Anderson's absolute objects and constant timelike vector hidden in Dirac matrices.

Yuri A. Rylov

Institute for Problems in Mechanics, Russian Academy of Sciences
101-1 ,Vernadskii Ave., Moscow, 117526, Russia
email: rylov@ipmnet.ru
Web site: *http://rsfq1.physics.sunysb.edu/˜rylov/yrylov.htm*
or mirror Web site: *http://194.190.131.172/˜rylov/yrylov.htm*

Abstract

Anderson's theorem asserting, that symmetry of dynamic equations written in the relativisitically covariant form is determined by symmetry of its absolute objects, is applied to the free Dirac equation. γ-matrices are the only absolute objects of the Dirac equation. There are two ways of the γ-matrices transformation: (1) γ^i is a 4-vector and ψ is a scalar, (2) γ^i are scalars and ψ is a spinor. In the first case the Dirac equation is nonrelativistic, in the second one it is relativistic. Transforming Dirac equation to another scalar–vector variables, one shows that the first way of transformation is valid, and the Dirac equation is not relativistic completely.

Key words: absolute object, relativistic covariance, Dirac equation

1 Introduction

Relativistic covariance of dynamic equations and its role in relativistic physics was discussed intensively in the sixth decade of XX century. It seems now that all problems of relativistic description of relativistic dynamical systems have been discussed and solved. Unfortunately, it is not so. Some problems remain. In particular, there is a problem, connected with application of so called absolute objects.

Concept of the absolute object was introduced, apparently, by J.L. Anderson [1], who divided all objects, connected with dynamic systems, into two sorts: dynamical objects and absolute objects.

Dynamical objects (variables) are such objects, which are different for different solutions of dynamic equations. The absolute object is such an object, which is the same for all solutions of the dynamic equations [1].

For instance, let us consider a system of Maxwell equations, describing electromagnetic field tensor F^{ik}, generated by a given 4-current J^i.

$$\partial_k F^{ik} = 4\pi J^i, \qquad \varepsilon_{iklm} g^{jm} \partial_j F^{kl} = 0, \qquad \partial_k \equiv \frac{\partial}{\partial x^k} \tag{1.1}$$

The 4-current is considered to be a given function $J^i = J^i(x)$ of coordinates x of inertial coordinate system K. Here the electromagnetic field tensor F^{ik} is a dynamical object. The Levi-Chivita pseudotensor ε_{iklm}, the metric tensor g^{jm} and external 4-current $J^i(x)$ are absolute objects, because they are the same for all solutions of dynamic equations (1.1).

If one considers the metric tensor g^{ik} to be a solution of the gravitation equation (but not as a fixed quantity), the metric tensor stops to be an absolute object and becomes to be a dynamical object. Similarly, if the 4-current J^i is determined by charged particles, whose motion is described by some dynamic equations, J^i becomes to be a dynamical object.

It is a common practice to think that if dynamic equations of a system can be written in the relativistically covariant form, such a possibility provides automatically a relativistic character of considered dynamic system, described by these equations. In general, it is valid only in the case, when dynamic equations do not contain absolute objects, or these absolute objects has the Poincare group as a group of their symmetry [1]. More exactly, J.L. Anderson shows that the symmetry group of a system of dynamic equations, written in the relativistically covariant form, coincides with the group of symmetry of all absolute objects of this system.

The absolute object, which is a constant unit timelike vector l_k, $k = 0, 1, 2, 3$

$$g^{ik} l_i l_k = 1, \qquad l_k = \text{const.} \tag{1.2}$$

is of most interest. Such a vector l_k is to be interpreted, as a vector, describing a preferred direction in the space-time. Existence of a preferred direction in the space-time is incompatible with the relativity principles.

Let us consider equations

$$m\frac{d^2 x^\alpha}{dt^2} = eF^\alpha_{.0} + eF^\alpha_{.\beta}\frac{dx^\beta}{dt}, \qquad \alpha = 1, 2, 3; \tag{1.3}$$

$$\frac{m}{2}\frac{d}{dt}\left(\frac{dx^\alpha}{dt}\frac{dx^\alpha}{dt}\right) = eF^\alpha_{.0}\frac{dx^\alpha}{dt},$$

describing motion of a nonrelativistic particle of the mass m and of the charge e in the given electromagnetic field F^{ik}. The speed of the light is chosen $c = 1$ Equations (1.3) are written in the non-covariant form, and they are incompatible with the relativity principles. Introducing a constant unit vector l_k, one can write four equations (1.3) in the relativistically covariant form

