

SUBSONIC LAMB WAVES IN ANISOTROPIC PLATES

by

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Annotation

A six-dimensional complex formalism for analysis of Lamb waves propagating with subsonic speed in anisotropic plates is formulated. Conditions for non-existence of certain Lamb waves in anisotropic plates are obtained. An example of transversely isotropic plate having “forbidden” speed at which no subsonic Lamb wave propagates, is presented.

1. Introduction

Beginning from Lamb’s pioneering work [1] in which the governing equations for harmonic waves propagating in isotropic plates with traction-free boundaries were first derived, in most of the subsequent theoretical works on Lamb waves in plates it was assumed that such a wave consists of several partial waves of the form

$$\mathbf{u}_k(\mathbf{x}) = \mathbf{m}_k e^{ir\gamma_k \mathbf{x} \cdot \mathbf{v}} e^{ir(\mathbf{n} \cdot \mathbf{x} - ct)} \quad (1.1)$$

where \mathbf{u}_k is the displacement field of the k -th partial wave; \mathbf{m}_k is vectorial, in general, complex amplitude determined by the Christoffel equation (this equation will be introduced in Sec. 2); γ_k is a root of the Christoffel equation; r is the wave number; \mathbf{v} is the unit normal to the middle plane of a plate; \mathbf{n} is the unit vector determining direction of propagation of Lamb wave; c is the phase speed; and t is time. For existence of the resulting Lamb wave all partial waves should have the same wave number and phase speed. Bearing in mind that the Christoffel equation has six roots, representation for Lamb wave takes the form

$$\mathbf{u}(\mathbf{x}) = \left(\sum_{k=1}^6 C_k \mathbf{m}_k e^{ir\gamma_k \mathbf{x} \cdot \mathbf{v}} \right) e^{ir(\mathbf{n} \cdot \mathbf{x} - ct)} \quad (1.2)$$

where C_k are arbitrary complex coefficients determined up to a multiplier by boundary conditions.

REMARK 1.1. a) Representation (1.1) for partial waves composing Lamb wave is also used for analysis of Rayleigh waves propagating on the traction-free plane boundary of an elastic half space, see [2-8]; and for Stoneley interfacial waves propagating on the plane boundary between contacting dissimilar elastic half spaces, see [9-12].

b) For Rayleigh waves roots γ_k in representation (1.1) should be complex with $\text{Im}(\gamma_k) < 0$, this ensures attenuation of Rayleigh wave in the “lower” half-space $(\mathbf{v} \cdot \mathbf{x}) < 0$. If $\text{Re}(\gamma_k) = 0$ for all partial waves composing Rayleigh wave, then such a wave is called genuine Rayleigh wave; if $\text{Re}(\gamma_k) \neq 0$ for some k , then it is sometimes called generalized Rayleigh wave [8]. For Lamb waves cases $\text{Re}(\gamma_k) = 0$ and $\text{Re}(\gamma_k) \neq 0$ usually are not distinguished.

The following analysis indicates that for Lamb waves propagating with *subsonic* phase speed in anisotropic plates (*subsonic* phase speed does not exceed the minimal speed of all bulk waves propagating in the same direction) representation (1.2) needs in correction: in some cases dependent on anisotropy Lamb wave may consist in fewer components, than it is assumed in traditional approaches [13 - 16]. This phenomenon results in a statement asserting possibility to exist “forbidden” subsonic speed, at which no Lamb wave can propagate. An example of the transversely isotropic plate having “forbidden” speed is constructed.

2. Basic notations

In the absence of body forces the equation of motion for anisotropic medium can be written in the form

$$\mathbf{A}(\partial_x, \partial_t)\mathbf{u} \equiv \text{div}_x \mathbf{C} \cdot \nabla_x \mathbf{u} - \rho \ddot{\mathbf{u}} = 0 \quad (2.1)$$

where ρ is the material density; \mathbf{C} is the fourth order elasticity tensor assumed to be positive definite:

$$\forall \mathbf{B} \in \text{sym}(R^3 \otimes R^3), \mathbf{B} \neq 0 \quad (\mathbf{B} \cdot \mathbf{C} \cdot \mathbf{B}) \equiv \sum_{i,j,m,n} B_{ij} C^{ijmn} B_{mn} > 0 \quad (2.2)$$

