Asymmetric Nondegenerate Geometry

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Abstract

Nondegenerate geometry (T-geometry) with nonsymmetric world function is considered. In application to the space-time geometry the asymmetry of world function means that the past and the future are not equivalent geometrically. T-geometry is described in terms of finite point subspaces and world function between pairs of points of these subsets, i.e. in the language which is immanent to geometry and free of external means of description (coordinates, curves). Such a description appears to be simple and effective even in the case of complicated T-geometry. Antisymmetric component of the world function generates appearance of additional metric fields. This leads to appearance of three sorts of Christoffel symbols and three sorts of geodesics. Three sorts of the first order tubes (future, past and neutral) appear. If the fields connected with the antisymmetric component are strong enough the timelike first order tube has a finite length in the timelike direction. It was shown earler that the symmetric T-geometry explains non-relativistic quantum effects without a reference to principles of quantum mechanics. One should expect that nonsymmetric space-time T-geometry is also characteristic for microcosm, and it will be useful in the elementary particle theory, because there is a series of unexpected associations. First order tubes are associated with world tubes of closed strings. Antisymmetry is associated with supersymmetry. Three sorts of tubes are associated with three sorts of quarks. Limitation of the tube in time direction is associated with confinement. At any rate, the space-time T-geometry with additional parameters has more capacities, than usual Riemannian geometry.

1 Introduction

The most general geometry is obtained, if for its construction one uses the purely metric conception of geometry. The conception of geometry (CG) is defined as a method of construction of the standard (Euclidean) geometry. There are several different conceptions of geometry, (i.e. methods of the Euclidean geometry construction). They differ in amounts of numerical information on geometry.

title of CG	*non-numerical information*	*numerical information*
Euclidean CG	Euclidean axioms	$\emptyset$
Riemannian CG	Manifold, curve coordinate system	$n, \quad g_{ik}(x)$
topology-metric CG	topological space, curve	$\rho(P,Q) \geq 0,$ $\rho(P,Q) = 0, \quad \text{iff} \quad P = Q$ $\rho(P,Q) + \rho(Q,R) \geq \rho(P,R)$
purely metric CG	$\emptyset$	$\sigma(P,Q) = \frac{1}{2}\rho^2(P,Q) \in \mathbb{R}$

Varying the numerical information at the fixed non-numerical one, we obtain different geometries in the scope of the same conception of geometry. Varying continuously numbers and functions, constituting numerical information of CG, one obtains a continuous set of geometries, each of them differs slightly from the narrow one. Any admissible value of numerical information is attributed to some geometry in the scope of the given CG. The purely metric CG contains only numerical information, and it generates the most general class of geometries, suitable for applying them as the space-time geometries.

Information on geometry is contained in the world function $\Sigma(P,Q) = \frac{1}{2}\rho^2(P,Q)$, where $\rho(P,Q)$ is the distance between the points P and Q. Conventionally the world function Σ is considered to be a symmetric real function of points P and Q of the point set Ω.

$$\Sigma(P,Q) = \Sigma(Q,P), \qquad P,Q \in \Omega \tag{1.1}$$

In the papers [1, 2, 3] a nondegenerate geometry with the symmetric world function was considered. It is a very general conception of geometry. It admits one to construct such a space-time geometry, which explains quantum phenomena without referring to principles of quantum mechanics, i.e. as a geometric properties of the space-time.

The purely metric CG generates a set of geometries each of them is labelled by the form of the world function Σ. Any such a geometry is called nondegenerate geometry. T-geometry (tubular geometry) is another name of nondegenerate geometry. In general, T-geometries are not axiomatized. Only one of T-geometries (the standard geometry) has been axiomatized. The proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ plays the role of this standard geometry. It is labelled by the world function $\Sigma_{\rm E}$. The standard geometry has been studied very well. All geometric objects and all statements of the standard geometry $\mathcal{G}_{\rm E}$ are expressed in terms and only in terms of the world function $\Sigma_{\rm E}$. To obtain geometric objects and statements of a T-geometry $\mathcal{G}$, labelled by the world function Σ, it is sufficient one to take geometric objects and statements of the standard geometry $\mathcal{G}_{\rm E}$, labelled by $\Sigma_{\rm E}$, and to replace $\Sigma_{\rm E}$ by Σ there. As far as all statements of $\mathcal{G}$ are expressed in terms of Σ, one obtains statements of $\mathcal{G}$ as a modification of statements of $\mathcal{G}_{\rm E}$.

Such an approach admits one to describe any T-geometry $\mathcal{G}$ without introducing axiomatization of $\mathcal{G}$. Let us yield necessary definitions.

Definition 1.1 *T-geometry is the set of all statements about properties of all geometric objects .*

The T-geometry is constructed on the point set Ω by giving the world function Σ. The Σ-space $V=\{\Sigma,\Omega\}$ is obtained from the metric space after removal of the constraints, imposed on the metric and introduction of the world function Σ

$$\Sigma\left(P,Q\right)=\frac{1}{2}\rho^{2}\left(P,Q\right),\qquad P,Q\in\Omega \tag{1.2}$$

instead of the metric ρ:

Definition 1.2 *Σ-space $V=\{\Sigma,\Omega\}$ is nonempty set Ω of points P with given on $\Omega\times\Omega$ real function Σ*

$$\Sigma:\quad\Omega\times\Omega\rightarrow\mathbb{R},\qquad\Sigma(P,P)=0,\qquad\forall P\in\Omega. \tag{1.3}$$

The function Σ is known as the world function [4], or Σ-function. The metric ρ may be introduced in Σ-space by means of the relation (1.2). If Σ is positive, the metric ρ is also positive, but if Σ is negative, the metric is imaginary.

Definition 1.3 . *Nonempty point set $\Omega'\subset\Omega$ of Σ-space $V=\{\Sigma,\Omega\}$ with the world function $\Sigma'=\Sigma|_{\Omega'\times\Omega'}$, which is a contraction Σ on $\Omega'\times\Omega'$, is called Σ-subspace $V'=\{\Sigma',\Omega'\}$ of Σ-space $V=\{\Sigma,\Omega\}$.*

Further the world function $\Sigma'=\Sigma|_{\Omega'\times\Omega'}$, which is a contraction of Σ will be denoted as Σ. Any Σ-subspace of Σ-space is a Σ-space. In T-geometry a geometric object $\mathcal{O}$ is described by means of skeleton-envelope method. It means that any geometric object $\mathcal{O}$ is defined as follows.

Definition 1.4 *Geometric object $\mathcal{O}$ is some Σ-subspace of Σ-space, which can be represented as a set of intersections and joins of elementary geometric objects (EGO).*

Definition 1.5 *Elementary geometric object $\mathcal{E} \subset \Omega$ is a set of zeros of the envelope function*

$$f_{\mathcal{P}^n}: \qquad \Omega \rightarrow \mathbb{R}, \qquad \mathcal{P}^n \equiv \{P_0, P_1, ...P_n\} \in \Omega^{n+1} \tag{1.4}$$

i.e.

$$\mathcal{E} = \mathcal{E}_f(\mathcal{P}^n) = \{R | f_{\mathcal{P}^n}(R) = 0\} \tag{1.5}$$

The finite set $\mathcal{P}^n \subset \Omega$ of parameters of the envelope function $f_{\mathcal{P}^n}$ is the skeleton of elementary geometric object (EGO). The set $\mathcal{E} \subset \Omega$ of points forming EGO is called the envelope of its skeleton $\mathcal{P}^n$. The envelope function $f_{\mathcal{P}^n}$ is an algebraic function of s arguments $w = \{w_1, w_2, ...w_s\}$, $s = (n+2)(n+1)$. Each of arguments $w_k = \Sigma(Q_k, L_k)$ is a Σ-function of two arguments $Q_k, L_k \in \{R, \mathcal{P}^n\}$.

For continuous T-geometry the envelope $\mathcal{E}$ is usually a continual set of points. The envelope function $f_{\mathcal{P}^n}$, determining EGO is a function of the running point $R \in \Omega$ and of parameters $\mathcal{P}^n \in \Omega^{n+1}$. Thus, any elementary geometric object is determined by its skeleton $\mathcal{P}^n$ and by the form of the envelope function $f_{\mathcal{P}^n}$.

Let one investigates T-geometry on the Σ-space $V = \{\Sigma, \Omega\}$. For some special choice Σ_{E} of Σ-function, the Σ-space V turns to a Σ-subspace $V'_{\mathrm{E}} = \{\Sigma_{\mathrm{E}}, \Omega\}$ of a n-dimensional proper Euclidean space $V_{\mathrm{E}} = \{\Sigma_{\mathrm{E}}, \Omega_{\mathrm{E}}\}$, $\Omega \subset \Omega_{\mathrm{E}}$. (It will be shown). Then all relations between geometric objects in V'_{E} are relations of proper Euclidean geometry. Replacement of Σ_{E} by Σ means a deformation of V'_{E}, because world function Σ describes distances between two points, and change of these distances is a deformation of the space. We shall use concept of deformation in a wide meaning, including in this term any increase and any reduction of number of points in the set Ω. Then any transition from $\{\Sigma_{\mathrm{E}}, \Omega_{\mathrm{E}}\}$ to $\{\Sigma, \Omega\}$ is a deformation of $\{\Sigma_{\mathrm{E}}, \Omega_{\mathrm{E}}\}$.

Let us write Euclidean relations between geometric objects in V'_{E} in the σ-immanent form (i.e. in the form, which contains references only to geometrical objects and Σ-function). Changing the world function Σ_{E} by Σ in these relations, one obtains the relations between the geometric objects in the Σ-space $V = \{\Sigma, \Omega\}$.

Thus, geometry in the proper Euclidean space is known very well, and one uses deformation, described by world function, to establish T-geometry in arbitrary Σ-space. Considering deformations of Euclidean space, one goes around the problem of axiomatics in the Σ-space $V = \{\Sigma, \Omega\}$. One uses only Euclidean axiomatics. T-geometry of arbitrary Σ-space is obtained as a result of "deformation of proper Euclidean geometry". This point is very important, because axiomatics of arbitrary T-geometry is very complicated. It is relatively simple only for highly symmetric spaces. Investigation of arbitrary deformations is much simpler, than investigations of arbitrary axiomatics. Formally, a work with deformations of Σ-spaces is manipulations with the world function. These manipulations may be carried out without mention of space deformations.

Description of EGOs by means (1.4) is carried out in the deform-invariant form (invariant with respect to Σ-space deformations). The envelope function $f_{\mathcal{P}^n}$ as a function of arguments $w_k = \Sigma(Q_k, L_k)$, $Q_k, L_k \in \{R, \mathcal{P}^n\}$ does not depend on the form of the world function Σ. Thus, definition of the envelope function is ivariant with respect to deformations (deform-invariant), and the envelope function determines any EGO in all Σ-spaces at once.

Let $\mathcal{E}_{\mathrm{E}}$ be EGO in the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$. Let $\mathcal{E}_{\mathrm{E}}$ be described by the skeleton $\mathcal{P}^n$ and the envelope function $f_{\mathcal{P}^n}$ in the Σ-space $V_{\mathrm{E}} = \{\Sigma_{\mathrm{E}}, \Omega\}$. Then the EGO $\mathcal{E}$ in the T-geometry $\mathcal{G}$, described by the same skeleton $\mathcal{P}^n$ and the same envelope function $f_{\mathcal{P}^n}$ in the Σ-space $V = \{\Sigma, \Omega\}$, is an analog in $\mathcal{G}$ of the Euclidean EGO $\mathcal{E}_{\mathrm{E}}$. T-geometry $\mathcal{G}$ may be considered to be a result of deformation of the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$, when distances $\sqrt{\Sigma(P,Q)+\Sigma(Q,P)}$ between the pairs of points P and Q are changed. At such a deformation the Euclidean EGO $\mathcal{E}_{\mathrm{E}}$ transforms to its analog $\mathcal{E}$.

The Euclidean space has the most powerful group of motion, and the same envelope $\mathcal{E}_{\mathrm{E}}$ may be generated by the envelope function $f_{\mathcal{P}^n}$ with different values $\mathcal{P}^n_{(1)}$, $\mathcal{P}^n_{(2)}$, ... of the skeleton $\mathcal{P}^n$, or even by another envelope function $f_{(1)\mathcal{Q}^m}$. It means that the Euclidean EGO $\mathcal{E}_{\mathrm{E}}$ may have several analogs $\mathcal{E}_{(1)}, \mathcal{E}_{(2)}$, ...in the geometry $\mathcal{G}$. In other words, deformation of the Euclidean space may split EGOs, (but not only deform them). Note that the splitting may be interpreted as a kind of deformation.

In the topologo-metric conception of geometry [5, 6, 7] one uses concept of curve as the continuous mapping

$$\mathcal{L}: \quad [0,1] \to \Omega, \qquad [0,1] \subset \mathbb{R}, \tag{1.6}$$

The curve $\mathcal{L}([0,1]) \subset \Omega$ is considered to be an important geometrical object of geometry. From point of view of T-geometry the set of points $\mathcal{L}([0,1]) \subset \Omega$ cannot be considered to be EGO, because the mapping (1.6) is not deform-invariant. Indeed, let us consider a sphere $\mathcal{S}_{P_0P_1}$, passing through the point P_1 and having its center at the point P_0. It is described by the envelope function

$$f_{P_0P_1}(R) = \sqrt{\Sigma(P_0,R)+\Sigma(R,P_0)} - \sqrt{\Sigma(P_1,R)+\Sigma(R,P_1)} \tag{1.7}$$

In the two-dimensional proper Euclidean space the envelope function (1.7) describes a one-dimensional circumference $\mathcal{L}_1$, whereas in the three-dimensional proper Euclidean space the envelope function (1.7) describes a two-dimensional sphere $\mathcal{S}_2$. The point set $\mathcal{L}_1$ can be represented as the continuous mapping (1.6), whereas the surface $\mathcal{S}_2$ cannot. Transition from two-dimensional Euclidean spac3e to three-dimensional Euclidean space is a space deformation. Thus, deformation of the Σ-space may destroy the property of EGO of being a curve (1.6).

Application of objects, defined by the property (1.6) for investigation of T-geometries is inconvenient, because the T-geometry investigation is founded on deform-invariant methods. Formally, one cannot choose appropriate envelope function for description of the set (1.6), because the envelope function is deform-invariant, whereas the set (1.6) is not. Hence, (1.6) is incompatible with the definition 1.5 of EGO. In other words, the point set (1.6) is not deform-invariant.

There are at least two approach to geometry:

(1) The geometry is supposed to be completely described by the world function. The standard (proper Euclidean) geometry is such a kind of geometry. Then any possible geometry is one of T-geometries.

(2) The geometry is supposed to be described by the world function incompletely. There are functions of three and more number of points, which describe such a geometry (for instance, functions $\Sigma_3(P_0, P_2, P_2)$,...). We shall refer to such a geometry as the fortified geometry (FG). The standard (proper Euclidean) geometry is described by the world function Σ. In the Euclidean geometry the function Σ_3 and other functions have some trivial form (for instance, $\Sigma_{3\mathrm{E}} \equiv 0$).

The T-geometry pretends only to description of usual geometry, but not to fortified geometry (FG). It is not quite clear, whether the papers [5, 6, 7] pretend to description of the fortified geometry, or they are restricted by the usual one. If they pretend only to a description of usual geometry, then the concept of the curve (1.6) is only a method for description of geometry (something like a coordinate system), but not an object of geometry. All real properties of geometry (not those of the description method) should be formulated in the form invariant with respect to a change of the description method (in particular, in the form, where the concept of the curve is not mentioned at all).

If the papers [5, 6, 7] pretend to a description of some special cases of FG, which cannot be described by T-geometry, then a use of the concept (1.6) may be justified. But such an approach seems to be too complicated. It seems to be simpler one to have investigated the usual geometry completely, and thereafter to begin investigate a more complicated case of fortified geometry. The next step in the investigation of the usual geometry is a refuse from the constraint (1.1). Such a geometry will be referred to as a nonsymmetric T-geometry.

The nonsymmetric T-geometry can be investigated by the same methods, as the symmetric one. The world function Σ in the nonsymmetric T-geometry is presented in the form

$$\begin{aligned}
\Sigma(P,Q) &= G(P,Q) + A(P,Q), \qquad P,Q \in \Omega && (1.8)\\
G(P,Q) &= G(Q,P), \qquad A(P,Q) = -A(Q,P) && (1.9)\\
G(P,Q) &= \frac{1}{2}\left(\Sigma(P,Q) + \Sigma(Q,P)\right), && (1.10)\\
A(P,Q) &= \frac{1}{2}\left(\Sigma(P,Q) - \Sigma(Q,P)\right) && (1.11)
\end{aligned}$$

where G denotes the symmetric part of the world function Σ, whereas A denotes its antisymmetric part.

Motives for consideration of nonsymmetric T-geometry are as follows. In the symmetric T-geometry the distance from the point P to the point Q is the same as the distance from the point Q to the point P. In the asymmetric T-geometry it is not so. Apparently, it is not important for spacelike distances in the space-time, because it can be tested experimentally for spacelike distances. In the case, when interval between points P and Q is timelike, one uses watch to measure this interval. But the watch can measure the time interval only in one direction, and one cannot be sure that the time interval is the same in opposite direction.

If the antisymmetric part A of the world function does not vanish, it means that the future and the past are not equivalent geometrically. We do not insist that this fact takes place, but we admit this. It is useful to construct a nonsymmetric T-geometry, to apply it to the space-time and to obtain the corollaries of asymmetry which could be tested experimentally. The symmetrical part of the world function generates the field of the metric tensor g_{ik}. In a like way the antisymmetric part generates some vector force filed a_i. Maybe, existence of this field can be tested experimentally. For construction of nonsymmetric T-geometry one does not need to make any additional supposition. It is sufficient to remove the constraint (1.1) and to apply mathematical technique developed for the symmetric T-geometry with necessary modifications.

Besides, there is a hope that nonsymmetric T-geometry will be useful in the elementary particle theory, where the main object is a superstring. The first order tubes (main objects of T-geometry) are associated with world tubes of strings. In the nonsymmetric T-geometry antisymmetric variables appear. They are absent in the conventional symmetric T-geometry, but antisymmetric variables are characteristic for the superstring theory.

Two important general remarks.

1. Nonsymmetric T-geometry, as well as the symmetric one, is considered on an arbitrary set Ω of points P. It is formulated in the scope of the purely metric conception of geometry, which is very simple, because it uses only very simple tools for the geometry description. The T-geometry formulated in terms of the world function Σ and finite subsets $\mathcal{P}^n \equiv \{P_0, P_1, ..., P_n\}$ of the set Ω. Mathematically it means, that the purely metric CG uses only mappings

$$m_n: \quad I_n \rightarrow \Omega, \qquad I_n \equiv \{0, 1, ...n\} \subset \{0\} \cup \mathbb{N} \tag{1.12}$$

whereas the topology-metric CG uses much more complicated mappings (1.6), known as curves $\mathcal{L}$. Both mappings (1.12) and (1.6) are methods of the geometry description (and construction). But the method (1.12) is much simpler. It can be studied exhaustively, whereas the set of mappings (1.6) cannot.

2. The nonsymmetric T-geometry will be mainly interpreted as a symmetric T-geometry determined by the two-point scalar $G(P,Q)$ with some additional metric structures, introduced to the symmetric geometry by means of the additional two-point scalar $A(P,Q)$. For instance, in the symmetric space-time T-geometry the world line of a free particle is described by a geodesic. In the nonsymmetric space-time T-geometry there are, in general, several different types of geodesics. This fact may be interpreted in the sense, that a free particle has some internal degrees of freedom, and it may be found in different states. In these different states the free particle interacts with the force fields, generated by the two-point scalar $A(P,Q)$, differently. Several different types of geodesics are results of this interaction.

In the second section formulation of the asymmetric T-geometry in a coordinate-free form is presented. In the third section properties of tubes in the Σ-space are considered. The fourth section is devoted to consideration of Σ-space, given on a manifold. Derivatives of the world function at the coincidence points $x = x'$ are considered in the fifth section. The two-point curvature tensor is investigated in the sixth section. The seventh and eighth sections are devoted to gradient lines and their σ-immanent description. Condition of the neutral first order tube degeneration are considered in the ninth section. In the tenth section examples of the first order tubes are considered.

