The Euclidean geometry deformations and capacities of their application to microcosm space-time geometry

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Abstract

Usually a Riemannian geometry is considered to be the most general geometry, which could be used as a space-time geometry. In fact, any Riemannian geometry is a result of some deformation of the Euclidean geometry. Class of these Riemannian deformations is restricted by a series of unfounded constraints. Eliminating these constraints, one obtains a more wide class of possible space-time geometries (T-geometries). Any T-geometry is described by the world function completely. T-geometry is a powerful tool for the microcosm investigations due to three its characteristic features: (1) Any geometric object is defined in all T-geometries at once, because its definition does not depend on the form of world function. (2) Language of T-geometry does not use external means of description such as coordinates and curves; it uses only primordially geometrical concepts: subspaces and world function. (3) There is no necessity to construct the complete axiomatics of T-geometry, because it uses deformed Euclidean axiomatics, and one can investigate only interesting geometric relations. Capacities of T-geometries for the microcosm description are discussed in the paper. When the world function is symmetric and T-geometry is nondegenerate, the particle mass is geometrized, and nonrelativistic quantum effects are described as geometric ones, i.e. without a reference to principles of quantum theory. When world function is asymmetric, the future is not geometrically equivalent to the past, and capacities of T-geometry increase multiply. Antisymmetric component of the world function generates some metric fields, whose influence on geometry properties is especially strong in the microcosm.

Key words: space-time geometry, nondegenerate geometry, geometrization of mass, quantum effects geometrization, world function, future-past geometric nonequivalence.

1 Introduction

In the case, when the existing theory cannot explain observed physical phenomena, a choice of an appropriate space-time geometry is the most effective and simplest method of solution of arising problems. The most reliable and doubtless conceptions of contemporary theoretical physics: the special relativity theory and the general relativity theory were created by means of a change of the space-time geometry. It is a common practice to consider the Riemannian conception of geometry to be the most general conception of geometry. It is common practice to think that further development of usual geometry is impossible. To increase the geometry capacities, some authors tries to provide geometry by such unusual properties as stochasticity and noncommutativity.

In reality the Riemannian conception of geometry is not the most general conception of geometry. List of geometries, generated by the Riemannian conception of geometry, is restricted by a series of unfounded constraints. Imposition of these constraints was generated by a series of historical reasons and cannot be justified. For instance, there is no necessity for introducing stochasticity in geometry. It is sufficient to eliminate some constraints, imposed on the Riemannian geometry. After elimination of these constraints the motion of particles in such a space-time geometry becomes to be stochastic automatically [1], although the geometry in itself remains to be deterministic.

To understand this, let us consider the problem, what is the geometry, in general, and the Riemannian geometry, in particular. Well known mathematician Felix Klein [2] supposed that only such a construction on the point set is a geometry, where all points of the set have the same properties. For instance, Felix Klein insisted that Euclidean geometry and Lobachevsky geometry are geometries, whereas the Riemannian geometries are not geometries at all. As a rule the Riemannian geometries are not uniform, and their points have different properties. According to the Felix Klein opinion, they should be called as "Riemannian topographies" or as "Riemannian geographies".

It is hardly relevant now to discuss the question what is the correct name for the Riemannian geometry, but it is very important to understand, why Felix Klein insisted on different names for the Euclidean geometry and for the Riemannian one. The fact is that one can formulate axiomatics (system of axioms), determining the Euclidean geometry as a self-sufficient construction, which does not need auxiliary means for its construction. There is no axiomatics for the Riemannian geometry. First, it is very difficult (technically complicated) to construct axiomatics for each of possible Riemannian geometries. Second, there is no necessity in such axiomatics. In applications of the Riemannian geometry to the space-time model only relations between the physical objects (world lines of particles, vectors, etc.) are important. The geometry in itself is less interesting. All relations of the Riemannian geometry are obtained as a result of modification (deformation) of corresponding Euclidean relations.

Practically, this deformation is realized by a replacement of infinitesimal Euclidean interval $ds_{\mathrm{E}}^2 = \eta_{ik}dx^i dx^k$, η_{ik} =const by the infinitesimal interval $ds_{\mathrm{R}}^2 = g_{ik}dx^i dx^k$, where g_{ik} is a function of the point x. Such a replacement is a change of distances between the points of the space-time, what is a space-time deformation by definition. Thus, the Riemannian geometry is not a self-sufficient construction (it has not its own axiomatics). The Riemannian geometry is a deformed Euclidean geometry. The Riemannian deformation of the space-time, converting Euclidean geometry to the Riemannian one, form a class of deformations, restricted by a series of constraints.

In general, any deformation is described by a change of distances ρ between all pairs of space points. In the case of the space-time this distance may be real (timelike), or imaginary (spacelike). It is more reasonable to use the quantity $\Sigma(P,Q) = \frac{1}{2}\rho^2(P,Q)$, known as world function [3]. Here $\rho(P,Q)$ is the distance between the points P and Q. The world function is real always, and it is very convenient at description of geometry. The world function contains complete information on geometry. This property is the most remarkable property of the world function. In application to the Euclidean geometry, as a special case of Riemannian geometry, this property is formulated in the form of a theorem [4, 5, 6], which states that, if and only if the world function satisfies some Euclideaness conditions, formulated in terms of the world function, the corresponding geometry is Euclidean. These conditions will be written down, as soon as corresponding mathematical technique is developed. Now it is important only that the Euclideaness conditions contain references only to the world function and finite subspaces of the whole space. The dimension of the space, and all other parameters of the Euclidean geometry are determined by the form of the world function.

In the case of Euclidean geometry all information on geometry is contained in the world function. This property remains to be valid also, if the world function does not satisfy the Euclideaness conditions and the geometry is not Euclidean. Then any choice of the world function Σ corresponds to some geometry $\mathcal{G}_\Sigma$. This circumstance can be interpreted in the sense, that the world function Σ describes deformation of the Euclidean space, and any deformation Σ corresponds to some geometry $\mathcal{G}_\Sigma$, which can be interpreted as a result of the Euclidean geometry deformation.

From this viewpoint the Riemannian geometry is a result of the Euclidean geometry deformation, when the world function between the points x and x' is determined by the relation

$$\Sigma(x,x') = \frac{1}{2}\left(\int\limits_{\mathcal{L}_g} \sqrt{g_{ik}dx^i dx^k}\right)^2 \tag{1.1}$$

where integration is produced along the shortest curve (geodesic) $\mathcal{L}_g$ between the points x and x'.

The relation (1.1) describes the Riemannian deformation of the Euclidean space. This deformation is determined by the dimension n of the space and by the metric tensor g_{ik}, which is a set of $n(n+1)/2$ functions of one space point x. Information contained in these $n(n+1)/2$ functions is much less, than information included in one function Σ of two space points x and x'. In other words, the Riemannian deformation is a deformation of a very special form. This raises the question. What are foundations for consideration of the Riemannian deformation as the most general admissible deformation of the space-time? Why the Riemannian geometry is the most general possible geometry of the space-time?

There are no reasonable foundations for pretention of the Riemannian geometry to the role of the most general space-time geometry. Consideration of the Riemannian geometry as the unique possible space-time geometry is a delusion which should be rejected. The question what is the reason of this delusion is important and interesting. We shall not discuss it, restricting ourselves by the remark that this delusion is an associative delusion [7]. In other words, it is a delusion of the same sort, which stimulated the scientific community to believe the Ptolemaic doctrine for a long time. In other time (in the middle of XIX century) a delusion of the same sort stimulated rejection of the idea of non-Euclidean geometry.

As soon as we assume that there are non-Riemannian deformations of the Euclidean geometry, generating more general geometries, than Riemannian ones (we shall refer to them as T-geometries), the simple and evident idea arises, that many properties of particles in the microcosm can be explained as specific properties of the microcosm geometry. In this connection it is relevant to mention that the special relativity theory has solved problems of motion with large velocities by means of simple change of the space-time geometry. The general relativity has solved problems of relativistic gravitation, changing the Minkowski geometry by the most general Riemannian geometry, connecting the form of geometry with the matter distribution in the space-time.

Description of nonrelativistic quantum effects can be obtained also by means a simple change of the space-time geometry [1]. The arising space-time geometry depends explicitly on the quantum constant. The particle motion in such a geometry appears to be stochastic (quantum mechanics principles are not mentioned at such a description of quantum effects). Such an explanation of quantum effects differs from the conventional quantum-mechanical explanation by absence of any additional suppositions. One considers simply all possible geometries, generated by arbitrary deformations (but not only Riemannian ones) of Euclidean geometry. One chooses among these geometries the geometry which corresponds to the best advantage to the experimental data.

As far as the world function as a function of two space points contains much more information, than the metric tensor components, the capacities of explanation of different effects, reserved in application of T-geometries as space-time geometries, appear to be much more, than other capacities of explanation, used now in the elementary particle theory and in the quantum field theory. One does not need to make additional suppositions on properties of physical phenomena in the microcosm. It is sufficient to consider all possible T-geometries and to choose this one, which agrees with experimental data. Of course, the problem of choosing the appropriate geometry is not a simple problem, because for its solution one needs to study very large massive of data. But it is important that there is no necessity to invent anything. It is sufficient to investigate the existing data.

From viewpoint of common sense and logic the strategy of the microcosm investigation, based on the dominating role of geometry seems to be most encouraging. Besides, this strategy appeared to be successful at construction of special relativity, general relativity and explanation of nonrelativistic quantum effects.

Idea of the geometry description in terms of only distance is a very old idea. There were attempts to carry out this program, using so called distance geometry [8, 9], when some constraints, imposed on the metric of metrical space were removed. Unfortunately, T-geometry has not been constructed, because external means of description (in particular, concept of a curve) were used. In previous papers [4, 5, 6] one considered the symmetric T-geometry, i.e. T-geometry with symmetric world function $\Sigma(P,Q) = \Sigma(Q,P)$. Such a restriction on the world function $\Sigma(Q,P)$ seems to be usual and conventional. In the present paper one considers non-symmetric T-geometry, when the world function is asymmetric. Asymmetric world function is associated with the situation, when the future and the past are not equivalent geometrically. One cannot test experimentally, whether the future and the past are equivalent geometrically, because one can measure the time only from the past to the future. We do not insist that the future and the past are not equivalent geometrically. But investigation of geometrical description reserves, hidden in such an asymmetry, seems to be useful.

The main constraints on the Riemannian deformations are as follows: (1) fixed dimension, (2) continuity, (3) impossibility of deformation of one-dimensional curve into a surface, or into a point. These specific constraints are conditioned by application of a coordinate system at the description of Euclidean and Riemannian geometries. Indeed, any coordinate system has fixed number of coordinates (dimension). Coordinates change continuously, and this property is attributed to geometry, because coordinates label the space points. Finally, transformations of coordinates transform one-dimensional curve to one-dimensional curve, and this property of coordinate system is attributed to geometry in itself.

There are methods of separation of geometric properties from the properties of the coordinate system. One considers description of geometry in all possible coordinate systems. The properties common for all these descriptions are properties of the considered geometry. But there are no coordinate transformations from n coordinates to m coordinates ($m \neq n$). There are no coordinate transformations, which transform one-dimensional curve to n-dimensional surface ($n \neq 1$), and it is a common practice to attribute these properties of the coordinate systems to geometry in itself.

Thus, constraints on the means of description are attributed to the geometry in itself. To remove these constraints, generated by the means of description, one should remove all external means of descriptions and use the language, which uses only concepts which are attributes of the geometry in itself. It means that the geometry is to be described in terms of subspaces and world functions between points of these subspaces.

Practically a use of only finite subspaces of the whole space appears to be sufficient. As a result the description of geometry is carried out in terms of finite number of points and world function between pairs of them. Such a description, which does not contain any external means of description, will be referred to as σ-immanent description. The σ-immanent description is convenient in the sense that it admits one to deal with geometry directly. One does not need to consider coordinate systems and group of their transformations. Sometimes we shall use the coordinate description to connect σ-immanent description with conventional description of geometry. But construction of T-geometry is produced in the σ-immanent form.

In the second section the main statements of T-geometry are formulated. Concepts of a multivector, scalar Σ-product and collinearity are introduced in the third sectioon. The fourth section is devoted to investigation of the tube properties. In the fifth section one investigates a connection between the particle motion stochasticity and the T-geometry dondegeneracy. The particle dynamics in the nondegenerate space-time geometry is investigated in the sixth section. Asymmetric T-geometry on a manifold is investigated in the seventh section. The eighth section is devoted to of the world function properties in vicinity of coincidence of its arguments. Properties different sorts of curvature tensors are investigated in the nineth section. The tenth section is devoted to investigation of gradient lines. The nondegeneracy conditions of the neutral first order tube are investigated in the eleventh section. Examples of the first order tubes are considered in the twelfth section.

2 T-geometry and Σ-space. Coordinate-free description.

Let us yield necessary definitions.

Definition 2.1 *T-geometry is the set of all statements about properties of all geometric objects .*

The T-geometry is constructed on the point set Ω by giving the world function Σ. The Σ-space $V = \{\Sigma, \Omega\}$ is obtained from the metric space after removal of the constraints, imposed on the metric ρ, and introduction of the world function Σ

$$\Sigma\left(P, Q\right) = \frac{1}{2}\rho^2\left(P, Q\right), \qquad P, Q \in \Omega \tag{2.1}$$

instead of the metric ρ:

Definition 2.2 *Σ-space $V = \{\Sigma, \Omega\}$ is nonempty set Ω of points P with given on $\Omega \times \Omega$ real function Σ*

$$\Sigma: \quad \Omega \times \Omega \to \mathbb{R}, \qquad \Sigma(P, P) = 0, \qquad \forall P \in \Omega. \tag{2.2}$$

The function Σ is known as the world function [3], or Σ-function. The metric ρ may be introduced in Σ-space by means of the relation (2.1). If Σ is positive, the metric ρ is also positive, but if Σ is negative, the metric is imaginary.

Definition 2.3 . *Nonempty point set $\Omega' \subset \Omega$ of Σ-space $V = \{\Sigma, \Omega\}$ with the world function $\Sigma' = \Sigma|_{\Omega' \times \Omega'}$, which is a contraction Σ on $\Omega' \times \Omega'$, is called Σ-subspace $V' = \{\Sigma', \Omega'\}$ of Σ-space $V = \{\Sigma, \Omega\}$.*

Further the world function $\Sigma' = \Sigma|_{\Omega' \times \Omega'}$, which is a contraction of Σ will be denoted as Σ. Any Σ-subspace of Σ-space is a Σ-space. In T-geometry a geometric object $\mathcal{O}$ is described by means of skeleton-envelope method. It means that any geometric object $\mathcal{O}$ is defined as follows.

Definition 2.4 *Geometric object $\mathcal{O}$ is some Σ-subspace of Σ-space, which can be represented as a set of intersections and joins of elementary geometric objects (EGO).*

Definition 2.5 *Elementary geometric object $\mathcal{E} \subset \Omega$ is a set of zeros of the envelope function*

$$f_{\mathcal{P}^n}: \qquad \Omega \to \mathbb{R}, \qquad \mathcal{P}^n \equiv \{P_0, P_1, ...P_n\} \in \Omega^{n+1} \tag{2.3}$$

i.e.

$$\mathcal{E} = \mathcal{E}_f(\mathcal{P}^n) = \{R | f_{\mathcal{P}^n}(R) = 0\} \tag{2.4}$$

The finite set $\mathcal{P}^n \subset \Omega$ of parameters of the envelope function $f_{\mathcal{P}^n}$ is the skeleton of elementary geometric object (EGO). The set $\mathcal{E} \subset \Omega$ of points forming EGO is called the envelope of its skeleton $\mathcal{P}^n$. The envelope function $f_{\mathcal{P}^n}$ is an algebraic function of s arguments $w = \{w_1, w_2, ...w_s\}$, $s = (n+2)(n+1)$. Each of arguments $w_k = \Sigma(Q_k, L_k)$ is a Σ-function of two arguments $Q_k, L_k \in \{R, \mathcal{P}^n\}$.

For continuous T-geometry the envelope $\mathcal{E}$ is usually a continual set of points. The envelope function $f_{\mathcal{P}^n}$, determining EGO is a function of the running point $R \in \Omega$ and of parameters $\mathcal{P}^n \in \Omega^{n+1}$. Thus, any elementary geometric object is determined by its skeleton $\mathcal{P}^n$ and by the form of the envelope function $f_{\mathcal{P}^n}$.