Substituting partial wave (1.1) in Eq. (2.1) produces the Christoffel equation:

$$\left((\gamma_k \mathbf{v} + \mathbf{n}) \cdot \mathbf{C} \cdot (\mathbf{n} + \gamma_k \mathbf{v}) - \rho c^2 \mathbf{I} \right) \cdot \mathbf{m}_k = 0 \quad (2.3)$$

where \mathbf{I} is the unit diagonal matrix. Equation (2.3) admits an equivalent form

$$\det \left((\gamma_k \mathbf{v} + \mathbf{n}) \cdot \mathbf{C} \cdot (\mathbf{n} + \gamma_k \mathbf{v}) - \rho c^2 \mathbf{I} \right) = 0 \quad (2.4)$$

Left-hand side of Eq. (2.4) can be regarded as a polynomial of degree six with respect to the parameter γ_k .

REMARK 2.1. a) Since polynomial coefficients in the left-hand side of Eq. (2.4) are real, corresponding roots γ_k are either real or complex-conjugate.

b) It can be shown (see, for example, [5]) that Eq. (2.4) has no real roots if the phase speed is less than the so called (lowest) limiting speed c_3^{lim} . In its turn, c_3^{lim} does not exceed the lowest speed of all bulk waves propagating in the same. Hereinafter, the following condition will be imposed on the phase speed

$$0 < c < c_3^{\text{lim}} \quad (2.5)$$

Condition (2.5) ensures absence of the real roots of Eq. (2.4).

3. Six-dimensional formalism

Since it is not known in advance, whether representation (1.1) for partial wave is the only possible one, a more general representation for a harmonic wave with the plane wave front and non-homogeneous amplitude will be considered:

$$\mathbf{v}(x'') e^{i r(\mathbf{n} \cdot \mathbf{x} - ct)} \quad (3.1)$$

where $\mathbf{v}(x'')$ is a non-constant complex-valued vector-function; $x'' = ir \mathbf{v} \cdot \mathbf{x}$, thus, x'' is dimensionless (imaginary) coordinate in the direction determined by vector \mathbf{v} . Exponential multiplier in (3.1) corresponds to movement of the plane wave front in the direction of propagation with the phase speed c . At this stage, no *a priori* restrictions on smoothness are imposed on function $\mathbf{v}(x'')$.

Substituting representation (3.1) in Eq. (2.1) yields the following differential equation of the second order

$$\left((\mathbf{v} \cdot \mathbf{C} \cdot \mathbf{v}) \partial_{x''}^2 + (\mathbf{v} \cdot \mathbf{C} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{C} \cdot \mathbf{v}) \partial_{x''} + (\mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n} - \rho c^2 \mathbf{I}) \right) \mathbf{v}(x'') = 0 \quad (3.2)$$

Direct analysis of Eq. (3.2) is difficult. Situation can be simplified by introducing additional vector-function

$$\mathbf{w}(x'') = \partial_{x''} \mathbf{v}(x'') \quad (3.3)$$

Bearing in mind (3.3) and the positive definite condition for the tensor \mathbf{C} , Eq. (3.2) can be reduced to a matrix differential equation of the first order with respect to six-dimensional vector-function (\mathbf{v}, \mathbf{w})

$$\partial_{x^n} \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \mathbf{R}_6 \cdot \begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix},$$

$$\mathbf{R}_6 = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M} & -\mathbf{N} \end{pmatrix} \quad (3.4)$$

In (3.4) real matrices \mathbf{M} and \mathbf{N} are as follows

$$\mathbf{M} = (\mathbf{v} \cdot \mathbf{C} \cdot \mathbf{v})^{-1} \cdot (\mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n} - \rho c^2 \mathbf{I}), \quad (3.5)$$

$$\mathbf{N} = (\mathbf{v} \cdot \mathbf{C} \cdot \mathbf{v})^{-1} \cdot (\mathbf{v} \cdot \mathbf{C} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{C} \cdot \mathbf{v})$$

Taking into account the structure of matrix \mathbf{R}_6 , it is convenient to represent corresponding six-dimensional eigenvectors of \mathbf{R}_6 in the form $\mathbf{m}_6 \equiv (\mathbf{m}; \mathbf{m}')$, where $\mathbf{m}, \mathbf{m}' \in C^3$ (here C^k denotes k -dimensional complex vector space).