2 Σ-space and T-geometry. Coordinate-free description

Definition 2.1 . *Σ-space $V = \{\Sigma, \Omega\}$ is called isometrically embeddable in Σ-space $V' = \{\Sigma', \Omega'\}$, if there exists such a monomorphism $f : \Omega \to \Omega'$, that $\Sigma(P,Q) = \Sigma'(f(P), f(Q))$, $\quad \forall P, \forall Q \in \Omega, \quad f(P), f(Q) \in \Omega'$,*

Any Σ-subspace V' of Σ-space $V = \{\Sigma, \Omega\}$ is isometrically embeddable in it.

Definition 2.2 . *Two Σ-spaces $V = \{\Sigma, \Omega\}$ and $V' = \{\Sigma', \Omega'\}$ are called to be isometric (equivalent), if V is isometrically embeddable in V', and V' is isometrically embeddable in V.*

Definition 2.3 *The Σ-space $M = \{\Sigma, \Omega\}$ is called a finite Σ-space, if the set Ω contains a finite number of points.*

Definition 2.4 . *The Σ-subspace $M_n(\mathcal{P}^n) = \{\Sigma, \mathcal{P}^n\}$of the Σ-space $V = \{\Sigma, \Omega\}$, consisting of $n+1$ points $\mathcal{P}^n = \{P_0, P_1, ..., P_n\}$ is called the nth order Σ-subspace .*

The T-geometry is a set of all propositions on properties of Σ-subspaces of Σ-space $V = \{\Sigma, \Omega\}$. Presentation of T-geometry is produced in the language, containing only references to Σ-function and constituents of Σ-space, i.e. to its Σ-subspaces.

Definition 2.5 *A description is called σ-immanent, if it does not contain any references to objects or concepts other, than finite subspaces of the Σ-space and its world function (metric).*

σ-immanence of description provides independence of the description on the method of description. In this sense the σ-immanence of a description in T-geometry reminds the concept of covariance in Riemannian geometry. Covariance of some geometrical relation in Riemannian geometry means that the considered relation is valid in all coordinate systems and, hence, describes only the properties of the Riemannian geometry in itself. Covariant description provides cutting-off from the coordinate system properties, considering the relation in all coordinate systems at once. The σ-immanence provides truncation from the methods of description by absence of a reference to objects, which do not relate to geometry in itself (coordinate system, concept of curve, dimension).

The idea of constructing the T-geometry is very simple. All relations of proper Euclidean geometry are written in the σ-immanent form and declared to be valid for any Σ-function. This results that any relation of proper Euclidean geometry corresponds to some relation of T-geometry. It is important that in the relations, declared to be relations of T-geometry, only the properties (1.3) were used. The special properties of the Euclidean Σ-function are not to be taken into account. The metric part of these relations was formulated and proved by K. Menger [8]. Let us present this result in our designations in the form of the theorem

Theorem 1 *The symmetric Σ-space $V = \{\Sigma, \Omega\}$ is isometrically embeddable in n-dimensional proper Euclidean space E_n, if and only if any $(n+2)$th order Σ-subspace $M(\mathcal{P}^{n+2}) \subset \Omega$ is isometrically embeddable in E_n.*

Unfortunately, the formulation of this theorem is not σ-immanent, as far as it contains a reference to n-dimensional Euclidean space E_n which is not defined σ-immanently. A more constructive version of the Σ-space Euclideaness conditions will be formulated, as soon as we construct a necessary mathematical technique.

The basic elements of T-geometry are finite Σ-subspaces $M_n(\mathcal{P}^n)$, i.e. finite sets

$$\mathcal{P}^n = \{P_0, P_1, \dots, P_n\} \subset \Omega \tag{2.1}$$

The simplest finite subset is a nonzero vector $\overrightarrow{\mathcal{P}^1} = \mathbf{P}_0\mathbf{P}_1 \equiv \overrightarrow{P_0P_1}$. The vector $\overrightarrow{P_0P_1}$ is an ordered set of two points $\{P_0, P_1\}$. The scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \Sigma(P_0, Q_1) - \Sigma(P_1, Q_1) - \Sigma(P_0, Q_0) + \Sigma(P_1, Q_0) \tag{2.2}$$

is the main construction of T-geometry, and we substantiate this definition.

σ-immanent expression for scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ in the proper Euclidean space has the form (2.2). This relation can be easily proved as follows.

In the proper Euclidean space three vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0\mathbf{Q}_1$, and $\mathbf{P}_1\mathbf{Q}_1$ are coupled by the relation

$$|\mathbf{P}_1\mathbf{Q}_1|^2 = |\mathbf{P}_0\mathbf{Q}_1 - \mathbf{P}_0\mathbf{P}_1|^2 = |\mathbf{P}_0\mathbf{P}_1|^2 + |\mathbf{P}_0\mathbf{Q}_1|^2 - 2(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1) \tag{2.3}$$

where $(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1)$ denotes the scalar product of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{Q}_1$ in the proper Euclidean space, and $|\mathbf{P}_0\mathbf{P}_1|^2 \equiv (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1)$. It follows from (2.3)

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1) = \frac{1}{2}\left(|\mathbf{P}_0\mathbf{Q}_1|^2 + |\mathbf{P}_0\mathbf{P}_1|^2 - |\mathbf{P}_1\mathbf{Q}_1|^2\right) \tag{2.4}$$

Substituting the point Q_1 by Q_0 in (2.4), one obtains

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_0) = \frac{1}{2}\{|\mathbf{P}_0\mathbf{Q}_0|^2 + |\mathbf{P}_0\mathbf{P}_1|^2 - |\mathbf{P}_1\mathbf{Q}_0|^2\} \tag{2.5}$$

Subtracting (2.5) from (2.4) and using the properties of the scalar product in the proper Euclidean space, one obtains

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \frac{1}{2}\{|\mathbf{P}_0\mathbf{Q}_1|^2 + |\mathbf{P}_1\mathbf{Q}_0|^2 - |\mathbf{P}_0\mathbf{Q}_0|^2 - |\mathbf{P}_1\mathbf{Q}_1|^2\} \tag{2.6}$$

Taking into account that in the proper Euclidean geometry $|\mathbf{P}_0\mathbf{Q}_1|^2 = 2\Sigma(P_0, Q_1) = 2G(P_0, Q_1)$, one obtains the relation (2.2) from the relation (2.6).

In the Euclidean geometry the world function is symmetric, and the order of arguments in the rhs of (2.2) is not essential. In the asymmetric T-geometry the order of arguments in the rhs of (2.2) is essential. The order has been chosen in such a way that

$$\begin{aligned}(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)_{\mathrm{s}} &\equiv \frac{1}{2}\left((\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) + (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_1)\right) \\ &= G(P_0, Q_1) - G(P_1, Q_1) - G(P_0, Q_0) + G(P_1, Q_0) \qquad (2.7)\end{aligned}$$

$$\begin{aligned}(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)_{\mathrm{a}} &\equiv \frac{1}{2}\left((\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) - (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_1)\right) \\ &= A(P_0, Q_1) - A(P_1, Q_1) - A(P_0, Q_0) + A(P_1, Q_0) \qquad (2.8)\end{aligned}$$

It follows from (2.2) that

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = -(\mathbf{P}_1\mathbf{P}_0.\mathbf{Q}_0\mathbf{Q}_1), \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = -(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_1\mathbf{Q}_0) \qquad (2.9)$$

Thus, the scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is antisymmetric with respect to permutation $P_0 \leftrightarrow P_1$ of points determining the vector $\mathbf{P}_0\mathbf{P}_1$, as well as with respect to permutation $Q_0 \leftrightarrow Q_1$.

Definition 2.6 *The finite Σ-space $M_n(\mathcal{P}^n) = \{\Sigma, \mathcal{P}^n\}$ is called oriented $\overrightarrow{M_n(\mathcal{P}^n)}$, if the order of its points $\mathcal{P}^n = \{P_0, P_1, \ldots P_n\}$ is fixed.*

Definition 2.7 . *The nth order multivector m_n is the mapping*

$$m_n: \qquad I_n \to \Omega, \qquad I_n \equiv \{0, 1, ..., n\} \qquad (2.10)$$

The set I_n has a natural ordering, which generates an ordering of images $m_n(k) \in \Omega$ of points $k \in I_n$. The ordered list of images of points in I_n has one-to-one connection with the multivector and may be used as the multivector descriptor. Different versions of the point list will be used for writing the nth order multivector descriptor:

$$\overrightarrow{P_0P_1...P_n} \equiv \mathbf{P}_0\mathbf{P}_1...\mathbf{P}_n \equiv \overrightarrow{\mathcal{P}^n}$$

Originals of points P_k in I_n are determined by the order of the point P_k in the list of descriptor. Index of the point P_k has nothing to do with the original of P_k. Further we shall use descriptor $\overrightarrow{P_0P_1...P_n}$ of the multivector instead of the multivector. In this sense the nth order multivector $\overrightarrow{P_0P_1...P_n}$ in the Σ-space $V = \{\Sigma, \Omega\}$ may be defined as the ordered set $\{P_l\}$, $\quad l = 0, 1, \ldots n$ of $n+1$ points $P_0, P_1, ..., P_n$, belonging to the Σ-space V. Some points may be identical. The point P_0 is the origin of the multivector $\overrightarrow{P_0P_1...P_n}$. Image $m_n(I_n)$ of the set I_n contains k points ($k \leq n+1$). The set of all nth order multivectors m_n constitutes the set $\Omega^{n+1} = \bigotimes_{k=1}^{n+1} \Omega$, and any multivector $\overrightarrow{\mathcal{P}^n} \in \Omega^{n+1}$.

Definition 2.8 . *The scalar Σ-product $(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})$ of two nth order multivectors $\overrightarrow{\mathcal{P}^n}$ and $\overrightarrow{\mathcal{Q}^n}$ is the real number*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = \det \|(\mathbf{P}_0\mathbf{P}_i.\mathbf{Q}_0\mathbf{Q}_k)\|, \qquad i,k = 1,2,...n \tag{2.11}$$

$$\begin{aligned}(\mathbf{P}_0\mathbf{P}_i.\mathbf{Q}_0\mathbf{Q}_k) &\equiv \Sigma(P_0,Q_k) + \Sigma(P_i,Q_0) - \Sigma(P_0,Q_0) - \Sigma(P_i,Q_k), \\ P_0,P_i,Q_0,Q_k &\in \Omega, \qquad \overrightarrow{\mathcal{P}^n},\overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}\end{aligned} \tag{2.12}$$

Operation of permutation of the multivector points can be effectively defined in the Σ-space. Let us consider two nth order multivectors $\overrightarrow{\mathcal{P}^n} = \overrightarrow{P_0P_1P_2...P_n}$ and $\overrightarrow{\mathcal{P}^n_{(k\leftrightarrow l)}} = \overrightarrow{P_0P_1...P_{k-1}P_lP_{k+1}...P_{l-1}P_kP_{l+1}...P_n}$, $(n \geq 1)$, which is a result of permutation of points P_k, P_l, $(k < l)$. The scalar Σ-product $(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})$ is defined by the relation (2.11). If $k = 0$, then the permutation $P_0 \leftrightarrow P_l$ changes the sign of the lth row of the determinant (2.11), and

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = -(\overrightarrow{\mathcal{P}^n_{(0\leftrightarrow l)}}.\overrightarrow{\mathcal{Q}^n}) \qquad l = 1,2,...n, \qquad \forall\overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}, \tag{2.13}$$

If $k \neq 0$, and $k < l$, the permutation $P_k \leftrightarrow P_l$ transposes the lth row and the kth row of the determinant (2.11). The determinant changes its sign and, taking into account (2.13), one obtains

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = -(\overrightarrow{\mathcal{P}^n_{(k\leftrightarrow l)}}.\overrightarrow{\mathcal{Q}^n}) \qquad k \neq l, \qquad l,k = 0,1,2,...n, \qquad \forall\overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}, \tag{2.14}$$

As far as the relation (2.14) is valid for permutation of any two points of the multivector $\overrightarrow{\mathcal{P}^n}$ and for any multivector $\forall\overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}$, one may write

$$\overrightarrow{\mathcal{P}^n_{(i\leftrightarrow k)}} = -\overrightarrow{\mathcal{P}^n}, \qquad i,k = 0,1,...n, \qquad i \neq k, \qquad n \geq 1. \tag{2.15}$$

Thus, a change of the nth order multivector sign $(n \geq 1)$ (multiplication by the number $a = -1$) may be always defined as an odd permutation of points.

Let us consider the relation

$$\overrightarrow{\mathcal{P}^n}T\overrightarrow{\mathcal{R}^n}: \qquad (\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = (\overrightarrow{\mathcal{R}^n}.\overrightarrow{\mathcal{Q}^n}) \wedge (\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{P}^n}) = (\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{R}^n}), \qquad \forall\overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}, \tag{2.16}$$

between two nth order multivectors $\overrightarrow{\mathcal{P}^n} \in \Omega^{n+1}$ and $\overrightarrow{\mathcal{R}^n} \in \Omega^{n+1}$. The relation (2.16) is reflexive, symmetric and transitive, and it may be considered as an equivalence relation.

Definition 2.9 . *Two nth order multivectors $\overrightarrow{\mathcal{P}^n} \in \Omega^{n+1}$ and $\overrightarrow{\mathcal{R}^n} \in \Omega^{n+1}$ are equivalent $\overrightarrow{\mathcal{P}^n} = \overrightarrow{\mathcal{R}^n}$, if the relations (2.16) takes place.*

Definition 2.10 . *If the nth order multivector $\overrightarrow{\mathcal{N}^n}$ satisfies the relations*

$$(\overrightarrow{\mathcal{N}^n}.\overrightarrow{\mathcal{Q}^n}) = 0 \wedge (\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{N}}) = 0, \qquad \forall\overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}, \tag{2.17}$$

$\overrightarrow{\mathcal{N}^n}$ is the null nth order multivector.

Any nth order multivector $\overrightarrow{\mathcal{P}^n}$, having at least two similar points $P_k = P_l$, $(l \neq k)$, is the null nth order multivector $\overrightarrow{\mathcal{N}^n}$, because in this case the determinant in (2.11) has similar rows (kth and lth) and vanishes for $\forall \overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}$. The same is valid for determinant $\left(\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{N}^n}\right)$, but in this case the determinant has similar columns (kth and lth) and vanishes for $\forall \overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}$.

Any nth order multivector $\overrightarrow{\mathcal{P}^n}$ has either at least two similar points and belongs to the equivalence class $\xi\left(\mathcal{N}^n\right)$ of null multivector $\overrightarrow{\mathcal{N}^n}$, or has all different points and belongs to the equivalence class $\xi\left(\overrightarrow{M_n}(\mathcal{P}^n)\right)$, where $\overrightarrow{M_n}(\mathcal{P}^n) = \{\varepsilon, \Sigma, \mathcal{P}^n\}$ is the nth order oriented Σ-subspace of $V = \{\Sigma, \Omega\}$, consisting of $n+1$ points $\mathcal{P}^n$, and ε is the orientation of this subset which takes values $\varepsilon = \pm 1$. Thus, the set Ω^{n+1} of all nth order multivectors may be presented in the form

$$\Omega^{n+1} = \xi\left(\mathcal{N}^n\right) \bigcup_{\overrightarrow{M_n}(\mathcal{P}^n)} \xi\left(\overrightarrow{M_n}(\mathcal{P}^n)\right) \tag{2.18}$$

Any equivalence class $\xi\left(\overrightarrow{M_n}(\mathcal{P}^n)\right)$ contains $\frac{1}{2}(n+1)!$ elements. The equivalence class $\xi\left(+1, \Sigma, \mathcal{P}^n\right)$ can be obtained from the multivector $\overrightarrow{\mathcal{P}^n}$ by means of even permutation of points. Multivectors which are obtained from the multivector $\overrightarrow{\mathcal{P}^n}$ by means of the odd permutation belong to the equivalence class $\xi\left(-1, \Sigma, \mathcal{P}^n\right)$. The factor set Ω^{n+1}/T is formed by the equivalence classes $\xi\left(\mathcal{N}^n\right)$ of null multivectors $\overrightarrow{\mathcal{N}^n}$ and nth order oriented Σ-subspaces $\overrightarrow{M_n}(\mathcal{P}^n)$. It is easy to see that the scalar Σ-product $(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})$ of two non-null nth order multivectors $\overrightarrow{\mathcal{P}^n} \in \xi\left(\overrightarrow{M_n}(\mathcal{P}^n)\right)$ and $\overrightarrow{\mathcal{Q}^n} \in \xi\left(\overrightarrow{M_n}(\mathcal{Q}^n)\right)$ depends only on nth order oriented Σ-subspaces $\overrightarrow{M_n}(\mathcal{P}^n)$, $\overrightarrow{M_n}(\mathcal{Q}^n)$. Thus, in reality the scalar Σ-product $(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})$ describes mutual disposition of nth order oriented Σ-subspaces $\overrightarrow{M_n}(\mathcal{P}^n)$ and $\overrightarrow{M_n}(\mathcal{Q}^n)$, which are constituents of the Σ-space $V = \{\Sigma, \Omega\}$. This result shows that as well as Σ-subspaces, *the multivectors are natural constituents of the Σ-space*, and description in terms of multivectors is equivalent to a description in terms of Σ-subspaces.

This fact puts the question, if it is possible to express the scalar Σ-product of $\overrightarrow{M_n}(\mathcal{P}^n)$ and $\overrightarrow{M_n}(\mathcal{Q}^n)$ directly in terms of nth order oriented Σ-subspaces. In principle, this problem can be solved, but the solution will be very complicated and ineffective. The fact is that the multivectors are associated with integer numbers, whereas the oriented Σ-subspaces are associated with natural numbers which do not contain zero. Descriptions in terms of only natural numbers seems ineffective. Even a record of a number in Roman numerals looks very complicated, because Roman numerals associate with natural numbers and do not contain zero. Apparently, the rule for multiplication of two numbers, written in Roman numerals, looks very complicated. Multivectors are associated with Arabic numerals. They contain null multivectors. Reduced to Σ-subspaces, the multivectors generate the equivalence class $\xi\left(+1, M_n\right)$, which is an analog of positive number, the equivalence class $\xi(-1, M_n)$, which is an analog of negative numbers and the equivalence class $\xi(\mathcal{N}^n)$, which is an analog of zero.

Definition 2.11 . *The length* $|\overrightarrow{\mathcal{P}^n}|$ *of the multivector* $\overrightarrow{\mathcal{P}^n}$ *is the number*

$$|\overrightarrow{\mathcal{P}^n}| = \left\{ \begin{array}{ll} |\sqrt{(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n})}| = |\sqrt{F_n(\mathcal{P}^n)}|, & (\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n}) \geq 0 \\ i\,|\sqrt{(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n})}| = i|\sqrt{F_n(\mathcal{P}^n)}|, & (\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n}) < 0 \end{array} \right. \qquad \overrightarrow{\mathcal{P}^n} \in \Omega^{n+1} \tag{2.19}$$

where the quantity $F_n(\mathcal{P}^n)$ *is defined by the relations*

$$F_n: \quad \Omega^{n+1} \to \mathbb{R}, \qquad \Omega^{n+1} = \bigotimes_{k=1}^{n+1} \Omega, \qquad n = 1, 2, \ldots \tag{2.20}$$

$$F_n\left(\overrightarrow{\mathcal{P}^n}\right) = \det ||\left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k\right)||, \qquad P_0, P_i, P_k \in \Omega, \qquad i, k = 1, 2, \ldots n \tag{2.21}$$

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k) \equiv \Sigma(P_i, P_0) + \Sigma(P_0, P_k) - \Sigma(P_i, P_k), \qquad i, k = 1, 2, \ldots n, \tag{2.22}$$

The function (2.20) is a symmetric function of all its arguments $\mathcal{P}^n = \{P_0, P_1, ..., P_n,\}$, i.e. it is invariant with respect to permutation of any points P_i, P_k, $i, k = 0, 1, ...n$. It follows from representation

$$F_n\left(\overrightarrow{\mathcal{P}^n}\right) = F_n(\mathcal{P}^n) = \left(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n}\right)$$

and the relation (2.15). It means that the squared length $|\overrightarrow{\mathcal{P}^n}|^2 = |M(\mathcal{P}^n)|^2$ of any multivector $\overrightarrow{\mathcal{P}^n}$ does not depend on the order of points. The squared length of any finite subset $\mathcal{P}^n$ is unique.

