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(Translation into English)

A numerical method for solving axisymmetric problems for geometrically nonlinear elastic-plastic shells of revolution

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1. Introduction. Axisymmetric finite displacement elastic-plastic shells under quasi-static deformation until very recently studied exclusively on the basis of the deformation theory of plasticity of Hencki-Ilyushin (see review [1]). This preference, apparently, is based on the belief that this theory provides a solution in a simpler way than the theory of elastic-plastic flow, since it does not require consequent tracking the deformation process. However, in many cases, because of the nature of the problem (the construction dependences of load - "general" deformation in the study of buckling in the large) or due to computational reasons (ensuring the convergence of iterative processes) it needs to consistently follow the deformation process. This deprives the deformation theory of specified benefits. On the other hand, in most of geometrically nonlinear problems complex loadings is implemented, this requires the use of a more physically-based theory of elastic-plastic flow, taking into account the history of loading.
In the first papers [2-4], based on the theory of Prandtl-Reuss, used a simplified geometrically nonlinear theory of shells. The equations of this theory take into account the quadratic terms respectively to the angles of rotation of normal to the middle surface, while the parameters of the geometry correspond to initial non-deformed state.
The study of buckling of elastic spherical shells [5], [6] is based on more accurate theory of finite axisymmetric deformation of thin non-shallow shells [7], proposed by E. Reissner [7] and modified in [5]. In this theory the influence of the rotations of normal is fully considered, and geometry parameters assigned to the current deformed state. It was shown that the effect of use of more accurate theory on the values of the critical load can reach about 10%. Obviously, for post-buckling deformation the effect will be even more significant.
In [8] proposed a numerical method for calculating sub-critical deformation of shells of revolution made from materials with complex theology. It was based on a modified E. Reissner theory. The method was applied to calculate deformations of the round plates and cylindrical shells using the theory of elastic-plastic flow.
In this paper, we propose another method based on a modified E. Reissner theory and the theory of elastic-plastic flow, which is suitable for the study of quasi-static processes of axisymmetric deformation of shells of revolution, including supercritical (post-buckling) regimes.

2. Statement of problem. The equations of the modified E. Reissner theory of shells are as follows:

$$\beta=\varphi_0-\varphi;\ u=r-r_0,\ w=z-z_0,\ r'=\alpha\cos\varphi,\ z'=\alpha\sin\varphi,\ \varepsilon_{\xi 0}=\alpha/\alpha_0-1,\ \varepsilon_{\theta 0}=r/r_0-1$$

$$\kappa_\xi=\beta'/\alpha_0,\ \kappa_\theta=(\sin\varphi_0-\sin\varphi)/r_0,\ \varepsilon_\xi=\varepsilon_{\xi 0}+\eta\kappa_\xi,\ \varepsilon_\theta=\varepsilon_{\theta 0}+\eta\kappa_\theta$$

$$H=N_\xi\cos\varphi-Q\sin\varphi,\ V=N_\xi\sin\varphi+Q\cos\varphi,\ (rV)'+\alpha r p_V=0,\ (rH)'-\alpha N_\theta+\alpha r p_H=0$$

$$(rM_\xi)'-\alpha M_\theta\cos\varphi-\alpha r Q=0,\ p_V=p_V(r,\varphi,z,\xi,p),\ p_H=p_H(r,\varphi,z,\xi,p)$$

$$N_i=\int_{-h/2}^{h/2}\sigma_i d\eta,\qquad M_i=\int_{-h/2}^{h/2}\sigma_i\eta d\eta\quad (i=\xi,\theta) \tag{1}$$

Here the bar denotes a spatial derivation along ξ ($\xi_a\le\xi\le\xi_b$); r_0, z_0, φ_0,α_0 are known functions of ξ, describing the initial geometry of the shell (see Fig. 1); analogical geometrical parameters for deformed state of the shell are denoted as r, z, φ, α; α_0, α is a Lame coefficient along meridian; η is the coordinate along the normal to the middle surface; h is the thickness of the shell $(-h/2 \le \eta \le h/2)$; β is the angle of rotation of normal; u, w, H, V and p_H, p_V are components of middle surface displacement, components of main force vector in meridian plane and components of distributed load along x_1 and x_2.

