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HOMOGENIZED POISSON'S RATIO OF POROUS MEDIA

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Abstract:

Homogenized Poisson's ratio variation due to variation of porosity in porous media containing closed pores is analyzed by application of the two-scale asymptotic analysis combined with the periodic boundary integral equation method.

1 Introduction

The considered porous medium is modeled by a deterministic scheme based on a regular space lattice with inclusions located at the nodes. Porous media can have different kinds of lattices, each having pores of specific geometry and orientation placed at the corresponding nodes. Uniformly distributed pores are modeled by the spheres located at the nodes of the face-centered cubic lattice (*FCC*).

In deriving basic equations it is assumed that the medium is elastic and anisotropic, and that no restrictions on the specific kind of anisotropy is imposed. Numerical computations will be implemented for an isotropic medium with spherical pores. The other assumption concerns the displacement field, which is supposed to be infinitesimal, so equations of the linear theory of elasticity can be applied.

The main problem for a porous medium with uniformly distributed pores is in its effective characteristic determination; in the case of elasticity it means determination of the effective (or averaged) Young's and shear moduli, Poisson's ratios etc. Along with this main problem several others can be solved in parallel, namely determination of level of microstructural stresses in a matrix material, these re highly oscillating stresses which may have high magnitude and can initiate volume fracture, and determination of scattering cross sections by pores, this is related to the ratio of the incident wave energy to the energy scattered by these inclusions. The latter problem is interesting due its direct connection to non-destructive testing of porous materials with uniformly distributed pores.

Existing literature on mechanics of heterogeneous media and wave scattering is so vast that one has to confine himself to review only works which have a rather close relation to the matter discussed.

The closest solutions in mechanics of heterogeneous media, including porous media can be obtained by application of the two-scale asymptotic analysis [1-3]. In this method it is assumed that two fields exist: (i) the global field, which is described by "slow" variables; and, (ii) a local field, having high oscillations, which is described by "fast" variables. Application of the two-scale asymptotic analysis to the problem stated above will be considered in a more detail later on.

In the two-scale asymptotic method the effective elasticity tensor related to the porous medium can be represented by the following expression

$$\mathbf{C}_0 = f\,\mathbf{C} + \mathbf{K}, \tag{1.1}$$

where $\mathbf{C}_0$ is the effective (homogenized) elasticity tensor, f is the volume fraction of the pores, $\mathbf{C}$ is the elasticity tensor of the material without pores (matrix), and $\mathbf{K}$ is a correcting tensor, or "corrector". It is clear from Eq. (1.1), that the main difficulty in determination of the effective elasticity tensor is in finding the corrector.

Remark. It is interesting to note that Eq. (1.1) covers almost all the existing methods of homogenization by choosing different expressions for the corrector:

a) Thus, if $\mathbf{K} = 0$ the well known Voigt's homogenization is obtained.

b) Taking

$$\mathbf{K} = -f\,\mathbf{C}\,, \tag{1.2}$$

the Reuss homogenization for the elasticity tensor comes out (for porous medium the Reuss homogenization produces the zero homogenized elasticity tensor).

Determination of the corrector in the two-scale asymptotic method demands the solution of the cell problem, which in turn consists of (i) setting up a boundary-value problem on the internal boundaries between pore(s) and the matrix material in a cell; and, (ii) formulating a periodic boundary-value problem on the outer boundary of a cell. The latter one is of the non-classical type in the sense that it is formulated on the boundary, which due to periodicity must have angular points and edges.

Along with FEM and finite differences methods, the following other methods for obtaining the solution to the cell problem are known. In [4 - 6], methods based on the Eshelby's transformation strain were applied to analyses of isotropic media with ellipsoidal inclusions. The advantage of these methods resides in their principle possibility to analyze media with anisotropic components, while from the computational point of view these methods are not very convenient since they lead to the three-dimensional integral equations with weakly singular kernels, and the problem reduces to the solution of the ill-posed problem for the integral equations of the first order.

In [7, 8], media with isotropic components were studied by applying a method based on the periodic fundamental solution for isotropic medium, which originally was constructed in [9]. Because of multipolar expansions used for the solution of the inner boundary value problem this method is confined to inclusions of spherical form. A similar approach was also used in the case of isotropic composites, but it was based on the Galerkin technique for solution of the inner boundary value problem [10].