A surjective homomorphism $\mathfrak{S}: C^6 \rightarrow C^3$, such that

$$\mathfrak{S}(\mathbf{m}_6) = \mathbf{m} \quad (3.6)$$

will be needed for the subsequent analysis.

REMARK 3.1. a) Due to (3.4), (3.5), determinant of the matrix \mathbf{R}_6 can be represented in the form

$$\det \mathbf{R}_6 = \det \left((\mathbf{v} \cdot \mathbf{C} \cdot \mathbf{v})^{-1} (\mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n} - \rho c^2 \mathbf{I}) \right) \quad (3.7)$$

Right-hand side of (3.7) shows that matrix \mathbf{R}_6 is not degenerate at any phase speed with the exception when $\rho c^2 = \lambda_k (\mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n})$, $k = 1, 2, 3$ (here λ_k denotes an eigenvalue of the corresponding matrix); i.e. degeneracy occurs when the phase speed coincides with the speed of one of the bulk waves propagating in the same direction. It is clear that for Lamb wave with the phase speed satisfying (2.5), matrix \mathbf{R}_6 is not degenerate.

b) Since matrix \mathbf{R}_6 is not symmetric, its left and right eigenvectors generally are different. In the subsequent analysis term “eigenvector” will be referred to right eigenvector of matrix \mathbf{R}_6 .

c) Matrix \mathbf{R}_6 resembles one, which is used for constructing the “fundamental elasticity tensor”. Such a tensor and the corresponding matrix were introduced in [17] and were used later on for analysis of Rayleigh and Stoneley waves propagating in anisotropic halfspaces, see [4 - 8, 11, 12].

PROPOSITION 3.1. a) A set of all roots of the Christoffel equation (2.4) coincides with a set of all eigenvalues of matrix \mathbf{R}_6 ;

b) Spectral space of the Christoffel equation (2.3) coincides with surjection (3.6) of the spectral space of matrix \mathbf{R}_6 .

Proof. Let γ_k be a root of the characteristic polynomial (2.4) and \mathbf{m}_k be the corresponding eigenvector of Eq. (2.3). Substituting functions $\mathbf{v}(x'') = \mathbf{m}_k e^{\gamma_k x''}$ and $\mathbf{w}(x'') = \gamma_k \mathbf{m}_k e^{\gamma_k x''}$ in Eq. (3.4) yields

$$\gamma_k \begin{pmatrix} \mathbf{m}_k \\ \gamma_k \mathbf{m}_k \end{pmatrix} = \mathbf{R}_6 \cdot \begin{pmatrix} \mathbf{m}_k \\ \gamma_k \mathbf{m}_k \end{pmatrix} \quad (3.8)$$

Thus, every root of Eq. (2.4) is an eigenvalue of matrix \mathbf{R}_6 , and the corresponding eigenvector of Eq. (2.3) coincides with the vector $\mathfrak{I}(\mathbf{m}_6)$.

Further, let γ_k be an eigenvalue, and $(\mathbf{m}_k; \mathbf{m}'_k)$, $\mathbf{m}_k, \mathbf{m}'_k \in C^3$ be the corresponding eigenvector of matrix \mathbf{R}_6

$$\gamma_k \begin{pmatrix} \mathbf{m}_k \\ \mathbf{m}'_k \end{pmatrix} = \mathbf{R}_6 \cdot \begin{pmatrix} \mathbf{m}_k \\ \mathbf{m}'_k \end{pmatrix} \quad (3.9)$$

Relation (3.9) along with (3.4), (3.5) yield

$$\left(\gamma_k^2 (\mathbf{v} \cdot \mathbf{C} \cdot \mathbf{v}) + \gamma_k (\mathbf{v} \cdot \mathbf{C} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{C} \cdot \mathbf{v}) + (\mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n} - \rho c^2 \mathbf{I}) \right) \cdot \mathbf{m}_k = 0 \quad (3.10)$$

But (3.10) coincides with (2.3).

Remark 2.1 ensures

COROLLARY. Under condition (2.5) all eigenvalues of matrix \mathbf{R}_6 are complex and form spectrum of \mathbf{R}_6 by the complex-conjugate pairs.

PROPOSITION 3.2. Under condition (2.5) matrix \mathbf{R}_6 is not normal matrix, except may be one value of the phase speed c .

