What is geometry? Physical geometry and mathematical geometry as two aspects of the proper Euclidean geometry.

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Abstract

Primordially a geometry was a science on properties of geometrical objects and their mutual disposition. Such interpretation of the term "geometry" is qualified as physical geometry. A use of only Euclidean geometry generated another interpretation of the term "geometry", which was interpreted as a logical construction. Such interpretation of the term " geometry" is qualified as mathematical geometry. Mathematical geometry cannot use for description of the space-time, generally speaking. Nevertheless the mathematical geometry has been used for description of the space-time during the twentieth century. This circumstance lead to problems in general relativity.

Key words: Physical geometry; mathematical geometry; multivariant geometry; metric approach; tachyon gas; dark matter;

1 Introduction

Geometry has been arisen many years ago as a science on a shape of geometrical objects and on their mutual disposition in space. It was the proper Euclidean geometry, which could be constructed as a logical construction. In the Euclidean geometry all geometrical statements are deduced from several basic statements (axioms). Study of the Euclidean geometry consists of prove of numerous theorems. As far as the geometry has been studied for several centuries, an illusion appeared that the prove of theorems is a content of a geometry and there are no geometry without theorems.

In reality the prove of theorems is only a method of a geometry construction but not the geometry itself. Any geometry is a set of statements on properties $\mathcal{P}$ of geometrical objects. In principle these properties $\mathcal{P}$ can be obtained by other methods. But only Euclidean method of a geometry construction has been known, and the illusion arose that this method is the geometry itself.

According to such a representation about geometry any geometry is considered in contemporary mathematics as a logical construction. The primordial representation of geometry as a science on a shape of geometrical objects and on their mutual disposition in space appeared to be forgotten. Although geometry is applied for description of geometrical objects in space or in space-time, but this application is considered as secondary. The primary representation about geometry is the statement that geometry is a logical construction. For instance, the symplectic geometry which has no relation to description of geometrical objects is qualified as a geometry, because it is a logical construction which is very close to the proper Euclidean geometry.

Along with representation about geometry as a logical construction there exist primordial representation on geometry as a science on a shape of geometrical objects and on their mutual disposition in space or in space-time [1]. Precisely this representation of geometry is important in physical applications, and it will be referred to as a physical geometry. On the contrary, representation of a geometry as a logical construction will be referred to as a mathematical geometry. In the contemporary mathematics the mathematical geometry dominates, and it is applied as a geometry of the space-time. The physical geometry is practically absent in contemporary mathematics. Some mathematicians state even that the physical geometry (i.e. the geometry described completely by a metric) is not a division of mathematics, because calculations and theorems are absent in physical geometry.

The physical geometry is determined completely, if the distance $\rho(P,Q)$ between any pair of points P,Q is given. All geometrical relations of a physical geometry are defined in terms of distance ρ and only in terms of ρ. These relations are taken from the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$, which is a mathematical geometry and physical geometry at once. Using the fact that the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ is a logical construction, one can construct all geometrical relations and geometrical objects of the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$. Thereafter these relations are expressed in terms of the distance function ρ_{E} of the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$. Replacing the distance ρ_{E} in the geometrical relations of $\mathcal{G}_{\mathrm{E}}$ by the distance ρ of a physical geometry $\mathcal{G}$, one obtains these geometric relations in the physical geometry $\mathcal{G}$. Thus, construction of a physical geometry does not need a use of Euclidean method of the geometry construction, which uses complicated proves of numerous theorems. Construction of a physical geometry uses the already created Euclidean geometry instead of a use of the Euclidean method of the geometry construction.

Instead of the distance ρ the physical geometry can be described by the world function $\sigma = \frac{1}{2}\rho^2$. Such a description is more simple and effective, because in the space-time the world function σ is always real, whereas ρ is imaginary for spacelike distances.

Such a property of the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ as dimension cannot be introduced in some physical geometries, because introduction of dimension in the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ is connected with special properties of the world function σ_{E} of $\mathcal{G}_{\mathrm{E}}$.