$$m\frac{d}{d\tau}\left[\frac{\dot{x}^i}{l_k \dot{x}^k} - \frac{1}{2} g^{ik} l_k \frac{\dot{x}^s g_{sl} \dot{x}^l}{(l_j \dot{x}^j)^2}\right] = eF^{il} g_{lk} \dot{x}^k; \qquad i = 0, 1, 2, 3 \tag{1.4}$$

$$\dot{x}^k \equiv \frac{dx^k}{d\tau}, \qquad g_{ik} = \mathrm{diag}\left\{1,-1,-1,-1\right\}$$

where τ is a parameter along the world line $x^l = x^l(\tau)$, $l = 0,1,2,3$ of the particle, and F^{il} is some fixed function of coordinates x.

The equation (1.4) is relativistically covariant with respect to vectors x^i, l_i, and tensors F^{ik}, g_{ik}. A reference to the quantities x^i, l_i, F^{ik}, g_{ik} means that they are considered to be formal variables (but not functions of coordinates x).

Transforming quantities x^i, l_i, F^{ik}, g_{ik} from the coordinate K to the coordinate system $\tilde{K}$

$$x^i \to \tilde{x}^i = x^i + \omega^i_{.k}x^k + o(\omega), \qquad l_i \to \tilde{l}_i = \frac{\partial x^k}{\partial \tilde{x}^i}l_k \tag{1.5}$$

$$F^{ik} \to \tilde{F}^{ik} = \frac{\partial \tilde{x}^i}{\partial x^l}\frac{\partial \tilde{x}^k}{\partial x^m}F^{lm} \qquad g^{ik} \to \tilde{g}^{ik} = \frac{\partial \tilde{x}^i}{\partial x^l}\frac{\partial \tilde{x}^k}{\partial x^m}g^{lm} = g^{ik}, \tag{1.6}$$

one obtains instead of (1.4)

$$m\frac{d}{d\tau}\left[\left(\tilde{l}_k\frac{d\tilde{x}^k}{d\tau}\right)^{-1}\frac{d\tilde{x}^i}{d\tau} - \frac{1}{2}g^{ik}\tilde{l}_k\left(\tilde{l}_j\frac{d\tilde{x}^j}{d\tau}\right)^{-2}g_{sl}\frac{d\tilde{x}^s}{d\tau}\frac{d\tilde{x}^l}{d\tau}\right] = e\tilde{F}^{il}g_{lk}\frac{d\tilde{x}^k}{d\tau} \tag{1.7}$$

Equations (1.4) and (1.7) have the same form, provided the quantities x^i, l_i, F^{ik} are considered to be formal variables. If for instance, the electromagnetic field F^{ik} is considered to be a function of coordinates x, i.e. $F^{ik} = F^{ik}(x)$, then $F^{ik}(x)$ and $\tilde{F}^{ik}(\tilde{x})$ are different functions respectively of x and $\tilde{x}$. In this case the equations (1.4) and (1.7) have different form, because rhs of (1.4) and (1.7) are different function of x and $\tilde{x}$ respectively. In this case one must say that the equation (1.4) is not relativistically covariant with respect to the quantities x^i, l_i, (now a reference to the variable F^{ik} is absent, and it is considered to be a function of x).

Thus, the equation (1.4) is relativistically covariant with respect to the quantities x^i, l_i, F^{ik}. Nevertheless it is incompatible with the relativity principles. Now the reason of this incompatibility is an existence of the constant timelike unit vector l_k. This vector describes a preferred direction in space-time. Any 3-plane orthogonal to l_k may be considered as set of simultaneous events. If the coordinate system is chosen in such a way, that the vector l_k takes the form $l_k = \{1,0,0,0\}$, $t = x^0 = \tau$, the equation (1.4) takes the form (1.3).

Thus, the nonrelativistic character of the equation may be described either by non-covariant form of the equation, or by introducing the absolute object l_k, whose symmetry group is a subgroup of the Lorentz group and does not coincide with the Lorentz group. If a system of dynamic equation is written in a relativistically covariant form and contains a constant timelike unit vector l_k. This vector describes a split of the space-time into space and time, and the system of dynamic equations is incompatible with the relativity principles.