Proof. Definition of the normal (real) matrix gives

$$\mathbf{R}_6^t \cdot \mathbf{R}_6 = \mathbf{R}_6 \cdot \mathbf{R}_6^t \quad (3.11)$$

In view of (3.4) relation (3.11) implies

$$\mathbf{M}^t \cdot \mathbf{M} = \mathbf{I} \quad (3.12)$$

Taking into account (3.5), relation (3.12) requires

$$\mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n} - \mathbf{v} \cdot \mathbf{C} \cdot \mathbf{v} = \rho c^2 \mathbf{I} \quad (3.13)$$

This completes the proof, since the left-hand side of (3.13) is independent of c .

COROLLARY. Under condition (2.5) eigenvectors of matrix \mathbf{R}_6 do not form an orthogonal basis in C^6 , except may be one value of the phase speed c at which relation (3.13) holds.

4. Representations for Lamb waves

Structure of the general solution of system (3.4) is determined by the Jordan normal form of matrix \mathbf{R}_6 [18, Chap. IV, §5]. Due to Proposition 3.1 and corresponding Corollary, for the phase speed which satisfies (2.5) only three types of the Jordan normal forms of matrix \mathbf{R}_6 can occur

$$\mathbf{J}_6^{(I)} = \begin{pmatrix} \gamma_1 & & & & & \\ & \gamma_2 & & & & \\ & & \gamma_3 & & & \\ & & & \gamma_4 & & \\ & & & & \gamma_5 & \\ & & & & & \gamma_6 \end{pmatrix}, \quad \mathbf{J}_6^{(II)} = \begin{pmatrix} \begin{pmatrix} \gamma_1 & 1 \\ 0 & \gamma_1 \end{pmatrix} & & & & \\ & \begin{pmatrix} \gamma_2 & 1 \\ 0 & \gamma_2 \end{pmatrix} & & & \\ & & \gamma_3 & & \\ & & & \gamma_4 & \\ & & & & \gamma_5 \end{pmatrix},$$

$$\mathbf{J}_6^{(III)} = \begin{pmatrix} \begin{pmatrix} \gamma_1 & 1 & 0 \\ 0 & \gamma_1 & 1 \\ 0 & 0 & \gamma_1 \end{pmatrix} & & & & \\ & \begin{pmatrix} \gamma_2 & 1 & 0 \\ 0 & \gamma_2 & 1 \\ 0 & 0 & \gamma_2 \end{pmatrix} & & & \\ & & & & & \gamma_3 \\ & & & & & & \gamma_4 \\ & & & & & & & \gamma_5 \end{pmatrix} \quad (4.1)$$

In expressions (4.1) eigenvalues γ_k are renumbered in such a way that they satisfy the following condition

$$\gamma_{2k-1} = \overline{\gamma_{2k}}, \quad k = 1, 2, \dots \quad (4.2)$$

Moreover, since matrix \mathbf{R}_6 is real, a condition analogous to (4.2) is satisfied by eigenvectors of \mathbf{R}_6 :

$$(\mathbf{m}_{2k-1}; \mathbf{m}'_{2k-1}) = (\overline{\mathbf{m}_{2k}}; \overline{\mathbf{m}'_{2k}}) \quad (4.3)$$

Transition from the system of the first order (3.4) to the initial system (3.2) allows to represent the general solution (the fundamental matrix) in the form

$$\mathbf{v}^{(I)}(x'') = \sum_{k=1}^6 C_k \mathbf{m}_k e^{\gamma_k x''}$$

$$\mathbf{v}^{(II)}(x'') = \sum_{k=1}^2 \left((C_{2k-1} + C_{2k} x'') \mathbf{m}_k + C_{2k} \mathbf{m}_k^1 \right) e^{\gamma_k x''} + \sum_{k=3}^4 C_{k+2} \mathbf{m}_k e^{\gamma_k x''} \quad (4.4)$$

$$\mathbf{v}^{(III)}(x'') = \sum_{k=1}^2 \left(C_{3k-2} + C_{3k-1} x'' + \frac{1}{2} C_{3k} x''^2 \right) \mathbf{m}_k e^{\gamma_k x''} +$$

$$\sum_{k=1}^2 \left((C_{3k-1} + C_{3k} x'') \mathbf{m}_k^1 + C_{3k} \mathbf{m}_k^2 \right) e^{\gamma_k x''}$$

where C_k are unknown complex coefficients, and $\mathbf{m}_k^1, \mathbf{m}_k^2 \in C^3$ are the generalized eigenvectors associated with the eigenvector \mathbf{m}_k .