In $\mathcal{G}_{\mathrm{E}}$ the equivalence (equality) of two vectors $\mathbf{PQ}$ and $\mathbf{AB}$ is defined in terms of world function σ_{E} as follows

$$(\mathbf{AB}\mathrm{eqv}\mathbf{PQ}): \quad (\mathbf{AB}.\mathbf{PQ}) = |\mathbf{AB}| \cdot |\mathbf{PQ}| \wedge |\mathbf{AB}| = |\mathbf{PQ}| \tag{1.1}$$

where scalar product $(\mathbf{AB}.\mathbf{PQ})$ of vectors $\mathbf{AB}$ and $\mathbf{PQ}$ is defined by the relation

$$(\mathbf{AB}.\mathbf{PQ}) = \sigma_{\mathrm{E}}(A,Q) + \sigma_{\mathrm{E}}(B,P) - \sigma_{\mathrm{E}}(A,P) - \sigma_{\mathrm{E}}(B,Q) \tag{1.2}$$

The vector module $|\mathbf{AB}|$ is defined by the relation

$$|\mathbf{AB}| = \sqrt{2\sigma_{\mathrm{E}}(A,B)} \tag{1.3}$$

In $\mathcal{G}_{\mathrm{E}}$ n vectors $\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2, ...\mathbf{P}_0\mathbf{P}_n$ are linear dependent, if and only if the Gram determinant $F_n(\mathcal{P}^n)$ vanishes

$$F_n(\mathcal{P}^n) \equiv \det ||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)|| = 0, \quad i,k = 1,2,...n \tag{1.4}$$

where $\mathcal{P}^n = \{P_0, P_2, ...P_n\}$, and the scalar product $(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)$ is defined by the relation (1.2).

In the n-dimensional Euclidean space Ω_n the following conditions are fulfilled

$$\exists\mathcal{P}^n \equiv \{P_0, P_1, ...P_n\} \subset \Omega_n, \qquad F_n(\mathcal{P}^n) \neq 0, \qquad F_k\left(\Omega_n^{k+1}\right) = 0, \qquad k > n \tag{1.5}$$

which mean that n is the maximal number of linear independent vectors. This number n is the metric dimension of the space Ω_n, or the dimension of the physical geometry on Ω_n.

Conditions (1.5) are formulated in terms and only in terms of σ_{E}. In the case of arbitrary physical geometry $\mathcal{G}$ definitions of scalar product and linear dependence of vectors have the form (1.1) – (1.4) with the world function σ_{E} replaced by the world function σ of the physical geometry $\mathcal{G}$. These conditions are rather restrictive. They are not fulfilled for many physical geometries. In these physical geometries one cannot introduce dimension of a geometry.

Such a statement looks rather unexpected. For instance, construction of the Riemannian geometry begins from introduction of a manifold, having a fixed dimension n.

2 Space-time geometry of Minkowski

Space-time geometry of Minkowski is to be a physical geometry, because it is used for description of the space-time. It means that equality of two vectors $\mathbf{PQ}$ and $\mathbf{AB}$ is defined by the coordinateless relations (1.1) – (1.3). Usually one uses the mathematical geometry of Minkowski, where two vectors $\mathbf{PQ}$ and $\mathbf{AB}$ are equivalent, if and only if their coordinates $\mathbf{PQ}_k$ and $\mathbf{AB}_k$ are equal