Periodic fundamental solutions for media with arbitrary anisotropy were developed in [11]. In combination with the boundary integral equation method (BIEM) these fundamental solutions were applied to solution of the cell problem for composites with anisotropic inhomogeneities and porous media in [12, 13], analysis of microstructural stresses in the matrix material was considered in [14]. Problems of wave scattering by pores were studied in [15] by application of the same method. Some of obvious advantages of this method are due to potential possibility to reduce the solution of the inner boundary-value problem to summation of the rapidly convergent series, while periodic boundary conditions on the outer boundary are satisfied automatically due to periodicity of the fundamental solution.

The following analysis is targeted to obtaining values for the homogenized Poisson's ratio of porous medium, these data can be important for non-destructive evaluation of porous concentrations by analyzing the speed propagation of bulk and surface waves, as these are highly sensitive to variation of Poisson's ratio. Numerical data are obtained for isotropic porous material containing spherical pores, while the theoretical analysis is carried out for a material with arbitrary elastic anisotropy.

2 Basic notations

The equations of equilibrium for a homogeneous anisotropic medium can be written in the form:

$$\mathbf{A}(\partial_x)\mathbf{u} = -\,\mathrm{div}_{\mathrm{x}}\mathbf{C}\cdot\cdot\nabla_x\mathbf{u} = 0\,, \tag{2.1}$$

where u is a displacement field. It is assumed that the tensor of elasticity satisfies the condition of positive definiteness, which is generally adopted for problems of mechanics of inhomogeneous media.

Applying the Fourier transform

$$f^{\wedge}(\xi) = \int f(\mathbf{x})\exp(2\pi i\mathbf{x}\cdot\xi)dx,\ \xi\in R^3\,, \tag{2.2}$$

to Eqs. (2.1), gives the following symbol of the operator $\mathbf{A}$:

$$\mathbf{A}^{\wedge}(\xi) = (2\pi)^2\,\xi\cdot\mathbf{C}\cdot\xi\,. \tag{2.3}$$

Directly from the definition of the fundamental solution E, the following formula for the corresponding symbol can be written:

$$\mathbf{E}^{\wedge}(\xi) = \mathbf{A}^{\wedge}(\xi)^{-1}\,. \tag{2.4}$$

Formula (2.4) shows that the symbol $\mathbf{E}^{\wedge}$ is also strongly elliptic, positively homogeneous of degree -2 with respect to ξ, and analytical everywhere in $R^3\setminus 0$.

3 Spatially periodic fundamental solution

Consider a homogeneous anisotropic medium, loaded by the periodically distributed force singularities, located in nodes m of a spatial lattice Λ.

Let $\mathbf{a}_i,\ (i=1,2,3)$ be linearly independent vectors of the main periods of the lattice, so that each of the nodes can be represented in the form:

$$\mathbf{m} = \sum_i m_i\mathbf{a}_i\,, \tag{3.1}$$

where $m_i \in Z$ are the integer-valued coordinates of the node m in the basis $(\mathbf{a}_i)$. The adjoint basis $(\mathbf{a}_i *)$ is introduced in such a manner that $\mathbf{a}_i * \cdot \mathbf{m} = m_i$. The lattice corresponding to the adjoint basis will be denoted by $\Lambda *$.

Now, the periodic delta-function corresponding to the singularities located at the nodes of the lattice Λ can be represented the form:

$$\delta_p(\mathbf{x}) = V_Q^{-1} \sum_{\mathbf{m}^* \in \Lambda^*} \exp(-2\pi i \mathbf{x} \cdot \mathbf{m}^*), \tag{3.2}$$

where V_Q is the volume of the fundamental region (cell) Q. Formula (3.2) defines the periodic delta-function uniquely.