2 Free Dirac equation

Let $\mathcal{S}_{\mathrm{D}}$ be the dynamic system, described by the free Dirac equation

$$i\hbar\gamma^l\partial_l\psi - m\psi = 0 \tag{2.1}$$

which can be obtained from the action

$$\mathcal{S}_{\mathrm{D}}: \qquad \mathcal{A}_{\mathrm{D}}[\bar{\psi},\psi] = \int(-m\bar{\psi}\psi + \frac{i}{2}\hbar\bar{\psi}\gamma^l\partial_l\psi - \frac{i}{2}\hbar\partial_l\bar{\psi}\gamma^l\psi)d^4x \tag{2.2}$$

Here ψ is four-component complex wave function, $\bar{\psi} = \psi^*\gamma^0$ is conjugate wave function, and ψ^* is the Hermitian conjugate one. γ^i, $i = 0,1,2,3$ are 4×4 complex constant matrices, satisfying the relations

$$\gamma^l\gamma^k + \gamma^k\gamma^l = 2g^{kl}I, \qquad k,l = 0,1,2,3. \tag{2.3}$$

where I is unit 4×4 matrix. The speed of the light is chosen $c = 1$.The quantities γ^l form an absolute object, because they are similaar for all solutions of the Dirac equation (2.1)

There are two approaches to the Dirac equation. In the first approach [2, 3] the wave function ψ is considered to be a scalar function defined on the field of Clifford numbers γ^l,

$$\psi = \psi(x,\gamma)\Gamma, \qquad \overline{\psi} = \Gamma\overline{\psi}(x,\gamma), \tag{2.4}$$

where Γ is a constant nilpotent factor which has the property $\Gamma f(\gamma)\Gamma = a\Gamma$. Here $f(\gamma)$ is arbitrary function of γ^l and a is a complex number, depending on the form of the function f. Within such an approach ψ, $\bar{\psi}$ transform as scalars and γ^l transform as components of a 4-vector under the Lorentz transformations. In this case the symmetry group of γ^l is a subgroup of the Lorentz group, and $\mathcal{S}_{\mathrm{D}}$ is nonrelativistic dynamic system. Then the matrix vector γ^l describes some preferred direction in the space-time.

In the second (conventional) approach [4] ψ is considered to be a spinor, and γ^l, $\quad l = 0,1,2,3$ are scalars with respect to the transformations of the Lorentz group. In this case the symmetry group of the absolute objects γ^l is the Lorentz group, and dynamic system $\mathcal{S}_{\mathrm{D}}$ is considered to be a relativistic dynamic system.

Of course, the approaches leading to incompatible conclusions cannot be both valid. At least, one of them is wrong. Analyzing the two approaches, Sommerfeld [3] considered the first approach to be more reasonable. In the second case the analysis is rather difficult due to non-standard transformations of γ^l and ψ under linear coordinate transformations T. Indeed, the transformation T for the vector $j^l = \bar{\psi}\gamma^l\psi$ has the form

$$\tilde{\bar{\psi}}\tilde{\gamma}^l\tilde{\psi} = \frac{\partial\tilde{x}^l}{\partial x^s}\bar{\psi}\gamma^s\psi, \tag{2.5}$$

where quantities marked by tilde mean quantities at the transformed coordinate system. This transformation can be carried out by two different ways

$$1: \quad \tilde{\psi} = \psi, \qquad \tilde{\overline{\psi}} = \overline{\psi}, \qquad \tilde{\gamma}^l = \frac{\partial\tilde{x}^l}{\partial x^s}\gamma^s, \qquad l = 0,1,2,3 \tag{2.6}$$

$$2: \quad \tilde{\gamma}^l = \gamma^l, \qquad l = 0,1,2,3, \qquad \tilde{\psi} = S(\gamma,T)\psi, \qquad \tilde{\overline{\psi}} = \overline{\psi}S^{-1}(\gamma,T), \tag{2.7}$$

$$S^*(\gamma,T)\gamma^0 = \gamma^0 S^{-1}(\gamma,T) \tag{2.8}$$

The relations (2.6) correspond to the first approach and the relations (2.7) correspond to the second one. Both ways (2.6) and (2.7) lead to the same result, provided

$$S^{-1}(\gamma,T)\gamma^l S(\gamma,T) = \frac{\partial \tilde{x}^l}{\partial x^s}\gamma^s \tag{2.9}$$

In particular, for infinitesimal Lorentz transformation

$$x^i \to x^i + \delta\omega^i_{.k} x^k$$

$S(\gamma,T)$ has the form [4]

$$S(\gamma,T) = \exp\left(\frac{\delta\omega_{ik}}{8}\left(\gamma^i\gamma^k - \gamma^k\gamma^i\right)\right) \tag{2.10}$$

The second way (2.7) has a defect. The transformation law of ψ depends on γ, i.e. under linear coordinate transformation T the components of ψ transform through ψ and γ^l, but not only through ψ. Note that tensor components at a coordinate system transform only through tensor components at other coordinate system, and this transformation does not contain any absolute objects. (for instance, the relation (2.5)).