In the case, when multivector $\overrightarrow{\mathcal{P}^n}$ does not contain similar points, it coincides with the oriented finite Σ-subspace $\overrightarrow{M_n(\mathcal{P}^n)}$, and it is a constituent of Σ-space. In the case, when at least two points of multivector coincide, the multivector length vanishes, and the multivector is considered to be a null multivector. The null multivector $\overrightarrow{\mathcal{P}^n}$ is not a finite Σ-subspace $M_n(\mathcal{P}^n)$, or an oriented finite Σ-subspace $\overrightarrow{M_n(\mathcal{P}^n)}$, but a use of null multivectors assists in creation of a more simple technique, because the null multivectors $\overrightarrow{\mathcal{P}^n}$ play a role of zeros. Essentially, the multivectors are basic objects of T-geometry. As to continual geometric objects, which are analogs of planes, sphere, ellipsoid, etc., they are constructed by means of skeleton-envelope method (see [3]) with multivectors, or finite Σ-subspaces used as skeletons. As a consequence the T-geometry is presented σ-immanently, i.e. without references to objects, external with respect to Σ-space.

The usual vector $\mathbf{P}_0\mathbf{P}_1 \equiv \overrightarrow{P_0P_1} \equiv \overrightarrow{\mathcal{P}^1} = \{P_0, P_1\}$, $P_0, P_1 \in \Omega$ is a special case of multivector. The squared length $|\mathbf{P}_0\mathbf{P}_1|^2$ of the vector $\mathbf{P}_0\mathbf{P}_1$ is defined by the relation (2.2). This gives

$$|\mathbf{P}_0\mathbf{P}_1|^2 \equiv (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1) = \Sigma(P_0, P_1) + \Sigma(P_1, P_0) = 2G(P_0, P_1) \tag{2.23}$$

The following quantities are also associated with the vector $\mathbf{P}_0\mathbf{P}_1$

$$\begin{aligned}
|\mathbf{P}_1\mathbf{P}_0|^2 &\equiv (\mathbf{P}_1\mathbf{P}_0.\mathbf{P}_1\mathbf{P}_0) = 2G(P_1, P_0), &(2.24)\\
(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_1\mathbf{P}_0) &= -\Sigma(P_0, P_1) - \Sigma(P_1, P_0) = -2G(P_0, P_1), &(2.25)
\end{aligned}$$

It is rather unexpected that $|\mathbf{P}_0\mathbf{P}_1|^2 = 2G(P_0, P_1)$, but it is well that the vector $\mathbf{P}_0\mathbf{P}_1$ has only one length, but not two $\sqrt{2\Sigma(P_0, P_1)}$ and $\sqrt{2\Sigma(P_1, P_0)}$

Definition 2.12 *The squared length $|M(\mathcal{P}^n)|^2$ of the nth order Σ-subspace $M(\mathcal{P}^n) \subset \Omega$ of the Σ-space $V = \{\Sigma, \Omega\}$ is the real number.*

$$|M(\mathcal{P}^n)|^2 = F_n(\mathcal{P}^n),$$

where $M(\mathcal{P}^n) = \{P_0, P_1, ..., P_n,\} \subset \Omega$ with all different $P_i \in \Omega$, $i = 0, 1, ...n$, $\overrightarrow{\mathcal{P}^n} \in \Omega^{n+1}$, and the quantity $F_n(\mathcal{P}^n)$ is defined by the relations (2.21) – (2.22).

The meaning of the written relations is as follows. In the special case, when the Σ-space is Euclidean space, its Σ-function is symmetric and coincides with Σ-function of Euclidean space, any two points P_0, P_i determine the vector $\mathbf{P}_0\mathbf{P}_i$, and the relation (2.22) is a σ-immanent expression for the scalar Σ-product $(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)$ of two vectors. Then the relation (2.21) is the Gram's determinant for n vectors $\mathbf{P}_0\mathbf{P}_i$, $i = 1, 2, \ldots n$, and $\sqrt{F_n(\mathcal{P}^n)}$ is the Euclidean volume of the $(n+1)$-edr with vertices at the points $\mathcal{P}^n$.

Now we enable to formulate in terms of the world function the necessary and sufficient condition of that the Σ-space is the n-dimensional Euclidean space

I.

$$\Sigma(P, Q) = \Sigma(Q, P), \qquad P, Q \in \Omega \tag{2.26}$$

II.

$$\exists \mathcal{P}^n \subset \Omega, \qquad F_n(\mathcal{P}^n) \neq 0, \qquad F_{n+1}(\Omega^{n+2}) = 0, \tag{2.27}$$

III.

$$\Sigma(P, Q) = \frac{1}{2}\sum_{i,k=1}^{n} g^{ik}(\mathcal{P}^n)[x_i(P) - x_i(Q)][x_k(P) - x_k(Q)], \qquad \forall P, Q \in \Omega, \tag{2.28}$$

where the quantities $x_i(P)$, $x_i(Q)$ are defined by the relations

$$x_i(P) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}), \qquad x_i(Q) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{Q}), \qquad i = 1, 2, ...n \tag{2.29}$$

The contravariant components $g^{ik}(\mathcal{P}^n)$, $(i, k = 1, 2, \ldots n)$ of metric tensor are defined by its covariant components $g_{ik}(\mathcal{P}^n)$, $(i, k = 1, 2, \ldots n)$ by means of relations

$$\sum_{k=1}^{n} g_{ik}(\mathcal{P}^n)g^{kl}(\mathcal{P}^n) = \delta_i^l, \qquad i, l = 1, 2, \ldots n, \tag{2.30}$$

where covariant components $g_{ik}(\mathcal{P}^n)$ are defined by relations

$$g_{ik}(\mathcal{P}^n) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k), \qquad i,k = 1,2,\ldots n \tag{2.31}$$

IV. The relations

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}) = x_i, \qquad x_i \in \mathbb{R}, \qquad i = 1,2,\ldots n, \tag{2.32}$$

considered to be equations for determination of $P \in \Omega$, have always one and only one solution.
IVa. The relations (2.32), considered to be equations for determination of $P \in \Omega$, have always not more than one solution.

Remark 1 *The condition (2.27) is a corollary of the condition (2.28). It is formulated in the form of a special condition, in order that a determination of dimension were separated from determination of coordinate system.*

The condition II determines the space dimension. The condition III describes σ-immanently the scalar Σ-product properties of the proper Euclidean space. Setting $n+1$ points $\mathcal{P}^n$, satisfying the condition II, one determines n-dimensional basis of vectors in Euclidean space. Relations (2.31), (2.30) determine covariant and contravariant components of the metric tensor, and the relations (2.29) determine covariant coordinates of points P and Q at this basis. The relation (2.28) determines the expression for Σ-function for two arbitrary points in terms of coordinates of these points. Finally, the condition IV describes continuity of the set Ω and a possibility of the manifold construction on it. Necessity of conditions I – IV for Euclideaness of Σ-space is evident. One can prove their sufficiency [2]. The connection of conditions I – IV with the Euclideaness of the Σ-space can be formulated in the form of a theorem.

Theorem 2 *The Σ-space $V = \{\Sigma, \Omega\}$ is the n-dimensional Euclidean space, if and only if σ-immanent conditions I – IV are fulfilled.*

Remark 2 *For the σ-space were proper Euclidean, the eigenvalues of the matrix $g_{ik}(\mathcal{P}^n)$, $\quad i,k = 1,2,\ldots n$ must have the same sign, otherwise it is pseudoeuclidean.*

The theorem states that it is sufficient to know metric (world function) to construct the Euclidean geometry. Concepts of topological space and curve, which are used usually in metric geometry for increasing its informativity, appear to be excess in the sense that they are not needed for construction of geometry.

Proof of this theorem can be found in [2]. A similar theorem for another (but close) necessary and sufficient conditions has been proved in ref. [9]. Here we show only constructive character of conditions I – IV for proper Euclidean space. It means that starting from an abstract Σ-space, satisfying conditions I – IV, one can determine dimension n and construct a rectilinear coordinate system with conventional description of the proper Euclidean space in it. One construct sequentially straight, two-dimensional plane, etc...up to n-dimensional plane coincide with the set Ω. To construct all these objects, one needs to develop technique of T-geometry.

Definition 2.13 *Two nth order multivectors $\overrightarrow{\mathcal{P}^n}$, $\overrightarrow{\mathcal{Q}^n}$ are neutrally collinear (n-collinear) $\overrightarrow{\mathcal{P}^n} \parallel_{(\mathrm{n})} \overrightarrow{\mathcal{Q}^n}$, if*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})(\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{P}^n}) = |\overrightarrow{\mathcal{P}^n}|^2 \cdot |\overrightarrow{\mathcal{Q}^n}|^2 \tag{2.33}$$

Definition 2.14 *The nth order multivector $\overrightarrow{\mathcal{P}^n}$ is f-collinear to nth order multivector $\overrightarrow{\mathcal{Q}^n}$ $\left(\overrightarrow{\mathcal{P}^n} \parallel_{(\mathrm{f})} \overrightarrow{\mathcal{Q}^n}\right)$, if*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})^2 = |\overrightarrow{\mathcal{P}^n}|^2 \cdot |\overrightarrow{\mathcal{Q}^n}|^2 \tag{2.34}$$

Definition 2.15 *The nth order multivector $\overrightarrow{\mathcal{P}^n}$ is p-collinear to nth order multivector $\overrightarrow{\mathcal{Q}^n}$ $\left(\overrightarrow{\mathcal{P}^n} \parallel_{(\mathrm{p})} \overrightarrow{\mathcal{Q}^n}\right)$, if*

$$(\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{P}^n})^2 = |\overrightarrow{\mathcal{P}^n}|^2 \cdot |\overrightarrow{\mathcal{Q}^n}|^2 \tag{2.35}$$

Here indices "f" and "p" are associated with the terms "future" and "past" respectively.

In the symmetric T-geometry there is only one type of collinearity, because the three mentioned types of collinearity coincide in the symmetric T-geometry. The property of the neutral collinearity is commutative, i.e. if $\overrightarrow{\mathcal{P}^n} \parallel_{(\mathrm{n})} \overrightarrow{\mathcal{Q}^n}$, then $\overrightarrow{\mathcal{Q}^n} \parallel_{(\mathrm{n})} \overrightarrow{\mathcal{P}^n}$. The property of p-collinearity and f-collinearity are not commutative, in general. Instead, one has according to (2.34) and (2.35) that if $\overrightarrow{\mathcal{P}^n} \parallel_{(\mathrm{p})} \overrightarrow{\mathcal{Q}^n}$, then $\overrightarrow{\mathcal{Q}^n} \parallel_{(\mathrm{f})} \overrightarrow{\mathcal{P}^n}$.

Definition 2.16 . *The nth order multivector $\overrightarrow{\mathcal{P}^n}$ is f-parallel to the nth order multivector $\overrightarrow{\mathcal{Q}^n}$ $\left(\overrightarrow{\mathcal{P}^n} \uparrow\uparrow_{(\mathrm{f})} \overrightarrow{\mathcal{Q}^n}\right)$, if*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = |\overrightarrow{\mathcal{P}^n}| \cdot |\overrightarrow{\mathcal{Q}^n}| \tag{2.36}$$

The nth order multivector $\overrightarrow{\mathcal{P}^n}$ is f-antiparallel to the nth order multivector $\overrightarrow{\mathcal{Q}^n}$ $\left(\overrightarrow{\mathcal{P}^n} \uparrow\downarrow_{(\mathrm{f})} \overrightarrow{\mathcal{Q}^n}\right)$, if

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = -|\overrightarrow{\mathcal{P}^n}| \cdot |\overrightarrow{\mathcal{Q}^n}| \tag{2.37}$$

Definition 2.17 *The nth order multivector $\overrightarrow{\mathcal{P}^n}$ is p-parallel to the nth order multivector $\overrightarrow{\mathcal{Q}^n}$ $\left(\overrightarrow{\mathcal{P}^n} \uparrow\uparrow_{(\mathrm{p})} \overrightarrow{\mathcal{Q}^n}\right)$, if*

$$(\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{P}^n}) = |\overrightarrow{\mathcal{P}^n}| \cdot |\overrightarrow{\mathcal{Q}^n}| \tag{2.38}$$

The nth order multivector $\overrightarrow{\mathcal{P}^n}$ is p-antiparallel to the nth order multivector $\overrightarrow{\mathcal{Q}^n}$ $\left(\overrightarrow{\mathcal{P}^n} \uparrow\uparrow_{(\mathrm{p})} \overrightarrow{\mathcal{Q}^n}\right)$, if

$$(\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{P}^n}) = -|\overrightarrow{\mathcal{P}^n}| \cdot |\overrightarrow{\mathcal{Q}^n}| \tag{2.39}$$

The f-parallelism and the p-parallelism are connected as follows. If $\overrightarrow{\mathcal{P}^n} \uparrow\uparrow_{(\mathrm{p})} \overrightarrow{\mathcal{Q}^n}$, then $\overrightarrow{\mathcal{Q}^n} \uparrow\uparrow_{(\mathrm{f})} \overrightarrow{\mathcal{P}^n}$ and vice versa.

Vector $\mathbf{P}_0\mathbf{P}_1 = \overrightarrow{\mathcal{P}^1}$ as well as the vector $\mathbf{Q}_0\mathbf{Q}_1 = \overrightarrow{\mathcal{Q}^1}$ are the first order multivectors. If $\mathbf{P}_0\mathbf{P}_1 \uparrow\uparrow_{(\mathrm{f})} \mathbf{Q}_0\mathbf{Q}_1$, then $\mathbf{P}_1\mathbf{P}_0 \uparrow\downarrow_{(\mathrm{f})} \mathbf{Q}_0\mathbf{Q}_1$ and $\mathbf{P}_0\mathbf{P}_1 \uparrow\downarrow_{(\mathrm{f})} \mathbf{Q}_1\mathbf{Q}_0$

3 Geometrical objects in Σ-space. Tubes and their properties.

The simplest geometrical object in T-geometry is the nth order tube $\mathcal{T}(\mathcal{P}^n)$, which is determined by its skeleton $\mathcal{P}^n$. The tube is an analog of Euclidean n-dimensional plane, which is also determined by $n+1$ points $\mathcal{P}^n$, not belonging to a $(n-1)$-dimensional plane.

Definition 3.1 *nth order Σ-subspace $M(\mathcal{P}^n) = \mathcal{P}^n$ of nonzero length $|M(\mathcal{P}^n)|^2 = |\mathcal{P}^n|^2 = F_n(\mathcal{P}^n) \neq 0$ determines geometrical object (set of points) $\mathcal{T}(\mathcal{P}^n)$, called the nth order tube, by means of relation*

$$\mathcal{T}(\mathcal{P}^n) \equiv \mathcal{T}_{\mathcal{P}^n} = \left\{P_{n+1} | F_{n+1}\left(\mathcal{P}^{n+1}\right) = 0\right\}, \qquad P_i \in \Omega, \qquad i = 0, 1 \ldots n+1, \tag{3.1}$$

where the function F_n is defined by the relations (2.20) – (2.22)

The shape of the tube $\mathcal{T}(\mathcal{P}^n)$ does not depend on the order of points of the multivector $\overrightarrow{\mathcal{P}^n}$. The basic point $\mathcal{P}^n$, determining the tube $\mathcal{T}_{\mathcal{P}^n}$ belong to $\mathcal{T}_{\mathcal{P}^n}$.

The first order tube $\mathcal{T}_{P_0P_1}$ can be defined by means of concept of n-collinearity (2.33)

$$\begin{aligned}
\mathcal{T}\left(\mathcal{P}^1\right) &\equiv \mathcal{T}_{(\mathrm{n})P_0P_1} = \left\{P_2 | F_2\left(\mathcal{P}^2\right) = 0\right\} \equiv \left\{P_2 \left| \overrightarrow{P_0P_1} \|_{(\mathrm{n})} \overrightarrow{P_0P_2}\right.\right\} \\
&\equiv \left\{P_2 \left| \, |\overrightarrow{P_0P_1}|^2 |\overrightarrow{P_0P_2}|^2 - \left(\overrightarrow{P_0P_1}.\overrightarrow{P_0P_2}\right)\left(\overrightarrow{P_0P_2}.\overrightarrow{P_0P_1}\right) = 0\right.\right\}
\end{aligned} \tag{3.2}$$

As far as there are concepts of f-collinearity and of p-collinearity, one can define also the first order f-tube and p-tube on the basis of these collinearities. The first order f-tube is defined by the relation

$$\mathcal{T}_{(\mathrm{f})P_0P_1} = \left\{R \left| \overrightarrow{P_0P_1} \|_{(\mathrm{f})} \overrightarrow{P_0R}\right.\right\} = \left\{R \left| \, |\overrightarrow{P_0P_1}|^2 |\overrightarrow{P_0R}|^2 - \left(\overrightarrow{P_0P_1}.\overrightarrow{P_0R}\right)^2 = 0\right.\right\} \tag{3.3}$$

The first order p-tube is defined as follows

$$\mathcal{T}_{(\mathrm{p})P_0P_1} = \left\{R \left| \overrightarrow{P_0P_1} \|_{(\mathrm{p})} \overrightarrow{P_0R}\right.\right\} = \left\{R \left| \, |\overrightarrow{P_0P_1}|^2 |\overrightarrow{P_0R}|^2 - \left(\overrightarrow{P_0R}.\overrightarrow{P_0P_1}\right)^2 = 0\right.\right\} \tag{3.4}$$

In the symmetric T-geometry all three tubes (3.2) – (3.4) coincide. In the non-symmetric T-geometry they are different, in general. The tubes (3.2), (3.3), (3.4) can be divided into segments, each of them is determined by one of factors of expressions (3.2) – (3.4).

In all cases factorization of the expressions

$$F_{(\mathrm{f})}(P_0, P_1, P_2) = \left|\overrightarrow{P_0P_1}\right|^2 \left|\overrightarrow{P_0P_2}\right|^2 - \left(\overrightarrow{P_0P_1}.\overrightarrow{P_0P_2}\right)^2$$

$$F_{(\mathrm{p})}(P_0, P_1, P_2) = \left|\overrightarrow{P_0P_1}\right|^2 \left|\overrightarrow{P_0P_2}\right|^2 - \left(\overrightarrow{P_0P_2}.\overrightarrow{P_0P_1}\right)^2,$$

$$F_{(\mathrm{n})}\left(P_0,P_1,P_2\right)=\left|\overrightarrow{P_0P_1}\right|^2\left|\overrightarrow{P_0P_2}\right|^2-\left(\overrightarrow{P_0P_1}.\overrightarrow{P_0P_2}\right)\left(\overrightarrow{P_0P_2}.\overrightarrow{P_0P_1}\right) \tag{3.5}$$

have similar form

$$F_{(\mathrm{q})}\left(P_0,P_1,P_2\right)=-F_{(\mathrm{q}0)}F_{(\mathrm{q}1)}F_{(\mathrm{q}2)}F_{(\mathrm{q}3)} \tag{3.6}$$

Here index q runs values f,p,n, and factorization of expressions $F_{(\mathrm{q})}\left(P_0,P_1,P_2\right)$, $q=f,p,n$ has a similar form

$$F_{(\mathrm{q}0)}=F_{(\mathrm{q}0)}\left(P_0,P_1,P_2\right)=\sqrt{G_{02}}+\sqrt{G_{10}}+\sqrt{G_{12}-\eta_{\mathrm{q}}}$$

$$\begin{aligned}
F_{(\mathrm{q}1)} &= F_{(\mathrm{q}1)}\left(P_0,P_1,P_2\right)=\sqrt{G_{02}}-\sqrt{G_{10}}+\sqrt{G_{12}-\eta_{\mathrm{q}}}\\
F_{(\mathrm{q}2)} &= F_{(\mathrm{q}2)}\left(P_0,P_1,P_2\right)=\sqrt{G_{02}}+\sqrt{G_{10}}-\sqrt{G_{12}-\eta_{\mathrm{q}}}\\
F_{(\mathrm{q}3)} &= F_{(\mathrm{q}3)}\left(P_0,P_1,P_2\right)=\sqrt{G_{02}}-\sqrt{G_{10}}-\sqrt{G_{12}-\eta_{\mathrm{q}}}
\end{aligned} \tag{3.7}$$

where for brevity one uses designations

$$G_{ik}=G\left(P_i,P_k\right),\qquad A_{ik}=A\left(P_i,P_k\right),\qquad i,k=0,1,2$$

$$\eta_{\mathrm{f}}=-\eta_{\mathrm{p}}=A_{10}+A_{02}+A_{21},\qquad \eta_{\mathrm{n}}=\frac{\eta_{\mathrm{f}}^2}{\sqrt{4G_{01}G_{02}+\eta_{\mathrm{f}}^2}+2\sqrt{G_{01}G_{02}}} \tag{3.8}$$

In the symmetric geometry, when $A\left(P,Q\right)=0,\ \forall P,Q\in\Omega$, and $\eta=0$, all expressions (3.7), for $F_{(\mathrm{n}i)},F_{(\mathrm{f}i)},F_{(\mathrm{p}i)},\ \ i=0,1,2,3$ coincide.