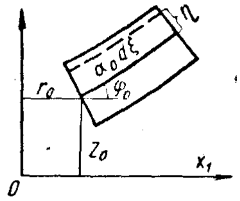

Рис. 1

We assume that in general case p_V and p_H may depend on spatial parameter ξ and loading p. The rest denotes are usual.
The elastic-plastic theory relations are represented as:

$$\partial_t \sigma = g_{11}\partial_t \varepsilon_\xi + g_{12}\partial_t \varepsilon_\theta,\ \ \partial_t \sigma = g_{11}\partial_t \varepsilon_\xi + g_{12}\partial_t \varepsilon_\theta \qquad (2)$$

Here ∂_t means time derivative (“time” parameter is defined below). Coefficients g_{ij} $(i, j = 1,2)$ are known functions of stresses, strains, material constants and hardening parameters.
The imagination about “time” may be clarified. In evolution problems always presented some parameter which plays the role of time. The independent monotonic growth of this parameter defines the development of such processes. For instance, in problems of sub-critical deformation the role of time plays the loading parameter. In problems of buckling and post-buckling deformations the role of time may be played in turn by the loading parameter, displacement, deformation and so on. In general case for quasi-static buckling problems the role of time may be passed to some functional, which depends on solution:

$$t = T(\overline{\Gamma}) \qquad (3)$$

Here $\overline{\Gamma} = (w, \beta, u, V, H, M_\xi, \sigma_\xi, o_\theta, p)$ are major unknowns. The rest unknown functions can be easily found using major unknowns and relations (1). Functions $\overline{\Gamma}$ represent the solution of the problem.
Functional (3) may have different representations for various stages of deformation process and is chosen so that the monotone growth of *t* reflects the physical nature of the process. Chosen functional (3) is suitable while solution time derivatives are less than some predefined value δ_* $(\delta_* > 0\!:\ |\overline{\Gamma}| < \delta_*)$. The violation of this condition signals about the necessity to change the time functional (3).
In some tasks of buckling it is possible to make so lucky choice of time functional that the necessity of its replacement does not appear at all. For instance, in tasks for spherical caps buckling under external pressure such choice is

$$T = \frac{1}{h}\int_{\xi_a}^{\xi_b} w d\xi / (\xi_b - \xi_a) \qquad (4)$$

Maximum and minimum of function $p(t)$ are defined during calculations and correspond to upper and lower critical loads.
The system of equation (1—3) are solved using initial and boundary conditions:

$$\overline{\Gamma}|_{t=0} = \overline{\Gamma}_0 \qquad (5)$$

$$U_a \overline{y}|_{\xi=\xi_a} = \overline{V}_a,\quad U_b \overline{y}|_{\xi=\xi_b} = \overline{V}_b \qquad (6)$$

Here $\overline{y} = (w, \beta, u, V, H, M_\xi)$. We assume that boundary conditions (6) satisfy to known restrictions, following from the principle of the virtual work [7]. Vectors V_a and V_b may depend on loading parameter (action of side forces and force moments).

3. **Method of solution.** The step-by-step procedure of solution is described below. During each small step along time parameter the quasi-linearization of primary nonlinear equations is used [9].
Let's differentiate the equations (1), (3) and conditions (6) with respect to «time». In result we get the relations, which are linear regarding «velocities» of unknowns (i.e. derivatives along «time»). Joining these relations with relations (2), we transform this system of relations so, that it contains only «velocities» of major unknowns $\overline{\Gamma}$**,** in result we have:

$$\overline{y}'_* = A\overline{y}_* + \overline{F}, \quad U_a \overline{y}_*|_{\xi=\xi_a} = \frac{\partial \overline{V}_a}{\partial p}, \quad U_b \overline{y}_*|_{\xi=\xi_b} = \frac{\partial \overline{V}_b}{\partial p} \tag{7}$$

$$\sigma_{\xi*} - (g_{11}\overline{E}_1 + g_{12}\overline{E}_2)\overline{y}_*, \quad \sigma_{\theta*} - (g_{21}\overline{E}_1 + g_{22}\overline{E}_2)\overline{y}_* \tag{8}$$

$$\dot{p} = \left[\sum_{i=1}^{9} (\partial T / \partial \gamma_i) \gamma_{i*} \right]^{-1} \tag{9}$$