Let us investigate T-geometry on the Σ-space $V = \{\Sigma, \Omega\}$. For some special choice $\Sigma_{\rm E}$ of Σ-function, the Σ-space V turns to a Σ-subspace $V'_{\rm E} = \{\Sigma_{\rm E}, \Omega\}$ of a n-dimensional proper Euclidean space $V_{\rm E} = \{\Sigma_{\rm E}, \Omega_{\rm E}\}$, $\Omega \subset \Omega_{\rm E}$. (It will be shown). Then all relations between geometric objects in $V'_{\rm E}$ are relations of proper Euclidean geometry. Replacement of $\Sigma_{\rm E}$ by Σ means a deformation of $V'_{\rm E}$, because world function Σ describes distances between two points, and change of these distances is a deformation of the space. We shall use concept of deformation in a wide meaning, including in this term any increase and any reduction of number of points in the set Ω. Then any transition from $\{\Sigma_{\rm E}, \Omega_{\rm E}\}$ to $\{\Sigma, \Omega\}$ is a deformation of $\{\Sigma_{\rm E}, \Omega_{\rm E}\}$.

Let us write Euclidean relations between geometric objects in $V'_{\rm E}$ in the σ-immanent form (i.e. in the form, which contains references only to geometrical objects and Σ-function). Replacing the world function $\Sigma_{\rm E}$ by Σ in these relations, one obtains the relations between the geometric objects in the Σ-space $V = \{\Sigma, \Omega\}$.

Geometry in the proper Euclidean space is known very well, and one uses deformation, described by world function, to establish T-geometry in arbitrary Σ-space. Considering deformations of Euclidean space, one goes around the problem of axiomatics in the Σ-space $V = \{\Sigma, \Omega\}$. One uses only Euclidean axiomatics.

T-geometry of arbitrary Σ-space is obtained as a result of "deformation of proper Euclidean geometry". This point is very important, because axiomatics of arbitrary T-geometry is very complicated. It is relatively simple only for highly symmetric spaces. Investigation of arbitrary deformations is much simpler, than investigations of arbitrary axiomatics. Formally, a work with deformations of Σ-spaces is manipulations with the world function. These manipulations may be carried out without mention of space deformations.

Description of EGOs by means (2.3) is carried out in the deform-invariant form (invariant with respect to Σ-space deformations). The envelope function $f_{\mathcal{P}^n}$ as a function of arguments $w_k = \Sigma(Q_k, L_k)$, $Q_k, L_k \in \{R, \mathcal{P}^n\}$ does not depend on the form of the world function Σ. Thus, definition of the envelope function is invariant with respect to deformations (deform-invariant), and the envelope function determines any EGO in all Σ-spaces at once.

Let $\mathcal{E}_{\mathrm{E}}$ be EGO in the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$. Let $\mathcal{E}_{\mathrm{E}}$ be described by the skeleton $\mathcal{P}^n$ and the envelope function $f_{\mathcal{P}^n}$ in the Euclidean Σ-space $V_{\mathrm{E}} = \{\Sigma_{\mathrm{E}}, \Omega\}$. Then the EGO $\mathcal{E}$ in the T-geometry $\mathcal{G}$, described by the same skeleton $\mathcal{P}^n$ and the same envelope function $f_{\mathcal{P}^n}$ in the Σ-space $V = \{\Sigma, \Omega\}$, is an analog in $\mathcal{G}$ of the Euclidean EGO $\mathcal{E}_{\mathrm{E}}$. T-geometry $\mathcal{G}$ may be considered to be a result of deformation of the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$, when distances $\sqrt{\Sigma(P,Q) + \Sigma(Q,P)}$ between the pairs of points P and Q are changed. At such a deformation the Euclidean EGO $\mathcal{E}_{\mathrm{E}}$ transforms to its analog $\mathcal{E}$.

The Euclidean space has the most powerful group of motion, and the same envelope $\mathcal{E}_{\mathrm{E}}$ may be generated by the envelope function $f_{\mathcal{P}^n}$ with different values $\mathcal{P}^n_{(1)}$, $\mathcal{P}^n_{(2)}$,... of the skeleton $\mathcal{P}^n$, or even by another envelope function $f_{(1)\mathcal{Q}^m}$. It means that the Euclidean EGO $\mathcal{E}_{\mathrm{E}}$ may have several analogs $\mathcal{E}_{(1)}, \mathcal{E}_{(2)}, \ldots$ in the geometry $\mathcal{G}$. In other words, deformation of the Euclidean space may split EGOs, (but not only deform them). Note that the splitting may be interpreted as a kind of deformation.

Concept of curve is defined as the continuous mapping

$$\mathcal{L}: \quad [0,1] \to \Omega, \qquad [0,1] \subset \mathbb{R}, \tag{2.5}$$

It is a common practice to consider the curve $\mathcal{L}([0,1]) \subset \Omega$ to be an important geometrical object of geometry. From point of view of T-geometry the set of points $\mathcal{L}([0,1]) \subset \Omega$ cannot be considered to be EGO, because the mapping (2.5) is not deform-invariant. Indeed, let us consider a sphere $\mathcal{S}_{P_0P_1}$, passing through the point P_1 and having its center at the point P_0. It is described by the envelope function

$$f_{P_0P_1}(R) = \sqrt{\Sigma(P_0,R) + \Sigma(R,P_0)} - \sqrt{\Sigma(P_1,R) + \Sigma(R,P_1)} \tag{2.6}$$

In the two-dimensional proper Euclidean space the envelope function (2.6) describes a one-dimensional circumference $\mathcal{L}_1$, whereas in the three-dimensional proper Euclidean space the envelope function (2.6) describes a two-dimensional sphere $\mathcal{S}_2$. The point set $\mathcal{L}_1$ can be represented as the continuous mapping (2.5), whereas the surface $\mathcal{S}_2$ cannot. Transition from two-dimensional Euclidean space to three-dimensional

Euclidean space is a space deformation. Thus, deformation of the Σ-space may destroy the property of EGO of being a curve (2.5).

Application of objects, defined by the property (2.5) for investigation of T-geometries is inconvenient, because the T-geometry investigation is founded on deform-invariant methods. Formally, one cannot choose appropriate envelope function for description of the set (2.5), because the envelope function is deform-invariant, whereas the set (2.5) is not. Hence, (2.5) is incompatible with the definition 2.5 of EGO.

The nonsymmetric T-geometry, considered in this paper can be investigated by the same methods, as the symmetric one. The world function Σ in the nonsymmetric T-geometry is presented in the form

$$\Sigma(P,Q) = G(P,Q) + A(P,Q), \qquad P,Q \in \Omega \tag{2.7}$$

$$G(P,Q) = G(Q,P), \qquad A(P,Q) = -A(Q,P) \tag{2.8}$$

$$G(P,Q) = \frac{1}{2}\left(\Sigma(P,Q) + \Sigma(Q,P)\right), \tag{2.9}$$

$$A(P,Q) = \frac{1}{2}\left(\Sigma(P,Q) - \Sigma(Q,P)\right) \tag{2.10}$$

where G denotes the symmetric part of the world function Σ, whereas A denotes its antisymmetric part.

Motives for consideration of nonsymmetric T-geometry are as follows. In the symmetric T-geometry the distance from the point P to the point Q is the same as the distance from the point Q to the point P. In the asymmetric T-geometry it is not so. Apparently, it is not important for spacelike distances in the space-time, because it can be tested experimentally for spacelike distances. In the case, when interval between points P and Q is timelike, one uses watch to measure this interval. But the watch can measure the time interval only in one direction, and one cannot be sure that the time interval is the same in opposite direction.

If the antisymmetric part A of the world function does not vanish, it means that the future and the past are not equivalent geometrically. We do not insist that this fact takes place, but we admit this. It is useful to construct a nonsymmetric T-geometry, to apply it to the space-time and to obtain the corollaries of asymmetry which could be tested experimentally. The symmetrical part of the world function generates the field of the metric tensor g_{ik}. In a like way the antisymmetric part generates some vector force filed a_i. Maybe, existence of this field can be tested experimentally. For construction of nonsymmetric T-geometry one does not need to make any additional supposition. It is sufficient to remove the constraint $\Sigma(P,Q) = \Sigma(Q,P)$ and to apply mathematical technique developed for the symmetric T-geometry with necessary modifications.

Besides, there is a hope that nonsymmetric T-geometry will be useful in the elementary particle theory, where the main object is a superstring. The first order tubes (main objects of T-geometry) are associated with world tubes of strings and branes. In the nonsymmetric T-geometry antisymmetric variables appear. They are absent in the conventional symmetric T-geometry, but antisymmetric variables are characteristic for the superstring theory.

Two important general remarks.

1. Nonsymmetric T-geometry, as well as the symmetric one, is considered on an arbitrary set Ω of points P. It is formulated in the scope of the purely metric conception of geometry [6], which is very simple, because it uses only very simple tools for the geometry description. The T-geometry formulated in terms of the world function Σ and finite subsets $\mathcal{P}^n \equiv \{P_0, P_1, ..., P_n\}$ of the set Ω. Mathematically it means, that the purely metric conception of geometry uses only mappings

$$m_n: \quad I_n \to \Omega, \qquad I_n \equiv \{0, 1, ...n\} \subset \{0\} \cup \mathbb{N} \tag{2.11}$$

whereas the topology-metric conception of geometry [10, 11, 12] uses much more complicated mappings (2.5), known as curves $\mathcal{L}$. Both mappings (2.11) and (2.5) are methods of the geometry description (and construction). But the method (2.11) is much simpler. It can be studied exhaustively, whereas the set of mappings (2.5) cannot.

2. The nonsymmetric T-geometry will be mainly interpreted as a symmetric T-geometry determined by the two-point scalar $G(P,Q)$ with some additional metric structures, introduced to the symmetric geometry by means of the additional two-point scalar $A(P,Q)$. For instance, in the symmetric space-time T-geometry the world line of a free particle is described by a geodesic. In the nonsymmetric space-time T-geometry there are, in general, several different types of geodesics. This fact may be interpreted in the sense, that a free particle has some internal degrees of freedom, and it may be found in different states. In these different states the free particle interacts differently with the force fields, generated by the two-point scalar $A(P,Q)$. Several different types of geodesics are results of this interaction.

Definition 2.6 . *Σ-space $V = \{\Sigma, \Omega\}$ is called isometrically embeddable in Σ-space $V' = \{\Sigma', \Omega'\}$, if there exists such a monomorphism $f: \Omega \to \Omega'$, that $\Sigma(P,Q) = \Sigma'(f(P), f(Q))$, $\forall P, \forall Q \in \Omega$, $f(P), f(Q) \in \Omega'$,*

Any Σ-subspace V' of Σ-space $V = \{\Sigma, \Omega\}$ is isometrically embeddable in it.

Definition 2.7 . *Two Σ-spaces $V = \{\Sigma, \Omega\}$ and $V' = \{\Sigma', \Omega'\}$ are called to be isometric (equivalent), if V is isometrically embeddable in V', and V' is isometrically embeddable in V.*

Definition 2.8 *The Σ-space $M = \{\Sigma, \Omega\}$ is called a finite Σ-space, if the set Ω contains a finite number of points.*

Definition 2.9 . *The Σ-subspace $M_n(\mathcal{P}^n) = \{\Sigma, \mathcal{P}^n\}$ of the Σ-space $V = \{\Sigma, \Omega\}$, consisting of $n+1$ points $\mathcal{P}^n = \{P_0, P_1, ..., P_n\}$ is called the nth order Σ-subspace .*

The T-geometry is a set of all propositions on properties of Σ-subspaces of Σ-space $V = \{\Sigma, \Omega\}$. Presentation of T-geometry is produced in the language, containing only references to Σ-function and constituents of Σ-space, i.e. to its Σ-subspaces.

Definition 2.10 *A description is called σ-immanent, if it does not contain any references to objects or concepts other, than finite subspaces of the Σ-space and its world function (metric).*

σ-immanence of description provides independence of the description on the method of description. In this sense the σ-immanence of a description in T-geometry reminds the concept of covariance in Riemannian geometry. Covariance of some geometric relation in Riemannian geometry means that the considered relation is valid in all coordinate systems and, hence, describes only the properties of the Riemannian geometry in itself. Covariant description provides cutting-off from the coordinate system properties, considering the relation in all coordinate systems at once. The σ-immanence provides truncation from the methods of description by absence of a reference to objects, which do not relate to geometry in itself (coordinate system, concept of curve, dimension).

The idea of constructing the T-geometry is very simple. Relations of proper Euclidean geometry are written in the σ-immanent form and declared to be valid for any Σ-function. This results that any relation of proper Euclidean geometry corresponds to some relation of T-geometry.

3 Multivectors as basic objects of T-geometry. Scalar Σ-product and concept of collinearity

The basic elements of T-geometry are finite Σ-subspaces $M_n(\mathcal{P}^n)$, i.e. finite sets

$$\mathcal{P}^n = \{P_0, P_1, \dots, P_n\} \subset \Omega \tag{3.1}$$

The simplest finite subset is a nonzero vector $\overrightarrow{\mathcal{P}^1} = \mathbf{P}_0\mathbf{P}_1 \equiv \overrightarrow{P_0P_1}$. The vector $\overrightarrow{P_0P_1}$ is an ordered set of two points $\{P_0, P_1\}$. The scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \Sigma(P_0, Q_1) - \Sigma(P_1, Q_1) - \Sigma(P_0, Q_0) + \Sigma(P_1, Q_0) \tag{3.2}$$

is the main construction of T-geometry, and we substantiate this definition.

σ-immanent expression for scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ in the proper Euclidean space has the form (3.2). This relation can be easily proved as follows.

In the proper Euclidean space three vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0\mathbf{Q}_1$, and $\mathbf{P}_1\mathbf{Q}_1$ are coupled by the relation

$$|\mathbf{P}_1\mathbf{Q}_1|^2 = |\mathbf{P}_0\mathbf{Q}_1 - \mathbf{P}_0\mathbf{P}_1|^2 = |\mathbf{P}_0\mathbf{P}_1|^2 + |\mathbf{P}_0\mathbf{Q}_1|^2 - 2(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1) \tag{3.3}$$

where $(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1)$ denotes the scalar product of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{Q}_1$ in the proper Euclidean space, and $|\mathbf{P}_0\mathbf{P}_1|^2 \equiv (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1)$. It follows from (3.3)

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1) = \frac{1}{2}\left(|\mathbf{P}_0\mathbf{Q}_1|^2 + |\mathbf{P}_0\mathbf{P}_1|^2 - |\mathbf{P}_1\mathbf{Q}_1|^2\right) \tag{3.4}$$

Substituting the point Q_1 by Q_0 in (3.4), one obtains

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_0) = \frac{1}{2}\{|\mathbf{P}_0\mathbf{Q}_0|^2 + |\mathbf{P}_0\mathbf{P}_1|^2 - |\mathbf{P}_1\mathbf{Q}_0|^2\} \tag{3.5}$$

Subtracting (3.5) from (3.4) and using the properties of the scalar product in the proper Euclidean space, one obtains

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \frac{1}{2}\{|\mathbf{P}_0\mathbf{Q}_1|^2 + |\mathbf{P}_1\mathbf{Q}_0|^2 - |\mathbf{P}_0\mathbf{Q}_0|^2 - |\mathbf{P}_1\mathbf{Q}_1|^2\} \tag{3.6}$$

Taking into account that in the proper Euclidean geometry $|\mathbf{P}_0\mathbf{Q}_1|^2 = 2\Sigma(P_0, Q_1) = 2G(P_0, Q_1)$, one obtains the relation (3.2) from the relation (3.6).

In the Euclidean geometry the world function is symmetric, and the order of arguments in the rhs of (3.2) is not essential. In the asymmetric T-geometry the order of arguments in the rhs of (3.2) is essential. The order has been chosen in such a way that

$$\begin{aligned}(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)_{\mathrm{s}} &\equiv \frac{1}{2}\left((\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) + (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_1)\right) \\ &= G(P_0, Q_1) - G(P_1, Q_1) - G(P_0, Q_0) + G(P_1, Q_0)\end{aligned} \tag{3.7}$$

$$\begin{aligned}(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)_{\mathrm{a}} &\equiv \frac{1}{2}\left((\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) - (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_1)\right) \\ &= A(P_0, Q_1) - A(P_1, Q_1) - A(P_0, Q_0) + A(P_1, Q_0)\end{aligned} \tag{3.8}$$

It follows from (3.2) that

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = -(\mathbf{P}_1\mathbf{P}_0.\mathbf{Q}_0\mathbf{Q}_1), \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = -(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_1\mathbf{Q}_0) \tag{3.9}$$

Thus, the scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is antisymmetric with respect to permutation $P_0 \leftrightarrow P_1$ of points determining the vector $\mathbf{P}_0\mathbf{P}_1$, as well as with respect to permutation $Q_0 \leftrightarrow Q_1$.