The fact that the symmetry group of a dynamic system coincides with the symmetry group of absolute objects was derived at the supposition, that under the coordinate transformation any object transforms only via its components. This condition is violated in the second case, and one cannot be sure that the symmetry group of dynamic system coincides with that of absolute objects.

3 The case of two-dimensional space-time

To determine which of the two approaches is valid, let us consider such a transformation of the dependent variable ψ, which eliminates the γ-matrices. At first, we consider a more simple case of the two-dimensional space-time. Let

$$\psi_{\mathrm{D}} = \binom{\psi_+}{\psi_-}, \qquad \bar{\psi}_{\mathrm{D}} = \psi^*_{\mathrm{D}}\gamma^0, \qquad \psi^*_{\mathrm{D}} = (\psi^*_+, \psi^*_-) \tag{3.1}$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.2}$$

Representation (3.2) of γ-matrices is chosen in such a way that the pseudoscalar matrix $\gamma^0\gamma^1$ be diagonal, and the wave functions $\psi_{\mathrm{D}} = \binom{\psi_+}{0}$, $\psi_{\mathrm{D}} = \binom{0}{\psi_-}$ be its eigenfunctions for any choice of ψ_+, ψ_-. In this case in virtue of (3.1) the Dirac equation

$$i\hbar\gamma^l\partial_l\psi_{\mathrm{D}} - m\psi_{\mathrm{D}} = 0 \tag{3.3}$$

takes the form

$$\psi_+ = i\lambda\partial_+\psi_-, \qquad \psi_- = i\lambda\partial_-\psi_+, \tag{3.4}$$

$$\lambda \equiv \hbar/m, \qquad \partial_\pm \equiv \partial_0 \pm \partial_1 \tag{3.5}$$

It follows from Eq.(3.4) that both wave functions $\psi_\pm$ satisfy the free Klein-Gordon equation

$$\lambda^2\partial_l\partial^l\psi_\pm + \psi_\pm = 0 \tag{3.6}$$

Let us introduce the two-component differential operator

$$\hat{\mathcal{L}}(w,\partial,\lambda) = \begin{pmatrix} \sqrt{w_+} + i\lambda\sqrt{w_-}\partial_+ \\ \sqrt{w_-} + i\lambda\sqrt{w_+}\partial_- \end{pmatrix}. \tag{3.7}$$

where $w_l = (w_0, w_1)$ is a constant timelike vector and

$$w_+ = w_0 + w_1, \qquad w_- = w_0 - w_1$$

Under the continuous Lorentz transformation

$$\begin{array}{c} x^0 \to \tilde{x}^0 = x^0\cosh\chi + x^1\sinh\chi \\ x^1 \to \tilde{x}^1 = x^1\cosh\chi + x^0\sinh\chi \end{array} \tag{3.8}$$

the components $w_\pm$ and $\partial_\pm$ transform as follows

$$\begin{array}{cc} w_+ \to \tilde{w}_+ = e^\chi w_+, & w_- \to \tilde{w}_- = e^{-\chi}w_- \\ \partial_+ \to \tilde{\partial}_+ = e^\chi\partial_+, & \partial_- \to \tilde{\partial}_- = e^{-\chi}\partial_- \end{array} \tag{3.9}$$

According to Eqs. (3.8), (3.9) the differential operator (3.7) transforms as follows

$$\hat{\mathcal{L}}(w,\partial,\lambda) \to \hat{\mathcal{L}}(\tilde{w},\tilde{\partial},\lambda) = \begin{pmatrix} \sqrt{\tilde{w}_+} + i\lambda\sqrt{\tilde{w}_-}\tilde{\partial}_+ \\ \sqrt{\tilde{w}_-} + i\lambda\sqrt{\tilde{w}_+}\tilde{\partial}_- \end{pmatrix} =$$

$$= \begin{pmatrix} e^{\chi/2}(\sqrt{w_+} + i\lambda\sqrt{w_-}\partial_+) \\ e^{-\chi/2}(\sqrt{w_-} + i\lambda\sqrt{w_+}\partial_-) \end{pmatrix} = e^{-\gamma^0\gamma^1\chi/2}\hat{\mathcal{L}}(w,\partial,\lambda) \tag{3.10}$$