Here

$$\overline{y}_* = \overline{y} / \dot{p}, \ \sigma_{\xi*} = \sigma_\xi / \dot{p}, \quad \sigma_{\theta*} = \sigma_\theta / \dot{p}, \quad \gamma_{i*} = \gamma_i / \dot{p}, \ \overline{\Gamma} = \{\gamma_i\} \tag{10}$$

Matrix A and vectors $\overline{F}, \overline{E}_1, \overline{E}_2$ in relations for «velocities» (7)--(9) are known functions of solution components. Detailed expressions for these functions are not presented here because they are quite complicated.
By using found major unknowns $\overline{\Gamma}$ it is possible to calculate the «velocities» $\dot{\overline{\Gamma}}$: 1) using (1) it is possible to find all unknowns. Then the coefficients g_{ij} in (2) may be found; 2) Then, by using mentioned above formulas the matrix A and vectors $\overline{F}, \overline{E}_1, \overline{E}_2$ may be found; 3) Solving the boundary value problem (7), we find vector $\overline{y}_*$, then using (8) we find values $\sigma_{\xi*}$, $\sigma_{\theta*}$; 4) Finally, using (9), we find the «velocity» of loading parameter and using (10) «velocities» of the rest unknowns $\overline{\Gamma}$.
The described operations for detection of «velocities» of major unknowns $\overline{\Gamma}$ are denoted as

$$\dot{\overline{\Gamma}} = \Pi(\overline{\Gamma}) \tag{11}$$

Cauchy problem (11), (5) is solved by the predictor-corrector method and Euler recalculation method. The stresses σ_ξ, σ_θ are calculated in grid nodes (ξ_i, η_j):

$$\begin{aligned} &\xi_i = \xi_a + ih_1, \quad h_1 = (\xi_b - \xi_a)/n, \quad i = 0,1,\ldots,n; \\ &\eta_j = \eta_j + jh_2, \quad h_2 = h/(2m), \quad i = -m,\ldots 0,1,\ldots,m. \end{aligned} \tag{12}$$

Solution of boundary value problems (7) are realized by two ways: a) using finite difference approximation and orthogonal sweep method [10]; auxiliary Cauchy problems are solved by Runge-Kutta method; б) using spline-approximation [11] and described below variant of spline method implementation.
Let's consider application of splines to the solution of boundary value problem (7). The solution is sought as spline of 2nd degree with coefficients $\overline{s}_{ij}$:

$$\bar{y}_*(\xi) = \bar{s}_{i,0} + \bar{s}_{i,1}(\xi - \xi_i) + \bar{s}_{i,2}(\xi - \xi_i)^2 / 2 \tag{13}$$

Here $\xi \in [\xi_{i-1}, \xi_i]$, $i = 0,1,...,n-1$. Polinomials (13) represent spline being subjected to continuity conditions:

$$\bar{s}_{i+1,0} = \bar{s}_{i,0} + \bar{s}_{i,1}h_1 + \bar{s}_{i,2}h_1^2 / 2, \quad \bar{s}_{i+1,1} = \bar{s}_{i,1} + \bar{s}_{i,2}h_1 \tag{14}$$

Here $i = 0,1,...,n-1$. This spline should satisfy boundary conditions:

$$U_a\bar{s}_{0,0} = \partial V_a / \partial p, \quad U_b\bar{s}_{n,0} = \partial V_b / \partial p \tag{15}$$