Definition 3.1 *The finite* Σ*-space* $M_n(\mathcal{P}^n) = \{\Sigma, \mathcal{P}^n\}$ *is called oriented* $\overrightarrow{M_n(\mathcal{P}^n)}$, *if the order of its points* $\mathcal{P}^n = \{P_0, P_1, \ldots P_n\}$ *is fixed.*

Definition 3.2 . *The* n*th order multivector* m_n *is the mapping*

$$m_n: \qquad I_n \to \Omega, \qquad I_n \equiv \{0, 1, ..., n\} \tag{3.10}$$

The set I_n has a natural ordering, which generates an ordering of images $m_n(k) \in \Omega$ of points $k \in I_n$. The ordered list of images of points in I_n has one-to-one connection with the multivector and may be used as the multivector descriptor. Different versions of the point list will be used for writing the nth order multivector descriptor:

$$\overrightarrow{P_0P_1...P_n} \equiv \mathbf{P}_0\mathbf{P}_1...\mathbf{P}_n \equiv \overrightarrow{\mathcal{P}^n}$$

Originals of points P_k in I_n are determined by the order of the point P_k in the list of descriptor. Index of the point P_k has nothing to do with the original of P_k. Further we shall use descriptor $\overrightarrow{P_0P_1...P_n}$ of the multivector instead of the multivector. In this sense the nth order multivector $\overrightarrow{P_0P_1...P_n}$ in the Σ-space $V=\{\Sigma,\Omega\}$ may be defined as the ordered set $\{P_l\}$, $\quad l=0,1,\dots n$ of $n{+}1$ points P_0, P_1, ..., P_n, belonging to the Σ-space V. Some points may be identical. The point P_0 is the origin of the multivector $\overrightarrow{P_0P_1...P_n}$. Image $m_n\left(I_n\right)$ of the set I_n contains k points $(k\leq n+1)$. The set of all nth order multivectors m_n constitutes the set $\Omega^{n+1}=\bigotimes\limits_{k=1}^{n+1}\Omega$, and any multivector $\overrightarrow{\mathcal{P}^n}\in\Omega^{n+1}$.

Definition 3.3 . *The scalar Σ-product $(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})$ of two nth order multivectors $\overrightarrow{\mathcal{P}^n}$ and $\overrightarrow{\mathcal{Q}^n}$ is the real number*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})=\det\|(\mathbf{P}_0\mathbf{P}_i.\mathbf{Q}_0\mathbf{Q}_k)\|,\qquad i,k=1,2,...n \tag{3.11}$$

$$\begin{aligned}(\mathbf{P}_0\mathbf{P}_i.\mathbf{Q}_0\mathbf{Q}_k)\ &\equiv\ \Sigma(P_0,Q_k)+\Sigma(P_i,Q_0)-\Sigma(P_0,Q_0)-\Sigma(P_i,Q_k),\\ P_0,P_i,Q_0,Q_k\ &\in\ \Omega,\qquad \overrightarrow{\mathcal{P}^n},\overrightarrow{\mathcal{Q}^n}\in\Omega^{n+1}\end{aligned} \tag{3.12}$$

Operation of permutation of the multivector points can be effectively defined in the Σ-space. Let us consider two nth order multivectors $\overrightarrow{\mathcal{P}^n}=\overrightarrow{P_0P_1P_2...P_n}$ and $\overrightarrow{\mathcal{P}^n_{(k\leftrightarrow l)}}=\overrightarrow{P_0P_1...P_{k-1}P_lP_{k+1}...P_{l-1}P_kP_{l+1}...P_n}$, $\quad(n\geq 1)$, which is a result of permutation of points P_k, P_l, $\ (k<l)$. The scalar Σ-product $(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})$ is defined by the relation (3.11). One can show that

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})=-(\overrightarrow{\mathcal{P}^n_{(k\leftrightarrow l)}}.\overrightarrow{\mathcal{Q}^n})\qquad k\neq l,\qquad l,k=0,1,2,...n,\qquad \forall\overrightarrow{\mathcal{Q}^n}\in\Omega^{n+1}, \tag{3.13}$$

As far as the relation (3.13) is valid for permutation of any two points of the multivector $\overrightarrow{\mathcal{P}^n}$ and for any multivector $\overrightarrow{\mathcal{Q}^n}\in\Omega^{n+1}$, one may write

$$\overrightarrow{\mathcal{P}^n_{(i\leftrightarrow k)}}=-\overrightarrow{\mathcal{P}^n},\qquad i,k=0,1,...n,\qquad i\neq k,\qquad n\geq 1. \tag{3.14}$$

Thus, a change of the nth order multivector sign $(n\geq 1)$ (multiplication by the number $a=-1$) may be always defined as an odd permutation of points.

Let us consider the relation

$$\overrightarrow{\mathcal{P}^n}T\overrightarrow{\mathcal{R}^n}:\qquad (\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})=(\overrightarrow{\mathcal{R}^n}.\overrightarrow{\mathcal{Q}^n})\wedge(\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{P}^n})=(\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{R}^n}),\qquad \forall\overrightarrow{\mathcal{Q}^n}\in\Omega^{n+1}, \tag{3.15}$$

between two nth order multivectors $\overrightarrow{\mathcal{P}^n}\in\Omega^{n+1}$ and $\overrightarrow{\mathcal{R}^n}\in\Omega^{n+1}$. The relation (3.15) is reflexive, symmetric and transitive, and it may be considered as an equivalence relation.

Definition 3.4 . *Two nth order multivectors $\overrightarrow{\mathcal{P}^n}\in\Omega^{n+1}$ and $\overrightarrow{\mathcal{R}^n}\in\Omega^{n+1}$ are equivalent $\overrightarrow{\mathcal{P}^n}=\overrightarrow{\mathcal{R}^n}$, if the relations (3.15) takes place.*

Definition 3.5 . *If the nth order multivector* $\overrightarrow{\mathcal{N}^n}$ *satisfies the relations*

$$(\overrightarrow{\mathcal{N}^n}.\overrightarrow{\mathcal{Q}^n}) = 0 \wedge (\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{N}}) = 0, \qquad \forall \overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}, \tag{3.16}$$

$\overrightarrow{\mathcal{N}^n}$ *is the null nth order multivector.*

Definition 3.6 . *The length* $|\overrightarrow{\mathcal{P}^n}|$ *of the multivector* $\overrightarrow{\mathcal{P}^n}$ *is the number*

$$|\overrightarrow{\mathcal{P}^n}| = \begin{cases} |\sqrt{(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n})}| = |\sqrt{F_n(\mathcal{P}^n)}|, & (\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n}) \geq 0 \\ i\,|\sqrt{(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n})}| = i|\sqrt{F_n(\mathcal{P}^n)}|, & (\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n}) < 0 \end{cases} \qquad \overrightarrow{\mathcal{P}^n} \in \Omega^{n+1} \tag{3.17}$$

where the quantity $F_n(\mathcal{P}^n)$ *is defined by the relations*

$$F_n: \quad \Omega^{n+1} \to \mathbb{R}, \qquad \Omega^{n+1} = \bigotimes_{k=1}^{n+1} \Omega, \qquad n = 1, 2, \ldots \tag{3.18}$$

$$F_n\left(\overrightarrow{\mathcal{P}^n}\right) = \det ||\,(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)\,||, \qquad P_0, P_i, P_k \in \Omega, \qquad i,k = 1,2,...n \tag{3.19}$$

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k) \equiv \Sigma\,(P_i, P_0) + \Sigma\,(P_0, P_k) - \Sigma\,(P_i, P_k)\,, \qquad i,k = 1,2,...n, \tag{3.20}$$

The function (3.18) is a symmetric function of all its arguments $\mathcal{P}^n = \{P_0, P_1, ..., P_n\}$, i.e. it is invariant with respect to permutation of any points P_i, P_k, $i,k = 0,1,...n$. It follows from representation

$$F_n\left(\overrightarrow{\mathcal{P}^n}\right) = F_n\left(\mathcal{P}^n\right) = \left(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n}\right)$$

and the relation (3.14). It means that the squared length $|\overrightarrow{\mathcal{P}^n}|^2 = |M\left(\mathcal{P}^n\right)|^2$ of any multivector $\overrightarrow{\mathcal{P}^n}$ does not depend on the order of points. The squared length of any finite subset $\mathcal{P}^n$ is unique.

In the case, when multivector $\overrightarrow{\mathcal{P}^n}$ does not contain similar points, it coincides with the oriented finite Σ-subspace $\overrightarrow{M_n(\mathcal{P}^n)}$, and it is a constituent of Σ-space. In the case, when at least two points of multivector coincide, the multivector length vanishes, and the multivector is considered to be a null multivector. The null multivector $\overrightarrow{\mathcal{P}^n}$ is not a finite Σ-subspace $M_n(\mathcal{P}^n)$, or an oriented finite Σ-subspace $\overrightarrow{M_n(\mathcal{P}^n)}$, but a use of null multivectors assists in creation of a more simple technique, because the null multivectors $\overrightarrow{\mathcal{P}^n}$ play a role of zeros. Essentially, the multivectors are basic objects of T-geometry. As to continual geometric objects, which are analogs of planes, sphere, ellipsoid, etc., they are constructed by means of skeleton-envelope method (see [6]) with multivectors, or finite Σ-subspaces used as skeletons. As a consequence the T-geometry is presented σ-immanently, i.e. without references to objects, external with respect to Σ-space.

The usual vector $\mathbf{P}_0\mathbf{P}_1 \equiv \overrightarrow{P_0P_1} \equiv \overrightarrow{\mathcal{P}^1} = \{P_0, P_1\}\,,\ \ P_0, P_1 \in \Omega$ is a special case of multivector. The squared length $|\mathbf{P}_0\mathbf{P}_1|^2$ of the vector $\mathbf{P}_0\mathbf{P}_1$ is defined by the relation (3.2). This gives

$$|\mathbf{P}_0\mathbf{P}_1|^2 \equiv (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1) = \Sigma\,(P_0, P_1) + \Sigma\,(P_1, P_0) = 2G\,(P_0, P_1) \tag{3.21}$$

The following quantities are also associated with the vector $\mathbf{P}_0\mathbf{P}_1$

$$\begin{aligned}
|\mathbf{P}_1\mathbf{P}_0|^2 &\equiv (\mathbf{P}_1\mathbf{P}_0.\mathbf{P}_1\mathbf{P}_0) = 2G(P_1, P_0), &(3.22)\\
(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_1\mathbf{P}_0) &= -\Sigma(P_0, P_1) - \Sigma(P_1, P_0) = -2G(P_0, P_1), &(3.23)
\end{aligned}$$

It is rather unexpected that $|\mathbf{P}_0\mathbf{P}_1|^2 = 2G(P_0, P_1)$, but it is well that the vector $\mathbf{P}_0\mathbf{P}_1$ has only one length, but not two $\sqrt{2\Sigma(P_0, P_1)}$ and $\sqrt{2\Sigma(P_1, P_0)}$, as one could expect.

Definition 3.7 *The squared length $|M(\mathcal{P}^n)|^2$ of the nth order Σ-subspace $M(\mathcal{P}^n) \subset \Omega$ of the Σ-space $V = \{\Sigma, \Omega\}$ is the real number.*

$$|M(\mathcal{P}^n)|^2 = F_n(\mathcal{P}^n),$$

where $M(\mathcal{P}^n) = \{P_0, P_1, ..., P_n,\} \subset \Omega$ with all different $P_i \in \Omega$, $i = 0, 1, ...n$, $\overrightarrow{\mathcal{P}^n} \in \Omega^{n+1}$, and the quantity $F_n(\mathcal{P}^n)$ is defined by the relations (3.19) – (3.20).

The meaning of the written relations is as follows. In the special case, when the Σ-space is Euclidean space, its Σ-function is symmetric. It coincides with Σ-function of Euclidean space. Any two points P_0, P_i determine the vector $\mathbf{P}_0\mathbf{P}_i$. The relation (3.20) is a σ-immanent expression for the scalar Σ-product $(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)$ of two vectors. Then the relation (3.19) is the Gram's determinant for n vectors $\mathbf{P}_0\mathbf{P}_i$, $i = 1, 2, \dots n$, and $\sqrt{F_n(\mathcal{P}^n)}/n!$ is the Euclidean volume of the $(n+1)$-hedron with vertices at the points $\mathcal{P}^n$.

Now we enable to formulate in terms of the world function the necessary and sufficient condition of that the Σ-space is the n-dimensional Euclidean space
I.

$$\Sigma(P, Q) = \Sigma(Q, P), \qquad P, Q \in \Omega \tag{3.24}$$

II.

$$\exists\mathcal{P}^n \subset \Omega, \qquad F_n(\mathcal{P}^n) \neq 0, \qquad F_{n+1}(\Omega^{n+2}) = 0, \tag{3.25}$$

III.

$$\Sigma(P, Q) = \frac{1}{2}\sum_{i,k=1}^{n} g^{ik}(\mathcal{P}^n)[x_i(P) - x_i(Q)][x_k(P) - x_k(Q)], \qquad \forall P, Q \in \Omega, \tag{3.26}$$

where the quantities $x_i(P)$, $x_i(Q)$ are defined by the relations

$$x_i(P) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}), \qquad x_i(Q) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{Q}), \qquad i = 1, 2, ...n \tag{3.27}$$

The contravariant components $g^{ik}(\mathcal{P}^n)$, $(i, k = 1, 2, \dots n)$ of metric tensor are defined by its covariant components $g_{ik}(\mathcal{P}^n)$, $(i, k = 1, 2, \dots n)$ by means of relations

$$\sum_{k=1}^{n} g_{ik}(\mathcal{P}^n) g^{kl}(\mathcal{P}^n) = \delta_i^l, \qquad i, l = 1, 2, \dots n, \tag{3.28}$$

where covariant components $g_{ik}(\mathcal{P}^n)$ are defined by relations

$$g_{ik}(\mathcal{P}^n) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k), \qquad i,k = 1,2,\ldots n \tag{3.29}$$

IV. The relations

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}) = x_i, \qquad x_i \in \mathbb{R}, \qquad i = 1,2,\ldots n, \tag{3.30}$$

considered to be equations for determination of $P \in \Omega$, have always one and only one solution.

Remark 1 *The condition (3.25) is a corollary of the condition (3.26). It is formulated in the form of a special condition, in order that a determination of dimension were separated from determination of a coordinate system.*

The condition II determines the space dimension. The condition III describes σ-immanently the scalar Σ-product properties of the proper Euclidean space. Setting $n+1$ points $\mathcal{P}^n$, satisfying the condition II, one determines n-dimensional basis of vectors in Euclidean space. Relations (3.29), (3.28) determine covariant and contravariant components of the metric tensor, and the relations (3.27) determine covariant coordinates of points P and Q at this basis. The relation (3.26) determines the expression for Σ-function for two arbitrary points in terms of coordinates of these points. Finally, the condition IV describes continuity of the set Ω and a possibility of the manifold construction on it. Necessity of conditions I – IV for Euclideaness of Σ-space is evident. One can prove their sufficiency [5]. The connection of conditions I – IV with the Euclideaness of the Σ-space can be formulated in the form of a theorem.

Theorem 1 *The Σ-space $V = \{\Sigma, \Omega\}$ is the n-dimensional Euclidean space, if and only if σ-immanent conditions I – IV are fulfilled.*

Remark 2 *For the Σ-space were proper Euclidean, the eigenvalues of the matrix $g_{ik}(\mathcal{P}^n)$, $\quad i,k = 1,2,\ldots n$ must have the same sign, otherwise it is pseudo-Euclidean.*

The theorem states that it is sufficient to know metric (world function) to construct the Euclidean geometry. Concepts of topological space and curve, which are used usually in metric geometry for increasing its informativity, appear to be excess in the sense that they are not needed for construction of geometry. Proof of this theorem can be found in [5].

Definition 3.8 *Two nth order multivectors $\overrightarrow{\mathcal{P}^n}$, $\overrightarrow{\mathcal{Q}^n}$ are neutrally collinear (n-collinear) $\overrightarrow{\mathcal{P}^n} \parallel_{(\mathrm{n})} \overrightarrow{\mathcal{Q}^n}$, if*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})(\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{P}^n}) = |\overrightarrow{\mathcal{P}^n}|^2 \cdot |\overrightarrow{\mathcal{Q}^n}|^2 \tag{3.31}$$

Definition 3.9 *The nth order multivector $\overrightarrow{\mathcal{P}^n}$ is f-collinear to nth order multivector $\overrightarrow{\mathcal{Q}^n}$,* $\left(\overrightarrow{\mathcal{P}^n} \parallel_{(\mathrm{f})} \overrightarrow{\mathcal{Q}^n}\right)$, *if*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})^2 = |\overrightarrow{\mathcal{P}^n}|^2 \cdot |\overrightarrow{\mathcal{Q}^n}|^2 \tag{3.32}$$

Definition 3.10 *The nth order multivector $\overrightarrow{\mathcal{P}^n}$ is p-collinear to nth order multivector $\overrightarrow{\mathcal{Q}^n}$,* $\left(\overrightarrow{\mathcal{P}^n} \parallel_{(\mathrm{p})} \overrightarrow{\mathcal{Q}^n}\right)$, *if*

$$(\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{P}^n})^2 = |\overrightarrow{\mathcal{P}^n}|^2 \cdot |\overrightarrow{\mathcal{Q}^n}|^2 \tag{3.33}$$

Here indices "f" and "p" are associated with the terms "future" and "past" respectively.