5. Disperse equations

Traction-free boundary conditions are formulated on the plate surfaces

$$\mathbf{t}_v \Big|_{\mathbf{x} \cdot \mathbf{v} = \pm h} \equiv \pm \mathbf{v} \cdot \mathbf{C} \cdot \nabla_{\mathbf{x}} \mathbf{u} \Big|_{\mathbf{x} \cdot \mathbf{v} = \pm h} = 0 \quad (5.1)$$

In (5.1) $2h$ is the plate thickness.

Substituting the displacement field (4.4) in the boundary conditions (5.1) and transition to the dimensionless coordinate $x'' = ir(\mathbf{v} \cdot \mathbf{x})$ gives

$$\sum_{k=1}^6 C_k \mathbf{t}_k e^{\gamma_k x''} \Big|_{x'' = \pm \xi} = 0 \quad (5.2)$$

where $\xi = irh$, and \mathbf{t}_k is (up to the exponential multiplier) the surface traction corresponding the coefficient C_k .

Solution of the boundary-value problem (5.2) can be treated as a non-trivial solution of the linear system (5.2) with respect to unknown coefficients C_k , $k = 1, \dots, 6$. The latter is equivalent to vanishing all the determinants of the sixth-order associated with the 6×6 -matrix:

$$\delta \equiv \det \begin{pmatrix} \mathbf{t}_1(\xi)e^{+\gamma_1\xi} & \dots & \mathbf{t}_6(\xi)e^{+\gamma_6\xi} \\ \mathbf{t}_1(-\xi)e^{-\gamma_1\xi} & \dots & \mathbf{t}_6(-\xi)e^{-\gamma_6\xi} \end{pmatrix} = 0 \quad (5.3)$$

Equations (5.3) are the disperse equations we are looking for. In the case of arbitrary elastic anisotropy solvability of Eqs. (5.3) has not been studied, however, the following propositions take place

PROPOSITION 5.1. Condition

$$\gamma_k = - \frac{\mathbf{v} \otimes \overline{\mathbf{m}_k} \cdot \mathbf{C} \cdot \mathbf{m}_k \otimes \mathbf{n}}{\mathbf{v} \otimes \overline{\mathbf{m}_k} \cdot \mathbf{C} \cdot \mathbf{m}_k \otimes \mathbf{v}} \quad (5.4)$$

is necessary for Lamb wave to be composed by a single partial wave (1.1).

Proof. If at some k corresponding partial wave satisfies boundary conditions, then Eqs. (5.2) yield

$$(\gamma_k \mathbf{v} \cdot \mathbf{C} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{C} \cdot \mathbf{n}) \cdot \mathbf{m}_k = 0 \quad (5.5)$$

Multiplication of both sides of Eq. (5.5) by $\overline{\mathbf{m}_k}$ gives

$$\gamma_k \mathbf{v} \otimes \overline{\mathbf{m}_k} \cdot \mathbf{C} \cdot \mathbf{m}_k \otimes \mathbf{v} + \mathbf{v} \otimes \overline{\mathbf{m}_k} \cdot \mathbf{C} \cdot \mathbf{m}_k \otimes \mathbf{n} = 0 \quad (5.6)$$

Then, in view of the positive definiteness of the elasticity tensor, it remains to note that Eq. (5.6) is equivalent to (5.4).

PROPOSITION 5.2. Condition (5.4) is necessary for Lamb wave to be composed by two partial waves (1.1) corresponding to the complex-conjugate eigenvalues.

Proof. Assume that (5.4) does not hold, but a nontrivial solution of Eqs. (5.2) exists at some eigenvalues γ_k and $\bar{\gamma}_k$. In this case Eqs. (5.3) reduce to two equations. The first one can be represented in the form

$$\mathbf{t}_k \times \bar{\mathbf{t}}_k = \mathbf{0}, \quad (5.7)$$

where $\mathbf{t}_k = [\gamma_k \mathbf{v} \cdot \mathbf{C} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{C} \cdot \mathbf{n}] \cdot \mathbf{m}_k$. Equation (5.7) means colinearity of vectors \mathbf{t}_k and $\bar{\mathbf{t}}_k$:

$$\mathbf{t}_k = \alpha \mathbf{e}, \quad \bar{\mathbf{t}}_k = \bar{\alpha} \mathbf{e} \quad (5.8)$$

where α is a complex constant ($\alpha \neq 0$, since (5.4) does not hold by the assumption); and $\mathbf{e} \in R^3$. By accounting (5.8), the second equation flowing out from (5.3) takes the form

$$\det \begin{pmatrix} \alpha e^{+\gamma_k \xi} & \bar{\alpha} e^{+\bar{\gamma}_k \xi} \\ \alpha e^{-\gamma_k \xi} & \bar{\alpha} e^{-\bar{\gamma}_k \xi} \end{pmatrix} = 0 \quad (5.9)$$

Direct verification shows that for $\text{Im}(\gamma_k) \neq 0$ and $\alpha \neq 0$ the left-hand side of (5.9) cannot vanish.

REMARK 5.1. a) Condition (5.4) is not sufficient for Lamb wave to be composed by a single partial wave, or two partial waves corresponding to the complex-conjugate eigenvalues. This is because of possibility for vectors $\bar{\mathbf{m}}_k$ and $\mathbf{t}_k \equiv [\gamma_k \mathbf{v} \cdot \mathbf{C} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{C} \cdot \mathbf{n}] \cdot \mathbf{m}_k \neq 0$ to be mutually orthogonal.

b) Proposition 5.1 remains valid for Rayleigh waves. In this case conditions (5.1), (5.2) are formulated on the free surface at $x'' = 0$. In an indirect proof, based on Stroh's formalism (see [4, 5]), it was shown that *subsonic* Rayleigh wave cannot be composed by a single partial wave. For *supersonic* Rayleigh wave propagating on a half-space with elastic symmetry a condition analogous to (5.4) was derived in [19].

6. Lamb waves in transversely isotropic plates

Let unit vectors \mathbf{e}_k , $k = 1, 2, 3$ form an orthogonal basis in R^3 , and vector \mathbf{e}_1 coincides with the normal vector \mathbf{v} to the median surface of a plate, while vectors \mathbf{e}_2 and \mathbf{e}_3 lie on the basal plane of a transversely isotropic medium. Corresponding elasticity tensor has following components

$$\begin{array}{cccccc}
 c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\
 & c_{22} & c_{23} & 0 & 0 & 0 \\
 & & c_{22} & 0 & 0 & 0 \\
 & & & c_{44} & 0 & 0 \\
 & & & & c_{55} & 0 \\
 & & & & & c_{55}
 \end{array} \tag{6.1}$$

where $c_{44} = \frac{1}{2}(c_{22} - c_{23})$. Condition of positive definiteness for the regarded elasticity tensor yields

$$c_{11} > 0; \quad c_{22} > 0; \quad c_{55} > 0; \quad c_{22} > |c_{23}|; \quad c_{11}c_{22} > c_{12}^2; \quad c_{11} > \frac{2c_{12}^2}{c_{22} + c_{23}} \tag{6.2}$$

Substituting the elasticity tensor (6.1) in (3.5) gives

$$\mathbf{M} = \left(\frac{c_{55} - \rho c^2}{c_{11}} \mathbf{v} \otimes \mathbf{v} + \frac{c_{22} - \rho c^2}{c_{55}} \mathbf{n} \otimes \mathbf{n} + \frac{c_{22} - c_{23} - \rho c^2}{2c_{55}} \mathbf{w} \otimes \mathbf{w} \right) \tag{6.3}$$

$$\mathbf{N} = \left(\frac{c_{12} + c_{55}}{c_{11}} \mathbf{v} \otimes \mathbf{n} + \frac{c_{12} + c_{55}}{c_{55}} \mathbf{n} \otimes \mathbf{v} \right)$$

where $\mathbf{w} = \mathbf{n} \times \mathbf{v}$.