The spline should also satisfy equations in nodes ξ_i $(i = 0,1,...,n)$

$$\bar{s}_{i,1} = A|_{\xi=\xi_i}\, \bar{s}_{i,0} + \bar{F}|_{\xi=\xi_i} \tag{16}$$

Relations (14—16) are the closed system of equations regarding the spline coefficients $\bar{s}_{ij}$, $(i = 0,1,...,n;\, j = 0,1)$, $(i = 0,1,...,n-1;\, j = 2)$, the number of algebraic equations is $(18n + 12)$. We decrease the number of equations using continuity conditions (14). By induction from continuity conditions it is not difficult to get the following relations between major spline coefficients $\bar{s}_{0,0}$, $\bar{s}_{0,1}$; $\bar{s}_{i,2}$ $(i = 0,1,...,n-1)$ and the rest its coefficients

$$\bar{s}_{i,0} = \bar{s}_{0,0} + ih_1\bar{s}_{0,1} + 0.5h_1^2\sum_{k=1}^{i-1}\bar{s}_{k,0}(2i - 2k - 1) \tag{17}$$

Relations (17), being substituted in to continuity conditions, convert them in to identities. Introducing relations (17) in equations (15, 16), we get closed system of algebraic equations for major spline coefficients:

$$U_a\bar{s}_{0,0} = \partial V_a / \partial p, \quad U_a\left\{\bar{s}_{0,0} + nh_1\bar{s}_{0,1} + 0.5h_1^2\sum_{k=1}^{n-1}\bar{s}_{k,2}(2n - 2k - 1)\right\} = \partial V_b / \partial p$$

$$-A|_{\xi=\xi_i}\, \bar{s}_{0,0} + [E - ih_1A|_{\xi=\xi_i}\,]\bar{s}_{0,1} + \sum_{k=0}^{i-1}\left\{h_1E - 0.5h_1^2A|_{\xi=\xi_1}\,(2i - 2k - 1)\right\}\bar{s}_{k,2} = \bar{F}|_{\xi=\xi_i}$$

Here E is the unitary matrix. This system of $6(n+2)$ equations has quasi-triangular matrix and is easy solved by Gaussian elimination method.

4. Results of calculations. Described algorithms are coded for BESM-6 computer. The codes are checked by comparison with known numerical solution for a) physically and geometrically nonlinear problems about sub-critical deformation of circular plates [12] (stress-strain relations corresponded to differentiated with respect to time equations of Hencki-Ilyushin theory of plasticity) and b) geometrically nonlinear problems of buckling and post-buckling behavior of elastic spherical shells [13]. Results calculated by using different codes are in good agreement between each other and with mentioned above known data.
The spline and orthogonal sweep methods are compared in calculations of toroidal shell deformations forced by internal pressure. The solutions are controlled by known results [14]. It is found that in the range of geometry parameter $\lambda = [3(1-\nu)]^{1/4} L_m / (Rh)^{1/2} \le 25 - 30$ the solutions of accuracy 1—3% may be calculated more economically by the spline method (here L_m is a meridian length; R is a minimal radius of curvature of middle surface; h is a thickness of shell; ν is Poisson ratio). In this case along meridian the number of grid intervals is

$n = 10 - 40$. Outside the pointed range and for solution with higher accuracy more economical is the orthogonal sweep method.

For detection of upper critical loads with accuracy 1—3% in calculations with constant «time» step it needs about 20—30 steps. In calculations of elastic-plastic shells the accuracy 1—3% it needs about 10—20 grid nodes along thickness ($m = 5 - 10$).