In the symmetric T-geometry there is only one type of collinearity, because the three mentioned types of collinearity coincide in the symmetric T-geometry. The property of the neutral collinearity is commutative, i.e. if $\overrightarrow{\mathcal{P}^n} \parallel_{(\mathrm{n})} \overrightarrow{\mathcal{Q}^n}$, then $\overrightarrow{\mathcal{Q}^n} \parallel_{(\mathrm{n})} \overrightarrow{\mathcal{P}^n}$. The property of p-collinearity and f-collinearity are not commutative, in general. Instead, one has according to (3.32) and (3.33) that, if $\overrightarrow{\mathcal{P}^n} \parallel_{(\mathrm{p})} \overrightarrow{\mathcal{Q}^n}$, then $\overrightarrow{\mathcal{Q}^n} \parallel_{(\mathrm{f})} \overrightarrow{\mathcal{P}^n}$.

Definition 3.11 . *The nth order multivector $\overrightarrow{\mathcal{P}^n}$ is f-parallel to the nth order multivector $\overrightarrow{\mathcal{Q}^n}$* $\left(\overrightarrow{\mathcal{P}^n} \uparrow\uparrow_{(\mathrm{f})} \overrightarrow{\mathcal{Q}^n}\right)$, *if*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = |\overrightarrow{\mathcal{P}^n}| \cdot |\overrightarrow{\mathcal{Q}^n}| \tag{3.34}$$

The nth order multivector $\overrightarrow{\mathcal{P}^n}$ is f-antiparallel to the nth order multivector $\overrightarrow{\mathcal{Q}^n}$ $\left(\overrightarrow{\mathcal{P}^n} \uparrow\downarrow_{(\mathrm{f})} \overrightarrow{\mathcal{Q}^n}\right)$, *if*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = -|\overrightarrow{\mathcal{P}^n}| \cdot |\overrightarrow{\mathcal{Q}^n}| \tag{3.35}$$

Definition 3.12 *The nth order multivector $\overrightarrow{\mathcal{P}^n}$ is p-parallel to the nth order multivector $\overrightarrow{\mathcal{Q}^n}$* $\left(\overrightarrow{\mathcal{P}^n} \uparrow\uparrow_{(\mathrm{p})} \overrightarrow{\mathcal{Q}^n}\right)$, *if*

$$(\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{P}^n}) = |\overrightarrow{\mathcal{P}^n}| \cdot |\overrightarrow{\mathcal{Q}^n}| \tag{3.36}$$

The nth order multivector $\overrightarrow{\mathcal{P}^n}$ is p-antiparallel to the nth order multivector $\overrightarrow{\mathcal{Q}^n}$ $\left(\overrightarrow{\mathcal{P}^n} \uparrow\uparrow_{(\mathrm{p})} \overrightarrow{\mathcal{Q}^n}\right)$, *if*

$$(\overrightarrow{\mathcal{Q}^n}.\overrightarrow{\mathcal{P}^n}) = -|\overrightarrow{\mathcal{P}^n}| \cdot |\overrightarrow{\mathcal{Q}^n}| \tag{3.37}$$

The f-parallelism and the p-parallelism are connected as follows. If $\overrightarrow{\mathcal{P}^n} \uparrow\uparrow_{(\mathrm{p})} \overrightarrow{\mathcal{Q}^n}$, then $\overrightarrow{\mathcal{Q}^n} \uparrow\uparrow_{(\mathrm{f})} \overrightarrow{\mathcal{P}^n}$ and vice versa.

Vector $\mathbf{P}_0\mathbf{P}_1 = \overrightarrow{\mathcal{P}^1}$ as well as the vector $\mathbf{Q}_0\mathbf{Q}_1 = \overrightarrow{\mathcal{Q}^1}$ are the first order multivectors. If $\mathbf{P}_0\mathbf{P}_1 \uparrow\uparrow_{(\mathrm{f})} \mathbf{Q}_0\mathbf{Q}_1$, then $\mathbf{P}_1\mathbf{P}_0 \uparrow\downarrow_{(\mathrm{f})} \mathbf{Q}_0\mathbf{Q}_1$ and $\mathbf{P}_0\mathbf{P}_1 \uparrow\downarrow_{(\mathrm{f})} \mathbf{Q}_1\mathbf{Q}_0$

4 Tubes in Σ-space and their properties.

The simplest geometrical object in T-geometry is the nth order tube $\mathcal{T}(\mathcal{P}^n)$, which is determined by its skeleton $\mathcal{P}^n$. The tube is an analog of Euclidean n-dimensional plane, which is also determined by $n+1$ points $\mathcal{P}^n$, not belonging to a $(n-1)$-dimensional plane.

Definition 4.1 *nth order Σ-subspace $M(\mathcal{P}^n) = \mathcal{P}^n$ of nonzero length $|M(\mathcal{P}^n)|^2 = |\mathcal{P}^n|^2 = F_n(\mathcal{P}^n) \neq 0$ determines the nth order tube $\mathcal{T}(\mathcal{P}^n)$ by means of relation*

$$\mathcal{T}(\mathcal{P}^n) \equiv \mathcal{T}_{\mathcal{P}^n} = \left\{P_{n+1} | F_{n+1}\left(\mathcal{P}^{n+1}\right) = 0\right\}, \qquad P_i \in \Omega, \qquad i = 0, 1 \ldots n+1, \tag{4.1}$$

where the function F_n is defined by the relations (3.18) – (3.20)

The shape of the tube $\mathcal{T}(\mathcal{P}^n)$ does not depend on the order of points of the multivector $\overrightarrow{\mathcal{P}^n}$. The basic point $\mathcal{P}^n$, determining the tube $\mathcal{T}_{\mathcal{P}^n}$, belong to $\mathcal{T}_{\mathcal{P}^n}$.

The first order tube $\mathcal{T}_{P_0P_1}$ can be defined by means of concept of n-collinearity (3.31)

$$\begin{aligned} \mathcal{T}(\mathcal{P}^1) &\equiv \mathcal{T}_{(\mathrm{n})P_0P_1} = \{R | F_2(P_0, P_1, R) = 0\} \equiv \left\{R \left| \overrightarrow{P_0P_1} \|_{(\mathrm{n})} \overrightarrow{P_0R} \right.\right\} \\ &\equiv \left\{R \left| \, |\overrightarrow{P_0P_1}|^2 |\overrightarrow{P_0R}|^2 - \left(\overrightarrow{P_0P_1}.\overrightarrow{P_0R}\right)\left(\overrightarrow{P_0R}.\overrightarrow{P_0P_1}\right) = 0 \right.\right\} \end{aligned} \tag{4.2}$$

As far as there are concepts of f-collinearity and of p-collinearity, one can define also the first order f-tube and p-tube on the basis of these collinearities. The first order f-tube is defined by the relation

$$\mathcal{T}_{(\mathrm{f})P_0P_1} = \left\{R \left| \overrightarrow{P_0P_1} \|_{(\mathrm{f})} \overrightarrow{P_0R} \right.\right\} = \left\{R \left| \, |\overrightarrow{P_0P_1}|^2 |\overrightarrow{P_0R}|^2 - \left(\overrightarrow{P_0P_1}.\overrightarrow{P_0R}\right)^2 = 0 \right.\right\} \tag{4.3}$$

The first order p-tube is defined as follows

$$\mathcal{T}_{(\mathrm{p})P_0P_1} = \left\{R \left| \overrightarrow{P_0P_1} \|_{(\mathrm{p})} \overrightarrow{P_0R} \right.\right\} = \left\{R \left| \, |\overrightarrow{P_0P_1}|^2 |\overrightarrow{P_0R}|^2 - \left(\overrightarrow{P_0R}.\overrightarrow{P_0P_1}\right)^2 = 0 \right.\right\} \tag{4.4}$$

In the symmetric T-geometry all three tubes (4.2) – (4.4) coincide. In the nonsymmetric T-geometry they are different, in general. The tubes (4.2), (4.3), (4.4) can be divided into segments, each of them is determined by one of factors of expressions (4.2) – (4.4).

In all cases the factorization of the expressions

$$F_{(\mathrm{f})}(P_0, P_1, R) = \left|\overrightarrow{P_0P_1}\right|^2 \left|\overrightarrow{P_0R}\right|^2 - \left(\overrightarrow{P_0P_1}.\overrightarrow{P_0R}\right)^2$$

$$F_{(\mathrm{p})}(P_0, P_1, R) = \left|\overrightarrow{P_0P_1}\right|^2 \left|\overrightarrow{P_0R}\right|^2 - \left(\overrightarrow{P_0R}.\overrightarrow{P_0P_1}\right)^2,$$

$$F_{(\mathrm{n})}(P_0, P_1, R) = \left|\overrightarrow{P_0P_1}\right|^2 \left|\overrightarrow{P_0R}\right|^2 - \left(\overrightarrow{P_0P_1}.\overrightarrow{P_0R}\right)\left(\overrightarrow{P_0R}.\overrightarrow{P_0P_1}\right) \tag{4.5}$$

has similar form

$$F_{(\mathrm{q})}\left(P_0,P_1,R\right)=-F_{(\mathrm{q}0)}F_{(\mathrm{q}1)}F_{(\mathrm{q}2)}F_{(\mathrm{q}3)} \tag{4.6}$$

Here index q runs values $f,p,n,$ and factorization of expressions $F_{(\mathrm{q})}\left(P_0,P_1,R\right)$, $q=f,p,n$ has a similar form

$$F_{(\mathrm{q}0)}=F_{(\mathrm{q}0)}\left(P_0,P_1,R\right)=\sqrt{G_{0R}}+\sqrt{G_{01}}+\sqrt{G_{1R}-\eta_{\mathrm{q}}} \tag{4.7}$$

$$\begin{aligned} F_{(\mathrm{q}1)} &= F_{(\mathrm{q}1)}\left(P_0,P_1,R\right)=\sqrt{G_{0R}}-\sqrt{G_{01}}+\sqrt{G_{1R}-\alpha_{\mathrm{q}}\eta_{\mathrm{q}}} \\ F_{(\mathrm{q}2)} &= F_{(\mathrm{q}2)}\left(P_0,P_1,R\right)=\sqrt{G_{0R}}+\sqrt{G_{01}}-\sqrt{G_{1R}-\eta_{\mathrm{q}}} \\ F_{(\mathrm{q}3)} &= F_{(\mathrm{q}3)}\left(P_0,P_1,R\right)=\sqrt{G_{0R}}-\sqrt{G_{01}}-\sqrt{G_{1R}-\alpha_{\mathrm{q}}\eta_{\mathrm{q}}} \end{aligned} \tag{4.8}$$

where for brevity one uses designations

$$\begin{aligned} G_{ik} &= G\left(P_i,P_k\right), \qquad A_{ik}=A\left(P_i,P_k\right), \qquad i,k=0,1 \\ G_{iR} &= G\left(P_i,R\right), \qquad A_{iR}=A\left(P_i,R\right), \qquad i=0,1 \end{aligned}$$

$$\begin{aligned} \eta_{\mathrm{f}} &= -\eta_{\mathrm{p}}=A_{10}+A_{0R}+A_{R1}, \qquad \eta_{\mathrm{n}}=\frac{\eta_{\mathrm{f}}^2}{\sqrt{4G_{01}G_{0R}+\eta_{\mathrm{f}}^2}+2\sqrt{G_{01}G_{0R}}} \\ \alpha_{\mathrm{p}} &= \alpha_{\mathrm{f}}=1, \qquad \alpha_{\mathrm{n}}=-1 \end{aligned} \tag{4.9}$$

In the symmetric T-geometry, when $A\left(P,Q\right)=0$, $\forall P,Q\in\Omega$, and $\eta=0$, all expressions (4.8), for $F_{(\mathrm{n}i)},F_{(\mathrm{f}i)},F_{(\mathrm{p}i)}$, $i=0,1,2,3$ coincide.

Factorizations (4.6) – (4.8) determine division of the tubes into segments. As it follows from (4.9) $\eta_{\mathrm{p}}=\eta_{\mathrm{f}}=\eta_{\mathrm{n}}=0$, provided $R=P_0$, or $R=P_1$. Then one can see that

$$\begin{aligned} P_0,P_1 &\in \mathcal{T}_{(\mathrm{q})[P_0P_1]}=\left\{R|F_{(\mathrm{q}1)}\left(P_0,P_1,R\right)=0\right\} \\ &= \left\{R|\sqrt{G_{0R}}-\sqrt{G_{01}}+\sqrt{G_{1R}-\alpha_{\mathrm{q}}\eta_{\mathrm{q}}}=0\right\} \end{aligned} \tag{4.10}$$

$$\begin{aligned} P_0 &\in \mathcal{T}_{(\mathrm{q})P_0]P_1}=\left\{R|F_{(\mathrm{q}2)}\left(P_0,P_1,R\right)=0\right\} \\ &= \left\{R|\sqrt{G_{0R}}+\sqrt{G_{01}}-\sqrt{G_{1R}-\eta_{\mathrm{q}}}=0\right\} \end{aligned} \tag{4.11}$$

$$\begin{aligned} P_1 &\in \mathcal{T}_{(\mathrm{q})P_0[P_1}=\left\{R|F_{(\mathrm{q}3)}\left(P_0,P_1,R\right)=0\right\} \\ &= \left\{R|\sqrt{G_{0R}}-\sqrt{G_{01}}-\sqrt{G_{1R}-\alpha_{\mathrm{q}}\eta_{\mathrm{q}}}=0\right\} \end{aligned} \tag{4.12}$$

The tube segment

$$\mathcal{T}_{(\mathrm{q}0)P_0P_1}=\left\{R|F_{(\mathrm{q}0)}\left(P_0,P_1,R\right)=0\right\} \tag{4.13}$$

determined (4.7), does not contain basic points P_0, P_1, in general. $\mathcal{T}_{(q0)P_0P_1} = \emptyset$ for any timelike tube $\mathcal{T}_{P_0P_1}$.

In the relations (4.10) – (4.13) index q runs values f, p, n. Values of η_q, α_q are determined by the relations (4.9).

The tube segments may be classified by the number of basic points P_0, P_1, belonging to the segment. The segment $\mathcal{T}_{(q)[P_0P_1]}$, containing two basic points will be referred to as the internal tube segment. The segments $\mathcal{T}_{(q)P_0[P_1}$ and $\mathcal{T}_{(q)P_0]P_1}$ contain one basic point. They will be referred to as external tube segments, or as tube rays, directed along the vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_1\mathbf{P}_0$ respectively. The segment $\mathcal{T}_{(q0)P_0P_1}$, which does not contain basic points, will be referred to as null segment. As a rule it is empty.

In the geometry of Minkowski the timelike tube $\mathcal{T}_{P_0P_1}$, determined by the timelike vector $\mathbf{P}_0\mathbf{P}_1$, is the straight, passing through the points P_0 and P_1. $\mathcal{T}_{[P_0P_1]}$ is the segment $[P_0, P_1]$ of the straight between the points P_0 and P_1. The tube rays $\mathcal{T}_{P_0[P_1}$, $\mathcal{T}_{P_0]P_1}$ are rays of the straight $[P_1, \infty)$ and $(-\infty, P_0]$. The null segment $\mathcal{T}_{(0)P_0P_1}$ is empty.

Definition 4.2 . *Section $\mathcal{S}_{n;P}$ of the tube $\mathcal{T}(\mathcal{P}^n)$ at the point $P \in \mathcal{T}(\mathcal{P}^n)$ is the set $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n))$ of points, belonging to the tube $\mathcal{T}(\mathcal{P}^n)$*

$$\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n)) = \{P' \mid \bigwedge_{l=0}^{l=n} \Sigma(P_l, P') = \Sigma(P_l, P)\}, \qquad P \in \mathcal{T}(\mathcal{P}^n), \qquad P' \in \Omega. \tag{4.14}$$

Let us note that $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n)) \subset \mathcal{T}(\mathcal{P}^n)$, because $P \in \mathcal{T}(\mathcal{P}^n)$. Indeed, whether the point P belongs to $\mathcal{T}(\mathcal{P}^n)$ depends only on values of $n+1$ quantities $\Sigma(P_l, P)$, $l = 0, 1, ...n$. In accordance with (4.14) these quantities are the same for both points P and P'. Hence, the running point $P' \in \mathcal{T}(\mathcal{P}^n)$, if $P \in \mathcal{T}(\mathcal{P}^n)$.

In the proper Euclidean space the nth order tube is n-dimensional plane, containing points $\mathcal{P}^n$, and its section $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n))$ at the point P consists of one point P.

Definition 4.3 *Section $\mathcal{S}_{n;P}$ of the tube $\mathcal{T}(\mathcal{P}^n)$ at the point $P \in \mathcal{T}(\mathcal{P}^n)$ is minimal, if $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n)) = \{P\}$.*

Definition 4.4 *The first order tube $\mathcal{T}_{P_0P_1}$ is degenerate, if its section at any point $P \in \mathcal{T}_{P_0P_1}$ is minimal.*

Minimality of the first tube section means that the first order tube degenerates to a curve, and any section of the tube consists of one point. It means that there is only one vector $\overrightarrow{P_0R}$, $R \in \mathcal{T}_{P_0P_1}$ of fixed length, which is parallel, or antiparallel to the vector $\overrightarrow{P_0P_1}$. As far as in the nonsymmetric T-geometry there is several types of parallelism, there is several types of degeneracy, in general. In the symmetric T-geometry there is only one type of the degeneracy.