With the account of (6.3) the following proposition flows out directly from the analysis of the spectral properties of matrix \mathbf{R}_6 :

PROPOSITION 6.1. a) Relation between the elastic constants, density, and phase speed

$$\rho c^2 = 2 \frac{|c_{12} + c_{55}|}{(c_{11} - c_{55})^2} \sqrt{c_{11}c_{55}(c_{11}c_{55} + c_{22}c_{55} + 2c_{12}c_{55} - c_{11}c_{22} + c_{12}^2) - \frac{2c_{12}c_{55}^2 + 2c_{11}c_{12}c_{55} - c_{11}^2c_{22} + c_{11}c_{22}c_{55} + c_{11}c_{12}^2 + c_{12}^2c_{55} + 2c_{11}c_{55}^2}{(c_{11} - c_{55})^2}} \tag{6.4}$$

is necessary and sufficient for rise of the Jordan normal form $\mathbf{J}_6^{(II)}$;

b) Different roots γ_k of the Christoffel equation (2.4) corresponding to (6.4) are

$$\gamma_1 = -i \left(\frac{c_{11}c_{22} - (c_{11} + c_{55})\rho c^2 - 2c_{12}c_{55} - c_{12}^2}{2c_{11}c_{55}} \right)^{1/2}, \quad \gamma_2 = -\gamma_1, \quad (6.5)$$

$$\gamma_3 = -i \left(\frac{c_{22} - c_{23} - 2\rho c^2}{2c_{55}} \right)^{1/2}, \quad \gamma_4 = -\gamma_3$$

where γ_1 and γ_2 correspond to the Jordan blocks;
 c) Corresponding amplitudes have the form

$$\mathbf{m}_1 = p \mathbf{v} \left(\frac{c_{22} - \rho c^2}{c_{11}} \right)^{1/4} + ip \mathbf{n} \left(1 - \frac{\rho c^2}{c_{55}} \right)^{1/4}, \quad \mathbf{m}_2 = \overline{\mathbf{m}_1},$$

$$\mathbf{m}_1^1 = ip \mathbf{v} \left(1 - \frac{\rho c^2}{c_{55}} \right)^{1/4} + p \mathbf{n} \left(\frac{c_{22} - \rho c^2}{c_{11}} \right)^{1/4}, \quad \mathbf{m}_2^1 = \overline{\mathbf{m}_1^1}, \quad (6.6)$$

$$\mathbf{m}_3 = \mathbf{m}_4 = \mathbf{w}$$

where p is the normalization factor:

$$p = \left(\left(\frac{c_{22} - \rho c^2}{c_{11}} \right)^{1/2} + \left(1 - \frac{\rho c^2}{c_{55}} \right)^{1/2} \right)^{-1/2} \quad (6.7)$$

REMARK 6.1. a) Natural requirement for the right-hand side of (6.4) to be real and positive leads to the following restrictions:

$$c_{11} \neq c_{55}; \quad c_{12}^2 + c_{11}c_{55} + c_{22}c_{55} + 2c_{12}c_{55} - c_{11}c_{22} \geq 0; \quad (6.8)$$

$$c_{12}^2 + 4c_{12}c_{55} + 4c_{55}^2 - c_{11}c_{22} > 0;$$

In (6.4) - (6.8) and hereinafter the case $c_{11} = c_{55}$ is not considered. Of course, conditions (6.8) should be completed with (6.2) and (2.5). Direct verification shows that all together conditions (6.8), (6.2) and (2.5) define non-empty region $\Omega \subset R^5$, of admissible values of elastic parameters.

Finally, the main result of this section can be proved:

THEOREM 6.1. No Lamb wave propagates in the transversely isotropic plate, if the phase speed satisfies condition (6.4).

Proof. Substituting eigenvalues (6.5) and corresponding amplitudes (6.6) in (5.3) gives two independent conditions

$$\det_4 \begin{pmatrix} (a_1 \mathbf{v} + b_1 \mathbf{n}) e^{+\gamma_1 \xi} & (\bar{a}_1 \mathbf{v} + \bar{b}_1 \mathbf{n}) e^{+\bar{\gamma}_1 \xi} & (a_2(\xi) \mathbf{v} + b_2(\xi) \mathbf{n}) e^{+\gamma_1 \xi} & (\overline{a_2(\xi) \mathbf{v} + b_2(\xi) \mathbf{n}}) e^{+\bar{\gamma}_1 \xi} \\ (a_1 \mathbf{v} + b_1 \mathbf{n}) e^{-\gamma_1 \xi} & (\bar{a}_1 \mathbf{v} + \bar{b}_1 \mathbf{n}) e^{-\bar{\gamma}_1 \xi} & (a_2(-\xi) \mathbf{v} + b_2(-\xi) \mathbf{n}) e^{-\gamma_1 \xi} & (\overline{a_2(-\xi) \mathbf{v} + b_2(-\xi) \mathbf{n}}) e^{-\bar{\gamma}_1 \xi} \end{pmatrix} = 0 \quad (6.9)$$