Under the space reflection

$$x^0 \to \tilde{x}^0 = x^0, \qquad x^1 \to \tilde{x}^1 = -x^1 \tag{3.11}$$

one has

$$\begin{array}{cc} w_+ \to \tilde{w}_+ = w_-, & w_- \to \tilde{w}_- = w_+ \\ \partial_+ \to \tilde{\partial}_+ = \partial_-, & \partial_- \to \tilde{\partial}_- = \partial_+ \end{array} \tag{3.12}$$

$$\hat{\mathcal{L}}(w,\partial,\lambda) \to \hat{\mathcal{L}}(\tilde{w},\tilde{\partial},\lambda) = c\gamma^0\hat{\mathcal{L}}(w,\partial,\lambda) \tag{3.13}$$

Under the time reflection

$$x^0 \to \tilde{x}^0 = -x^0, \qquad x^1 \to \tilde{x}^1 = x^1 \tag{3.14}$$

one can write

$$\begin{array}{ll} w_+ \to \tilde{w}_+ = e^{i\pi} w_-, & w_- \to \tilde{w}_- = e^{-i\pi} w_+ \\ \partial_+ \to \tilde{\partial}_+ = e^{i\pi}\partial_-, & \partial_- \to \tilde{\partial}_- = e^{-i\pi}\partial_+ \end{array} \tag{3.15}$$

$$\hat{\mathcal{L}}(w,\partial,\lambda) \to \hat{\mathcal{L}}(\tilde{w},\tilde{\partial},\lambda) = e^{i\pi/2}\gamma^1\hat{\mathcal{L}}(w,\partial,\lambda) \tag{3.16}$$

It means that the differential operator $\hat{\mathcal{L}}(w,\partial,\lambda)$ transforms as a spinor under all transformations of the Lorentz group.

Let us form the two-component quantity

$$\psi_{\mathrm{D}} = \binom{\psi_+}{\psi_-} = \hat{\mathcal{L}}(w,\partial,\lambda)\psi \tag{3.17}$$

If ψ is a scalar, satisfying the Klein-Gordon equation

$$\lambda^2\partial_l\partial^l\psi + \psi = 0, \tag{3.18}$$

then ψ_{D} is a spinor, satisfying the Dirac equation (3.3) for any choice of the timelike constant vector $w_l = (w_0, w_1)$. Vice versa, if the spinor ψ_{D} satisfies the Dirac equation (3.3), then the scalar ψ defined by Eq.(3.17) satisfies the Klein-Gordon equation (3.18) for any choice of the timelike vector w.

Let us compare equations (3.3) and (3.18). None of them contains the vector w explicitly, but connection (3.17) between ψ and ψ_{D} contains this vector w. This fact can be explained only by the fact that the vector w is "hidden" inside the γ-matrices. Eliminating γ-matrices by means of a changing of variables in (3.3), one discovers the constant timelike vector. Let us show this.

The action for the dynamic system $\mathcal{S}_{\mathrm{D}}$ has the form

$$\mathcal{S}_{\mathrm{D}}: \qquad \mathcal{A}_{\mathrm{D}}[\bar{\psi},\psi] = \int(-m\bar{\psi}\psi + \frac{i}{2}\hbar\bar{\psi}\gamma^l\partial_l\psi - \frac{i}{2}\hbar\partial_l\bar{\psi}\gamma^l\psi)d^2x \tag{3.19}$$

Let us substitute four real components of the two-component complex wave function ψ by four scalar-vector variables ρ, j^i, φ

$$\rho = \bar{\psi}\psi, \qquad j^i = \bar{\psi}\gamma^i\psi, \qquad i = 0,1 \tag{3.20}$$

Let us set

$$\gamma^0 = \sigma_1, \quad \gamma^1 = i\sigma_2 \tag{3.21}$$

where $\boldsymbol{\sigma} = \{\sigma_1,\sigma_2,\sigma_3\}$ are Pauli matrices, having the property

$$\sigma_\alpha\sigma_\beta = \sigma_0\delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma, \qquad \alpha,\beta = 1,2,3 \tag{3.22}$$