Factorizations (3.6), (3.7) determine division of the tubes into segments. For instance, segments $\mathcal{T}_{(\mathrm{f})[P_0P_1]},\mathcal{T}_{(\mathrm{p})[P_0P_1]},\mathcal{T}_{(\mathrm{n})[P_0P_1]}$ of the tubes (3.3), (3.4), (3.2) between the points P_0 and P_1 are determined respectively by the relations

$$\mathcal{T}_{(\mathrm{q})[P_0P_1]}=\left\{P_2\left|\sqrt{G_{02}}-\sqrt{G_{10}}+\sqrt{G_{12}-\eta_{\mathrm{q}}}=0\right.\right\} \tag{3.9}$$

where index q runs values f,p,n. Values of η_{q} are determined by the relations (3.8).

Definition 3.2 . *Section $\mathcal{S}_{n;P}$ of the tube $\mathcal{T}(\mathcal{P}^n)$ at the point $P\in\mathcal{T}(\mathcal{P}^n)$ is the set $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n))$ of points, belonging to the tube $\mathcal{T}(\mathcal{P}^n)$*

$$\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n))=\{P'\mid\bigwedge_{l=0}^{l=n}\Sigma(P_l,P')=\Sigma(P_l,P)\},\qquad P\in\mathcal{T}(\mathcal{P}^n),\qquad P'\in\Omega. \tag{3.10}$$

Let us note that $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n))\subset\mathcal{T}(\mathcal{P}^n)$, because $P\in\mathcal{T}(\mathcal{P}^n)$. Indeed, whether the point P belongs to $\mathcal{T}(\mathcal{P}^n)$ depends only on values of $n+1$ quantities $\Sigma(P_l,P),\ \ l=0,1,...n$. In accordance with (3.10) these quantities are the same for both points P and P'. Hence, the running point $P'\in\mathcal{T}(\mathcal{P}^n)$, if $P\in\mathcal{T}(\mathcal{P}^n)$.

In the proper Euclidean space the nth order tube is n-dimensional plane, containing points $\mathcal{P}^n$, and its section $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n))$ at the point P consists of one point P.

Definition 3.3 *Section $\mathcal{S}_{n;P}$ of the tube $\mathcal{T}(\mathcal{P}^n)$ at the point $P \in \mathcal{T}(\mathcal{P}^n)$ is minimal, if $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n)) = \{P\}$.*

Definition 3.4 *The first order tube $\mathcal{T}_{P_0P_1}$ is degenerate, if its section at any point $P \in \mathcal{T}_{P_0P_1}$ is minimal.*

Minimality of the first tube section means that the first order tube degenerates to a curve, and any section of the tube consists of one point. It means that there is only one vector $\overrightarrow{P_0R}$, $R \in \mathcal{T}_{P_0P_1}$ of fixed length, which is parallel, or antiparallel to the vector $\overrightarrow{P_0P_1}$. As far as in the nonsymmetric T-geometry there is several types of parallelism, there is several types of degeneration, in general. In the symmetric T-geometry there is only one type of the degeneration.

Definition 3.5 *The Σ-space $V = \{\Sigma, \Omega\}$ is degenerate on the set $\mathcal{T}$ of the first order tubes, if the set $\mathcal{T}$ contains only degenerate tubes $\mathcal{T}_{P_0P_1}$.*

Definition 3.6 *The Σ-space $V = \{\Sigma, \Omega\}$ is locally f-degenerate, if all first order tubes $\mathcal{T}_{(\mathrm{f})P_0P_1}$ are degenerate.*

Definition 3.7 *The Σ-space $V = \{\Sigma, \Omega\}$ is locally p-degenerate, if all first order tubes $\mathcal{T}_{(\mathrm{p})P_0P_1}$ are degenerate.*

Definition 3.8 *The Σ-space $V = \{\Sigma, \Omega\}$ is locally n-degenerate, if all first order tubes $\mathcal{T}_{(\mathrm{n})P_0P_1} \equiv \mathcal{T}_{P_0P_1}$ are degenerate.*

Note that the Riemannian space considered to be a Σ-space is locally degenerate.

4 Asymmetric T-geometry on manifold

We have considered T-geometry in the coordinate-free form. But to discover a connection between the T-geometry and usual differential geometry, one needs to introduce coordinates and to consider the T-geometry on a manifold. It is important also from the viewpoint of the asymmetric T-geometry application as a possible space-time geometry. The asymmetric T-geometry on the manifold may be considered to be a conventional symmetric geometry (for instance, Riemannian) with additional force fields $a_i(x)$, $a_{ikl}(x)$, generated on the manifold by the antisymmetric component A of the world function. Testing experimentally existence of these force fields, one can conclude whether the antisymmetric component A exists and how large it is.

Let us suppose that there exist a set of one-to-one mappings

$$\mu_P : \qquad V \to D_{P,n} \subset E_{P,n}, \qquad \forall P \in \Omega \tag{4.1}$$

where $V=\{\Sigma,\Omega\}$ is the Σ-space and $D_{P,n}=\left\{G_{(\mathrm{E})},\mathcal{M}_{P,n}\right\}$ is a continuous region of the n-dimensional Euclidean space $E_{P,n}=\left\{G_{(\mathrm{E})},\mathbb{R}^n\right\}$, $\mathcal{M}_{P,n}\subset\mathbb{R}^n$. The world function of $D_{P,n}$ has the form

$$G_{(\mathrm{E})}\left(x,x'\right)=\frac{1}{2}g_{ik}\left(x^i-x'^i\right)\left(x^k-x'^k\right),\qquad x^i,x'^i\in\mathbb{R},\qquad i,k=1,2,...n \tag{4.2}$$

Here g_{ik} =const is the metric tensor in the rectilinear coordinate system K, introduced in the Euclidean space $E_{P,n}$. It is easy to verify, that

$$G_{(\mathrm{E}),i}g^{ik}G_{(\mathrm{E}),k}=2G_{(\mathrm{E})},\qquad G_{(\mathrm{E}),i}G_{(\mathrm{E})}^{ik'}G_{(\mathrm{E}),k'}=2G_{(\mathrm{E})} \tag{4.3}$$

where the following designations are used

$$G_{(\mathrm{E}),i}\equiv\partial_iG_{(\mathrm{E})}\equiv\frac{\partial G_{(\mathrm{E})}}{\partial x^i},\qquad G_{(\mathrm{E}),i'}\equiv\partial_{i'}G_{(\mathrm{E})}\equiv\frac{\partial G_{(\mathrm{E})}}{\partial x'^i} \tag{4.4}$$

$$G_{(\mathrm{E})ik'}\equiv G_{(\mathrm{E}),ik'}\equiv\partial_i\partial_{k'}G_{(\mathrm{E})}\equiv\frac{\partial^2G_{(\mathrm{E})}}{\partial x^i\partial x'^k},\qquad G_{(\mathrm{E})ik'}G_{(\mathrm{E})}^{lk'}=\delta_k^l \tag{4.5}$$

$$g_{ik}=-\left[G_{(\mathrm{E})ik'}\right]_{x'=x},\qquad g^{ik}=-\left[G_{(\mathrm{E})}^{ik'}\right]_{x'=x}, \tag{4.6}$$

As far as all quantities in the relations (4.3) are tensors with respect to arbitrary coordinate transformations, the relations (4.3) are valid in any curvilinear coordinate system.

As far as the mapping μ_P is one-to-one, one can use coordinates $x=\{x^1,x^2,...,x^n\}$ for labelling of points in V. Let $x=\mu_P(P)\in\mathcal{M}_{P,n}$ and $x'=\mu_P(P')\in\mathcal{M}_{P,n}$ be images of points $P\in\Omega$ and $P'\in\Omega$ respectively at the mapping μ_P. At this mapping the symmetric component G of the world function Σ transforms as follows $G(P,P')\to G_P(x,x')=G(\mu_P(P),\mu_P(P'))$, and relations (4.3) do not take place for $G_P(x,x')$, in general. But if

$$\lim_{\substack{x'\to x,\\ G_P(x,x')>0}}\frac{G_{P,i}(x,x')g^{ik}(x)G_{P,k}(x,x')-2G_P(x,x')}{2G_P(x,x')}=0, \tag{4.7}$$

one may speak that in vicinity of the point $x=\mu_P(P)$ the Σ-space $\{G_P,\mathcal{M}_{P,n}\}$ has a structure close to that of the Euclidean space. If the relation (4.7) takes place for all point $P\in\Omega$, one may speak that the local structure of the Σ-space $\{G_P,\mathcal{M}_{P,n}\}$ is close to that of the Euclidean space.

Let the vector $\mathbf{u}_{xx'}=\{x,x'\}=\mu_P\left(\overrightarrow{PP'}\right)$ in $\mathcal{M}_{P,n}$ be the image of the vector $\overrightarrow{PP'}$ in Ω. Then $-G_{P,i}\equiv-G_{P,i}(x,x')$, $i=1,2,...n$ are covariant coordinates of the vector $\mathbf{u}_{xx'}$ at the coordinate system K. In the Euclidean space $E_{(\mathrm{P})}$ one can introduce a linear space $\mathcal{L}_{(\mathrm{P})}$ and consider the vector $\mathbf{u}_{xx'}$ to be a vector of $\mathcal{L}_{(\mathrm{P})}$ with coordinates $-G_{P,i}$ at the coordinate system K. The squared length $|\mathbf{u}_{xx'}|_{(\mathrm{E})}^2$ of $\mathbf{u}_{xx'}$ as a vector of $\mathcal{L}_{(\mathrm{P})}$ is

$$|\mathbf{u}_{xx'}|_{(\mathrm{E})}^2=G_{P,i}g^{ik}G_{P,k} \tag{4.8}$$

whereas the squared length $|\mathbf{PP}'|^2 = (\mathbf{PP}'.\mathbf{PP}') = |\mathbf{u}_{xx'}|^2$ as an original of $\mathbf{u}_{xx'}$ in $V = \{\Sigma, \Omega\}$ is equal to $2G(P, P')$. The numerator in relation (4.7) describes this difference.

In the special case, when Σ-space $V = \{\Sigma, \Omega\}$ is a Riemannian space, the condition

$$|\mathbf{u}_{xx'}|^2_{(\mathrm{E})} = G_{P,i} g^{ik} G_{P,k} = 2G = |\mathbf{PP}'|^2 \tag{4.9}$$

takes place [4, 10]. In this case the Euclidean space $D_{P,n} = \left\{G_{(\mathrm{E})}, \mathcal{M}_{P,n}\right\}$ is the Euclidean space tangent to $V = \{\Sigma, \Omega\}$ at the point P. The mapping (4.1) is a geodesic mapping, which transforms any vector $\overrightarrow{PP'}$ at the point P in the Riemannian space V to the tangent vector $\mathbf{u}_{xx'}$ of the same length in the tangent Euclidean space $D_{P,n}$. In this special case the quantities $-G_{P,i} \equiv -G_{P,i}(x, x')$, $i = 1, 2, ...n$ may be considered to be coordinates of the vector $\overrightarrow{PP'}$ in the coordinate system K. Numerator of the expression (4.7) vanishes identically. It means that the local structure of the Riemannian space is close to that of the Euclidean space.

Any mapping (4.1) determines the world function $\Sigma(P, P')$ as a function

$$\Sigma_{(x)}(x, x') = \Sigma\left(\mu_x^{-1}(x), \mu_x^{-1}(x')\right)$$

on the manifold $\mathcal{M}_{P,n} \times \mathcal{M}_{P,n}$, where $\mu_x^{-1}(x)$ is the function reverse to $x = \mu_P(P)$. If the local structure of the function $\Sigma_{(x)}(x, x')$ given on the manifold $\mathcal{M}_{P,n}$ is close to the Euclidean structure, one can differentiate the function $\Sigma_{(x)}(x, x')$ with respect to x and x'.

One may apply another approach which appears to be more convenient. Let it be possible to attribute $n+1$ real numbers $x = \{x^i\}$, $i = 0, 1, ...n$ to any point P in such a way, that there be one-to-one correspondence between the point P and the set x of $n+1$ coordinates $\{x^i\}$, $i = 0, 1, ...n$. All points x form a set $\mathcal{M}_{n+1}$. Then the world function $\Sigma(P, P')$ is a function $\Sigma(x, x')$

$$\Sigma: \quad \mathcal{M}_{n+1} \times \mathcal{M}_{n+1} \to \mathbb{R}, \qquad \Sigma(x, x) = 0, \qquad \forall x \in \mathcal{M}_{n+1} \tag{4.10}$$

of coordinates $x, x' \in \mathcal{M}_{n+1} \subset \mathbb{R}^{n+1}$ of points $P, P' \in \Omega$. Two-point quantities (Σ-function and their derivatives) are designed as a rule by capital characters. One-point quantities are designed by small characters.

Let the function $\Sigma(x, x')$ be multiply differentiable. Then the set $\mathcal{M}_{n+1} \subset \mathbb{R}^{n+1}$ may be called the $(n+1)$th order manifold. One can differentiate $\Sigma(x, x')$ with respect to x^i and with respect to x'^i, forming two-point tensors. For instance,

$$\begin{aligned}
\Sigma_{,k}(x, x') &\equiv \frac{\partial}{\partial x^k}\Sigma(x, x'), \qquad \Sigma_{,k'}(x, x') \equiv \frac{\partial}{\partial x'^k}\Sigma(x, x') \\
\Sigma_{,kl'}(x, x') &\equiv \Sigma_{,l'k}(x, x') \equiv \frac{\partial^2}{\partial x^k \partial x'^l}\Sigma(x, x'),
\end{aligned}$$

are two-point tensors. Here indices after comma mean differentiation with respect to x^k, if the index k has not a prime and with respect to x'^k, if the index k has a prime. The first argument of the two-point quantity is denoted by unprimed variable, whereas the second one is denoted by primed one. Primed indices relate to the second argument of the two-point quantity, whereas the unprimed ones relate to the first argument.

$\Sigma_k \equiv \Sigma_{,k} = \Sigma_{,k}(x,x')$ is a vector at the point x and a scalar at the point x'. Vice versa $\Sigma_{k'} \equiv \Sigma_{,k'} = \Sigma_{,k'}(x,x')$ is a vector at the point x' and a scalar at the point x. The quantity $\Sigma_{,kl'} = \Sigma_{,kl'}(x,x')$ is a vector at the point x and a vector at the point x'. Other derivatives are not tensors. For instance, $\Sigma_{,kl}(x,x') \equiv \Sigma_{,lk}(x,x') \equiv \frac{\partial^2}{\partial x^k \partial x^l}\Sigma(x,x')$ is a scalar at the point x', but it is not a tensor at the point x.

To construct tensors of higher rank by means of differentiation, let us introduce covariant derivatives. Let $\Sigma_{,kl'} \equiv \Sigma_{kl'} \equiv \Sigma_{l'k}$ and $\det||\Sigma_{kl'}|| \neq 0$. The quantity $\Sigma_{kl'}$ will be referred to as covariant fundamental metric tensor. One can introduce also contravariant fundamental metric tensor $\Sigma^{ik'} \equiv \Sigma^{k'i}$, defining it by the relation

$$\Sigma^{ik'}\Sigma_{lk'} = \delta^i_l \qquad \Sigma^{i'k}\Sigma_{l'k} = \delta^{i'}_{l'} \tag{4.11}$$

Let us note that the quantity

$$\tilde{\Gamma}^i_{kl}(x,x') \equiv \Sigma^{is'}\Sigma_{,kls'}, \qquad \Sigma_{,kls'} \equiv \frac{\partial^3 \Sigma}{\partial x^k \partial x^l \partial x'^s} \tag{4.12}$$

is a scalar at the point x' and the Christoffel symbol at the point x. Vice versa, the quantity

$$\tilde{\Gamma}^{i'}_{k'l'}(x,x') \equiv \Sigma^{si'}\Sigma_{,k'l's}, \qquad \Sigma_{,k'l's} \equiv \frac{\partial^3 \Sigma}{\partial x'^k \partial x'^l \partial x^s} \tag{4.13}$$

is a scalar at the point x and a Christoffel symbol at the point x'.

In the same way one can introduce two other Christoffel symbols on the basis of the function G

$$\Gamma^i_{kl}(x,x') \equiv G^{is'}G_{,kls'}, \qquad G_{,kls'} \equiv \frac{\partial^3 G}{\partial x^k \partial x^l \partial x'^s}, \qquad G^{is'}G_{,ks'} = \delta^i_k \tag{4.14}$$

$$\Gamma^{i'}_{k'l'}(x,x') \equiv G^{si'}G_{,k'l's}, \qquad G_{,k'l's} \equiv \frac{\partial^3 G}{\partial x'^k \partial x'^l \partial x^s} \tag{4.15}$$

Using Christoffel symbols (4.12) - (4.15), one can introduce two covariant derivatives $\tilde{\nabla}^{x'}_i$, $\nabla^{x'}_i$ with respect to x^i and two covariant derivatives $\tilde{\nabla}^x_{i'}$, $\nabla^x_{i'}$ with respect to x'^i. For instance, the quantities

$$\Sigma_{ik} \equiv \tilde{\nabla}^{x'}_k \tilde{\nabla}^{x'}_i \Sigma \equiv \Sigma_{,i||k} = \Sigma_{,ik} - \tilde{\Gamma}^s_{ik}(x,x')\Sigma_{,s} = \Sigma_{,ik} - \Sigma^{ls'}\Sigma_{,iks'}\Sigma_{,l} \tag{4.16}$$

$$G_{ik} \equiv \nabla^{x'}_k \nabla^{x'}_i G \equiv G_{,i|k} = G_{,ik} - \Gamma^s_{ik}(x,x')G_{,s} = G_{,ik} - G^{ls'}G_{,iks'}G_{,l} \tag{4.17}$$

are scalars at the point x' and a second rank tensors at the point x. Here two vertical strokes denote covariant derivative with the Christoffel symbol $\tilde{\Gamma}^s_{ik}$, and one vertical stroke denote covariant derivative with the Christoffel symbol Γ^s_{ik}. In the same way one obtains

$$\Sigma_{i'k'} \equiv \tilde{\nabla}^x_{k'} \tilde{\nabla}^x_{i'} \Sigma \equiv \Sigma_{,i'||k'} = \Sigma_{,i'k'} - \tilde{\Gamma}^{s'}_{i'k'}(x,x')\Sigma_{,s'} = \Sigma_{,i'k'} - \Sigma^{l's}\Sigma_{,i'k's}\Sigma_{,l'} \tag{4.18}$$

$$G_{i'k'} \equiv \nabla^x_{k'} \nabla^x_{i'} G \equiv G_{,i'|k'} = G_{,i'k'} - \Gamma^{s'}_{i'k'}(x,x')G_{,s'} = G_{,i'k'} - G^{l's}G_{,i'k's}G_{,l'} \tag{4.19}$$

Covariant derivatives $\tilde{\nabla}_k^{x'}$, $\tilde{\nabla}_i^{x'}$ with respect to x commute, as well as $\nabla_k^{x'}$, $\nabla_i^{x'}$, i.e.

$$\left(\tilde{\nabla}_k^{x'}\tilde{\nabla}_i^{x'}-\tilde{\nabla}_i^{x'}\tilde{\nabla}_k^{x'}\right)T_{ml'}^{sp'}\equiv 0,\qquad \left(\nabla_k^{x'}\nabla_i^{x'}-\nabla_i^{x'}\nabla_k^{x'}\right)T_{ml'}^{sp'}\equiv 0 \tag{4.20}$$

where $T_{ml'}^{sp'}$ is an arbitrary tensor at points x and x'. Unprimed indices are associated with the point x, and primed ones with the point x'. The covariant derivatives commute, because the Riemann-Christoffel curvature tensors $\tilde{R}_{i.kl}^{\cdot j}$, $R_{i.kl}^{\cdot j}$ constructed respectively of Christoffel symbols $\tilde{\Gamma}_{il}^s$ and Γ_{il}^s vanish identically

$$\tilde{R}_{i.lm}^{\cdot s}\equiv\tilde{\Gamma}_{il,m}^s-\tilde{\Gamma}_{im,l}^s+\tilde{\Gamma}_{il}^j\tilde{\Gamma}_{jm}^s-\tilde{\Gamma}_{im}^j\tilde{\Gamma}_{jl}^s\equiv 0 \tag{4.21}$$

$$R_{i.lm}^{\cdot s}\equiv\Gamma_{il,m}^s-\Gamma_{im,l}^s+\Gamma_{il}^j\Gamma_{jm}^s-\Gamma_{im}^j\Gamma_{jl}^s\equiv 0 \tag{4.22}$$

One can test the identity (4.21), substituting (4.12) into (4.21).