Substitution of the periodic fundamental solution $\mathbf{E}_p$ into Eq. (2.1) yields

$$\mathbf{A}(\partial_\mathbf{x})\mathbf{E}_p(\mathbf{x}) = \delta_p(\mathbf{x})\mathbf{I}, \tag{3.3}$$

where I is the identity matrix. Looking for $\mathbf{E}_p$ also in the form of harmonic series and taking into account representation (3.2), it is possible to obtain:

$$\mathbf{E}_p(\mathbf{x}) = V_Q^{-1} \sum_{\mathbf{m}^* \in \Lambda_0^*} \mathbf{E}\,\hat{}\,(\mathbf{m}^*) \exp(-2\pi i \mathbf{x} \cdot \mathbf{m}^*), \tag{3.4}$$

where $\Lambda_0 *$ is the adjoint lattice without the zero node. It should be noted that Eq. (3.4) defines the periodic fundamental solution up to an additive (tensorial) constant.

Lemma 1. The series on the right side of Eq. (3.4) is convergent in the L^1-topology, defining the fundamental solution of the class $\overline{L^1}(Q, R^3 \otimes R^3)$, where $\overline{L^1}$ is a class of integrable in Q functions with the zero mean value.

Proof of the lemma can be found in [11].

4 Effective elasticity tensor

For clarity and simplicity it will be assumed that the considered medium has the only one kind of uniformly distributed voids placed in the nodes of spatial lattice Λ. The region occupied by an individual void in a cell Q will be denoted by Ω.

The two-scale asymptotic analyses being applied to such a medium produces the following expression for the corrector [12]:

$$\mathbf{K} = -V_Q^{-1} \int_{\partial\Omega} \mathbf{C} \cdot\cdot (\nu_{\mathbf{Y}} \otimes \mathbf{H}(\mathbf{Y})) dY \,, \tag{4.1}$$

where Y are the "fast" variables, H is the third-order tensor field, being a solution of the following boundary value problem:

$$\begin{aligned} &\mathbf{A}(\partial_Y)\mathbf{H}(\mathbf{Y}) = 0, \ \mathbf{Y} \in Q \setminus \Omega \\ &\mathbf{T}(\nu_Y, \partial_Y)\mathbf{H}(\mathbf{Y}) \big|_{\partial\Omega} = -\nu_Y \cdot \mathbf{C} \end{aligned} . \tag{4.2}$$

In Eqs. (4.1) and (4.2) ν_Y represents field of the external unit normal to the boundary $\partial\Omega$, and the elasticity tensor C is referred to the matrix material (without pores)

Lemma 2. Under assumptions stated above, boundary-value problem (4.2) admits the unique solution.

Proof of the lemma can be found in [11, 12].

Remark. Supposition that the tensor C in Eq. (4.2) is not strong elliptic, violates proof of Lemma 2.

Now, the solution of the boundary value problem (4.2) for the third-order tensor traction field $-\nu_Y \cdot \mathbf{C}$ can be constructed by applying boundary integral equation method, giving the following representation for the desired solution [12]:

$$\left(\tfrac{1}{2}\mathbf{I}+\mathbf{S}\right)\mathbf{H}(\mathbf{Y}') = \mathbf{H}_c \quad \mathbf{Y}' \in \partial\Omega\,, \tag{4.3}$$

where $\mathbf{H}_c$ is a constant tensor, and S is a singular integral operator resulting from a restriction of the double-layer potential on the surface $\partial\Omega$. Some of the relevant properties of operator S are discussed in [13].

Substitution of Eq. (3.4) for periodic fundamental solutions into expression for the operator S allows us to obtain a lower (on energy) bound for the corrector; i.e.

$$\mathbf{K} = -8\pi^2 V_Q^{-2} \sum_{\mathbf{m}^* \in \Lambda_0^*} \left(\chi^\wedge{}_\Omega(\mathbf{m}^*)\right)^2 \mathbf{C}\cdot\cdot\mathbf{m}^* \otimes \mathbf{E}^\wedge(\mathbf{m}^*) \otimes \mathbf{m}^* \cdot\cdot\mathbf{C} \tag{4.4}$$

where $\chi^\wedge{}_\Omega$ is the Fourier image of the characteristic function of the region Ω. An expression for the upper bound can be obtained similarly [12, 13].

Theorem. Series appearing on the right side of Eq. (4.4) is absolutely convergent, provided Ω is a proper open region in *Q*.