Here $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Chivita 3-pseudotensor $\varepsilon_{123} = 1$, σ_0 is the unite matrix . Let us represent ψ in the form

$$\psi = A\,(\boldsymbol{\sigma}\mathbf{n})\,e^{i\varphi}\Pi, \qquad \bar{\psi} = A\Pi\,(\boldsymbol{\sigma}\mathbf{n})\,\sigma_1 e^{-i\varphi}, \qquad (\boldsymbol{\sigma}\mathbf{n}) \equiv \sigma_\alpha n_\alpha, \qquad \mathbf{n}^2 = n_\alpha n_\alpha = 1 \tag{3.23}$$

$$\Pi = \frac{1}{2}(1+\gamma^0) = \frac{1}{2}(1+\sigma_1), \tag{3.24}$$

where A, $\mathbf{n} = \{n_1, n_2, n_3\}$ are intermediate variables, which will be expressed via variables ρ, j^i. Π is the zero divisor. Using identity (3.22) and its corollary

$$(\boldsymbol{\sigma}\mathbf{n})\,\sigma_\alpha\,(\boldsymbol{\sigma}\mathbf{n}) \equiv -\mathbf{n}^2\sigma_\alpha + 2n_\alpha\,(\boldsymbol{\sigma}\mathbf{n}) \tag{3.25}$$

one obtains

$$\rho = \bar{\psi}\psi = A^2\left(-1 + 2n_1^2\right) \qquad j^0 = A^2 \qquad j^1 = -2A^2 n_{\mathbf{3}} n_{\mathbf{1}} \tag{3.26}$$

Resolving (3.26) with respect to components of the 3-vector $\mathbf{n}$ and taking into account that $\mathbf{n}^2 = 1$, one obtains

$$n_1 = \sqrt{\frac{j^0+\rho}{2j^0}}, \qquad n_2 = \sqrt{\frac{j^i j_i - \rho^2}{2j^0\left(j^0+\rho\right)}}, \qquad n_3 = -\frac{j^1}{\sqrt{2j^0\left(j^0+\rho\right)}}. \tag{3.27}$$

Let us calculate Lagrangian density $\mathcal{L}$ of the action (3.19) in terms of components of the vector $\mathbf{n}$. One obtains

$$\mathcal{L} = -m\rho - \hbar A^2 n_2^2\left(\partial_0\frac{n_3}{n_2} - \partial_1\frac{n_1}{n_2}\right) - \hbar j^i\partial_i\varphi \tag{3.28}$$

Substituting $A^2 = j^0$ and (3.27) into (3.28), one derives

$$\mathcal{L} = -m\rho + \hbar\frac{j^i j_i - \rho^2}{\left(j^0+\rho\right)}\left(\partial_0\frac{j^1}{\sqrt{\left(j^i j_i - \rho^2\right)}} + \partial_1\frac{j^0+\rho}{\sqrt{j^i j_i - \rho^2}}\right) - \hbar j^i\partial_i\varphi \tag{3.29}$$

The Lagrangian density is expressed in terms of two scalars ρ,φ and the vector j^i. $\mathcal{L}$ is written in non-covariant form. It is not clear, if it possible to transform it to relativistically covariant form. To show that it is possible, let us introduce the two-component quantities

$$q^l = \frac{j^l + \rho f^l}{\sqrt{j^i j_i - \rho^2}}, \qquad l = 0, 1, \qquad f^l = \{1, 0\} \tag{3.30}$$

Now resolving relations (3.30) with respect to j^i in the form

$$j^l = \frac{2\rho\left(q^s f_s\right)}{\left(q^k q_k - 1\right)}q^l - \rho f^l, \qquad l = 0, 1 \tag{3.31}$$

and substituting j^i in (3.29), one obtains expression for the action

$$\mathcal{S}_{\mathrm{D}}: \qquad \mathcal{A}_{\mathrm{D}}[\rho, \varphi, j] = \int(-m\rho - \hbar\frac{2\rho\left(\partial_0 q_1 - \partial_1 q_0\right)}{\left(q^s q_s - 1\right)} - \hbar j^i\partial_i\varphi)d^2x \tag{3.32}$$

where q^l is expressed via dependent dynamical variables ρ, φ, j^i by means of the relation (3.30). If f^i is a vector, then accorfng to (3.30) q^i is also vector and the Lagrangian density in (3.32) has the covariant form, $\mathcal{L}$ is an invariant.

Thus, eliminating γ-matrices, and writing the Lagrangian density in the relativistically covariant form, one discovers the constant timelike unite vector f^i. This vector is an absolute object, describing a preferred space-time direction, that is incompatible with the relativity principles.