Covariant derivatives $\tilde{\nabla}_{k'}^x$, $\tilde{\nabla}_{i'}^x$ with respect to x' commute as well as $\nabla_{k'}^x$, $\nabla_{i'}^x$. Commutativity of covariant derivatives $\tilde{\nabla}_i^{x'}$, $\tilde{\nabla}_k^{x'}$ with respect to x for all values of x' means that the covariant derivative $\tilde{\nabla}_i^{x'}$, $\tilde{\nabla}_k^{x'}$ are covariant derivatives in some flat spaces $\tilde{E}_{x'}$. The same is valid for covariant derivatives $\nabla_i^{x'}$, $\nabla_k^{x'}$ which are covariant derivatives in the flat spaces $E_{x'}$. The spaces $\tilde{E}_{x'}$, $E_{x'}$ are associated with the Σ-spaces $V=\{\Sigma,\mathcal{M}_{n+1}\}$ and $V_{\rm s}=\{G,\mathcal{M}_{n+1}\}$ respectively, given on $\mathcal{M}_{n+1}$ by means of the world function Σ and its symmetric part G. Any of two-point invariant quantities Σ and G with nonvanishing determinants $\det||\Sigma_{ik'}||\neq 0$, and $\det||G_{ik'}||\neq 0$ realize two sets of mappings. For instance, the quantity Σ generates mappings $V=\{\Sigma,\mathcal{M}_{n+1}\}\rightarrow\tilde{E}_{x'}$, $V\rightarrow\tilde{E}_x$ The two-point quantity G generates also two sets of mappings $V_{\rm s}=\{G,\mathcal{M}_{n+1}\}\rightarrow E_{x'}$, $V_{\rm s}\rightarrow E_x$. Mappings of any set are labelled by points x or x' of the manifold $\mathcal{M}_{n+1}$. In the case of G both sets of mappings $V\rightarrow E_{x'}$ and $V\rightarrow E_x$ coincide, but in the case of Σ the sets $V\rightarrow\tilde{E}_{x'}$ and $V\rightarrow\tilde{E}_x$ are different, in general.

It is easy to see that

$$\tilde{\nabla}_s^{x'}\Sigma_{ik'}=\Sigma_{ik'||s}=\Sigma_{,ik's}-\tilde{\Gamma}_{is}^p\Sigma_{pk'}=\Sigma_{,ik's}-\Sigma^{pl'}\Sigma_{isl'}\Sigma_{pk'}\equiv 0$$

The covariant derivatives have the following properties

$$\tilde{\nabla}_k^{x'}t_{l'}^{i'}(x')=0,\qquad\nabla_k^{x'}t_{l'}^{i'}(x')=0,\qquad\tilde{\nabla}_{k'}^{x}t_l^i(x)=0\qquad\nabla_{k'}^{x}t_l^i(x)=0 \tag{4.23}$$

$$\tilde{\nabla}_s^{x'}\Sigma_{ik'}=\Sigma_{ik'||s}=0\qquad\Sigma_{ik'||s'}=0,\qquad G_{ik'|s}=0\qquad G_{ik'|s'}=0, \tag{4.24}$$

where $t_{l'}^{i'}(x')$ is an arbitrary tensor at the point x', and $t_l^i(x)$ is an arbitrary tensor at the point x.

The mappings $V_{\rm s}\rightarrow E_{x'}$, $V_{\rm s}\rightarrow E_x$ are associated with the mappings (4.1). It means that description of Σ-space, given on a manifold, involves the set of mappings (4.1) and some other mappings of the Σ-space onto Euclidean spaces. These mappings can serve as a powerful tool for description of the Σ-space properties. Let us note in this connection, that the Riemannian space may be considered to be a set of infinitesimal pieces of Euclidean spaces glued in some way between themselves. The way of gluing determines the character of the Riemannian space in the sense, that different ways of gluing generate different Riemannian spaces. The way of gluing is determined by the difference of the metric tensor at the point x and at the narrow point $x+dx$, where it has the forms $g_{ik}(x)$ and $g_{ik}(x+dx)$ respectively. The metric tensor depends on a point and on a choice of the coordinate system. It is rather difficultly one to separate dependence on the way of gluing from that on the choice of the coordinate system. Nevertheless the procedure of separation has been well developed. It leads to the curvature tensor, which is an indicator of the way of gluing.

In the case of the Σ-space one considers a set of *finite Euclidean spaces* $E_{x'}$ (instead of its infinitesimal pieces) and a set of mappings $\Sigma \rightarrow E_{x'}$. Here the "way of gluing" is determined by the dependence of mapping on the parameter x' and does not depend on a choice of the coordinate system. This circumstance simplifies investigation. Differentiating the mappings with respect to parameters x', one derives local characteristics of the "way of gluing", which are modifications of the curvature tensor. For instance, considering commutators of derivatives $\tilde{\nabla}_i^{x'}$ and $\tilde{\nabla}_{i'}^{x}$, one can introduce two-point curvature tensor for the Σ-space, as it have been made for the Riemannian space [10, 11]. We shall see this in the sixth section.

Let $\tilde{G}_{(x')ik}$ be metric tensor in the Euclidean space $\tilde{E}_{x'}$ at the point x. Then the Christoffel symbol $\tilde{\Gamma}_{kl}^i = \Sigma^{is'}\Sigma_{,kls'}$ in the space $\tilde{E}_{x'}$ can be written in the form

$$\tilde{\Gamma}_{kl}^i = \Sigma^{is'}\Sigma_{,kls'} = \frac{1}{2}\tilde{G}_{(x')}^{im}\left(\tilde{G}_{(x')km,l} + \tilde{G}_{(x')lm,k} - \tilde{G}_{(x')kl,m}\right) \tag{4.25}$$

where $\tilde{G}_{(x')}^{im}$ are contravariant components of the metric tensor $\tilde{G}_{(x')ik}$.

Let us consider the set of equations (4.25) as a system of linear differential equations for determination of the metric tensor components $\tilde{G}_{(x')ik}$, which is supposed to be symmetric. Solution of this system has the form

$$\tilde{G}_{(x')ik} = \Sigma_{ip'}\tilde{g}_{(x')}^{p'q'}\Sigma_{kq'} \tag{4.26}$$

where $\tilde{g}_{(x')}^{p'q'} = \tilde{g}_{(x')}^{p'q'}(x')$ is some symmetric tensor at the point x'. This fact can be tested by a direct substitution of (4.26) in (4.25). Taking the relation (4.26) at the coinciding points $x = x'$ and denoting coincidence of points x and x' by means of square brackets, one obtains from (4.26)

$$\tilde{g}_{(x')}^{p'q'}(x') = \left[\Sigma^{lp'}\right]_{x'}\left[\tilde{G}_{(x')lm}\right]_{x'}\left[\Sigma^{mq'}\right]_{x'} \tag{4.27}$$

or

$$\tilde{G}_{(x')ik} = \Sigma_{ip'}\left[\Sigma^{lp'}\right]_{x'}\left[\tilde{G}_{(x')lm}\right]_{x'}\left[\Sigma^{mq'}\right]_{x'}\Sigma_{kq'} \tag{4.28}$$

The equation (4.28) can be written in the form

$$\tilde{G}_{(x')ik}(x,x') = \tilde{P}_{(x')i.}^{\ \ l'}\tilde{G}_{(x')lm}(x',x')\tilde{P}_{(x')k.}^{\ \ m'}, \tag{4.29}$$

$$\tilde{P}_{(x')k.}^{\ \ m'} \equiv \tilde{P}_{(x')k.}^{\ \ m'}(x,x') \equiv \Sigma_{kq'}(x,x')\Sigma^{mq'}(x',x') \equiv \Sigma_{kq'}\left[\Sigma^{mq'}\right]_{x'} \tag{4.30}$$

The relation (4.29) means that the metric tensor $\tilde{G}_{(x')ik}$ of the Euclidean space $\tilde{E}_{x'}$ at the point x can be obtained as a result of the parallel transport of the tensor from the point x' in $\tilde{E}_{x'}$ by means of the parallel transport tensor $\tilde{P}_{(x')k.}^{\;\;\cdot m'}$. The parallel transport of the vector $b_{k'}$ from the point x' to the point x is defined by the relation

$$b_k = \tilde{P}_{(x')k.}^{\;\;\cdot m'} b_{m'}.$$

The parallel transport tensor has evident properties

$$\tilde{\nabla}_i^{x'} \tilde{P}_{(x')k.}^{\;\;\cdot m'} \equiv \tilde{P}_{(x')k.||i}^{\;\;\cdot m'} = 0, \tag{4.31}$$

$$\left[\tilde{P}_{(x')k.}^{\;\;\cdot m'}\right]_{x'} \equiv \tilde{P}_{(x')k.}^{\;\;\cdot m'}(x', x') = \delta_{k'}^{m'}. \tag{4.32}$$

In the same way one can obtain the parallel transport tensor $\tilde{P}_{(x)k'.}^{\;\;\cdot m}$ in the Euclidean space $\tilde{E}_x$

$$\left(\tilde{P}_{(x)}\right)_{k'.}^{\;\cdot m} \equiv \Sigma_{k'q}\left[\Sigma^{mq'}\right]_x \equiv \Sigma_{k'q}(x, x')\,\Sigma^{mq'}(x, x) \tag{4.33}$$

describing a parallel transport from the point x to the point x' in $\tilde{E}_x$.

In the same way one can obtain the parallel transport tensors $P_{(x')k.}^{\;\;\cdot m'}$ and $P_{(x)k'.}^{\;\;\cdot m}$ respectively in Euclidean spaces $E_{x'}$ and E_x

$$P_{(x')k.}^{\;\;\cdot m'} \equiv G_{kq'}\left[G^{mq'}\right]_{x'}, \qquad P_{(x)k'.}^{\;\;\cdot m} \equiv G_{k'q}\left[G^{mq'}\right]_x \tag{4.34}$$

Thus, the world function Σ of the Σ-space $V = \{\Sigma, \mathcal{M}_{n+1}\}$ and its symmetric component G determine Euclidean spaces $\tilde{E}_{x'}$, $\tilde{E}_x$, $E_{x'}$, E_x, mappings of V on them and the parallel transport of vectors and tensors in these Euclidean spaces independently of that, whether or not the Σ-space $V = \{\Sigma, \mathcal{M}_{n+1}\}$ is degenerate in the sense of definitions 3.6 – 3.8.

5 Derivatives of the world function at coincidence of points x and x'.

Let us represent the world function in the form

$$\Sigma(x, x') = G(x, x') + A(x, x'), \tag{5.1}$$

$$G(x, x') = G(x', x), \qquad A(x, x') = -A(x', x) \tag{5.2}$$

Here G is the symmetric component of the world function Σ, and A is its antisymmetric component.

Let us expand the function G and A with respect to powers of $\xi^i = x^i - x'^i$. Taking into account the symmetry relations (5.1), (5.2), one obtains

$$G(x, x') = \frac{1}{2}g_{ik}(x')\,\xi^i\xi^k + \frac{1}{6}g_{ikl}(x')\,\xi^i\xi^k\xi^l + \frac{1}{24}g_{iklm}(x')\,\xi^i\xi^k\xi^l\xi^m + \ldots \tag{5.3}$$

$$= \frac{1}{2}g_{ik}(x)\,\xi^i\xi^k - \frac{1}{6}g_{ikl}(x)\,\xi^i\xi^k\xi^l + \frac{1}{24}g_{iklm}(x)\,\xi^i\xi^k\xi^l\xi^m + \ldots \tag{5.4}$$

$$\begin{aligned} A(x,x') &= a_i(x')\,\xi^i + \frac{1}{2}a_{ik}(x')\,\xi^i\xi^k + \frac{1}{6}a_{ikl}(x')\,\xi^i\xi^k\xi^l + \ldots && (5.5)\\ &= a_i(x)\,\xi^i - \frac{1}{2}a_{ik}(x)\,\xi^i\xi^k + \frac{1}{6}a_{ikl}(x)\,\xi^i\xi^k\xi^l - \ldots && (5.6) \end{aligned}$$

In relations (5.3) and (5.5) the functions $G(x,x')$ and $A(x,x')$ are expanded at the point x'. In the relations (5.4) and (5.6) one has the same expansions after transposition $x \leftrightarrow x'$.

Differentiating relations (5.3) - (5.6) with respect to x and x' and setting $x = x'$ thereafter, one obtains relations between the expansion coefficients and expressions for derivatives of functions Σ, G, A at the limit of coincidence $x = x'$.

After calculations one obtains

$$\begin{aligned} g_{ikl} &= \frac{1}{2}(g_{ik,l} + g_{li,k} + g_{kl,i}), \qquad g_{ik,l} \equiv g_{ik,l}(x) \equiv \frac{\partial}{\partial x^l}g_{ik}(x) && (5.7)\\ a_{ik} &= \frac{1}{2}(a_{i,k} + a_{k,i}), \qquad a_{i,k} \equiv a_{i,k}(x) \equiv \frac{\partial}{\partial x^k}a_i(x) && (5.8) \end{aligned}$$

$$a_{iklm} = \frac{1}{2}(a_{ikl,m} + a_{klm,i} + a_{lmi,k} + a_{mik,l} - a_{ik,lm} - a_{lm,ik})$$

Coefficients g_{ik}, a_i, a_{ikl} are arbitrary and symmetric with respect to transposition of indices. Using square brackets for designation of coincidence $x = x'$ and relations (5.7), (5.8), one obtains

$$\begin{aligned} [G_{,i}(x,x')] &\equiv [G_{,i}] = [G_{,i'}] = 0,\\ [G_{,ik}(x,x')] &\equiv [G_{,ik}] = [G_{,i'k'}] = g_{ik}, && (5.9)\\ [G_{,i'k}(x,x')] &\equiv [G_{,i'k}] = -g_{ik} \end{aligned}$$

$$\begin{aligned} [G_{,ikl}(x,x')] &\equiv [G_{,ikl}] = [G_{,i'k'l'}] = \frac{1}{2}(g_{ik,l} + g_{li,k} + g_{kl,i})\\ [G_{,ikl'}(x,x')] &\equiv [G_{,ikl'}] = \frac{1}{2}(g_{ik,l} - g_{li,k} - g_{kl,i}) && (5.10)\\ [G_{,ik'l'}(x,x')] &\equiv [G_{,ik'l'}] = \frac{1}{2}(g_{kl,i} - g_{li,k} - g_{ik,l}) \end{aligned}$$

$$\begin{aligned} [G_{,iklm}(x,x')] &\equiv [G_{,iklm}] = [G_{,i'k'l'm'}] = g_{iklm}\\ [G_{,iklm'}(x,x')] &\equiv [G_{,iklm'}] = [G_{,i'k'l'm}] = -g_{iklm} + g_{ikl,m} && (5.11)\\ [G_{,ikl'm'}(x,x')] &\equiv [G_{,ikl'm'}] = g_{iklm} - g_{ikl,m} - g_{ikm,l} + g_{ik,ml} \end{aligned}$$

$$[A_{,i}(x,x')] \equiv [A_{,i}] = a_i, \qquad [A_{,i'}(x,x')] \equiv [A_{,i'}] = -a_i \qquad (5.12)$$

$$
\begin{aligned}
\left[A_{,ik}\left(x,x'\right)\right] &\equiv \left[A_{,ik}\right] = a_{ik} = \frac{1}{2}\left(a_{i,k}+a_{k,i}\right) \\
\left[A_{,ik'}\left(x,x'\right)\right] &\equiv \left[A_{,ik'}\right] = \frac{1}{2}\left(a_{i,k}-a_{k,i}\right) \\
\left[A_{,i'k'}\left(x,x'\right)\right] &\equiv \left[A_{,i'k'}\right] = -a_{ik} = -\frac{1}{2}\left(a_{i,k}+a_{k,i}\right)
\end{aligned} \tag{5.13}
$$

$$
\begin{aligned}
\left[A_{,ikl}\left(x,x'\right)\right] &\equiv \left[A_{,ikl}\right] = a_{ikl} \\
\left[A_{,ikl'}\left(x,x'\right)\right] &\equiv \left[A_{,ikl'}\right] = \frac{1}{2}a_{i,kl}+\frac{1}{2}a_{k,il}-a_{ikl} \\
\left[A_{,ik'l'}\left(x,x'\right)\right] &\equiv \left[A_{,ik'l'}\right] = -\frac{1}{2}a_{l,ik}-\frac{1}{2}a_{k,il}+a_{ikl}
\end{aligned} \tag{5.14}
$$

$$
\begin{aligned}
\left[A_{,iklm}\left(x,x'\right)\right] &\equiv \left[A_{,iklm}\right] = a_{iklm} \\
\left[A_{,iklm'}\left(x,x'\right)\right] &\equiv \left[A_{,iklm'}\right] = -a_{iklm}+a_{ikl,m} \\
\left[A_{,ikl'm'}\left(x,x'\right)\right] &\equiv \left[A_{,ikl'm'}\right] = a_{iklm}-a_{ikl,m}-a_{ikm,l}+a_{ik,lm} \\
\left[A_{,ik'l'm'}\left(x,x'\right)\right] &\equiv \left[A_{,ik'l'm'}\right] = -a_{iklm}-a_{ikl,m} \\
\left[A_{,i'k'l'm'}\left(x,x'\right)\right] &\equiv \left[A_{,i'k'l'm'}\right] = -a_{iklm}
\end{aligned} \tag{5.15}
$$

The first order coefficient $a_k\left(x\right)$ is a covariant vector at the point x. The second order coefficients $g_{ik}\left(x\right)$ is the second rank covariant tensors at the point x. The second order coefficient $a_{ik}\left(x\right)$ and the third order coefficients $g_{ikl}\left(x\right)$, $a_{ikl}\left(x\right)$ are not tensors, in general. The law of their transformation at the coordinate transformation is more complicated.

According to (5.1), (5.9) and (5.12)

$$
\left[\Sigma_{,i}\right] = a_i\left(x\right), \qquad \left[\Sigma_{,i'}\right] = -a_i\left(x\right) \tag{5.16}
$$

According to (5.1), (5.9) and (5.13)

$$
\begin{aligned}
\left[\Sigma_{,ik}\right] &= \sigma_{(\mathrm{f})ik} = g_{ik}+\frac{1}{2}\left(a_{i,k}+a_{k,i}\right) \\
\left[\Sigma_{,i'k'}\right] &= \sigma_{(\mathrm{p})ik} = g_{ik}-\frac{1}{2}\left(a_{i,k}+a_{k,i}\right) \\
\left[\Sigma_{ik'}\right] &\equiv \left[\Sigma_{,ik'}\right] = -\tilde{g}_{ik} = -g_{ik}+\frac{1}{2}\left(a_{i,k}-a_{k,i}\right)
\end{aligned} \tag{5.17}
$$

For the value $\left[\Sigma^{ik'}\right]$ of the quantity $\Sigma^{ik'}$ one obtains

$$
\left[\Sigma^{ik'}\right] = -\tilde{g}^{ik}, \tag{5.18}
$$

where $\tilde{g}^{ik}$ is determined by the relation

$$
\tilde{g}^{il}\tilde{g}_{ik} = \tilde{g}^{il}\left(g_{ik}-\frac{1}{2}\left(a_{i,k}-a_{k,i}\right)\right) = \delta_k^l \tag{5.19}
$$

The quantity a_i is a one point vector, and g_{ik} is a one-point tensor. Then it follows from (5.17), (5.19), that $\tilde{g}_{ik}$ and $\tilde{g}^{ik}$ are also a one-point tensors, whereas $\sigma_{(\mathrm{f})ik}$ and $\sigma_{(\mathrm{p})ik}$ are not tensors, in general.