Definition 4.5 *The Σ-space $V = \{\Sigma, \Omega\}$ is degenerate on the set $\mathcal{T}$ of the first order tubes, if the set $\mathcal{T}$ contains only degenerate tubes $\mathcal{T}_{P_0P_1}$.*

Definition 4.6 *The Σ-space $V = \{\Sigma, \Omega\}$ is locally f-degenerate, if all first order tubes $\mathcal{T}_{(\mathrm{f})P_0P_1}$ are degenerate.*

Definition 4.7 *The Σ-space $V = \{\Sigma, \Omega\}$ is locally p-degenerate, if all first order tubes $\mathcal{T}_{(\mathrm{p})P_0P_1}$ are degenerate.*

Definition 4.8 *The Σ-space $V = \{\Sigma, \Omega\}$ is locally n-degenerate, if all first order tubes $\mathcal{T}_{(\mathrm{n})P_0P_1} \equiv \mathcal{T}_{P_0P_1}$ are degenerate.*

Note that the Riemannian space considered to be a Σ-space is locally degenerate.

5 Nondegenerate tubes in the space-time and their interpretation

Nondegeneracy of the first order tube $\mathcal{T}_{P_0P_1}$ means that the tube is not a one-dimensional curve, although it is an analog of the Euclidean straight line. In the Minkowski space-time geometry $\mathcal{G}_{\mathrm{M}}$ the timelike straight line describes the world line of a free particle. One should expect that the nondegenerate first order tube $\mathcal{T}_{P_0P_1}$ describes also the free particle in the nondegenerate space-time geometry $\mathcal{G}_{\mathrm{D}}$.

Let us describe σ-immanently a particle of the mass m in $\mathcal{G}_{\mathrm{M}}$. World line of the particle is a broken line $\mathcal{T}_{\mathrm{br}}$ consisting of rectilinear internal segments $\mathcal{T}_{[P_iP_{i+1}]}$, $i = 0, \pm 1, \pm 2...$

$$\mathcal{T}_{\mathrm{br}} = \bigcup_i \mathcal{T}_{[P_iP_{i+1}]} \tag{5.1}$$

It is supposed that all segments $\mathcal{T}_{[P_iP_{i+1}]}$ has the same length μ, and the quantity μ is proportional to the particle mass m.

$$m = b\mu, \qquad b = \mathrm{const}, \tag{5.2}$$

where b is some universal constant transforming the length of a segment to its mass. The particle momentum $\mathbf{p}_i$ on the segment $\mathcal{T}_{[P_iP_{i+1}]}$ is defined by the relation

$$(\mathbf{p}_i.\mathbf{Q}_0\mathbf{Q}_1) = bc\left(\mathbf{P}_i\mathbf{P}_{i+1}.\mathbf{Q}_0\mathbf{Q}_1\right), \qquad \forall Q_0, Q_1 \in \mathbb{R}^4 \tag{5.3}$$

where c is the speed of the light. It means that the momentum $\mathbf{p}_i$ is proportional to the vector $\mathbf{P}_i\mathbf{P}_{i+1}$, determining the segment $\mathcal{T}_{[P_iP_{i+1}]}$. This can be written symbolically in the form

$$\mathbf{p}_i = bc\mathbf{P}_i\mathbf{P}_{i+1}, \qquad |\mathbf{p}_i|^2 = b^2c^2\mu^2 = m^2c^2, \qquad i = 0, \pm 1, \pm 2... \tag{5.4}$$

Segment $\mathcal{T}_{[P_iP_{i+1}]}$ is defined by the equation (4.10). In the case of the Minkowski geometry $\mathcal{G}_{\mathrm{M}}$, as well as in the case of any symmetric T-geometry one obtains

$$\mathcal{T}_{[P_iP_{i+1}]} = \left\{ R | \sqrt{G(P_i, P_{i+1})} - \sqrt{G(P_i, R)} - \sqrt{G(R, P_{i+1})} \right\} \tag{5.5}$$

Formulae (5.1) – (5.5) carry out σ-immanent description of the world line of a particle. It means that these relations (5.1) – (5.5) describe the particle world tube in any symmetric T-geometry.

If the particle is free, one should add the parallelism condition $\mathbf{P}_i\mathbf{P}_{i+1} \uparrow\uparrow \mathbf{P}_{i+1}\mathbf{P}_{i+2}$:

$$(\mathbf{P}_i\mathbf{P}_{i+1}.\mathbf{P}_{i+1}\mathbf{P}_{i+2}) = |\mathbf{P}_i\mathbf{P}_{i+1}| \cdot |\mathbf{P}_{i+1}\mathbf{P}_{i+2}|, \qquad i = 0, \pm 1, \pm 2... \tag{5.6}$$

In $\mathcal{G}_{\mathrm{M}}$ relations (5.1) – (5.6) describe σ-immanently the world line of a free particle of mass m. It is a timelike straight line, because in the Minkowski geometry there is only one timelike vector $\mathbf{P}_{i+1}\mathbf{P}_{i+2}$ of length μ, which is parallel to the vector $\mathbf{P}_i\mathbf{P}_{i+1}$. Then, if the vector $\mathbf{P}_i\mathbf{P}_{i+1}$ is fixed, the point $\mathbf{P}_{i+2}$ is determined uniquely. It means, that if one segment, for instance $\mathcal{T}_{[P_0P_1]}$, is fixed, positions of all other segments $\mathcal{T}_{[P_iP_{i+1}]}$, $i = 0, \pm 1, \pm 2...$ and the whole broken tube $\mathcal{T}_{\mathrm{br}}$ are determined uniquely. In other words, motion of the free particle in the Minkowski geometry $\mathcal{G}_{\mathrm{M}}$ is deterministic.

Equations (5.1) – (5.6), written in the σ-immanent form, determine the free particle world tube in the case of any symmetric T-geometry $\mathcal{G}_{\mathrm{D}}$. Let us consider the case, when the space-time geometry is symmetric and nondegenerate. Then there are many timelike vectors $\mathbf{P}_{i+1}\mathbf{P}_{i+2}$ of length μ, which are parallel to the vector $\mathbf{P}_i\mathbf{P}_{i+1}$. At fixed vector $\mathbf{P}_i\mathbf{P}_{i+1}$, the point P_{i+2} is not determined uniquely. Then at fixed segment $\mathcal{T}_{[P_0P_1]}$ the positions of all other segments $\mathcal{T}_{[P_iP_{i+1}]}$, $i = 0, \pm 1, \pm 2...$ and the whole broken tube $\mathcal{T}_{\mathrm{br}}$ are not determined uniquely. It means that the world tube of a free particle is stochastic.

Let us consider the case of geometry $\mathcal{G}_{\mathrm{D}}$ with the symmetric world function

$$\mathcal{G}_{\mathrm{D}}: \qquad G = \Sigma = \Sigma_{\mathrm{M}} + D(\Sigma_{\mathrm{M}}) \tag{5.7}$$

$$D(\Sigma_{\mathrm{M}}) = \begin{cases} d & \text{if } \sigma_0 < \Sigma_{\mathrm{M}} \\ \frac{d}{\sigma_0}\Sigma_{\mathrm{M}} & \text{if } 0 < \Sigma_{\mathrm{M}} < \sigma_0 \\ 0 & \text{if } \Sigma_{\mathrm{M}} < 0 \end{cases}, \qquad d = \frac{\hbar}{2bc} = \text{const} \tag{5.8}$$

where Σ_{M} is the world function in $\mathcal{G}_{\mathrm{M}}$, c is the speed of the light, $\hbar$ is the quantum constant, and the constant b is defined by the relation (5.2).The quantity $\sigma_0 \approx d \approx 10^{-20}\mathrm{cm}^2$. The function D is the distortion function, and the constant d is an integral distortion. The distortion function is the quantity, responsible for nondegeneracy of the space-time geometry $\mathcal{G}_{\mathrm{D}}$. Geometry $\mathcal{G}_{\mathrm{D}}$, described by the world function (5.7), (5.8), is uniform and isotropic. The tube segment $\mathcal{T}_{[P_0P_1]}$ has the shape of a hallow tube. Radius R of the tube is approximately $R \approx \sqrt{3d/2}$. More exactly the shape of the tube is described by the relation [13]

$$R^2(\tau) = \frac{3}{2}d + \frac{2\mu^2 d}{\mu^2 - 2d}\left(\tau - \frac{1}{2}\right)^2, \qquad \frac{\sqrt{2d}}{\mu} < \tau < 1 - \frac{\sqrt{2d}}{\mu} \tag{5.9}$$

where $R(\tau)$ is radius of the tube segment $\mathcal{T}_{[P_0P_1]}$ as a function of the parameter τ along the axis of this segment ($\tau = 0$ at the point P_0, and $\tau = 1$ at the point P_1). μ is the length of the segment $\mathcal{T}_{[P_0P_1]}$.

Vectors $\mathbf{P}_i\mathbf{P}_{i+1}$ and $\mathbf{P}_{i+1}\mathbf{P}_{i+2}$ are parallel in $\mathcal{G}_{\rm D}$, But they are not parallel in $\mathcal{G}_{\rm M}$. The angle $\vartheta_{\rm D}$ between $\mathbf{P}_i\mathbf{P}_{i+1}$ and $\mathbf{P}_{i+1}\mathbf{P}_{i+2}$ is equal to 0, because

$$\cosh\vartheta_{\rm D} = \frac{(\mathbf{P}_i\mathbf{P}_{i+1}.\mathbf{P}_{i+1}\mathbf{P}_{i+2})}{\mu^2} = 1 \tag{5.10}$$

In $\mathcal{G}_{\rm M}$ the angle $\vartheta_{\rm M}$ between $\mathbf{P}_i\mathbf{P}_{i+1}$ and $\mathbf{P}_{i+1}\mathbf{P}_{i+2}$ is determined by the relation

$$\cosh\vartheta_{\rm M} = \frac{(\mathbf{P}_i\mathbf{P}_{i+1}.\mathbf{P}_{i+1}\mathbf{P}_{i+2})_{\rm M}}{|\mathbf{P}_i\mathbf{P}_{i+1}|_{\rm M}\cdot|\mathbf{P}_{i+1}\mathbf{P}_{i+2}|_{\rm M}} = \frac{(\mathbf{P}_i\mathbf{P}_{i+1}.\mathbf{P}_{i+1}\mathbf{P}_{i+2})+d}{\mu^2-2d} \tag{5.11}$$

where index "M" means that the corresponding quantity is calculated in $\mathcal{G}_{\rm M}$. Taking into account (5.10) and supposing that $\sqrt{d}\ll\mu$, one obtains from (5.11)

$$\cosh\vartheta_{\rm M} = 1+\frac{3d}{\mu^2}, \qquad \vartheta_{\rm M}^2 = \frac{6d}{\mu^2} \tag{5.12}$$

The angle $\vartheta_{\rm M}$ describes intensity of stochasticity of the particle motion. The diffusion displacement λ of a particle determined this stochasticity is described by the quantity

$$\lambda = \mu\left\langle\vartheta_{\rm M}^2\right\rangle \approx \frac{3\hbar}{b\mu c} = \frac{3\hbar}{mc}$$

This is rather close to the particle Compton wave length.

One can see from (5.12), that this stochasticity is large for the particle of small mass $m = b\mu$. It is rather unexpected, because, dealing with general relativity, one thinks that influence of space-time geometry on the particle motion does not depend on its mass. This dependence (5.12) on the particle mass is a corollary of geometrization of the mass in the nondegenerate T-geometry. Indeed, the geometrical mass μ of the particle can be determined from the shape of the world tube (5.1). The geometrical mass μ is the distance between the adjacent points P_i and P_{i+1}, where the tube radius vanishes. In $\mathcal{G}_{\rm M}$ this radius vanishes everywhere, and the mass cannot be determined from the shape of the world tube (line).

Geometrization of the particle mass is very important phenomenon, which is essential for effective description of physical phenomena of microcosm.

6 Particle dynamics in the nondegenerate space-time geometry

In the Minkowski space-time geometry $\mathcal{G}_{\rm M}$ the particle motion is deterministic, and one can describe a single particle, writing dynamic equations for its world line. In $\mathcal{G}_{\rm D}$ it is impossible due to the world line stochasticity. In $\mathcal{G}_{\rm D}$ one uses statistical method of the particle motion description, when one describes motion of many identical particles. This method is described in details in papers [14, 15]. We consider here only characteristic features of this method, which are essential for understanding of geometric origin of nonrelativistic quantum phenomena.

Let us consider in $\mathcal{G}_\mathrm{M}$ a deterministic dynamic system $\mathcal{S}_\mathrm{d}$, consisting of deterministic particle. The dynamic system $\mathcal{S}_\mathrm{d}$ is described by the Lagrangian function $L(t, \mathbf{x}, \dot{\mathbf{x}})$, where $\mathbf{x} = \{x^1, x^2, x^3\}$ are coordinates of the particle in some inertial coordinate system, and $\dot{\mathbf{x}}$ is its velocity. By definition a pure statistical ensemble $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]$ of dynamic systems $\mathcal{S}_\mathrm{d}$ is such an ensemble, whose distribution function $F_\mathrm{p}(t, \mathbf{x}, \mathbf{p})$ may be represented in the form

$$\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]: \qquad F_\mathrm{p}(t, \mathbf{x}, \mathbf{p}) = \rho(t, \mathbf{x})\, \delta(\mathbf{p} - \mathbf{P}(t, \mathbf{x})) \tag{6.1}$$

where $\rho(t, \mathbf{x})$ and $\mathbf{P}(t, \mathbf{x}) = \{P_\alpha(t, \mathbf{x})\}$, $\alpha = 1, 2, 3$ are functions of only time t and $\mathbf{x}$. In other words, the pure ensemble $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]$ is a dynamic system, considered in the configuration space of coordinates $\mathbf{x}$. It is a fluidlike continuous dynamic system, which can be described by the action [15]

$$\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]: \quad \mathcal{A}[j, \varphi, \boldsymbol{\xi}] = \int \{L(x^0, \mathbf{x}, \mathbf{j}/j^0)\, j^0 - b_0 j^i [\partial_i \varphi + g^\alpha(\boldsymbol{\xi}) \partial_i \xi_\alpha]\} \mathrm{d}^4 x, \tag{6.2}$$

where $j^i = \{j^0, \mathbf{j}\}, \varphi, \boldsymbol{\xi}$ are dependent variables, which are considered to be functions of $x = \{x^0, \mathbf{x}\} = \{t, \mathbf{x}\}$. $L(x^0, \mathbf{x}, \mathbf{j}/j^0) = L(x^0, \mathbf{x}, d\mathbf{x}/dx^0)$ is the Lagrangian function of $\mathcal{S}_\mathrm{d}$. b_0 is an arbitrary constant and $g^\alpha(\boldsymbol{\xi})$, $\alpha = 1, 2, 3$ are arbitrary functions of the argument $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}$. These functions describe initial state of the statistical ensemble $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]$. The 4-current j^i, describes the fluid flow. The action (6.2) as well as dynamic equations, generated by this action, contain derivatives $j^k \partial_k$ only in the direction of the vector j^i. It means that the system of dynamic equations, which are partial differential equations, form essentially a system of ordinary differential equations, describing a single dynamic system $\mathcal{S}_\mathrm{d}$. Thus, Lagrangian function L describes both dynamic systems $\mathcal{S}_\mathrm{d}$ and $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]$.

If the dynamic system $\mathcal{S}_\mathrm{d}$ is subjected influence of some stochastic agent, it turns to stochastic system $\mathcal{S}_\mathrm{st}$, and parameters of the statistical ensemble $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{st}]$ stops to be constant. They become to be functions of the ensemble state j^i and of derivatives $\partial_k j^i$. The action for $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{st}]$ stops to depend on only $j^k \partial_k$. In this case dynamic equations contain derivatives in all directions, and the system of dynamic equations cannot be reduced to a system of ordinary equations. Physically it means that there is no dynamic equation for the single system $\mathcal{S}_\mathrm{st}$, although they exist for the statistical ensemble $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{st}]$.

Thus, if we want to describe a deterministic dynamic system and a stochastic system as different partial cases of a dynamic system and to describe their dynamics by similar method, we should describe dynamics of a pure statistical ensemble, but not dynamics of a single dynamic system. In this sense the concept of dynamics of the pure statistical ensemble is more general and fundamental, than concept of the single system dymanics. Such an idea is not new [16].