and

$$\det_2 \begin{pmatrix} (d\mathbf{w}) e^{+\gamma_3 \xi} & (\bar{d}\mathbf{w}) e^{+\bar{\gamma}_3 \xi} \\ (d\mathbf{w}) e^{-\gamma_3 \xi} & (\bar{d}\mathbf{w}) e^{-\bar{\gamma}_3 \xi} \end{pmatrix} = 0 \quad (6.10)$$

where a_1, b_1, a_2, b_2, d are scalar (nonzero) complex coefficients determined by (6.5), (6.6) and (5.3). Finally, direct computation of the determinants in the left-hand sides of (6.9) and (6.10) reveals that both these determinants do not vanish at nonzero ξ , this completes the proof.

REMARK 6.2. a) A more detailed analysis shows that “forbidden” speed for a transversely isotropic plate does not depend upon the wave frequency and thickness of a plate. As it follows from (6.4) such a speed is determined only by physical properties of a material.

b) It can be highly important for practical applications (for example, design of the delay lines and filters in electronics) that some transversely isotropic materials which only slightly differ from isotropic ones can have “forbidden” speed for Lamb waves. For instance, taking in (6.2), (6.8)

$$c_{11} = c_{22} = 1; \quad c_{55} = \frac{1}{2}; \quad c_{23} = 0 \quad (6.11)$$

we arrive to the following restriction imposed on c_{12} :

$$\frac{\sqrt{2}}{2} > c_{12} > 0 \quad (6.12)$$

Let $c_{12} = \frac{1}{10}$ and $\rho = 1$ (the material with elastic constants (6.11) and $c_{12} = 0$ is isotropic). Computation of “forbidden” speed by (6.4) yields

$$c = \sqrt{\frac{12\sqrt{22}-33}{50}} \approx 0.6824 \quad (6.13)$$

It remains to check whether condition (2.5) is satisfied. But, for waves propagating in the isotropic plane of arbitrary transversely isotropic material

$$c_3^{\lim} = c_3 \tag{6.14}$$

where c_3 denotes the minimal speed of all bulk waves propagating in the isotropic plane. For the regarded case computation of c_3 gives

$$c_3 = \sqrt{\frac{c_{55}}{\rho}} = \frac{\sqrt{2}}{2} \approx 0.7071 \tag{6.15}$$

Comparison of the right-hand sides of (6.13) and (6.15) shows that condition (2.5) is satisfied.

c) For the transversely anisotropic material considered in the preceding remark, matrix \mathbf{R}_6 can be regarded as one parametric with respect to the phase speed c . Analysis of the spectral properties of the matrix \mathbf{R}_6 in a small vicinity V of the phase speed (6.13) with the account of notations (4.1) - (4.3) reveals that at $c \rightarrow 0.6824$

$$\gamma_1 \rightarrow \gamma_3, \quad \gamma_2 \rightarrow \gamma_4, \tag{6.16}$$

$$(\mathbf{m}_1, \mathbf{m}'_1) \rightarrow (\mathbf{m}_3, \mathbf{m}'_3), \quad (\mathbf{m}_2, \mathbf{m}'_2) \rightarrow (\mathbf{m}_4, \mathbf{m}'_4).$$

It should also be noted that everywhere in V matrix \mathbf{R}_6 has the Jordan normal form $\mathbf{J}_6^{(I)}$, except the limiting value $c = 0.6824$ where the Jordan normal form $\mathbf{J}_6^{(II)}$ arises.

At the same time, boundary conditions (5.2) with the account of (6.16) lead to the following relations (at $c \rightarrow 0.6824$) between coefficients C_k :

$$C_1 \rightarrow -C_3, \quad C_2 \rightarrow -C_4, \quad C_5 = C_6 = 0 \tag{6.17}$$

Combining (6.16), (6.17) with representation (1.2) (which is valid everywhere in V , except the limiting value) yields:

$$\mathbf{u}(\mathbf{x}) \rightarrow 0 \tag{6.18}$$

uniformly with respect to \mathbf{x} at $c \rightarrow 0.6824$. In turn, expression (6.18) ensures that both strain and kinetic specific energy functions along with the energy fluxes tend to zero at $c \rightarrow 0.6824$.

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