For the quantities $\left[\Gamma_{kl}^{i}\right]$, $\left[\Gamma_{k'l'}^{i'}\right]$ and $\left[\tilde{\Gamma}_{kl}^{i}\right]$, $\left[\tilde{\Gamma}_{k'l'}^{i'}\right]$ one obtains

$$\left[\Gamma_{kl}^{i}\right] = \left[\Gamma_{k'l'}^{i'}\right] = \gamma_{kl}^{i}\left(x\right) = \frac{1}{2}g^{si}\left(g_{ks,l}+g_{sl,k}-g_{lk,s}\right), \qquad g^{ik}g_{lk}=\delta_{l}^{i} \quad (5.20)$$

$$\left[\tilde{\Gamma}_{kl}^{i}\right] = \tilde{\gamma}_{(\mathrm{f})}{}_{kl}^{i} = \tilde{g}^{is}g_{ps}\left(\gamma_{kl}^{p}+\beta_{kl}^{p}\right), \quad (5.21)$$

$$\left[\tilde{\Gamma}_{k'l'}^{i'}\right] = \tilde{\gamma}_{(\mathrm{p})}{}_{kl}^{i} = \tilde{g}^{si}g_{ps}\left(\gamma_{kl}^{p}-\beta_{kl}^{p}\right) \quad (5.22)$$

where

$$\gamma_{kl}^{i}\left(x\right) = \frac{1}{2}g^{si}\left(g_{ks,l}+g_{sl,k}-g_{lk,s}\right), \qquad g^{ik}g_{lk}=\delta_{l}^{i} \quad (5.23)$$

$$\beta_{kl}^{i}\left(x\right) = g^{si}\left(-\frac{1}{2}\left(a_{k,ls}+a_{l,ks}\right)+a_{kls}\right) \quad (5.24)$$

Here $\gamma_{kl}^{i}\left(x\right)$ is the Christoffel symbol for the symmetric case, when $A\left(x,x'\right)\equiv 0$.

Note that the tensors $\tilde{g}^{ik}$, $\tilde{g}_{ik}$ are not symmetric with respect to transposition indices, in general, whereas $\tilde{\gamma}_{(\mathrm{f})}{}_{kl}^{i} = \left[\tilde{\Gamma}_{kl}^{i}\right]$, $\tilde{\gamma}_{(\mathrm{p})}{}_{kl}^{i} = \left[\tilde{\Gamma}_{k'l'}^{i'}\right]$, γ_{kl}^{i} and β_{kl}^{i} are symmetric with respect to transposition of indices k and l. Besides it follows from (5.21), (5.22), that

$$\beta_{kl}^{i}\left(x\right) = g^{si}\left(-\frac{1}{2}\left(a_{k,ls}+a_{l,ks}\right)+a_{kls}\right) = \frac{1}{2}g^{ip}\tilde{g}_{ps}\left(\tilde{\gamma}_{(\mathrm{f})}{}_{kl}^{s}-\tilde{\gamma}_{(\mathrm{p})}{}_{kl}^{s}\right) \quad (5.25)$$

It means that the quantity β_{kl}^{i} is a one-point tensor, because a difference of two Christoffel symbols $\tilde{\gamma}_{(\mathrm{f})}{}_{kl}^{s}-\tilde{\gamma}_{(\mathrm{p})}{}_{kl}^{s}$ is a tensor.

6 Curvature tensors

In the Riemannian geometry the Riemann-Christoffel curvature tensor $\tilde{r}_{(\mathrm{q})}{}_{s.ik}^{.l}$ is defined as a commutator of covariant derivatives $\tilde{D}_{(\mathrm{q})i}$ with the Christoffel symbol $\tilde{\gamma}_{(\mathrm{q})}{}_{kl}^{i}$

$$\left(\tilde{D}_{(\mathrm{q})i}\tilde{D}_{(\mathrm{q})k}-\tilde{D}_{(\mathrm{q})k}\tilde{D}_{(\mathrm{q})i}\right)t_{s} = \tilde{r}_{(\mathrm{q})}{}_{s.ik}^{.l}t_{l}$$

where $\tilde{D}_{(\mathrm{q})i}$ is the usual covariant derivative in the Riemannian space with the Christoffel symbol $\tilde{\gamma}_{(\mathrm{q})}{}_{si}^{l}$,

$$\tilde{r}_{(\mathrm{q})}{}_{s.ik}^{.l} = \tilde{\gamma}_{(\mathrm{q})}{}_{si,k}^{l}-\tilde{\gamma}_{(\mathrm{q})}{}_{sk,i}^{l}+\tilde{\gamma}_{(\mathrm{q})}{}_{si}^{p}\tilde{\gamma}_{(\mathrm{q})}{}_{pk}^{l}-\tilde{\gamma}_{(\mathrm{q})}{}_{sk}^{p}\tilde{\gamma}_{(\mathrm{q})}{}_{pi}^{l} \quad (6.1)$$

and t_l is an arbitrary vector at the point x. Index q runs the values p and f.

In the Σ-space one can consider commutator of covariant derivatives $\tilde{\nabla}_i^{x'}$ and $\tilde{\nabla}_{k'}^{x}$ with respect to x^i and x'^k respectively. Calculations gives

$$\begin{aligned}
&\left(\tilde{\nabla}_i^{x'}\tilde{\nabla}_{s'}^{x}-\tilde{\nabla}_{s'}^{x}\tilde{\nabla}_i^{x'}\right)T_{k\ldots l'\ldots}^{j\ldots m'\ldots}\\
=\;&\tilde{F}_{ika's'}\Sigma^{ba'}T_{b\ldots l'\ldots}^{j\ldots m'\ldots}+\ldots-\Sigma^{ja'}\tilde{F}_{iba's'}T_{k\ldots l'\ldots}^{b\ldots m'\ldots}\\
&-\ldots+\Sigma^{am'}\tilde{F}_{iab's'}T_{k\ldots l'\ldots}^{j\ldots b'\ldots}+\ldots-\tilde{F}_{ial's'}\Sigma^{ab'}T_{k\ldots b'\ldots}^{j\ldots m'\ldots}-\ldots
\end{aligned}\tag{6.2}$$

where $T_{k\ldots l'\ldots}^{j\ldots m'\ldots}$ is a two-point tensor. $\tilde{F}$-tensor, defined by the relation

$$\tilde{F}_{ilk'j'}\equiv\Sigma_{iq'}\tilde{\Gamma}_{k'j'||l}^{q'}=\Sigma_{pj'}\tilde{\Gamma}_{il||k'}^{p}=\Sigma_{,ilj'||k'}=\Sigma_{,ilj'k'}-\Sigma_{,sj'k'}\Sigma^{sm'}\Sigma_{,ilm'}\tag{6.3}$$

is a two-point analog of the one-point curvature tensor $r_{slik}=g_{lp}r_{s.ik}^{.p}$. To test that the quantity (6.3) is a tensor, let us represent it in one of two forms

$$\tilde{F}_{ilk'j'}\equiv\Sigma_{pj'}\tilde{\Gamma}_{il||k'}^{p}=\Sigma_{pj'}\left(\tilde{\Gamma}_{il}^{p}-\tilde{\gamma}_{(\mathrm{f})}{}_{il}^{p}\right)_{||k'}\tag{6.4}$$

$$\tilde{F}_{ilk'j'}\equiv\Sigma_{iq'}\tilde{\Gamma}_{k'j'||l}^{q'}=\Sigma_{iq'}\left(\tilde{\Gamma}_{k'j'}^{q'}-\tilde{\gamma}_{(\mathrm{p})}{}_{k'j'}^{q'}\right)_{||l}\tag{6.5}$$

As far as the difference

$$\tilde{Q}_{(\mathrm{f})}{}_{il}^{p}=\tilde{\gamma}_{(\mathrm{f})}{}_{il}^{p}-\tilde{\Gamma}_{il}^{p}\qquad\tilde{Q}_{(\mathrm{p})}{}_{k'j'}^{q'}=\tilde{\gamma}_{(\mathrm{p})}{}_{k'j'}^{q'}-\tilde{\Gamma}_{k'j'}^{q'}\tag{6.6}$$

of two Christoffel symbols is a tensor it follows from (6.4) and (6.6) that $\tilde{F}_{ilk'j'}$ is a tensor. $\tilde{F}$-tensor can be presented as a result of covariant differentiation of the Σ-function. Indeed

$$\begin{aligned}
\tilde{Q}_{(\mathrm{f})}{}_{il}^{s}&=-\Sigma^{sj'}(\Sigma_{,ilj'}-\tilde{\gamma}_{(\mathrm{f})}{}_{il}^{m}\Sigma_{mj'})=-\Sigma^{sj'}\tilde{D}_{(\mathrm{f})l}\Sigma_{ij'}=-\Sigma^{sj'}\tilde{D}_{(\mathrm{f})l}\tilde{D}_{(\mathrm{p})j'}\Sigma_i\\
&=-\Sigma^{sj'}\tilde{D}_{(\mathrm{p})j'}\tilde{D}_{(\mathrm{f})l}\Sigma_i=-\Sigma^{sj'}\left(\tilde{D}_{(\mathrm{f})l}\Sigma_i\right)_{||j'}=-\left(\Sigma^{sj'}\tilde{D}_{(\mathrm{f})l}\Sigma_i\right)_{||j'}
\end{aligned}\tag{6.7}$$

$$\begin{aligned}
\tilde{Q}_{(\mathrm{p})}{}_{k'j'}^{q'}&=-\Sigma^{sq'}\left(\Sigma_{,sk'j'}-\tilde{\gamma}_{(\mathrm{p})}{}_{k'j'}^{m'}\Sigma_{sm'}\right)=-\Sigma^{sq'}\tilde{D}_{(\mathrm{p})j'}\Sigma_{sk'}\\
&=-\Sigma^{sq'}\tilde{D}_{(\mathrm{p})j'}\tilde{D}_{(\mathrm{f})s}\Sigma_{k'}=-\Sigma^{sq'}\tilde{D}_{(\mathrm{f})s}\tilde{D}_{(\mathrm{p})j'}\Sigma_{k'}=-\left(\Sigma^{sq'}\tilde{D}_{(\mathrm{p})j'}\Sigma_{k'}\right)_{||s}
\end{aligned}\tag{6.8}$$

Then according to (6.4) – (6.8), one obtains

$$\tilde{F}_{ilk'j'}=\left(\tilde{D}_{(\mathrm{f})l}\Sigma_i\right)_{||j'||k'}=\left(\tilde{D}_{(\mathrm{p})j'}\Sigma_{k'}\right)_{||i||l}\tag{6.9}$$

The commutator of covariant derivatives $\nabla_i^{x'}$ and $\nabla_{k'}^{x}$, connected with the symmetric component G of the world function, has the property

$$T_{k\ldots l'\ldots|s'|i}^{j\ldots m'\ldots}-T_{k\ldots l'\ldots|i|s'}^{j\ldots m'\ldots}=F_{ika's'}G^{ba'}T_{b\ldots l'\ldots}^{j\ldots m'\ldots}+\ldots-G^{ja'}F_{iba's'}T_{k\ldots l'\ldots}^{b\ldots m'\ldots}$$

$$-\ldots+G^{am'}F_{iab's'}T^{j\ldots b'\ldots}_{k\ldots l'\ldots}+\ldots-F_{ial's'}G^{ab'}T^{j\ldots m'\ldots}_{k\ldots b'\ldots}-\ldots \qquad (6.10)$$

where the curvature F-tensor has the form

$$F_{ilk'j'}\equiv G_{pj'}\Gamma^{p}_{il|k'}=G_{pj'}\left(\Gamma^{p}_{il}-\gamma^{p}_{il}\right)_{|k'}=G_{i;l|k'|j'} \qquad (6.11)$$

Here (;) denotes the usual covariant derivative with the Christoffel symbol γ^{i}_{kl}, and the Q-tensor is written as follows

$$Q^{i}_{kl}=\gamma^{i}_{kl}-\Gamma^{i}_{kl}=-G^{sj'}G_{i;l|j'} \qquad (6.12)$$

Let us discover a connection between the F-tensor at the coincidence limit $[F_{ilk'j'}]$ and the curvature tensor $r^{\cdot l}_{s.ik}$, constructed of the Christoffel symbols γ^{l}_{si} by means of formula (6.1)

$$r^{\cdot l}_{s.ik}=\gamma^{l}_{si,k}-\gamma^{l}_{sk,i}+\gamma^{p}_{si}\gamma^{l}_{pk}-\gamma^{p}_{sk}\gamma^{l}_{pi}. \qquad (6.13)$$

Let us take into account that

$$\left[\left(\nabla^{x'}_{i}+\nabla^{x}_{i'}\right)T_{sp'}\left(x,x'\right)\right]_{x}=D_{i}t_{sp}=t_{sp;i},\qquad t_{sp}\equiv\left[T_{sp'}\left(x,x'\right)\right]_{x} \qquad (6.14)$$

Then, using (6.10), (6.11), one obtains

$$\begin{aligned}&\left[\left(\nabla^{x'}_{i}+\nabla^{x}_{i'}\right)\left(\nabla^{x'}_{k}+\nabla^{x}_{k'}\right)T_{s}\left(x,x'\right)-\left(\nabla^{x'}_{k}+\nabla^{x}_{k'}\right)\left(\nabla^{x'}_{i}+\nabla^{x}_{i'}\right)T_{s}\left(x,x'\right)\right]_{x}\\ =\;&\left(D_{i}D_{k}-D_{k}D_{i}\right)t_{s}=\left[\Gamma^{a}_{is|k'}-\Gamma^{a}_{ks|i'}\right]_{x}t_{a} \qquad (6.15)\end{aligned}$$

where D_i is the usual covariant derivative in the Riemannian space with the Christoffel symbol $\gamma^{i}_{kl}=[\Gamma^{i}_{kl}]_{x}$

On the other hand the relation

$$\left(D_{i}D_{k}-D_{k}D_{i}\right)t_{s}=r^{\cdot l}_{s.ik}t_{l} \qquad (6.16)$$

takes place. Comparison of relations (6.15) and (6.16) gives

$$r^{\cdot l}_{s.ik}=\left[\Gamma^{l}_{is|k'}-\Gamma^{l}_{ks|i'}\right]_{x}=\left[\Gamma^{l}_{is,k'}-\Gamma^{l}_{ks,i'}\right]_{x}=-g^{lp}f_{ispk}+g^{lp}f_{kspi} \qquad (6.17)$$

where

$$f_{ispk}\equiv\left[F_{isp'k'}\right]_{x}=\left[G_{lp'}\Gamma^{l}_{is|k'}\right]_{x}=-g_{lp}\left[\Gamma^{l}_{is|k'}\right]_{x} \qquad (6.18)$$

According to (6.11) the one-point tensor f_{ispk} is symmetric with respect to transposition indices $i\leftrightarrow s$ and $p\leftrightarrow k$ separately.

$$f_{ispk}=f_{sipk},\qquad f_{ispk}=f_{iskp} \qquad (6.19)$$

Equation (6.17) can be written in the form

$$g_{lp}r^{\cdot l}_{s.ik}=-f_{ispk}+f_{kspi} \qquad (6.20)$$

The metric tensor g_{ik} is symmetric, and f_{ispk} has the following symmetry properties

$$f_{ispk}=f_{sipk},\qquad f_{ispk}=f_{iskp},\qquad f_{ispk}=f_{pkis} \qquad (6.21)$$

One can obtain connection of the type (6.17), (6.18) between the $\tilde{F}$-tensor and the Riemannian -Christoffel curvature tensor in the case of nonsymmetric T-geometry. Taking into account (6.4), evident identity

$$\frac{\partial}{\partial x^k}\left[\tilde{\Gamma}_{il}^p\right]_x \equiv \left[\tilde{\Gamma}_{il,k}^p\right]_x + \left[\tilde{\Gamma}_{il,k'}^p\right]_x \tag{6.22}$$

and using relations (5.17), (5.21), one obtains

$$\left[\tilde{F}_{ilk'j'}\right]_x = -\tilde{g}_{pj}\left[\tilde{\Gamma}_{il,k'}^p\right]_x = -\tilde{g}_{pj}\left(\tilde{\gamma}_{(\mathrm{f})}{}_{il,k}^{p} - \left[\tilde{\Gamma}_{il,k}^p\right]_x\right) \tag{6.23}$$

Let us take into account identity

$$\Sigma^{pr'}\Sigma_{,sr'k} + \left(\Sigma^{pr'}\right)_{,k}\Sigma_{sr'} = 0 \tag{6.24}$$

obtained by differentiation of (4.11). Then using relations (5.17), (5.21), one obtains from (6.23)

$$\begin{aligned}\left[\tilde{F}_{ilk'j'}\right]_x &= -\tilde{g}_{pj}\gamma_{(\mathrm{f})}{}_{il,k}^{p} + \tilde{g}_{pj}\left[\Sigma^{pq'}\Sigma_{,ilkq'} - \Sigma^{pr'}\Sigma_{,sr'k}\Sigma^{sq'}\Sigma_{,ilq'}\right]_x \\ &= -\tilde{g}_{pj}\gamma_{(\mathrm{f})}{}_{il,k}^{p} + \tilde{g}_{pj}\left[-\tilde{g}^{pq}\Sigma_{,ilkq'} - \tilde{\Gamma}_{sk}^p\tilde{\Gamma}_{il}^s\right]_x\end{aligned}$$

$$\tilde{f}_{ilkj} \equiv \left[\tilde{F}_{ilk'j'}\right]_x = -\left[\Sigma_{,ilkj'}\right]_x - \tilde{g}_{pj}\left(\gamma_{(\mathrm{f})}{}_{il,k}^{p} + \gamma_{(\mathrm{f})}{}_{il}^{s}\gamma_{(\mathrm{f})}{}_{sk}^{p}\right) \tag{6.25}$$

Alternating with respect to indices k, l, one obtains

$$\tilde{f}_{ilkj} - \tilde{f}_{iklj} = \tilde{g}_{pj}\tilde{r}_{(\mathrm{f})}{}_{i.kl}^{\cdot p} \tag{6.26}$$

where $\tilde{r}_{(\mathrm{f})}{}_{i.kl}^{\cdot s}$ is the Riemann-Christoffel curvature tensor, constructed on the base of the Christoffel symbol $\gamma_{(\mathrm{f})}{}_{ik}^{s}$

$$\tilde{r}_{(\mathrm{f})}{}_{i.kl}^{\cdot p} = \tilde{\gamma}_{(\mathrm{p})}{}_{ik,l}^{p} - \tilde{\gamma}_{(\mathrm{f})}{}_{il,k}^{p} + \tilde{\gamma}_{(\mathrm{f})}{}_{ik}^{s}\tilde{\gamma}_{(\mathrm{f})}{}_{sl}^{p} - \tilde{\gamma}_{(\mathrm{f})}{}_{il}^{p}\tilde{\gamma}_{(\mathrm{f})}{}_{sk}^{k} \tag{6.27}$$

In the same way one can express $\tilde{f}_{ilkj} - \tilde{f}_{iklj}$ via the Riemann-Christoffel curvature tensor $\tilde{r}_{(\mathrm{p})}{}_{i.kl}^{\cdot s}$, constructed on the base of the Christoffel symbol $\gamma_{(\mathrm{p})}{}_{ik}^{s}$

$$\tilde{f}_{ilkj} - \tilde{f}_{iklj} = \tilde{g}_{lp}\tilde{r}_{(\mathrm{p})}{}_{k.ij}^{\cdot p} \tag{6.28}$$

where

$$\tilde{r}_{(\mathrm{p})}{}_{i.kl}^{\cdot p} = \tilde{\gamma}_{(\mathrm{f})}{}_{ik,l}^{p} - \tilde{\gamma}_{(\mathrm{p})}{}_{il,k}^{p} + \tilde{\gamma}_{(\mathrm{p})}{}_{ik}^{s}\tilde{\gamma}_{(\mathrm{p})}{}_{sl}^{p} - \tilde{\gamma}_{(\mathrm{p})}{}_{il}^{p}\tilde{\gamma}_{(\mathrm{p})}{}_{sk}^{k} \tag{6.29}$$

To obtain representation (6.28), let us use another representation (6.5)

$$\tilde{F}_{ilk'j'} = \Sigma_{iq'}\tilde{\Gamma}_{k'j',l}^{q'}$$

of the $\tilde{F}$-tensor, which differs from the representation (6.4) by a change $x \leftrightarrow x'$. Producing the same operations (6.23) – (6.25), one obtains (6.28) instead of (6.26).

