated as the probability of detecting one of the particles in the volume dq_1 at the moment t_1 and another in the volume dq_2 at the moment t_2. Besides an ensemble state j^{ab} has to be invariant with respect to transposition of arguments of the first and second particles, i.e.

$$j^{(i)(k)}_{(1)(2)}(t_1,q_1;t_2,q_2) = j^{(k)(i)}_{(1)(2)}(t_2,q_2;t_1,q_1) \tag{5.1}$$

Due to (2.22), (2.23), (3.7) and (3.16) it follows from (5.1) that

$$\rho(t_1,q_1;t_2,q_2) = \rho(t_2,q_2;t_1,q_1)$$

$$\frac{\partial}{\partial q_1}\phi(t_1,q_1;t_2,q_2) = \frac{\partial}{\partial q_1}\phi(t_2,q_2;t_1,q_1) \tag{5.2}$$

In the case $t = t_1 = t_2$ it follows from (5.2) that

$$\rho(t,q_1,q_2) = \rho(t,q_2,q_1)$$
$$\phi(t,q_1,q_2) = \phi(t,q_2,q_1) \tag{5.3}$$

Due to (5.3) the wave function (3.22) satisfies the condition

$$\psi(t,q_1,q_2) = \pm\psi(t,q_2,q_1) \tag{5.4}$$

(5.4) has a double sign because (3.22) contains a radical. In quantum mechanics the particle's identity is known to be taken into account by means of relation (5.4). The sign of the right-hand side depends on a spin of particle. At this stage spin of particle is not yet introduced, and the sign in (5.4) remains indefinite.

6. *Conclusions*

It was shown in the example of two free particles that the basic relations of quantum mechanics can be obtained by proceeding from the conception of classical non-deterministic particles. Non-relativistic quantum mechanics are obtained from statistics of world-lines. It is remarkable that, from this point of view, non-relativistic quantum mechanics is a relativistic theory, because the use of relativistic notion of state (r-state) is a matter of principle. From this point of view the quantum properties are the result of relativism and statistics. The relativism is displayed in quantum mechanics on the level of notions, where transition to non-relativistical approximation is impossible.

I hope that non-relativistical quantum mechanics of interacting particles can be represented as a non-relativistical approximation of the relativistic statistics of world-lines. Then, non-relativistical quantum mechanics will be grounded in the same sense as thermodynamics was grounded by means of statistical theory. The region of applicability of quantum mechanics will be found as comprehensively as that for thermodynamics. On the other hand, using the explicit relation of the world-line statistics can bring about the creation of 'relativistical quantum mechanics' on the new base.

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