If the dynamic system $\mathcal{S}_\mathrm{d}$ is a free particle, its Lagrangian function has the form

$$L(\dot{\mathbf{x}}) = -mc^2 \sqrt{1 - \frac{\dot{\mathbf{x}}^2}{c^2}}, \qquad \dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} \tag{6.3}$$

The Lagrangian function depends on the only parameter m. In the case of space-time geometry $\mathcal{G}_{\mathrm{D}}$ the mass m depends on the ensemble state $\rho = j^0$. The mass m is modified as follows [1]

$$m^2 \to m_{\mathrm{q}}^2 = m^2 + \frac{\hbar^2}{4c^2}\left(\mathbf{\nabla} \ln \rho\right)^2, \qquad \rho = j^0 \tag{6.4}$$

Let us substitute (6.3), (6.4) in the action (6.2). One obtains in the nonrelativistic approximation

$$\mathcal{A}[j,\varphi,\boldsymbol{\xi}] = \int \{-\rho mc^2 + \frac{m\mathbf{j}^2}{2\rho} - \frac{\hbar^2\rho}{8m}\left(\mathbf{\nabla} \ln \rho\right)^2 - b_0 j^i[\partial_i\varphi + g^\alpha(\boldsymbol{\xi})\partial_i\xi_\alpha]\}\mathrm{d}^4x, \tag{6.5}$$

Any ideal fluid can be described in terms of wave function [14]. Describing the action (6.5) in terms of wave function and considering the special case, when the fluid flow is irrotational and the wave function has only one component, one obtains instead of (6.5)

$$\begin{aligned}\mathcal{A}[\psi,\psi^*] &= \int \left\{\frac{ib_0}{2}(\psi^*\partial_0\psi - \partial_0\psi^*\cdot\psi) - \frac{b_0^2}{2m}\mathbf{\nabla}\psi^*\cdot\mathbf{\nabla}\psi - mc^2\psi^*\psi\right.\\ &\left.+\frac{b_0^2}{8m\psi^*\psi}(\nabla(\psi^*\psi))^2 - \frac{\hbar^2}{8m\psi^*\psi}(\nabla(\psi^*\psi))^2\right\}\mathrm{d}^4x,\end{aligned} \tag{6.6}$$

The last term in the action (6.6) describes influence of geometrical stochasticity. This term contains the quantum constant $\hbar$. If one sets $\hbar = 0$ in (6.6), the action becomes to describe statistical ensemble of free deterministic particles. The action (6.6) generates nonlinear dynamic equation for the wave function ψ. The dynamic equation becomes to be a linear differential equation, provided one sets $b_0 = \hbar$, because in this case two last terms in (6.6) compensate each other. Note that the constant b_0 is an arbitrary integration constant, which can take any value, in particular $b_0 = \hbar$. After this substitution the action (6.6) takes the form

$$\mathcal{A}[\psi,\psi^*] = \int \left\{\frac{i\hbar}{2}(\psi^*\partial_0\psi - \partial_0\psi^*\cdot\psi) - \frac{\hbar^2}{2m}\mathbf{\nabla}\psi^*\cdot\mathbf{\nabla}\psi - mc^2\psi^*\psi\right\}\mathrm{d}^4x \tag{6.7}$$

Now all terms in the action becomes to be quantum, because they contain the quantum constant. One cannot obtain the action for the statistical ensemble of deterministic (classical) particles, setting $\hbar = 0$, as it is possible to make in the action (6.6). One cannot suppress the geometrical (quantum) stochasticity in (6.7), setting $\hbar = 0$. In the action (6.7) quantum properties (the constant $\hbar$) are attributed to those terms, which are purely dynamic in (6.6), and one cannot indicate the term, responsible for quantum effects. This is the price which is paid for linearity of the dynamic equation.

Possibility of compensation of two last terms (dynamic and quantum) in the action (6.6) is connected with the nonrelativistic character of the Hamiltonian function for a classical particle, which has the form $H_{\text{nr}} = mc^2 + \mathbf{p}^2/(2m)$. For instance, if the Hamiltonian function H_{nr} is replaced by the relativistic one $H_{\text{r}} = \sqrt{m^2c^4 + \mathbf{p}^2c^2}$ identification $b_0 = \hbar$ does not lead to the linear dynamic equation for the wave function.

From geometric viewpoint, the linearity of the dynamic equation for the wave function is valid only for nonrelativistic case. From this viewpoint it seems to be doubtful, that the linearity should be used as a principle of the relativistic quantum theory, although from practical viewpoint the linearity is a very useful property of dynamic equation.

Thus, removing unfounded constraints on geometry and using space-time geometry $\mathcal{G}_{\text{D}}$, one can freely explain nonrelativistic quantum effects. There is no necessity to invent and to use quantum principles. Quantum effects appear to be corollary of quantum (or geometric) stochasticity, generated by nondegenerate character of the space-time geometry.

Properties of the world function (5.8) at the coinciding points ($0 < \Sigma_{\text{M}} < \sigma_0$) are of no importance for stochastic behavior of particles, because it depends only on integral distortion d. From this viewpoint application of methods of differential geometry to investigation of the world function properties is useless, because differential geometry studies properties of the world function $\Sigma(x, x')$ in the limit, when $x \to x'$. But methods of differential geometry are useful for investigation of antisymmetric component $A(x, x')$ of the nonsymmetric world function. We shall see that the antisymmetric component $A(x, x')$ generates fields in the space-time and generates as a rule additional nondegeneracy of geometry. In other words, $A(x, x')$ generates not only interaction between particles, but also their additional stochasticity. In this sense the fields, generated by $A(x, x')$, are quantum fields.

7 Asymmetric T-geometry on manifold

We have considered T-geometry in the coordinate-free form. But to discover a connection between the T-geometry and usual differential geometry, one needs to introduce coordinates and to consider the T-geometry on a manifold. It is important also from the viewpoint of the asymmetric T-geometry application as a possible space-time geometry. The asymmetric T-geometry on the manifold may be considered to be a conventional symmetric geometry (for instance, Riemannian) with additional force fields $a_i(x)$, $a_{ikl}(x)$, generated on the manifold by the antisymmetric component A of the world function. Testing experimentally existence of these force fields, one can conclude whether the antisymmetric component A exists and how large it is.

Let it be possible to attribute $n+1$ real numbers $x = \{x^i\}$, $i = 0, 1, ...n$ to any point P in such a way, that there be one-to-one correspondence between the point P and the set x of $n+1$ coordinates $\{x^i\}$, $i = 0, 1, ...n$. All points x form a set $\mathcal{M}_{n+1}$. Then the world function $\Sigma(P, P')$ is a function $\Sigma(x, x')$

$$\Sigma: \quad \mathcal{M}_{n+1} \times \mathcal{M}_{n+1} \to \mathbb{R}, \qquad \Sigma(x, x) = 0, \qquad \forall x \in \mathcal{M}_{n+1} \tag{7.1}$$

of coordinates $x, x' \in \mathcal{M}_{n+1} \subset \mathbb{R}^{n+1}$ of points $P, P' \in \Omega$. Two-point quantities (Σ-function and their derivatives) are designed as a rule by capital characters. One-point quantities are designed by small characters.

Let the function $\Sigma(x, x')$ be multiply differentiable. Then the set $\mathcal{M}_{n+1} \subset \mathbb{R}^{n+1}$ may be called the $(n+1)$-dimensional manifold. One can differentiate $\Sigma(x, x')$ with respect to x^i and with respect to x'^i, forming two-point tensors. For instance,

$$\begin{aligned} \Sigma_{,k}(x, x') &\equiv \frac{\partial}{\partial x^k}\Sigma(x, x'), \qquad \Sigma_{,k'}(x, x') \equiv \frac{\partial}{\partial x'^k}\Sigma(x, x') \\ \Sigma_{,kl'}(x, x') &\equiv \Sigma_{,l'k}(x, x') \equiv \frac{\partial^2}{\partial x^k \partial x'^l}\Sigma(x, x'), \end{aligned}$$

are two-point tensors. Here indices after comma mean differentiation with respect to x^k, if the index k has not a prime, and differentiation with respect to x'^k, if the index k has a prime. The first argument of the two-point quantity is denoted by unprimed variable, whereas the second one is denoted by primed one. Primed indices relate to the second argument of the two-point quantity, whereas the unprimed ones relate to the first argument.

$\Sigma_k \equiv \Sigma_{,k} = \Sigma_{,k}(x, x')$ is a vector at the point x and a scalar at the point x'. Vice versa $\Sigma_{k'} \equiv \Sigma_{,k'} = \Sigma_{,k'}(x, x')$ is a vector at the point x' and a scalar at the point x. The quantity $\Sigma_{,kl'} = \Sigma_{,kl'}(x, x')$ is a vector at the point x and a vector at the point x'. Other derivatives are not tensors. For instance, $\Sigma_{,kl}(x, x') \equiv \Sigma_{,lk}(x, x') \equiv \frac{\partial^2}{\partial x^k \partial x^l}\Sigma(x, x')$ is a scalar at the point x', but it is not a tensor at the point x.

To construct tensors of higher rank by means of differentiation, let us introduce covariant derivatives. Let $\Sigma_{,kl'} \equiv \Sigma_{kl'} \equiv \Sigma_{l'k}$ and $\det ||\Sigma_{kl'}|| \neq 0$. The quantity $\Sigma_{kl'}$ will be referred to as covariant fundamental metric tensor. One can introduce also contravariant fundamental metric tensor $\Sigma^{ik'} \equiv \Sigma^{k'i}$, defining it by the relation

$$\Sigma^{ik'}\Sigma_{lk'} = \delta^i_l \qquad \Sigma^{i'k}\Sigma_{l'k} = \delta^{i'}_{l'} \tag{7.2}$$

Let us note that the quantity

$$\tilde{\Gamma}^i_{kl}(x, x') \equiv \Sigma^{is'}\Sigma_{,kls'}, \qquad \Sigma_{,kls'} \equiv \frac{\partial^3 \Sigma}{\partial x^k \partial x^l \partial x'^s} \tag{7.3}$$

is a scalar at the point x' and a Christoffel symbol at the point x. Vice versa, the quantity

$$\tilde{\Gamma}^{i'}_{k'l'}(x, x') \equiv \Sigma^{si'}\Sigma_{,k'l's}, \qquad \Sigma_{,k'l's} \equiv \frac{\partial^3 \Sigma}{\partial x'^k \partial x'^l \partial x^s} \tag{7.4}$$

is a scalar at the point x and a Christoffel symbol at the point x'.

In the same way one can introduce two other Christoffel symbols on the basis of the function G

$$\begin{aligned} \Gamma^i_{kl}(x, x') &\equiv G^{is'}G_{,kls'}, \qquad G_{,kls'} \equiv \frac{\partial^3 G}{\partial x^k \partial x^l \partial x'^s}, \qquad G^{is'}G_{,ks'} = \delta^i_k &(7.5) \\ \Gamma^{i'}_{k'l'}(x, x') &\equiv G^{si'}G_{,k'l's}, \qquad G_{,k'l's} \equiv \frac{\partial^3 G}{\partial x'^k \partial x'^l \partial x^s} &(7.6) \end{aligned}$$

Using Christoffel symbols (7.3) - (7.6), one can introduce two covariant derivatives $\tilde{\nabla}_i^{x'}$, $\nabla_i^{x'}$ with respect to x^i and two covariant derivatives $\tilde{\nabla}_{i'}^{x}$, $\nabla_{i'}^{x}$ with respect to x'^i. For instance, the quantities

$$\Sigma_{ik} \equiv \tilde{\nabla}_k^{x'}\tilde{\nabla}_i^{x'}\Sigma \equiv \Sigma_{,i||k} = \Sigma_{,ik} - \tilde{\Gamma}_{ik}^s(x,x')\Sigma_{,s} = \Sigma_{,ik} - \Sigma^{ls'}\Sigma_{,iks'}\Sigma_{,l} \quad (7.7)$$

$$G_{ik} \equiv \nabla_k^{x'}\nabla_i^{x'}G \equiv G_{,i|k} = G_{,ik} - \Gamma_{ik}^s(x,x')G_{,s} = G_{,ik} - G^{ls'}G_{,iks'}G_{,l} \quad (7.8)$$

are scalars at the point x' and second rank tensors at the point x. Here symbol "$||$" before the index denotes covariant derivative with the Christoffel symbol $\tilde{\Gamma}_{ik}^s$, and the symbol "$|$" before the index denotes covariant derivative with the Christoffel symbol Γ_{ik}^s. In the same way one obtains

$$\Sigma_{i'k'} \equiv \tilde{\nabla}_{k'}^{x}\tilde{\nabla}_{i'}^{x}\Sigma \equiv \Sigma_{,i'||k'} = \Sigma_{,i'k'} - \tilde{\Gamma}_{i'k'}^{s'}\Sigma_{,s'} = \Sigma_{,i'k'} - \Sigma^{l's}\Sigma_{,i'k's}\Sigma_{,l'} \quad (7.9)$$

$$G_{i'k'} \equiv \nabla_{k'}^{x}\nabla_{i'}^{x}G \equiv G_{,i'|k'} = G_{,i'k'} - \Gamma_{i'k'}^{s'}G_{,s'} = G_{,i'k'} - G^{l's}G_{,i'k's}G_{,l'} \quad (7.10)$$

Covariant derivatives $\tilde{\nabla}_k^{x'}$, $\tilde{\nabla}_i^{x'}$ with respect to x commute, as well as $\nabla_k^{x'}$, $\nabla_i^{x'}$, i.e.

$$\left(\tilde{\nabla}_k^{x'}\tilde{\nabla}_i^{x'} - \tilde{\nabla}_i^{x'}\tilde{\nabla}_k^{x'}\right)T_{ml'}^{sp'} \equiv 0, \qquad \left(\nabla_k^{x'}\nabla_i^{x'} - \nabla_i^{x'}\nabla_k^{x'}\right)T_{ml'}^{sp'} \equiv 0 \quad (7.11)$$

where $T_{ml'}^{sp'}$ is an arbitrary tensor at points x and x'. Unprimed indices are associated with the point x, and primed ones with the point x'. The covariant derivatives commute, because the Riemann-Christoffel curvature tensors $\tilde{R}_{i.kl}^{\cdot j}$, $R_{i.kl}^{\cdot j}$ constructed respectively of Christoffel symbols $\tilde{\Gamma}_{il}^s$ and Γ_{il}^s vanish identically

$$\tilde{R}_{i.lm}^{\cdot s} \equiv \tilde{\Gamma}_{il,m}^s - \tilde{\Gamma}_{im,l}^s + \tilde{\Gamma}_{il}^j\tilde{\Gamma}_{jm}^s - \tilde{\Gamma}_{im}^j\tilde{\Gamma}_{jl}^s \equiv 0 \quad (7.12)$$

$$R_{i.lm}^{\cdot s} \equiv \Gamma_{il,m}^s - \Gamma_{im,l}^s + \Gamma_{il}^j\Gamma_{jm}^s - \Gamma_{im}^j\Gamma_{jl}^s \equiv 0 \quad (7.13)$$

One can test the identity (7.12), substituting (7.3) into (7.12).

Covariant derivatives $\tilde{\nabla}_{k'}^{x}$, $\tilde{\nabla}_{i'}^{x}$ with respect to x' commute as well as $\nabla_{k'}^{x}$, $\nabla_{i'}^{x}$. Commutativity of covariant derivatives $\tilde{\nabla}_i^{x'}$, $\tilde{\nabla}_k^{x'}$ with respect to x for all values of x' means that the covariant derivative $\tilde{\nabla}_i^{x'}$, $\tilde{\nabla}_k^{x'}$ are covariant derivatives in some flat spaces $\tilde{E}_{x'}$. The same is valid for covariant derivatives $\nabla_i^{x'}$, $\nabla_k^{x'}$ which are covariant derivatives in the flat spaces $E_{x'}$. The spaces $\tilde{E}_{x'}$, $E_{x'}$ are associated with the Σ-spaces $V = \{\Sigma, \mathcal{M}_{n+1}\}$ and $V_{\rm s} = \{G, \mathcal{M}_{n+1}\}$ respectively, given on $\mathcal{M}_{n+1}$ by means of the world function Σ and its symmetric part G. Any of two-point invariant quantities Σ and G with nonvanishing determinants $\det||\Sigma_{ik'}|| \neq 0$, and $\det||G_{ik'}|| \neq 0$ realize two sets of mappings. For instance, the quantity Σ generates mappings $V = \{\Sigma, \mathcal{M}_{n+1}\} \rightarrow \tilde{E}_{x'}$, $V \rightarrow \tilde{E}_x$ The two-point quantity G generates also two sets of mappings $V_{\rm s} = \{G, \mathcal{M}_{n+1}\} \rightarrow E_{x'}$, $V_{\rm s} \rightarrow E_x$. Mappings of any set are labelled by points x or x' of the manifold $\mathcal{M}_{n+1}$. In the case of G both sets of mappings $V \rightarrow E_{x'}$ and $V \rightarrow E_x$ coincide, but in the case of Σ the sets $V \rightarrow \tilde{E}_{x'}$ and $V \rightarrow \tilde{E}_x$ are different, in general.