Note that relations (6.26) and (6.28) are different, because the tensor $\tilde{g}_{ik}$ is not symmetric. In (6.26) summation is produced over the first index, whereas in (6.28) it is produced over the second index. In the symmetric T-geometry, when $\tilde{g}_{ik}$ is symmetric, three expressions (6.17) (6.26) and (6.28) coincide.

There are two essentially different cases of antisymmetric T-geometry:

1. Rough antisymmetry, when the field $a_i(x) \neq 0$. In this case at small distances $x - x'$ the field $a_i(x)$ dominates, and the world function is determined by the linear form

$$\Sigma(x, x') = a_i(x)\left(x^i - x'^i\right) + ...$$

and antisymmetry is the main phenomenon at small distances.

2. Fine antisymmetry, when the field $a_i(x) \equiv 0$. In this case the antisymmetric effects are described by the field a_{ikl}. At small distances $x - x'$ the symmetric structure dominates, and the world function is determined by the quadratic form

$$\Sigma(x, x') = \frac{1}{2} g_{ik}(x)\left(x^i - x'^i\right)\left(x^k - x'^k\right) + ...$$

as in the symmetric T-geometry. In this case the antisymmetric effects may be considered as corrections to gravitational effects. This corrections may be essential at large distances $\xi^i = x^i - x'^i$, when the form $\frac{1}{6} a_{ikl}\xi^i\xi^k\xi^l$ becomes of the same order as the form $\frac{1}{2} g_{ik}\xi^i\xi^k$,

The asymmetric T-geometry with fine antisymmetry is simpler, because it is rather close to the usual symmetric T-geometry.

7 Gradient lines on the manifold.

Let us consider a one-dimensional line $\mathcal{L}_{(\mathrm{f})}$, passing through points x' and x''. This line is defined by the relations

$$\mathcal{L}_{(\mathrm{f})}: \quad \Sigma_{,i'}(x, x') = \tau \Sigma_{,i'}(x'', x') = \tau b_{i'}, \qquad i = 0, 1, ...n \tag{7.1}$$

Let us suppose that $\det \left|\left|\Sigma_{,i'k}(x, x')\right|\right| \neq 0$. Then $n + 1$ equations (7.1) can be resolved with respect to x in the form

$$\mathcal{L}_{(\mathrm{f})}: \quad x^i = x^i(\tau), \qquad i = 0, 1, ...n \tag{7.2}$$

where τ is a parameter along the line $\mathcal{L}_{(\mathrm{f})}$. As it follows from (7.1), this line passes through the point x' at $\tau = 0$ and through the point x'' at $\tau = 1$. Such a line will be referred to as gradient line (curve) from the future. Let us derive differential equation for the gradient curve $\mathcal{L}_{(\mathrm{f})}$.

Differentiating (7.1) with respect to τ, one obtains

$$\Sigma_{,ki'}(x, x') \frac{dx^k}{d\tau} = \Sigma_{,i'}(x'', x') = b_{i'}, \qquad i = 0, 1, ...n \tag{7.3}$$

Differentiating once more, one obtains

$$\Sigma_{,ki'}\left(x,x'\right)\frac{d^2x^k}{d\tau^2}+\Sigma_{,kli'}\left(x,x'\right)\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}=0,\qquad i=0,1,...n \tag{7.4}$$

Using relation (4.12), one can write equations (7.4) in the form

$$\frac{d^2x^i}{d\tau^2}+\tilde{\Gamma}_{kl}^i\left(x,x'\right)\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}=0,\qquad i=0,1,...n \tag{7.5}$$

The equation (7.5) may be interpreted as an equation for a geodesic in some $(n+1)$-dimensional Euclidean space with the Christoffel symbol $\tilde{\Gamma}_{kl}^i\left(x,x'\right)$. This geodesic passes through the points x' and x''.

Let the points x' and x'' are infinitesimally close. Then equation (7.5) can be written in the form

$$\mathcal{L}_{(\mathrm{f})}:\quad \frac{d^2x^i}{d\tau^2}+\tilde{\gamma}_{(\mathrm{f})}{}_{kl}^i\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}=0,\qquad i=0,1,...n \tag{7.6}$$

where $\tilde{\gamma}_{(\mathrm{f})}{}_{kl}^i=\tilde{\gamma}_{(\mathrm{f})}{}_{kl}^i\left(x\right)=\left[\tilde{\Gamma}_{kl}^i\right]_x$. Dividing the gradient line $\mathcal{L}_{(\mathrm{f})}$ into infinitesimal segments and writing equations (7.5) in the form (7.6) on each segment, one obtains that the gradient line $\mathcal{L}_{(\mathrm{f})}$ is described by the equations (7.6) everywhere.

The equation (7.6) does not contain a reference to the point x', and any gradient line (7.1), (7.2) is to satisfy this equation.

In the case of fine antisymmetry, when $a_i\equiv 0$, the equation (7.6) can be written in other form. Using relations (5.21), and taking into account that $a_i\equiv 0$, one obtains instead of (7.6)

$$\mathcal{L}_{(\mathrm{f})}:\qquad \frac{d^2x^i}{d\tau^2}+\left(\gamma_{kl}^i+\beta_{kl}^i\right)\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}=0,\qquad a_i\equiv 0 \tag{7.7}$$

where

$$\gamma_{kl}^i=\gamma_{kl}^i\left(x\right)=\frac{1}{2}g^{si}\left(g_{ks,l}+g_{sl,k}-g_{lk,s}\right), \tag{7.8}$$

$$\beta_{kl}^i=\beta_{kl}^i\left(x\right)=g^{si}a_{kls} \tag{7.9}$$

If $a_{ikl}=0$, the equations (7.7) may be considered to be the equations for a geodesic in a Riemannian space with the metric tensor g_{ik}.

Now let us consider another type of gradient line $\mathcal{L}_{(\mathrm{p})}$, passing through the points x and x''. Let the gradient line $\mathcal{L}_{(\mathrm{p})}$ be described by the equations

$$\mathcal{L}_{(\mathrm{p})}:\qquad \Sigma_{,i}\left(x,x'\right)=\tau\Sigma_{,i}\left(x,x''\right)=\tau b_i,\qquad i=0,1,...n \tag{7.10}$$

which determine

$$\mathcal{L}_{(\mathrm{p})}:\qquad x'^i=x'^i\left(\tau\right),\qquad i=0,1,...n \tag{7.11}$$

Equation (7.10) distinguishes from the equation (7.1) only in transposition of the first and second arguments of the world function $\Sigma\left(x,x'\right)$. The gradient line $\mathcal{L}_{(\mathrm{p})}$, determined by the relation (7.10), may be referred to as the gradient line from the past. Manipulating with the equation (7.10) in the same way as with (7.1), one obtains instead of (7.6)

$$\mathcal{L}_{(\mathrm{p})}: \qquad \frac{d^2x^i}{d\tau^2}+\tilde{\gamma}_{(\mathrm{p})}{}^{i}_{kl}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}=0, \qquad i=0,1,...n \tag{7.12}$$

In the case of fine antisymmetry, when $a_i \equiv 0$, the equation (7.12) can be written in other form. Using relation (5.22), and taking into account that $a_i \equiv 0$, one obtains instead of (7.12)

$$\mathcal{L}_{(\mathrm{p})}: \qquad \frac{d^2x^i}{d\tau^2}+\left(\gamma^i_{kl}-\beta^i_{kl}\right)\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}=0, \qquad a_i\equiv 0 \tag{7.13}$$

where γ^i_{kl} and β^i_{kl} are defined by the relations (7.8), (7.9).

In the case of symmetric T-geometry, when $a_{ikl}=0$ and $\beta^i_{kl}=0$, differential equations (7.7) and (7.13) respectively for gradient line $\mathcal{L}_{(\mathrm{f})}$ and for gradient line $\mathcal{L}_{(\mathrm{p})}$ coincide.

In the case of asymmetric T-geometry the quantities $[\Gamma^i_{kl}]_x$ and $\left[\Gamma^{i'}_{k'l'}\right]_x$ do not coincide, in general. In this case the equations (7.6) and (7.13) determine, in general, different gradient curves, passing through the same points x' and x''. Differential equations for the gradient curves $\mathcal{L}_{(\mathrm{p})}$ and $\mathcal{L}_{(\mathrm{f})}$ differ in the sign of the "antisymmetric force"

$$\beta^i_{kl}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}=g^{si}a_{kls}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau} \tag{7.14}$$

Finally, one can introduce the neutral gradient line $\mathcal{L}_{(\mathrm{n})}$, defining it by the relations

$$\mathcal{L}_{(\mathrm{n})}: \quad G_{,i'}(x,x')=\tau G_{,i'}(x'',x')=\tau b_{i'}, \qquad i=0,1,...n \tag{7.15}$$

which determine

$$\mathcal{L}_{(\mathrm{n})}: \quad x^i=x^i(\tau), \qquad i=0,1,...n \tag{7.16}$$

Equation (7.15) distinguishes from the equation (7.1) only in replacement of the world function $\Sigma(x,x')$ by its symmetric component $G(x,x')$. Manipulating with the equation (7.15) in the same way as with (7.1), one obtains instead of (7.7)

$$\mathcal{L}_{(\mathrm{n})}: \qquad \frac{d^2x^i}{d\tau^2}+\gamma^i_{kl}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}=0 \tag{7.17}$$

where the "antisymmetric force" is absent.

The gradient lines (7.1) and (7.10) are insensitive to transformation of the world function of the form

$$\Sigma\rightarrow\tilde{\Sigma}=f(\Sigma), \qquad |f'(\Sigma)|>0 \tag{7.18}$$

where f is an arbitrary function, because for determination of the gradient line only direction of the gradient Σ_i or $\Sigma_{i'}$ is important, but not its module. Indeed, after substitution of $\tilde{\Sigma}$ from (7.18) in (7.1) one obtains the equation

$$\Sigma_{,i'}(x,x')=\tau'\Sigma_{,i'}(x'',x'), \qquad i=0,1,...n, \qquad \tau'=\tau\frac{f'(\Sigma(x'',x'))}{f'(\Sigma(x,x'))} \tag{7.19}$$

which describes the same gradient line, but with another parametrization.

8 σ-immanent description of gradient lines.

The particle motion in the space-time geometry should be described σ-immanently, i.e. without a reference to a coordinate system and to a manifold. The motion of a particle is described by its world tube, which is a broken tube

$$\mathcal{T}_{(\mathrm{f})\mathrm{br}} = \bigcup_i \mathcal{T}_{(\mathrm{f})[P_iP_{i+1}]}, \tag{8.1}$$

consisting of segments $\mathcal{T}_{(\mathrm{f})[P_iP_{i+1}]}$, defined by the relation (3.9). Each segment $\mathcal{T}_{(\mathrm{f})[P_iP_{i+1}]}$ is associated with the vector $\overrightarrow{P_iP_{i+1}}$ of the particle momentum, having the length μ

$$\mu = |\overrightarrow{P_iP_{i+1}}| = \sqrt{2G\left(P_i, P_{i+1}\right)} \tag{8.2}$$

The length μ is supposed to be similar for all segments. It is associated with the particle mass m

$$m = b\mu, \qquad b = \mathrm{const} \tag{8.3}$$

where b is some universal constant. The broken tube (8.1), (8.2) will be referred to as the future broken tube.

In the asymmetric T-geometry there is the past broken tube

$$\mathcal{T}_{(\mathrm{p})\mathrm{br}} = \bigcup_i \mathcal{T}_{(\mathrm{p})[P_iP_{i+1}]}, \tag{8.4}$$

consisting of segments $\mathcal{T}_{(\mathrm{p})[P_iP_{i+1}]}$, defined by the relation (3.9) at $q = p$.

There is a neutral broken tube, defined by the relation

$$\mathcal{T}_{(\mathrm{n})\mathrm{br}} = \bigcup_i \mathcal{T}_{(\mathrm{n})[P_iP_{i+1}]}, \tag{8.5}$$

and the relation (3.9) at $q = n$. In the symmetric T-geometry definitions of all broken tubes $\mathcal{T}_{(\mathrm{f})\mathrm{br}}$, $\mathcal{T}_{(\mathrm{p})\mathrm{br}}$, $\mathcal{T}_{(\mathrm{n})\mathrm{br}}$ coincide.

The gradient line $\mathcal{L}_{(\mathrm{f})}$ may be defined as a broken tube (8.1) with the segment length $\mu \to 0$. If segment $\overrightarrow{P_iP_{i+1}}$ is fixed, the next segment $\overrightarrow{P_{i+1}P_{i+2}}$ is defined by its end point P_{i+2}, which provides extremum to the expression $\sqrt{2\Sigma\left(P_i, P_{i+2}\right)}$ under additional constraints

$$\sqrt{2\Sigma\left(P_i, P_{i+1}\right)} = \mu, \qquad \sqrt{2\Sigma\left(P_{i+1}, P_{i+2}\right)} = \mu.$$

The gradient line $\mathcal{L}_{(\mathrm{p})}$ may be defined as a broken tube (8.4) with the segment length $\mu \to 0$. If segment $\overrightarrow{P_iP_{i+1}}$ is fixed, the next segment $\overrightarrow{P_{i+1}P_{i+2}}$ is defined by its end point P_{i+2}, which provides extremum to the expression $\sqrt{2\Sigma\left(P_{i+2}, P_i\right)}$ under additional constraints

$$\sqrt{2\Sigma\left(P_{i+1}, P_i\right)} = \mu, \qquad \sqrt{2\Sigma\left(P_{i+2}, P_{i+1}\right)} = \mu.$$

If extremum is achieved at one point P_{i+2}, then position of P_{i+2} is determined uniquely. In this case the broken tubes $\mathcal{T}_{(\mathrm{f})\mathrm{br}}$ and $\mathcal{T}_{(\mathrm{p})\mathrm{br}}$ turn to the gradient lines at $\mu \to 0$.

Thus, using this procedure, one can define gradient lines σ-immanently. In the space-time geometry a gradient line is associated with a world line of a free massless particle, whereas the broken tubes with fixed length $\left|\mathcal{T}_{[P_iP_{i+1}]}\right| = \mu$ of segments $\mathcal{T}_{[P_iP_{i+1}]}$ are associated with world tubes of massive particles.

If the particle is free, adjacent segments are parallel, i.e. for the broken tube (8.1) $\overrightarrow{P_iP_{i+1}} \uparrow\uparrow_{(\mathrm{f})} \overrightarrow{P_{i+1}P_{i+2}}$, $i = ...0, 1, 2, ...$and

$$\left|\overrightarrow{P_iP_{i+1}}\right| \cdot \left|\overrightarrow{P_{i+1}P_{i+2}}\right| - \left(\overrightarrow{P_iP_{i+1}}.\overrightarrow{P_{i+1}P_{i+2}}\right) = 0 \tag{8.6}$$

For the broken tubes $\mathcal{T}_{(\mathrm{p})\mathrm{br}}$, $\mathcal{T}_{(\mathrm{n})\mathrm{br}}$, defined respectively by the relations (8.4), and (8.5), the condition of parallelism has the form $\overrightarrow{P_iP_{i+1}} \uparrow\uparrow_{(\mathrm{p})} \overrightarrow{P_{i+1}P_{i+2}}$, $i = ...0, 1, 2, ...$ or

$$\left|\overrightarrow{P_iP_{i+1}}\right| \left|\overrightarrow{P_{i+1}P_{i+2}}\right| - \left(\overrightarrow{P_{i+1}P_{i+2}}.\overrightarrow{P_iP_{i+1}}\right) = 0 \tag{8.7}$$

and $\overrightarrow{P_iP_{i+1}} \uparrow\uparrow_{(\mathrm{n})} \overrightarrow{P_{i+1}P_{i+2}}$, $i = ...0, 1, 2, ...$ or

$$\left|\overrightarrow{P_iP_{i+1}}\right| \left|\overrightarrow{P_{i+1}P_{i+2}}\right| - \sqrt{\left(\overrightarrow{P_{i+1}P_{i+2}}.\overrightarrow{P_iP_{i+1}}\right)\left(\overrightarrow{P_iP_{i+1}}.\overrightarrow{P_{i+1}P_{i+2}}\right)} = 0 \tag{8.8}$$

In general segments $\mathcal{T}_{(\mathrm{f})[P_iP_{i+1}]}$ and $\mathcal{T}_{(\mathrm{p})[P_iP_{i+1}]}$ are inswept surfaces. Under some conditions they degenerate into a one-dimensional line, and the first order tubes $\mathcal{T}_{(\mathrm{f})\mathrm{br}}$ and $\mathcal{T}_{(\mathrm{p})\mathrm{br}}$ degenerate correspondently into gradient lines $\mathcal{L}_{(\mathrm{f})}$ and $\mathcal{L}_{(\mathrm{p})}$. Let us investigate, when this fact takes place.

9 Conditions of degeneration of the neutral first order tube

In general, the asymmetric T-geometry is nondegenerate geometry, even if it is given on n-dimensional manifold $\mathcal{M}_n$. In general case the first order tube $\mathcal{T}_{(\mathrm{n})\mathbf{x}'\mathbf{x}''}$, passing through points x' and x'', does not coincide with gradient line $\mathcal{L}_{(\mathrm{f})}$ or $\mathcal{L}_{(\mathrm{p})}$, passing through the points x' and x'' and defined by the relations (7.1) and (7.10) respectively.

Let us investigate, under what conditions the neutral first order tube $\mathcal{T}_{(\mathrm{n})\mathbf{x}'\mathbf{x}''}$ degenerates into gradient line $\mathcal{L}_{(\mathrm{f})}$ or $\mathcal{L}_{(\mathrm{p})}$. The first order tube $\mathcal{T}_{(\mathrm{n})\mathbf{P}_0\mathbf{P}_1}$, passing through the points $P_0 = \{x\}$, $P_1 = \{x'\}$ is defined by the relation

$$F_2(P_0, P_1, R) = \begin{vmatrix} \left(\overrightarrow{P_0P_1}.\overrightarrow{P_0P_1}\right) & \left(\overrightarrow{P_0P_1}.\overrightarrow{P_0R}\right) \\ \left(\overrightarrow{P_0R}.\overrightarrow{P_0P_1}\right) & \left(\overrightarrow{P_0R}.\overrightarrow{P_0R}\right) \end{vmatrix} = 0 \tag{9.1}$$

where $R = \{x' + dx'\}$ is a running point on the tube $\mathcal{T}_{P_0P_1}$.