$$\mathbf{PQ}_k = \mathbf{AB}_k, \quad k = 0, 1, 2, 3 \tag{2.1}$$

For timelike vectors $\mathbf{PQ}$ and $\mathbf{AB}$ relations (1.1) and (2.1) coincide, but for spacelike vectors $\mathbf{PQ}$ and $\mathbf{AB}$ the relations (1.1) and (2.1) are different. According to (2.1) at the point A there is one and only one vector $\mathbf{AB}$, which is equivalent to spacelike vector $\mathbf{PQ}$. According to (1.1) at the point A there are many spacelike vectors $\mathbf{AB}$, $\mathbf{AB}'$, $\mathbf{AB}''$..., which are equivalent to vector $\mathbf{PQ}$, but vectors $\mathbf{AB}$, $\mathbf{AB}'$, $\mathbf{AB}''$...are not equivalent between themselves. For instance, in physical geometry vectors $\mathbf{AB} = \{r, r\sin\phi, r\cos\phi, z\}$, $\mathbf{AB}' = \{r', r'\sin\phi', r'\cos\phi', z\}$ are equivalent to the spacelike vector $\mathbf{PQ} = \{0, 0, 0, z\}$ for arbitrary values of quantities $r, r'\phi, \phi'$, but vectors $\mathbf{AB}$ and $\mathbf{AB}'$are not equivalent, generally speaking. The physical geometry, where vectors have the property, that there are many vectors $\mathbf{AB}$, $\mathbf{AB}'$, $\mathbf{AB}''$...are equivalent to some vector $\mathbf{PQ}$, but vectors $\mathbf{AB}$, $\mathbf{AB}'$, $\mathbf{AB}''$...are not equivalent between themselves, is called multivariant physical geometry. In the multivariant geometry the equivalency relation is intransitive, and this geometry is nonaxiomatizable, i.e. it cannot be presented as a logical construction, because in any logical construction the equivalence relation is transitive.

Thus, if one uses mathematical geometry of Minkowski for description of the space-time, the description of spacelike vectors and spacelike world lines will be incorrect. The reason of this incorrectness is the coordinate definition (2.1) of the vector equivalence. The true definition of the vector equivalence must be coordinateless. This definition must be formulated in terms of the world function and only in terms of the world function (1.1).

Describing tachyons, i.e. particles with spacelike world lines, in the mathematical geometry of Minkowski, one concludes that the tachyon world lines are smooth. As far such particles are not discovered experimentally, one concludes, that tachyons do not exist.

Describing tachyons in the physical geometry of Minkowski, one obtains that tachyon world line is not smooth. It wobbles with infinite amplitude, because of the geometry is multivariant with respect to spacelike vectors. A single tachyon cannot be discovered, because of the its world line wobbling. The fact, that single tachyons were not discovered experimentally, does not mean that tachyons do not exist. A single tachyon cannot be discovered, but the tachyon gas can be discovered by its gravitational field. The tachyon gas may form so called dark matter, which has been discovered in cosmos [2, 3].

Of course, existence of the tachyon gas does not mean that there are no other particles of the dark matter. But existence of the tachyon gas resolves the problem of dark matter, even if there are no other particles of the dark matter. It should note, that the explanation of the dark matter does not need any special hypotheses. A use of physical space-time geometry solves the problem freely.

3 Riemannian space-time geometry

Describing the general relativity, one uses mathematical Riemannian geometry of space-time as the most general kind of geometry. However, the pseudo-Riemannian geometry, which is used for description of the space-time is not a mathematical geometry, because it is multivariant with respect to spacelike vectors as the geometry of Minkowski. Besides, it is multivariant with respect to vectors $\mathbf{PQ}$ and $\mathbf{AB}$, if the points P and A do not coincide $P \neq A$. This phenomenon is known as absence fernparallelism in the Riemannian geometry, where parallelism is considered in the concepts of mathematical geometry. Although problem of the fernparallelism is solved by means of the so called parallel transport, nevertheless inconsistency of the Riemannian space-time geometry as a mathematical geometry remains.

But the main defect of the Riemannian geometry as a space-time geometry is conditioned by the fact, that the Riemannian geometry is not the most general kind of the space-time geometry. The set of physical geometries is much more powerful, than the set of Riemannian geometries. It means, that describing an arbitrary space-time, one should use a physical geometry.

The general relativity equations, written in terms of a physical geometry are coordinateless. They are written for the world function directly. Besides, they differ from the conventional gravitational equations, written for metric tensor on the basis of mathematical space-time geometry.

Dynamic equations of general relativity GR can be extended on the case of non-Riemannian (physical) space-time geometry [4]. As a result one obtains the extended general relativity (EGR). Dynamic equations of EGR are written in coordinateless form for the world function σ of the space-time directly, but not for metric tensor g_{ik} as in GR. Dynamic equations of EGR admits one to generate induced antigravitation which prevents from formation of black holes [5].

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