Let's consider the buckling of spherical cap under action of external pressure. The cap borders are fixed (no displacement and no rotation). Initial data we assume as follows:

$$\varphi_0(\xi) = 1.570796 + \xi, \quad r_0(\xi) = R\cos\xi, \quad z_0(\xi) = R\sin\xi, \quad \alpha_0(\xi) = R, \quad \xi_a = 0.785400, \quad \xi_b = 1.570796,$$
$$p_V = -p\cos\varphi, \; p_H = -p\sin\varphi, \; R/h = 100, \; \nu = 0.3, \; \varepsilon_s = 0.268_{10} - 2$$

here E is Young's modulus; that range of ξ corresponds to the angle 45°; expressions for external pressure account the changes of shape of the shell during deformation; R is the radius of curvature of meridian in initial state; ε_s deformation corresponding to yield limit σ_s. The strain-stress diagram of material is shown in table below

σ/σ_s	0 0	!.0	1,13	1.35	1,46	1,54	1.57	1.63	1,66
$\varepsilon/\varepsilon_s$	0.0	1.0	1,49	2,24	2,98	3,74	4,48	5,96	7,48

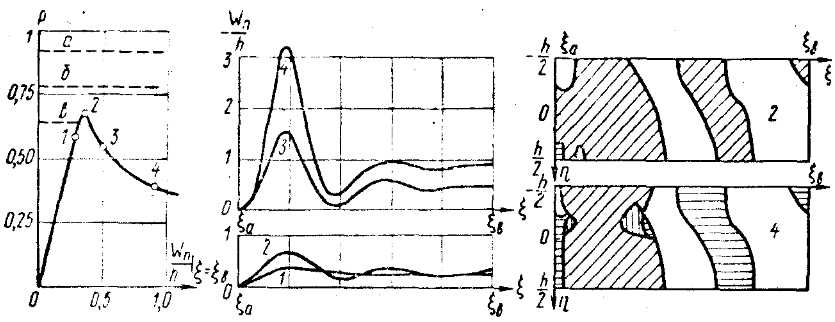

Fig. 2. 3. 4

Expression of «time» parameter is used as (4). Choice of parameter (4) as «time» is confirmed by experimental results [15]. For direct use of this parameter during solution process the expression (4) should be substituted in to general equation (9). It is easy to see that in this particular case the equation (9) is:

$$\dot{p} = \left[\frac{1}{h} \int_{\xi_a}^{\xi_b} w_* d\xi / (\xi_b - \xi_b) \right]^{-1}, \quad w_* = \dot{w} / \dot{p}$$

Considered example was studied earlier in [4] with the help of numerical algorithm using the load as «time». In this study also was used the elastic-plastic flow theory, but equations of geometrical non-linear theory of shells was used in simplified form. Upper critical load found in [4] by divergence of solution algorithm is equal to $\bar{p} = p / p_* = 0.78$ (where $p_* = 2E / (3(1-\nu^2))^{1/2} (h/R)^2$ is a critical load, predicted for elastic spherical cap by the linear bifurcation analysis). This load level is depicted by dashed line in Fig. 2. The dashed line "a" in figure corresponds to upper critical load, detected for the case under consideration by using geometrically nonlinear theory without plastic effects ($p = 0.92$).

The dependence "load—central displacement» which is found in our study, is drawn in Fig. 2 by solid line. Maximum on this curve corresponds to upper critical load $p = 0.67$, it is about 16% less than results of work [4]. It shows that the influence of more accurate terms in equations of geometrically nonlinear theory of shells in the case of elastic-plastic shells is similar to that in the case of elastic shells. Also it shows that the critical loads detected by the fact of convergence of algorithms should be treated as quite approximate.

Fig. 3 shows the displacement in the direction of normal to the middle surface of spherical shell for various time instants (this time instants are numbered in Fig. 2 by digits 1, 2, 3, 4). Fig. 4 demonstrates the situation of zone with elastic (no hatching) and plastic (angular hatching) states of shell material for instants of «time» 2 and 4. Horizontal hatching indicated zones of unloading from plastic state. Vertical hatching marks the zones of secondary plastic deformations.

Conclusion. Suggested method allows to study in more accurate settings the class of important practically and theoretically problems of nonlinear shell theory.

The numerical results indicate the complex character of loading during buckling of elastic-plastic shells. Therefore the use of elastic-plastic flow theory in studies of this class of problems is necessary.

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