One has the following expansion for the scalar Σ-products $\left(\overrightarrow{P_0P_1}.\overrightarrow{P_0R}\right)$ and $\left(\overrightarrow{P_0R}.\overrightarrow{P_0P_1}\right)$

$$\left(\overrightarrow{P_0P_1}.\overrightarrow{P_0R}\right) = \Sigma\left(P_1,P_0\right)+\Sigma\left(P_0,R\right)-\Sigma\left(P_1,R\right)$$

$$\begin{aligned} &= \Sigma\left(x',x\right)+\Sigma\left(x,x'+dx'\right)-\Sigma\left(x',x'+dx'\right) \\ &= 2G+\left(\Sigma_{,i'}-\left[\Sigma_{,i'}\right]_{x'}\right)dx'^{i'}+\frac{1}{2}\left(\Sigma_{,i'k'}-\left[\Sigma_{,i'k'}\right]_{x'}\right)dx'^{i'}dx'^{k'} \end{aligned} \tag{9.2}$$

$$\left(\overrightarrow{P_0R}.\overrightarrow{P_0P_1}\right) = \Sigma\left(P_0,P_1\right)+\Sigma\left(R,P_0\right)-\Sigma\left(R,P_1\right)$$

$$\begin{aligned} &= \Sigma\left(x,x'\right)+\Sigma\left(x'+dx',x\right)-\Sigma\left(x'+dx',x'\right) \\ &= 2G+\left(G_{,i'}-A_{,i'}-\left[\Sigma_{,i}\right]_{x'}\right)dx'^{i'}+\frac{1}{2}\left(G_{,i'k'}-A_{,i'k'}-\left[\Sigma_{,ik}\right]_{x'}\right)dx'^{i}dx'^{k} \end{aligned} \tag{9.3}$$

where unprimed indices associate with the first argument and the primed ones with the second argument of the Σ-function.

$$\left(\overrightarrow{P_0P_1}.\overrightarrow{P_0P_1}\right) = 2G \tag{9.4}$$

$$\left(\overrightarrow{P_0R}.\overrightarrow{P_0R}\right) = 2G\left(x,x'+dx'\right) = 2G+2G_{,i'}dx'^{i'}+G_{,i'k'}dx'^{i'}dx'^{k'} \tag{9.5}$$

Using relations (9.2) – (9.5) and (5.16), (5.17), one reduces equation (9.1) to the form

$$\left(\left(G_{,i'}-A_{,i'}-a_i\left(x'\right)\right)dx'^{i'}\right)\left(\left(G_{,l'}+A_{,l'}+a_l\left(x'\right)\right)dx'^{l'}\right) = 2Gg_{l'm'}\left(x'\right)dx'^{l'}dx'^{m'} \tag{9.6}$$

We suppose that the world function is such, that the tube $\mathcal{T}_{\mathbf{x}'\mathbf{x}}$ degenerates to a line. Then the solution of (9.6) has either the form

$$dx'^{k'} = d\tau g^{k'l'}\left(x'\right)\left(G_{,i'}-A_{,i'}-a_{i'}\left(x'\right)\right) \tag{9.7}$$

or the form

$$dx'^{k'} = g^{k'l'}\left(x'\right)\left(G_{,k'}+A_{,k'}+a_{k'}\left(x'\right)\right)d\tau \tag{9.8}$$

where $d\tau$ is an infinitesimal parameter. The relations (9.7), (9.8) describe a one-dimensional lines in vicinity of the point x'. Both expressions (9.7), (9.8) are solutions of the equation (9.6), provided the relation

$$\left(G_{,i'}-A_{,i'}-a_{i'}\left(x'\right)\right)g^{i'k'}\left(x'\right)\left(G_{,k'}+A_{,k'}+a_{k'}\left(x'\right)\right) = 2G \tag{9.9}$$

is fulfilled.

There is only one solution, provided solutions (9.7) and (9.8) coincide. It means that

$$A_{,k'}+a_{k'}\left(x'\right) = 0 \tag{9.10}$$

and the condition (9.9) transforms to the equation

$$G_{,i'}g^{i'k'}(x')\,G_{,k'} = 2G \tag{9.11}$$

This is well known equation for the world function of a Riemannian space [4]. The Riemannian geometry is locally degenerate in the sense of definition 3.8, and the equation (9.11) describes this property of Riemannian geometry.

According to expansion (5.5) the condition (9.10) may take place, only if

$$A(x,x') = a_i\left(x^i - x'^i\right), \qquad a_i = \text{const} \tag{9.12}$$

In this case the quantities (3.8) vanish, i.e.

$$\eta_{\mathrm{f}} = A(x,x') + A(x',y) + A(y,x) = 0, \qquad \forall x,x',y \in \mathcal{M}_n \tag{9.13}$$

and the first order tubes are similar in symmetric and nonsymmetric geometries.

For the case of past first order tube and the future one the conditions of degeneration are also rather rigid. In this case instead of (9.6) one obtains two conditions

$$4G\left(A_{,i'} + a_{i'}\right)dx'^{i'} = 0 \tag{9.14}$$

$$\left(2G\left(A_{,i'k'} - [\Sigma_{,i'k'}]_{x'}\right) - G_{,i'}G_{,k'}\right)dx'^{i'}dx'^{k'} = 0 \tag{9.15}$$

In the case (9.12) the condition (9.14) is fulfilled, and (9.15) reduces to (9.11).

If the first order tube is nondegenerate in symmetric T-geometry, it cannot degenerate after addition of antisymmetric component, because the local degeneration condition (9.11) remains to be not fulfilled.

Thus, practically any antisymmetric component of the world function destroys degeneration of the neutral first order tube. If one connects quantum effects with the first order tube degeneration [12], one concludes that the possible asymmetry of the space-time geometry is connected with quantum effects.

10 Examples of the first order tubes

To imagine the possible corollaries of asymmetry in T-geometry, let us construct the first order tube $\mathcal{T}_{P_0P_1}$ in the Σ-space. Let us consider Σ-space on the 4-dimensional manifold with the world function

$$\begin{aligned}\Sigma(x,x') &= a_i\xi^i + \frac{1}{2}g_{ki}\xi^i\xi^k, \qquad a_i = b_i\left(1 + \alpha f\left(\xi^2\right)\right), \\ \xi^i &= x^i - x'^i, \qquad \xi^2 \equiv g_{ki}\xi^i\xi^k, \qquad \alpha, b_i, g_{ik} = \text{const},\end{aligned} \tag{10.1}$$

where f is some function of ξ^2 and summation is made over repeating indices from 0 to 3. One can interpret the relation (10.1) as a Euclidean space with a linear structure $a_i\xi^i$ given on it. Such a Σ-space is uniform, but not isotropic, because there is a vector a_i, describing some preferable direction in the Σ-space.

In the given case the characteristic quantity (3.8) has the form

$$\eta_{\mathrm{f}}=\alpha\left(-b_ix^if\left(x^2\right)+b_ix'^if\left(x'^2\right)+b_i\xi^if\left(\xi^2\right)\right) \tag{10.2}$$

It does not depend on the constant component of the vector a_i. Then according to (3.2) - (3.7) the shape of the tube does not depend on the constant component of the vector a_i. We omit the constant component of the function f and consider the cases, when it has the form

$$\begin{aligned} 1 \quad &: \qquad f\left(\xi^2\right)=\xi^2, \qquad 2: \quad f\left(\xi^2\right)=\frac{1}{1+\beta\xi^2}, \qquad \beta=\mathrm{const} \\ \xi^2 \quad &\equiv \quad g_{ik}\xi^i\xi^k, \qquad \xi^i\equiv x^i-x'^i \end{aligned} \tag{10.3}$$

In the first case the antisymmetric structure is essential at large distances $\xi=x-x'$. In the second case the antisymmetric structure vanishes at large ξ.

Now let us construct the first order neutral tube $\mathcal{T}_{x'y}$, determined by two points x' and y. For simplicity the coordinate system is chosen in such a way, that $x'=0$. The equation (3.2) determining the shape of the tube has the form

$$\left|\begin{array}{cc} 2G\left(x',y\right) & (\mathbf{x'y}.\mathbf{x'x}) \\ (\mathbf{x'x'}.\mathbf{x'y}) & 2G\left(x',x\right) \end{array}\right| = 4G\left(x',y\right)G\left(x',x\right)-(\mathbf{x'y}.\mathbf{x'x})\left(\mathbf{x'x}.\mathbf{x'y}\right)=0 \tag{10.4}$$

In the first case, when $f\left(\xi^2\right)=\xi^2$, calculation gives for (10.4)

$$\left(x_iy^i\right)^2-x^2y^2=\alpha^2\left[\left(x_iy^i\right)\left(-2\left(b_ky^k\right)+2\left(b_kx^k\right)\right)-\left(b_ix^i\right)y^2+\left(b_iy^i\right)x^2\right]^2 \tag{10.5}$$

where $x^2=x^ix_i$, $y^2=y_iy^i$. In the case, when the metric tensor g_{ik} is the metric tensor of the proper Euclidean space, $x^2y^2\geq\left(x_iy^i\right)^2$, and the equation (10.5) has an interesting solution only if $\alpha=0$. Then

$$x^2y^2=\left(x_iy^i\right)^2, \qquad \mathcal{T}_{0y}=\left\{x\left|\bigwedge_{i=0}^{i=3}x^i=y^i\tau\right.\right\}, \qquad \alpha=0 \tag{10.6}$$

In the case $\alpha\neq0$, the first order tube $\mathcal{T}_{0y}$ degenerates to the set of basic points $\{0,y\}$, because substitution of $x^i=y^i\tau$ in the square bracket in (10.5) shows that the bracket vanishes only at $\tau=0$ or $\tau=1$. Thus, in the case of proper Euclidean metric tensor g_{ik} the first order tube shape does not depend on a_i, provided a_i =const.

Let us consider a more interesting case, when the metric tensor g_{ik} of Σ-space is the Minkowski one. Then $x^2y^2\leq\left(x_iy^i\right)^2$, provided the $\mathbf{0y}$ is timelike $|\mathbf{0y}|^2=2G(0,y)>0$. In this case the equation (10.5) has the solution (10.6), if $\alpha=0$.

If $\alpha\neq0$, let us introduce new variables z,w by means of relations

$$x^i=\zeta^iz\,|y|\,, \qquad y^i=\eta^i\,|y|\,, \qquad |y|=\sqrt{y_iy^i}, \qquad w=\left(\zeta^i\eta_i\right), \tag{10.7}$$

where η^i and ζ^i are unit vectors

$$\zeta^i\zeta_i=\nu=\pm1, \qquad \eta_i\eta^i=1 \tag{10.8}$$

Let us suppose that the vector b_i, which determines the antisymmetric structure, is a unit timelike vector, and the vector y^i determining the tube is chosen in such a way, that

$$b_i = \varepsilon \eta_i, \quad \varepsilon = \pm 1 \tag{10.9}$$

Then equation (10.5) takes the form

$$0 = z^2 w^2 \left[\left(1 - \frac{\nu}{w^2}\right) - \alpha^2 |y|^2 \left(\left(2 + \frac{\nu}{w^2}\right) zw - 3 \right)^2 \right] \tag{10.10}$$

One can see that equation (10.10) depends essentially only on combinations $\tau = zw$ and $s = \nu/w^2$ of variables z, w. Instead of variables z, w let us introduce variables r, τ, connected with z, w by means of relations.

$$\tau = zw, \qquad r = z\sqrt{w^2 - \nu} = \tau\sqrt{1-s}, \qquad s = \frac{\nu}{w^2}$$

Then the equation (10.10) transforms to the form

$$z^2 \left[\frac{r^2}{\tau^2} - \alpha^2 |y|^2 \left(\left(3 - \frac{r^2}{\tau^2}\right) \tau - 3 \right)^2 \right] = 0 \tag{10.11}$$

Trivial solution $z = 0$ of equation (10.11) describes vector $\mathbf{0x} = \mathbf{0}$. It means that the coordinate origin belongs to the tube. The tube $\mathcal{T}_{0y}$ is a result of all possible rotation of the curve $r = r(\tau)$ around the vector $\mathbf{0y}$. This curve is described by its coordinate $\tau |y|$ along the vector $\mathbf{0y}$ and coordinate $r |y|$ in the orthogonal direction. Solution of equation (10.11) has the form

$$r = \pm \frac{1}{2\alpha |y|} \left(-1 \pm \sqrt{\left(1 + 12\alpha^2 |y|^2 \tau (\tau - 1)\right)} \right) \tag{10.12}$$

The three-dimensional surface $\mathcal{T}_{0y}$ is a spherically symmetric surface, whose radius $r = r(\tau)$ is determined by the relation (10.12). Equation (10.12) gives four values of r for any value of τ, but only two of them are essential, because radii r and $-r$ describe the same surface. The two surfaces, determined by (10.12) cross between themselves at $\tau = 0$ and $\tau = 1$, when $r = 0$.

It follows from (10.12) that

$$\lim_{\tau \to \infty} \frac{r}{\tau} = \pm\sqrt{3},$$

It means the tube $\mathcal{T}_{0y}$ is infinite only in spacelike directions. In the timelike directions the tube size is restricted.

In the vicinity of the vector $\mathbf{0y}$, generating the tube, the shape of the tube depends on interrelation between the intensity of the antisymmetry, described by the constant α, and the length of the vector $\mathbf{0y}$. α appears in the equation (10.12) only in the combination $g = \alpha |y|$. In any case, at $\alpha \neq 0$ the tube $\mathcal{T}_{0y}$ does not degenerate into a one-dimensional curve. The tube $\mathcal{T}_{0y}$ is symmetric with respect to the reflection $\tau \to 1-\tau$. (See figures 1,2). If the antisymmetric structure is strong enough, and $\alpha|y| > 1/\sqrt{3}$, the tube $\mathcal{T}_{0y}$ is empty in its center in the sense that intersection of $\mathcal{T}_{0y}$ with the plane $\tau = 0.5$ is empty. If $\alpha|y| < 1/\sqrt{3}$ intersection of $\mathcal{T}_{0y}$ with the plane $\tau = 0.5$ forms two concentric spheres of radii

$$r_1 = \frac{3\alpha|y|}{2\left(\sqrt{\left(1-3\alpha^2|y|^2\right)}+1\right)}, \qquad r_2 = \frac{3\alpha|y|}{2\left(\sqrt{\left(1-3\alpha^2|y|^2\right)}-1\right)} \tag{10.13}$$

If $\alpha|y| \ll 1$, one of radii is small $r_1 = 1.5\alpha|y|$ and another one is large $r_2 = 1/(\alpha|y|)$.

Let us consider now the second case (10.3), when the antisymmetric structure is essential only at small distances. In this case one obtains instead of equation (10.10)

$$\sqrt{(w^2-\nu)}z|y| = \varepsilon\alpha\left(\frac{1}{1+\beta|y|^2} - \frac{zw}{1+\beta\nu z^2|y|^2} + \frac{wz-1}{1+\beta\left(\nu z^2 - 2\varepsilon zw + 1\right)|y|^2}\right) \tag{10.14}$$

where the same designations (10.7) - (10.9) are used. At large z the equation (10.14) transforms to the equation

$$\sqrt{(w^2-\nu)} = \frac{\varepsilon\alpha}{z|y|\left(1+\beta|y|^2\right)} \tag{10.15}$$

Let $\nu = 1$. Then at $z \to \infty$

$$r = z\sqrt{(w^2-1)} = \frac{\varepsilon\alpha}{|y|\left(1+\beta|y|^2\right)} \tag{10.16}$$

$$\tau = zw = \sqrt{z^2 + \frac{\alpha^2}{|y|^2\left(1+\beta|y|^2\right)^2}} \to \infty \tag{10.17}$$

It means that the tube is unlimited in the timelike direction **0y** and has a finite radius. If $\nu = -1$, the quantity z is restricted $z^2 < \frac{\alpha^2}{|y|^2\left(1+\beta|y|^2\right)^2}$, as it follows from (10.15). The tube is limited in spacelike directions.

Thus, the local antisymmetric structure produces only local perturbation of the tube shape. At the timelike infinity this perturbation reduces to a nonvanishing radius of the tube. Any asymmetry of the world function generates nondegeneracy of T-geometry. On the other hand, nondegeneracy of T-geometry is connected with the particle mass geometrization and with quantum effects [12].

11 Concluding remarks.

The main goal of the nonsymmetric T-geometry development is its possible application as a space-time geometry, especially as a space-time geometry of microcosm. Approach and methods of T-geometry distinguish from those of the Riemannian (pseudo-Riemannian) geometry, which is used now as a space-time geometry. The

Riemannian geometry imposes on the space-time geometry a series of constraints. These restrictions are generated by methods used at the description of the Riemannian geometry. Let us list some of them.

1. The continuity of space-time. This is very fine property which cannot be tested by a direct experiment. T-geometry is insensitive to continuity, and it is free of this constraint.

2. The Riemannian geometry is a geometry with fixed dimension. It is very difficult to imagine a geometry with variable dimension in the scope of the Riemannian geometry. Such a problem is absent in T-geometry.

3. For the Riemannian geometry description one uses a coordinate system. Considering all possible coordinate systems, one can truncate this restriction essentially. But the Riemannian geometry does not admit a coordinate-free description. It is connected with the circumstance, that in the scope of the Riemannian CG one does not construct axiomatics for any special Riemannian geometry. Axiomatics of the standard (Euclidean or pseudo-Euclidean) geometry is used. Relations (and axiomatics) of any special geometry are obtained as a result of the standard geometry deformation. The coordinate system is a means of such a deformation description, and it cannot be omitted in the Riemannian geometry. T-geometry uses another method of description of the standard geometry deformation, and T-geometry admits a coordinate-free description.

4. The Riemannian geometry uses a concept of a curve, which is essentially a method of the Riemannian geometry description. The curve is considered conventionally to be a geometrical object (but not as a method of the geometry description), and nobody is interested in separation of properties of geometry from properties imported by a use of the description in terms of curves. In particular, in the Riemannian geometry the absolute parallelism is absent, in general. Parallelism of two vectors in remote points is established by means of a reference to a curve, along which the vector was transported parallelly. In other words, geometrical property of parallelism of two vectors is formulated in terms of the method of description, and nobody knows how to remove this dependence on the methods of description. T-geometry is free of this defect. The concept of a curve is not used at the T-geometry construction. There is an absolute parallelism in T-geometry.

5. T-geometry uses a special geometrical language, which contains only concepts immanent to the geometry in itself (Σ-function and finite subspaces). One does not need to eliminate means of the geometry description.

6. This geometrical language admits one to consider and to investigate effectively such a situation, when the future and the past are not geometrically equivalent.

7. The means of the Riemannian geometry description suppress such an important property of geometry as nondegeneracy. As a corollary the particle mass geometrization appears to be impossible in the scope of Riemannian geometry. Geometrization of the particle mass is important, when the mass of a particle is unknown and must be determined from some geometrical relations. It may appear to be important for determination of the mass spectrum of elementary particles. T-geometry admits geometrization of the particle mass.

8. Consideration of nondegeneracy and geometrization of the particle mass have admitted one to make the important step in understanding of the microcosm space-time geometry. One succeeded in explanation of non-relativistic quantum effects as geometrical effects, generated by nondegeneracy of the space-time geometry. There is a hope that asymmetry of the space-time geometry will admit one to explain important characteristics of elementary particles.

References

[1] Yu. A. Rylov, Metric space: classification of finite subspaces instead of constraints on metric. *Proceedings on analysis and geometry*, Novosibirsk, Publishing House of Mathematical institute, 2000. pp. 481-504, (in Russian). English version: e-print math.MG/9905111

[2] Yu. A. Rylov, Description of metric space as a classification of its finite subspaces. *Funndamental'aya i Prikladnaya Matematika,* (in print). (in Russian).

[3] Yu. A. Rylov, Geometry without topology as a new conception of geometry. *Int. J. Math. Math. Sci.* (in print), eprint *math.MG/0103002.*

[4] J. L. Synge, *Relativity: The General Theory*, North-Holland, Amsterdam, 1960.

[5] V. A. Toponogov, Riemannian spaces with curvature restricted below, Uspechi Matematicheskich Nauk. **14**, 87, (1959). (in Russian).

[6] A. D. Alexandrov, V. N. Berestovski, I. G. Nikolayev, Generalized Riemannian spaces, *Uspechi Matematicheskich Nauk,* **41**, iss.3, 1, (1986). (in Russian).

[7] Yu. Burago, M. Gromov, G. Perelman, Alexandrov spaces with curvatures restricted below, *Uspechi Matematicheskich Nauk,* **47**, iss. 2, 3, (1992). (in Russian).

[8] K. Menger, Untersuchen über allgemeine Metrik, *Mathematische Annalen,* **100**, 75-113, (1928).

[9] Yu. A. Rylov, Extremal properties of Synge's world function and discrete geometry. *J. Math. Phys.* **31**, 2876-2890, (1990).

[10] Yu. A. Rylov, Description of Riemannian space by means of finite interval. *Izvestiya Vysshich Uchebnych Zavedeniyi* ser. fis.mat. 132, (1962). (in Russian).

[11] Yu. A. Rylov, Relative gravitational field and conservation laws in general relativity. *Ann. Phys.* (Leipzig) **12**, 329, (1964).

[12] Yu. A. Rylov, Non-Riemannian model of space-time responsible for quantum effects. *J. Math. Phys.* **32**, 2092-2098, (1991).