4 The case of four-dimensional space-time

A similar elimination of Dirac matrices can be made in the case of the four-dimensional space-time. The state of dynamic system $\mathcal{S}_{\mathrm{D}}$ (2.2) is described by eight real dependent variables (eight real components of four-component complex wave function ψ). It is possible to transform the variables ψ and to describe this system in terms of scalar-vector variables j^l, S^l, $(l = 0,1,2,3)$, φ, κ. The current 4-vector j^l and the spin 4-pseudovector S^l are defined by the relations

$$j^l = \bar{\psi}\gamma^l\psi, \qquad l = 0,1,2,3, \qquad \bar{\psi} = \psi^*\gamma^0;$$

$$S^l = i\bar{\psi}\gamma_5\gamma^l\psi, \qquad l = 0,1,2,3, \qquad \gamma_5 = \gamma^{0123} \equiv \gamma^0\gamma^1\gamma^2\gamma^3; \tag{4.1}$$

The scalar φ and pseudoscalar κ are defined implicitly via the wave function ψ by the relations

$$\psi = Ae^{i\varphi+\frac{1}{2}\gamma_5\kappa}e^{-\frac{i}{2}\gamma_5\boldsymbol{\sigma\eta}}e^{\frac{i\pi}{2}\boldsymbol{\sigma}\mathbf{n}}\Pi \tag{4.2}$$

$$\psi^* = A\Pi e^{-\frac{i\pi}{2}\boldsymbol{\sigma}\mathbf{n}}e^{-\frac{i}{2}\gamma_5\boldsymbol{\sigma\eta}}e^{-i\varphi-\frac{1}{2}\gamma_5\kappa} \tag{4.3}$$

where (*) means the Hermitian conjugation.

One uses the following designations

$$\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_3,\} = \{-i\gamma^2\gamma^3, -i\gamma^3\gamma^1, -i\gamma^1\gamma^2\} \tag{4.4}$$

$$\Pi = \frac{1}{4}(1+\gamma^0)(1+\mathbf{z}\boldsymbol{\sigma}), \qquad \mathbf{z} = \{z^\alpha\} = \mathrm{const}, \qquad \alpha = 1,2,3; \qquad \mathbf{z}^2 = 1 \tag{4.5}$$

The quantities A, κ, φ, $\boldsymbol{\eta} = \{\eta^\alpha\}$, $\mathbf{n} = \{n^\alpha\}$, $\alpha = 1,2,3$, $\mathbf{n}^2 = 1$ are eight real parameters, determining the wave function ψ. These parameters may be considered as new dependent variables, describing the state of dynamic system $\mathcal{S}_{\mathrm{D}}$. The quantity φ is a scalar, and κ is a pseudoscalar.

Six remaining variables A, $\boldsymbol{\eta} = \{\eta^\alpha\}$, $\mathbf{n} = \{n^\alpha\}$, $\alpha = 1,2,3$, $\mathbf{n}^2 = 1$ are intermediate. They can be expressed through the current 4-vector $j^l = \bar{\psi}\gamma^l\psi$ and spin 4-pseudovector S^i, defined by the relation (4.1). Because of two identities

$$S^l S_l \equiv -j^l j_l, \qquad j^l S_l \equiv 0. \tag{4.6}$$

there is only six independent components among eight components of quantities j^l, and S^l, $l = 0,1,2,3$. Connection between the 4-vector j^i and intermediate parameters A, η^α has the form

$$\begin{aligned} j^0 &= A^2\cosh\eta \\ j^\alpha &= A^2 v^\alpha \sinh\eta, \qquad \alpha = 1,2,3 \end{aligned} \tag{4.7}$$

where

$$\mathbf{v} = \{v^\alpha\}, \qquad v^\alpha = \eta^\alpha/\eta, \qquad \alpha = 1,2,3; \qquad \mathbf{v}^2 = 1. \tag{4.8}$$

The unit 3-pseudovector $\boldsymbol{\xi}$ is connected with the spin 4-pseudovector S^l by means of the relations

$$\xi^\alpha = \rho^{-1}\left[S^\alpha - \frac{j^\alpha S^0}{(j^0+\rho)}\right], \qquad \alpha = 1,2,3; \qquad \rho \equiv \sqrt{j^l j_l} \tag{4.9}$$

$$S^0 = \mathbf{j}\boldsymbol{\xi}, \qquad S^\alpha = \rho\xi^\alpha + \frac{(\mathbf{j}\boldsymbol{\xi})j^\alpha}{\rho + j^0}, \qquad \alpha = 1,2,3 \tag{4.10}$$