It is easy to see that

$$\tilde{\nabla}_s^{x'}\Sigma_{ik'} = \Sigma_{ik'||s} = \Sigma_{,ik's} - \tilde{\Gamma}_{is}^p\Sigma_{pk'} = \Sigma_{,ik's} - \Sigma^{pl'}\Sigma_{isl'}\Sigma_{pk'} \equiv 0$$

The covariant derivatives have the following properties

$$\tilde{\nabla}_k^{x'} t_{l'}^{i'}(x') = 0, \qquad \nabla_k^{x'} t_{l'}^{i'}(x') = 0, \qquad \tilde{\nabla}_{k'}^{x} t_l^i(x) = 0 \qquad \nabla_{k'}^{x} t_l^i(x) = 0 \tag{7.14}$$

$$\tilde{\nabla}_s^{x'} \Sigma_{ik'} = \Sigma_{ik'||s} = 0 \qquad \Sigma_{ik'||s'} = 0, \qquad G_{ik'|s} = 0 \qquad G_{ik'|s'} = 0, \tag{7.15}$$

where $t_{l'}^{i'}(x')$ is an arbitrary tensor at the point x', and $t_l^i(x)$ is an arbitrary tensor at the point x.

The considered mappings onto Euclidean spaces can serve as a powerful tool for description of the Σ-space properties. Let us note in this connection, that the Riemannian space may be considered to be a set of infinitesimal pieces of Euclidean spaces glued in some way between themselves. The way of gluing determines the character of the Riemannian space in the sense, that different ways of gluing generate different Riemannian spaces. The way of gluing is determined by the difference of the metric tensor at the point x and at the narrow point $x + dx$, where it has the forms $g_{ik}(x)$ and $g_{ik}(x + dx)$ respectively. The metric tensor depends on a point and on a choice of the coordinate system. It is rather difficult one to separate dependence on the way of gluing from that on the choice of the coordinate system. Nevertheless the procedure of separation has been well developed. It leads to the curvature tensor, which is an indicator of the way of gluing.

In the case of the Σ-space one considers a set of *finite Euclidean spaces* $E_{x'}$ (instead of its infinitesimal pieces) and a set of mappings $\Sigma \to E_{x'}$. Here the "way of gluing" is determined by the dependence of mapping on the parameter x'. It does not depend on a choice of the coordinate system. This circumstance simplifies investigation. Differentiating the mappings with respect to parameters x', one derives local characteristics of the "way of gluing", which are modifications of the curvature tensor. For instance, considering commutators of derivatives $\tilde{\nabla}_i^{x'}$ and $\tilde{\nabla}_{i'}^{x}$, one can introduce two-point curvature tensor for the Σ-space, as it have been made for the Riemannian space [17, 18]. We shall see this further.

Let $\tilde{G}_{(x')ik}$ be the metric tensor in the Euclidean space $\tilde{E}_{x'}$ at the point x. Then the Christoffel symbol $\tilde{\Gamma}_{kl}^i = \Sigma^{is'}\Sigma_{,kls'}$ in the space $\tilde{E}_{x'}$ can be written in the form

$$\tilde{\Gamma}_{kl}^i = \Sigma^{is'}\Sigma_{,kls'} = \frac{1}{2}\tilde{G}_{(x')}^{im}\left(\tilde{G}_{(x')km,l} + \tilde{G}_{(x')lm,k} - \tilde{G}_{(x')kl,m}\right) \tag{7.16}$$

where $\tilde{G}_{(x')}^{im}$ are contravariant components of the metric tensor $\tilde{G}_{(x')ik}$.

Let us consider the set of equations (7.16) as a system of linear differential equations for determination of the metric tensor components $\tilde{G}_{(x')ik}$, which is supposed to be symmetric. Solution of this system has the form

$$\tilde{G}_{(x')ik} = \Sigma_{ip'}\tilde{g}_{(x')}^{p'q'}\Sigma_{kq'} \tag{7.17}$$

where $\tilde{g}_{(x')}^{p'q'} = \tilde{g}_{(x')}^{p'q'}(x')$ is some symmetric tensor at the point x'. This fact can be tested by a direct substitution of (7.17) in (7.16). Taking the relation (7.17) at the coinciding points $x = x'$ and denoting coincidence of points x and x' by means of square brackets, one obtains from (7.17)

$$\tilde{g}_{(x')}^{p'q'}(x') = \left[\Sigma^{lp'}\right]_{x'}\left[\tilde{G}_{(x')lm}\right]_{x'}\left[\Sigma^{mq'}\right]_{x'} \tag{7.18}$$

or

$$\tilde{G}_{(x')ik} = \Sigma_{ip'} \left[\Sigma^{lp'}\right]_{x'} \left[\tilde{G}_{(x')lm}\right]_{x'} \left[\Sigma^{mq'}\right]_{x'} \Sigma_{kq'} \tag{7.19}$$

The equation (7.19) can be written in the form

$$\tilde{G}_{(x')ik}\left(x,x'\right) = \tilde{P}_{(x')i.}^{\ .l'} \tilde{P}_{(x')k.}^{\ .m'} \tilde{G}_{(x')lm}\left(x',x'\right), \tag{7.20}$$

$$\tilde{P}_{(x')k.}^{\ .m'} \equiv \tilde{P}_{(x')k.}^{\ .m'}\left(x,x'\right) \equiv \Sigma_{kq'}\left(x,x'\right)\Sigma^{mq'}\left(x',x'\right) \equiv \Sigma_{kq'}\left[\Sigma^{mq'}\right]_{x'} \tag{7.21}$$

The relation (7.20) means that the metric tensor $\tilde{G}_{(x')ik}$ of the Euclidean space $\tilde{E}_{x'}$ at the point x can be obtained as a result of the parallel transport of the metric tensor from the point x' in $\tilde{E}_{x'}$ by means of the parallel transport tensor $\tilde{P}_{(x')k.}^{\ .m'}$. The parallel transport of the vector $b_{k'}$ from the point x' to the point x is defined by the relation

$$b_k = \tilde{P}_{(x')k.}^{\ .m'} b_{m'}.$$

The parallel transport tensor has evident properties

$$\tilde{\nabla}_i^{x'} \tilde{P}_{(x')k.}^{\ .m'} \equiv \tilde{P}_{(x')k.||i}^{\ .m'} = 0, \tag{7.22}$$

$$\left[\tilde{P}_{(x')k.}^{\ .m'}\right]_{x'} \equiv \tilde{P}_{(x')k.}^{\ .m'}\left(x',x'\right) = \delta_{k'}^{m'}. \tag{7.23}$$

In the same way one can obtain the parallel transport tensor $\tilde{P}_{(x)k'.}^{\ .m}$ in the Euclidean space $\tilde{E}_x$

$$\tilde{P}_{(x)k'.}^{\ .m} \equiv \Sigma_{k'q}\left[\Sigma^{mq'}\right]_{x} \equiv \Sigma_{k'q}\left(x,x'\right)\Sigma^{mq'}\left(x,x\right) \tag{7.24}$$

describing a parallel transport from the point x to the point x' in $\tilde{E}_x$.

In the same way one can obtain the parallel transport tensors $P_{(x')k.}^{\ .m'}$ and $P_{(x)k'.}^{\ .m}$ respectively in Euclidean spaces $E_{x'}$ and E_x

$$P_{(x')k.}^{\ .m'} \equiv G_{kq'}\left[G^{mq'}\right]_{x'}, \qquad P_{(x)k'.}^{\ .m} \equiv G_{k'q}\left[G^{mq'}\right]_{x} \tag{7.25}$$

Thus, the world function Σ of the Σ-space $V = \{\Sigma, \mathcal{M}_{n+1}\}$ and its symmetric component G determine Euclidean spaces $\tilde{E}_{x'}$, $\tilde{E}_x$, $E_{x'}$, E_x, mappings of V on them and the parallel transport of vectors and tensors in these Euclidean spaces independently of that, whether or not the Σ-space $V = \{\Sigma, \mathcal{M}_{n+1}\}$ is degenerate in the sense of definitions 4.6 – 4.8.

8 Derivatives of the world function at coincidence of points x and x'.

Let the world function be represented in the form (2.7), (2.8). Let us expand the function G and A with respect to powers of $\xi^i = x^i - x'^i$. Taking into account the symmetry relations (2.8), one obtains

$$\begin{aligned} G(x,x') &= \frac{1}{2}g_{ik}(x')\xi^i\xi^k + \frac{1}{6}g_{ikl}(x')\xi^i\xi^k\xi^l + \frac{1}{24}g_{iklm}(x')\xi^i\xi^k\xi^l\xi^m + ... && (8.1)\\ &= \frac{1}{2}g_{ik}(x)\xi^i\xi^k - \frac{1}{6}g_{ikl}(x)\xi^i\xi^k\xi^l + \frac{1}{24}g_{iklm}(x)\xi^i\xi^k\xi^l\xi^m + ... && (8.2)\end{aligned}$$

$$\begin{aligned} A(x,x') &= a_i(x')\xi^i + \frac{1}{2}a_{ik}(x')\xi^i\xi^k + \frac{1}{6}a_{ikl}(x')\xi^i\xi^k\xi^l + ... && (8.3)\\ &= a_i(x)\xi^i - \frac{1}{2}a_{ik}(x)\xi^i\xi^k + \frac{1}{6}a_{ikl}(x)\xi^i\xi^k\xi^l - ... && (8.4)\end{aligned}$$

In relations (8.1) and (8.3) the functions $G(x,x')$ and $A(x,x')$ are expanded at the point x'. In the relations (8.2) and (8.4) one has the same expansions after transposition $x \leftrightarrow x'$.

Differentiating relations (8.1) - (8.4) with respect to x and x' and setting $x = x'$ thereafter, one obtains relations between the expansion coefficients and expressions for derivatives of functions Σ, G, A at the limit of coincidence $x = x'$.

After calculations one obtains

$$\begin{aligned} g_{ikl} &= \frac{1}{2}(g_{ik,l} + g_{li,k} + g_{kl,i}), \qquad g_{ik,l} \equiv g_{ik,l}(x) \equiv \frac{\partial}{\partial x^l}g_{ik}(x) && (8.5)\\ a_{ik} &= \frac{1}{2}(a_{i,k} + a_{k,i}), \qquad a_{i,k} \equiv a_{i,k}(x) \equiv \frac{\partial}{\partial x^k}a_i(x) && (8.6)\end{aligned}$$

$$a_{iklm} = \frac{1}{2}(a_{ikl,m} + a_{klm,i} + a_{lmi,k} + a_{mik,l} - a_{ik,lm} - a_{lm,ik})$$

Coefficients g_{ik}, a_i, a_{ikl} are arbitrary and symmetric with respect to transposition of indices. Using square brackets for designation of coincidence $x = x'$ and relations (8.5), (8.6), one obtains

$$\begin{aligned} [G_{,i}(x,x')] &\equiv [G_{,i}] = [G_{,i'}] = 0,\\ [G_{,ik}(x,x')] &\equiv [G_{,ik}] = [G_{,i'k'}] = g_{ik}, && (8.7)\\ [G_{,i'k}(x,x')] &\equiv [G_{,i'k}] = -g_{ik}\end{aligned}$$

$$\begin{aligned} [G_{,ikl}(x,x')] &\equiv [G_{,ikl}] = [G_{,i'k'l'}] = \frac{1}{2}(g_{ik,l} + g_{li,k} + g_{kl,i})\\ [G_{,ikl'}(x,x')] &\equiv [G_{,ikl'}] = \frac{1}{2}(g_{ik,l} - g_{li,k} - g_{kl,i}) && (8.8)\\ [G_{,ik'l'}(x,x')] &\equiv [G_{,ik'l'}] = \frac{1}{2}(g_{kl,i} - g_{li,k} - g_{ik,l})\end{aligned}$$

$$\begin{aligned} [G_{,iklm}(x,x')] &\equiv [G_{,iklm}] = [G_{,i'k'l'm'}] = g_{iklm}\\ [G_{,iklm'}(x,x')] &\equiv [G_{,iklm'}] = [G_{,i'k'l'm}] = -g_{iklm} + g_{ikl,m} && (8.9)\\ [G_{,ikl'm'}(x,x')] &\equiv [G_{,ikl'm'}] = g_{iklm} - g_{ikl,m} - g_{ikm,l} + g_{ik,ml}\end{aligned}$$

$$[A_{,i}(x,x')] \equiv [A_{,i}] = a_i, \qquad [A_{,i'}(x,x')] \equiv [A_{,i'}] = -a_i \tag{8.10}$$

$$\begin{aligned}
[A_{,ik}(x,x')] &\equiv [A_{,ik}] = a_{ik} = \frac{1}{2}(a_{i,k} + a_{k,i}) \\
[A_{,ik'}(x,x')] &\equiv [A_{,ik'}] = \frac{1}{2}(a_{i,k} - a_{k,i}) \\
[A_{,i'k'}(x,x')] &\equiv [A_{,i'k'}] = -a_{ik} = -\frac{1}{2}(a_{i,k} + a_{k,i})
\end{aligned} \tag{8.11}$$

$$\begin{aligned}
[A_{,ikl}(x,x')] &\equiv [A_{,ikl}] = a_{ikl} \\
[A_{,ikl'}(x,x')] &\equiv [A_{,ikl'}] = \frac{1}{2}a_{i,kl} + \frac{1}{2}a_{k,il} - a_{ikl} \\
[A_{,ik'l'}(x,x')] &\equiv [A_{,ik'l'}] = -\frac{1}{2}a_{l,ik} - \frac{1}{2}a_{k,il} + a_{ikl}
\end{aligned} \tag{8.12}$$

$$\begin{aligned}
[A_{,iklm}(x,x')] &\equiv [A_{,iklm}] = a_{iklm} \\
[A_{,iklm'}(x,x')] &\equiv [A_{,iklm'}] = -a_{iklm} + a_{ikl,m} \\
[A_{,ikl'm'}(x,x')] &\equiv [A_{,ikl'm'}] = a_{iklm} - a_{ikl,m} - a_{ikm,l} + a_{ik,lm} \\
[A_{,ik'l'm'}(x,x')] &\equiv [A_{,ik'l'm'}] = -a_{iklm} - a_{klm,i} \\
[A_{,i'k'l'm'}(x,x')] &\equiv [A_{,i'k'l'm'}] = -a_{iklm}
\end{aligned} \tag{8.13}$$

The first order coefficient $a_k(x)$ is a covariant vector at the point x. The second order coefficients $g_{ik}(x)$ is the second rank covariant tensors at the point x. The second order coefficient $a_{ik}(x)$ and the third order coefficients $g_{ikl}(x)$, $a_{ikl}(x)$ are not tensors, in general. The law of their transformation at the coordinate transformation is more complicated.

According to (2.7), (8.7) and (8.10)

$$[\Sigma_{,i}] = a_i(x), \qquad [\Sigma_{,i'}] = -a_i(x) \tag{8.14}$$

According to (2.7), (8.7) and (8.11)

$$\begin{aligned}
[\Sigma_{,ik}] &= \sigma_{(\mathrm{f})ik} = g_{ik} + \frac{1}{2}(a_{i,k} + a_{k,i}) \\
[\Sigma_{,i'k'}] &= \sigma_{(\mathrm{p})ik} = g_{ik} - \frac{1}{2}(a_{i,k} + a_{k,i}) \\
[\Sigma_{ik'}] &\equiv [\Sigma_{,ik'}] = -\tilde{g}_{ik} = -g_{ik} + \frac{1}{2}(a_{i,k} - a_{k,i})
\end{aligned} \tag{8.15}$$

One obtains for the value $\left[\Sigma^{ik'}\right]$ of the quantity $\Sigma^{ik'}$

$$\left[\Sigma^{ik'}\right] = -\tilde{g}^{ik}, \tag{8.16}$$

where $\tilde{g}^{ik}$ is determined by the relation

$$\tilde{g}^{il}\tilde{g}_{ik} = \tilde{g}^{il}\left(g_{ik} - \frac{1}{2}(a_{i,k} - a_{k,i})\right) = \delta_k^l \tag{8.17}$$

The quantity a_i is a one-point vector, and g_{ik} is a one-point tensor. Then it follows from (8.15), (8.17), that $\tilde{g}_{ik}$ and $\tilde{g}^{ik}$ are also a one-point tensors, whereas $\sigma_{(\mathrm{f})ik}$ and $\sigma_{(\mathrm{p})ik}$ are not tensors, in general.

