Restricted linear properties of scalar product in T-geometry

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Abstract

It is shown that scalar product of two vectors can be introduced in any geometry (metric space) independently of possibility of the linear space introduction. Linear properties of scalar product are restricted. Domain of linearity increases in homogeneous geometry.

T-geometry [1] is a very general geometry giving on an arbitrary set Ω of points by setting the world function σ:

$$\sigma: \quad \Omega \times \Omega \to \mathbb{R}, \qquad \sigma(P,P) = 0, \qquad \forall P, Q \in \Omega \tag{1}$$

T-geometry is symmetric, if additionally

$$\sigma(P,Q) = \sigma(Q,P) \tag{2}$$

The T-geometry has no its own axiomatics. It is obtained as a result of a deformation of the proper Euclidean geometry. In other words, any relation of T-geometry is obtained as a result of some deformation of the corresponding relation of the proper Euclidean geometry. The relation of Euclidean geometry is written in σ-immanent form (i.e. in terms of only world function σ). Thereafter the world function of the proper Euclidean space is replaced by the world function of the T-geometry in question. Corresponding relation of T-geometry arises. Representation of a proper Euclidean relation in σ-immanent form is always possible, because the proper Euclidean geometry can be formulated σ-immanently (in terms of only world function) [1].

Vector $\mathbf{P}_0\mathbf{P}_1 \equiv \overrightarrow{P_0P_1}$ in T-geometry is the ordered set of two points $\mathbf{P}_0\mathbf{P}_1 = \{P_0, P_1\}, \quad P_0, P_1 \in \Omega$. The scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is defined by the relation

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \sigma(P_0,Q_1) + \sigma(P_1,Q_0) - \sigma(P_0,Q_0) - \sigma(P_1,Q_1), \tag{3}$$

for all $P_0, P_1, Q_0, Q_1 \in \Omega$. As it follows from (1) – (3), in the symmetric T-geometry

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_1), \qquad P_0, P_1, Q_0, Q_1 \in \Omega \tag{4}$$

When the world function σ is such one [1] that the σ-space $V = \{\sigma, \Omega\}$ is the n-dimensional proper Euclidean space E_n the scalar product (3) turns to scalar product of two vectors in E_n. Besides as it follows from (1), (3) that in any T-geometry

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = -(\mathbf{P}_1\mathbf{P}_0.\mathbf{Q}_1\mathbf{Q}_0), \qquad \forall P_0, P_1, Q_0, Q_1 \in \Omega \tag{5}$$

$$(\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_1) = -(\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_1\mathbf{P}_0), \qquad \forall P_0, P_1, Q_0, Q_1 \in \Omega \tag{6}$$

$$\begin{aligned}
(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) + (\mathbf{P}_1\mathbf{P}_2.\mathbf{Q}_0\mathbf{Q}_1) &= (\mathbf{P}_0\mathbf{P}_2.\mathbf{Q}_0\mathbf{Q}_1), &(7)\\
(\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_1) + (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_1\mathbf{P}_2) &= (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_2.), &(8)
\end{aligned}$$

for all $P_0, P_1, P_2, Q_0, Q_1 \in \Omega$

Let us consider relations between two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{R}_0\mathbf{R}_1$

$$\mathbf{P}_0\mathbf{P}_1 \mathcal{T} \mathbf{R}_0\mathbf{R}_1 : \quad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = (\mathbf{R}_0\mathbf{R}_1.\mathbf{Q}_0\mathbf{Q}_1) \wedge (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_1) = (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{R}_0\mathbf{R}_1) \tag{9}$$

for all $Q_0, Q_1 \in \Omega$. The relation (9) is reflexive, symmetric and transitive. It can be considered to be equivalence relation. Thus, two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{R}_0\mathbf{R}_1$ are equivalent ($\mathbf{P}_0\mathbf{P}_1 = \mathbf{R}_0\mathbf{R}_1$), if the conditions

$$\begin{aligned}
(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) &= (\mathbf{R}_0\mathbf{R}_1.\mathbf{Q}_0\mathbf{Q}_1), \qquad \forall Q_0, Q_1 \in \Omega &(10)\\
(\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_1) &= (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{R}_0\mathbf{R}_1), \qquad \forall Q_0, Q_1 \in \Omega &(11)
\end{aligned}$$

take place. Then one obtains from (5) – (8)

$$\mathbf{P}_1\mathbf{P}_0 = -\mathbf{P}_0\mathbf{P}_1 = \alpha \mathbf{P}_0\mathbf{P}_1, \qquad \alpha = -1 \tag{12}$$

$$\mathbf{P}_0\mathbf{P}_2 = \mathbf{P}_0\mathbf{P}_1 + \mathbf{P}_1\mathbf{P}_2, \qquad \mathbf{P}_0\mathbf{P}_1 = \mathbf{P}_0\mathbf{P}_2 - \mathbf{P}_1\mathbf{P}_2 \tag{13}$$