Proof of the theorem can be found in [12, 13]

5 Homogenized Poisson's ratio for porous medium with isotropic matrix and spherical voids in nodes of *FCC*-lattice

Elasticity tensor for an isotropic matrix material (without pores) has the following components

$$\begin{aligned} &c^{11} = c^{22} = c^{33} = \lambda + 2\mu, \\ &c^{12} = c^{23} = c^{31} = \lambda, \\ &c^{44} = c^{55} = c^{66} = \mu \end{aligned} \tag{5.1}$$

where λ and μ are Lame constants, satisfying the following condition, which ensures positive definiteness of the elasticity tensor

$$3\lambda + 2\mu > 0, \quad \mu > 0 \tag{5.2}$$

Substituting elasticity tensor (5.1) into the expression (4.4), and taking into account that for the unit ball $\Omega \subset R^3$, the corresponding Fourier image of the characteristic function has the form

$$\chi_\Omega^\wedge(\xi) = \frac{1}{\pi|\xi|^2}\left(\frac{\sin(2\pi|\xi|)}{2\pi|\xi|} - \cos(2\pi|\xi|)\right), \tag{5.3}$$

we arrive to expression (4.5) for the corrector, where for an isotropic medium the symbol of the fundamental solution takes the form:

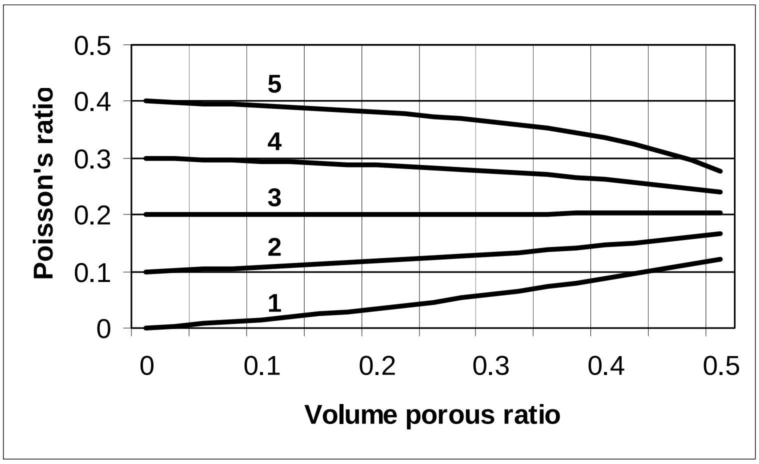
$$\mathbf{E}^\wedge(\xi) = \frac{1}{\left(2\pi|\xi|\right)^2 \mu}\left(\mathbf{I} - \frac{\lambda+\mu}{\lambda+2\mu}\frac{\xi\otimes\xi}{|\xi|^2}\right). \tag{5.4}$$

Now, the homogenized Poison's ratio ν_0 can be obtained by the following relation

$$\nu_0 = 1 - \frac{c_0^{11}}{2(c_0^{12} + c_0^{44})}, \tag{5.5}$$

where c_0^{11}, c_0^{12}, c_0^{44} are referred to the components of the homogenized elasticity tensor.

The obtained numerical data for the homogenized Poisson's ratio are presented in fig.1



These demonstrate decreasing of the overall homogenized Poisson's ratio ν_0 with the increase of the porous ratio, provided the corresponding value for the undisturbed matrix material (without pores) is roughly more than 0.2, while for lesser values of the Poisson's ratio of the undisturbed matrix material, the homogenized Poisson's ratio is increasing.

Fig. 1 Homogenized Poisson's ratio ν_0 for *FCC* spatial lattice

and spherical pores vs. volume porous ratio:

1 – Homogenized Poisson's ratio for the matrix material

with the initial Poisson's ratio (without pores) 0.0;

2 – 0.1; 3 – 0.2; 4 – 0.3; 5 – 0.4

It should be noted that the limiting value of the porous ratio for spherical pores and *FCC* lattice is about 0.740 (this lattice does not lead to the induced anisotropy due to regular package of pores). It can be expected that tending to the limiting value 0.740 for the porous ratio, all the curves will merge at Poisson's ratio value ~0.2.

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