Let us make a change of variables in the action (2.2), using substitution (4.2), (4.3), (4.5). Calculations are rather bulky, and we omit them (detailed calculation one can find in [5], or in [6]). Result of substitution has the form

$$\mathcal{S}_\mathrm{D}: \qquad \mathcal{A}_D[j,\varphi,\kappa,\boldsymbol{\xi}] = \int \mathcal{L}d^4x, \qquad \mathcal{L} = \mathcal{L}_{cl} + \mathcal{L}_{q1} + \mathcal{L}_{q2} \tag{4.11}$$

$$\mathcal{L}_{cl} = -m\rho - \hbar j^i\partial_i\varphi + \hbar j^s\varepsilon_{iklm}\mu^i\partial_s\mu^k z^l f^m, \qquad \rho \equiv \sqrt{j^l j_l} \tag{4.12}$$

$$\mathcal{L}_{q1} = 2m\rho\sin^2(\frac{\kappa}{2}) - \frac{\hbar}{2}S^l\partial_l\kappa, \tag{4.13}$$

$$\mathcal{L}_{q2} = -\hbar\rho\varepsilon_{iklm}q^i(\partial^k q^l)\nu^m \tag{4.14}$$

where the following designations are used

$$f^i = \{1,0,0,0\}, \qquad z^i = \{0,z^1,z^2,z^3\} \tag{4.15}$$

$$\nu^i = \xi^i - (\xi^s f_s)f^i, \qquad i = 0,1,2,3; \qquad \nu^i\nu_i = -1, \tag{4.16}$$

$$\mu^i \equiv \frac{\nu^i}{\sqrt{-(\nu^l + z^l)(\nu_l + z_l)}} = \frac{\nu^i}{\sqrt{2(1 - \nu^l z_l)}} = \frac{\nu^i}{\sqrt{2(1 + \boldsymbol{\xi}\mathbf{z})}}. \tag{4.17}$$

$$q^i \equiv \frac{j^i + f^i\rho}{\sqrt{(j^l + f^l\rho)(j_l + f_l\rho)}} = \frac{j^i + f^i\rho}{\sqrt{2\rho(\rho + j^l f_l)}}, \qquad q_s q^s = 1 \tag{4.18}$$

The 4-pseudovector S^i is defined by the relation (4.10).

Lagrangian density (4.11) – (4.14) appears to be relativistically invariant with respect to quantities $f^i, z^i, \xi^i, \nu^i, j^i, q^i, \kappa, \varphi$, provided the quantity f^i, $i = 0,1,2,3$ are considered to be components of a constant timelike unit 4-vector and z^i the constant 4-pseudovector orthognal to f^i. Then quantities ξ^i and ν^i, defined by (4.16) form 4-pseudovectors. Quantities μ^i and q^i appear to be respectively 4-vector and 4-pseudovector. The quantities φ, κ are scalar and pseudoscalar respectively. The component of ξ^i parallel to vector f^i appears to be arbitrary and unessential. The 4-vector z^i appears to be fictitious.

After eliminating γ-matrices and representing the Lagrangian density in a relativistically invariant form (4.11) – (4.14), one discovers an additional constant timelike unit 4-vector f^i. There is only one possible interpretation of this 4-vector. It describes the space-time split into space and time. It means that the system $\mathcal{S}_\mathrm{D}$, described by the free Dirac equation, is not relativistic, i.e. it is incompatible with the relativity principles. This result agrees with the result of the two-dimensional space-time consideration.

5 Concluding remark

Dirac equation is a very important equation. It is one of fundamental equations of quantum electrodynamics. Nonrelativistic character of Dirac equation means that it is not correct, and one needs a revision. Fortunately, nonrelativistic description concerns only internal degrees of freedom connected with the spin variables $\boldsymbol{\xi}$ [6]. At the low energy processes these degrees of freedom are not excited, and variables $\boldsymbol{\xi}$, describing them, are considered to be constants. At the high energy processes correction of the Dirac equation may appear to be essential.

References

[1] J. L. Anderson, *Principles of relativity physics.* Academic Press, New-York, 1967, pp 75-88.

[2] F. Sauter, *Zs. Phys.* **63,** 803, (1930), **64**, 295, (1930).

[3] A. Sommerfeld, *Atombau and Spektrallinien.* bd.2, Braunschweig, 1951.

[4] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory.* New York, 1961, chp. 4, sec.3.

[5] Yu. A. Rylov, Adv. Appl. Cliff. Alg. **5**, No. 1, 1, (1995).

[6] Yu. A. Rylov, eprint quant-ph/0011044.