Relations (12), (13) determine multiplication of a vector by a real number $\alpha = \pm 1$ and summation of vectors. But vectors in the σ-space $V = \{\sigma, \Omega\}$ are not free vectors and two vectors may be added, provided end of one vector is the origin of other vector. At such a definition of the vector summation the scalar product of vectors have linear properties

$$\begin{aligned}
(\alpha \mathbf{P}_0\mathbf{P}_1 + \boldsymbol{\beta} \mathbf{R_0R_1}.\mathbf{Q}_0\mathbf{Q}_1) &= \alpha (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) + \boldsymbol{\beta} (\mathbf{R_0R_1}.\mathbf{Q}_0\mathbf{Q}_1) &(14)\\
(\mathbf{Q}_0\mathbf{Q}_1.\alpha \mathbf{P}_0\mathbf{P}_1 + \boldsymbol{\beta} \mathbf{R_0R_1}) &= \alpha (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_1) + \boldsymbol{\beta} (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{R_0R_1}) &(15)
\end{aligned}$$

$$\alpha = \pm 1, \beta = \pm 1, \qquad P_0, P_1, R_0, R_1 \in \Omega, \ \ \forall Q_0, Q_1 \in \Omega$$

The linear properties of the scalar product are restricted as compared with those in the Euclidean space, because according to (5) – (8) the linear operations on vectors are always defined only in the following cases:

1. $\alpha = 0 \vee \beta = 0$

2. $\alpha = \beta = \pm 1$, $(P_1 = R_0) \vee (R_1 = P_0)$

3. $\alpha = -\beta = \pm 1$, $(P_1 = R_1) \vee (P_0 = R_0)$

In other cases the linear operations on vectors are not defined, in general, but they may be defined in some special T-geometries, when relations (14), (15)

$$\begin{array}{c} (\mathbf{S_0S_1}.\mathbf{Q}_0\mathbf{Q}_1) = \alpha\,(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) + \boldsymbol{\beta}\,(\mathbf{R_0R_1}.\mathbf{Q}_0\mathbf{Q}_1) \\ (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{S_0S_1}) = \alpha\,(\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_1) + \boldsymbol{\beta}\,(\mathbf{Q}_0\mathbf{Q}_1.\mathbf{R_0R_1}) \end{array} \qquad (16)$$

take place for all vectors $\mathbf{Q_0Q_1}$ and some $\alpha, \beta \in \mathbb{R}$. Then by definition

$$\mathbf{S_0S_1} = \alpha\mathbf{P}_0\mathbf{P}_1 + \beta\mathbf{R_0R_1} \qquad (17)$$

In general case, there is no vector $\mathbf{S_0S_1}$, satisfying (16), for all vectors $\mathbf{Q_0Q_1}$, and linear operations are defined only in the case, when (5) – (8) are fulfilled.

Let $E_n = \{\sigma_\mathrm{E}, \mathbb{R}^n\}$ be n-dimensional proper Euclidean space. Remove some set Ω' of points from $\mathbb{R}^n$. The σ-space $V = \{\sigma_\mathrm{E}, \Omega\}$, $\Omega = \mathbb{R}^n \backslash \Omega'$ is a result of deformation of Euclidean space E_n, because deletion of points is a special kind of deformation. Linear operation (17) is defined in E_n for all points P_0, P_1, R_0, R_1 and all real α, β. Linear operation (17) is defined in V for some real α, β and for $P_0, P_1, R_0, R_1 \in \Omega$, provided there are such points $S_0, S_1 \in \Omega$ that relations (16) are fulfilled for all $Q_0, Q_1 \in \Omega$. Thus, deletion of points from Euclidean space destroys its linear structure, but not completely. A change of the world function plays the same role. Nevertheless, there is a minimal linear structure (12), (13) which remains at any deformation of the Euclidean space.

References

[1] Yu. A. Rylov, Geometry without topology as a new conception of geometry. *Int. J. Math. Math. Sci.*, **30**, 733-760, (2002).