One obtains for the quantities $[\Gamma^i_{kl}]$, $[\Gamma^{i'}_{k'l'}]$ and $\left[\tilde{\Gamma}^i_{kl}\right]$, $\left[\tilde{\Gamma}^{i'}_{k'l'}\right]$ the following expressions

$$\left[\Gamma^i_{kl}\right] = \left[\Gamma^{i'}_{k'l'}\right] = \gamma^i_{kl}(x) = \frac{1}{2}g^{si}\left(g_{ks,l} + g_{sl,k} - g_{lk,s}\right), \qquad g^{ik}g_{lk} = \delta^i_l \tag{8.18}$$

$$\left[\tilde{\Gamma}^i_{kl}\right] = \tilde{\gamma}_{(\mathrm{f})}{}^i_{kl} = \tilde{g}^{is}g_{ps}\left(\gamma^p_{kl} + \beta^p_{kl}\right), \tag{8.19}$$

$$\left[\tilde{\Gamma}^{i'}_{k'l'}\right] = \tilde{\gamma}_{(\mathrm{p})}{}^i_{kl} = \tilde{g}^{si}g_{ps}\left(\gamma^p_{kl} - \beta^p_{kl}\right) \tag{8.20}$$

where

$$\gamma^i_{kl}(x) = \frac{1}{2}g^{si}\left(g_{ks,l} + g_{sl,k} - g_{lk,s}\right), \qquad g^{ik}g_{lk} = \delta^i_l \tag{8.21}$$

$$\beta^i_{kl}(x) = g^{si}\left(-\frac{1}{2}\left(a_{k,ls} + a_{l,ks}\right) + a_{kls}\right) \tag{8.22}$$

Here $\gamma^i_{kl}(x)$ is the Christoffel symbol for the symmetric case, when $A(x,x') \equiv 0$.

Note that the tensors $\tilde{g}^{ik}$, $\tilde{g}_{ik}$ are not symmetric with respect to transposition indices, in general, whereas $\tilde{\gamma}_{(\mathrm{f})}{}^i_{kl} = \left[\tilde{\Gamma}^i_{kl}\right]$, $\tilde{\gamma}_{(\mathrm{p})}{}^i_{kl} = \left[\tilde{\Gamma}^{i'}_{k'l'}\right]$, γ^i_{kl} and β^i_{kl} are symmetric with respect to transposition of indices k and l. Besides it follows from (8.19), (8.20), that

$$\beta^i_{kl}(x) = g^{si}\left(-\frac{1}{2}\left(a_{k,ls} + a_{l,ks}\right) + a_{kls}\right) = \frac{1}{2}g^{ip}\left(\tilde{g}_{sp}\tilde{\gamma}_{(\mathrm{f})}{}^s_{kl} - \tilde{g}_{ps}\tilde{\gamma}_{(\mathrm{p})}{}^s_{kl}\right) \tag{8.23}$$

In the case, when $a_i \equiv 0$ and tensor $\tilde{g}_{ik}$ is symmetric, the quantity β^i_{kl} is one-point tensor because difference of two Christoffel symbols $\tilde{\gamma}_{(\mathrm{f})}{}^s_{kl} - \tilde{\gamma}_{(\mathrm{p})}{}^s_{kl}$ is a tensor.

9 Curvature tensors

In the Riemannian geometry the Riemann-Christoffel curvature tensor $\tilde{r}_{(\mathrm{q})}{}^{\cdot l}_{s.ik}$ is defined as a commutator of covariant derivatives $\tilde{D}_{(\mathrm{q})i}$ with the Christoffel symbol $\tilde{\gamma}_{(\mathrm{q})}{}^i_{kl}$

$$\left(\tilde{D}_{(\mathrm{q})i}\tilde{D}_{(\mathrm{q})k} - \tilde{D}_{(\mathrm{q})k}\tilde{D}_{(\mathrm{q})i}\right)t_s = \tilde{r}_{(\mathrm{q})}{}^{\cdot l}_{s.ik}t_l$$

where $\tilde{D}_{(\mathrm{q})i}$ is the usual covariant derivative in the Riemannian space with the Christoffel symbol $\tilde{\gamma}_{(\mathrm{q})}{}^l_{si}$,

$$\tilde{r}_{(\mathrm{q})}{}^{\cdot l}_{s.ik} = \tilde{\gamma}_{(\mathrm{q})}{}^l_{si,k} - \tilde{\gamma}_{(\mathrm{q})}{}^l_{sk,i} + \tilde{\gamma}_{(\mathrm{q})}{}^p_{si}\tilde{\gamma}_{(\mathrm{q})}{}^l_{pk} - \tilde{\gamma}_{(\mathrm{q})}{}^p_{sk}\tilde{\gamma}_{(\mathrm{q})}{}^l_{pi} \tag{9.1}$$

and t_l is an arbitrary vector at the point x. Index q runs the values p and f.

In the Σ-space one can consider commutator of covariant derivatives $\tilde{\nabla}_i^{x'}$ and $\tilde{\nabla}_{k'}^{x}$ with respect to x^i and x'^k respectively. Calculation gives

$$\begin{aligned}
&\left(\tilde{\nabla}_i^{x'}\tilde{\nabla}_{s'}^{x}-\tilde{\nabla}_{s'}^{x}\tilde{\nabla}_i^{x'}\right)T_{k\ldots l'\ldots}^{j\ldots m'\ldots}\\
=\;&\tilde{F}_{ika's'}\Sigma^{ba'}T_{b\ldots l'\ldots}^{j\ldots m'\ldots}+\ldots-\Sigma^{ja'}\tilde{F}_{iba's'}T_{k\ldots l'\ldots}^{b\ldots m'\ldots}\\
&-\ldots+\Sigma^{am'}\tilde{F}_{iab's'}T_{k\ldots l'\ldots}^{j\ldots b'\ldots}+\ldots-\tilde{F}_{ial's'}\Sigma^{ab'}T_{k\ldots b'\ldots}^{j\ldots m'\ldots}-\ldots
\end{aligned}\tag{9.2}$$

where $T_{k\ldots l'\ldots}^{j\ldots m'\ldots}$ is an arbitrary two-point tensor. $\tilde{F}$-tensor, defined by the relation

$$\tilde{F}_{ilk'j'}\equiv\Sigma_{iq'}\tilde{\Gamma}_{k'j'||l}^{q'}=\Sigma_{pj'}\tilde{\Gamma}_{il||k'}^{p}=\Sigma_{,ilj'||k'}=\Sigma_{,ilj'k'}-\Sigma_{,sj'k'}\Sigma^{sm'}\Sigma_{,ilm'}\tag{9.3}$$

is a two-point analog of the one-point curvature tensor $r_{slik}=g_{lp}r_{s.ik}^{.p}$. To test that the quantity (9.3) is a tensor, let us represent it in one of two forms

$$\tilde{F}_{ilk'j'}\equiv\Sigma_{pj'}\tilde{\Gamma}_{il||k'}^{p}=\Sigma_{pj'}\left(\tilde{\Gamma}_{il}^{p}-\tilde{\gamma}_{(\mathrm{f})}{}_{il}^{p}\right)_{||k'}\tag{9.4}$$

$$\tilde{F}_{ilk'j'}\equiv\Sigma_{iq'}\tilde{\Gamma}_{k'j'||l}^{q'}=\Sigma_{iq'}\left(\tilde{\Gamma}_{k'j'}^{q'}-\tilde{\gamma}_{(\mathrm{p})}{}_{k'j'}^{q'}\right)_{||l}\tag{9.5}$$

As far as the difference

$$\tilde{Q}_{(\mathrm{f})}{}_{il}^{p}=\tilde{\gamma}_{(\mathrm{f})}{}_{il}^{p}-\tilde{\Gamma}_{il}^{p}\qquad\tilde{Q}_{(\mathrm{p})}{}_{k'j'}^{q'}=\tilde{\gamma}_{(\mathrm{p})}{}_{k'j'}^{q'}-\tilde{\Gamma}_{k'j'}^{q'}\tag{9.6}$$

of two Christoffel symbols is a tensor, it follows from (9.4) and (9.6) that $\tilde{F}_{ilk'j'}$ is a tensor. $\tilde{F}$-tensor can be presented as a result of covariant differentiation of the Σ-function. Indeed

$$\begin{aligned}
\tilde{Q}_{(\mathrm{f})}{}_{il}^{s}&=-\Sigma^{sj'}(\Sigma_{,ilj'}-\tilde{\gamma}_{(\mathrm{f})}{}_{il}^{m}\Sigma_{mj'})=-\Sigma^{sj'}\tilde{D}_{(\mathrm{f})l}\Sigma_{ij'}=-\Sigma^{sj'}\tilde{D}_{(\mathrm{f})l}\tilde{D}_{(\mathrm{p})j'}\Sigma_i\\
&=-\Sigma^{sj'}\tilde{D}_{(\mathrm{p})j'}\tilde{D}_{(\mathrm{f})l}\Sigma_i=-\Sigma^{sj'}\left(\tilde{D}_{(\mathrm{f})l}\Sigma_i\right)_{||j'}=-\left(\Sigma^{sj'}\tilde{D}_{(\mathrm{f})l}\Sigma_i\right)_{||j'}
\end{aligned}\tag{9.7}$$

$$\begin{aligned}
\tilde{Q}_{(\mathrm{p})}{}_{k'j'}^{q'}&=-\Sigma^{sq'}\left(\Sigma_{,sk'j'}-\tilde{\gamma}_{(\mathrm{p})}{}_{k'j'}^{m'}\Sigma_{sm'}\right)=-\Sigma^{sq'}\tilde{D}_{(\mathrm{p})j'}\Sigma_{sk'}\\
&=-\Sigma^{sq'}\tilde{D}_{(\mathrm{p})j'}\tilde{D}_{(\mathrm{f})s}\Sigma_{k'}=-\Sigma^{sq'}\tilde{D}_{(\mathrm{f})s}\tilde{D}_{(\mathrm{p})j'}\Sigma_{k'}=-\left(\Sigma^{sq'}\tilde{D}_{(\mathrm{p})j'}\Sigma_{k'}\right)_{||s}
\end{aligned}\tag{9.8}$$

Then according to (9.4) – (9.8), one obtains

$$\tilde{F}_{ilk'j'}=\left(\tilde{D}_{(\mathrm{f})l}\Sigma_i\right)_{||j'||k'}=\left(\tilde{D}_{(\mathrm{p})j'}\Sigma_{k'}\right)_{||i||l}\tag{9.9}$$

The commutator of covariant derivatives $\nabla_i^{x'}$ and $\nabla_{k'}^{x}$, connected with the symmetric component G of the world function, has the property

$$T_{k\ldots l'\ldots|s'|i}^{j\ldots m'\ldots}-T_{k\ldots l'\ldots|i|s'}^{j\ldots m'\ldots}=F_{ika's'}G^{ba'}T_{b\ldots l'\ldots}^{j\ldots m'\ldots}+\ldots-G^{ja'}F_{iba's'}T_{k\ldots l'\ldots}^{b\ldots m'\ldots}$$

$$-\ldots+G^{am'}F_{iab's'}T^{j\ldots b'\ldots}_{k\ldots l'\ldots}+\ldots-F_{ial's'}G^{ab'}T^{j\ldots m'\ldots}_{k\ldots b'\ldots}-\ldots \tag{9.10}$$

where the curvature F-tensor has the form

$$F_{ilk'j'}\equiv G_{pj'}\Gamma^{p}_{il|k'}=G_{pj'}\left(\Gamma^{p}_{il}-\gamma^{p}_{il}\right)_{|k'}=G_{i;l|k'|j'} \tag{9.11}$$

Here (;) denotes the usual covariant derivative with the Christoffel symbol γ^{i}_{kl}, and the Q-tensor is written as follows

$$Q^{i}_{kl}=\gamma^{i}_{kl}-\Gamma^{i}_{kl}=-G^{sj'}G_{i;l|j'} \tag{9.12}$$

Let us discover a connection between the F-tensor at the coincidence limit $[F_{ilk'j'}]$ and the curvature tensor $r^{\cdot l}_{s.ik}$, constructed of the Christoffel symbols γ^{l}_{si} by means of formula (9.1)

$$r^{\cdot l}_{s.ik}=\gamma^{l}_{si,k}-\gamma^{l}_{sk,i}+\gamma^{p}_{si}\gamma^{l}_{pk}-\gamma^{p}_{sk}\gamma^{l}_{pi}. \tag{9.13}$$

Let us take into account that

$$\left[\left(\nabla^{x'}_{i}+\nabla^{x}_{i'}\right)T_{sp'}\left(x,x'\right)\right]_{x}=D_{i}t_{sp}=t_{sp;i},\qquad t_{sp}\equiv\left[T_{sp'}\left(x,x'\right)\right]_{x} \tag{9.14}$$

Then, using (9.10), (9.11), one obtains

$$\begin{aligned}&\left[\left(\nabla^{x'}_{i}+\nabla^{x}_{i'}\right)\left(\nabla^{x'}_{k}+\nabla^{x}_{k'}\right)T_{s}\left(x,x'\right)-\left(\nabla^{x'}_{k}+\nabla^{x}_{k'}\right)\left(\nabla^{x'}_{i}+\nabla^{x}_{i'}\right)T_{s}\left(x,x'\right)\right]_{x}\\ =\ &\left(D_{i}D_{k}-D_{k}D_{i}\right)t_{s}=\left[\Gamma^{a}_{is|k'}-\Gamma^{a}_{ks|i'}\right]_{x}t_{a}\end{aligned} \tag{9.15}$$

where D_i is the usual covariant derivative in the Riemannian space with the Christoffel symbol $\gamma^{i}_{kl}=[\Gamma^{i}_{kl}]_{x}$

On the other hand, the relation

$$\left(D_{i}D_{k}-D_{k}D_{i}\right)t_{s}=r^{\cdot l}_{s.ik}t_{l} \tag{9.16}$$

takes place. Comparison of relations (9.15) and (9.16) gives

$$r^{\cdot l}_{s.ik}=\left[\Gamma^{l}_{is|k'}-\Gamma^{l}_{ks|i'}\right]_{x}=\left[\Gamma^{l}_{is,k'}-\Gamma^{l}_{ks,i'}\right]_{x}=-g^{lp}f_{ispk}+g^{lp}f_{kspi} \tag{9.17}$$

where

$$f_{ispk}\equiv\left[F_{isp'k'}\right]_{x}=\left[G_{lp'}\Gamma^{l}_{is|k'}\right]_{x}=-g_{lp}\left[\Gamma^{l}_{is|k'}\right]_{x} \tag{9.18}$$

According to (9.11) the one-point tensor f_{ispk} is symmetric with respect to transposition indices $i\leftrightarrow s$ and $p\leftrightarrow k$ separately.

$$f_{ispk}=f_{sipk},\qquad f_{ispk}=f_{iskp} \tag{9.19}$$

Equation (9.17) can be written in the form

$$g_{lp}r^{\cdot l}_{s.ik}=-f_{ispk}+f_{kspi} \tag{9.20}$$

The metric tensor g_{ik} is symmetric, and f_{ispk} has the following symmetry properties

$$f_{ispk}=f_{sipk},\qquad f_{ispk}=f_{iskp},\qquad f_{ispk}=f_{pkis} \tag{9.21}$$

One can obtain connection of the type (9.17), (9.18) between the $\tilde{F}$-tensor and the Riemannian -Christoffel curvature tensor in the case of nonsymmetric T-geometry. Taking into account (9.4), evident identity

$$\frac{\partial}{\partial x^k}\left[\tilde{\Gamma}_{il}^{p}\right]_x \equiv \left[\tilde{\Gamma}_{il,k}^{p}\right]_x + \left[\tilde{\Gamma}_{il,k'}^{p}\right]_x \tag{9.22}$$

and using relations (8.15), (8.19), one obtains

$$\left[\tilde{F}_{ilk'j'}\right]_x = -\tilde{g}_{pj}\left[\tilde{\Gamma}_{il,k'}^{p}\right]_x = -\tilde{g}_{pj}\left(\tilde{\gamma}_{(\mathrm{f})}{}_{il,k}^{p} - \left[\tilde{\Gamma}_{il,k}^{p}\right]_x\right) \tag{9.23}$$

Let us take into account identity

$$\Sigma^{pr'}\Sigma_{,sr'k} + \left(\Sigma^{pr'}\right)_{,k}\Sigma_{sr'} = 0 \tag{9.24}$$

obtained by differentiation of (7.2). Then using relations (8.15), (8.19), one obtains from (9.23)

$$\begin{aligned}\left[\tilde{F}_{ilk'j'}\right]_x &= -\tilde{g}_{pj}\tilde{\gamma}_{(\mathrm{f})}{}_{il,k}^{p} + \tilde{g}_{pj}\left[\Sigma^{pq'}\Sigma_{,ilkq'} - \Sigma^{pr'}\Sigma_{,sr'k}\Sigma^{sq'}\Sigma_{,ilq'}\right]_x \\ &= -\tilde{g}_{pj}\tilde{\gamma}_{(\mathrm{f})}{}_{il,k}^{p} + \tilde{g}_{pj}\left[-\tilde{g}^{pq}\Sigma_{,ilkq'} - \tilde{\Gamma}_{sk}^{p}\tilde{\Gamma}_{il}^{s}\right]_x\end{aligned}$$

$$\tilde{f}_{ilkj} \equiv \left[\tilde{F}_{ilk'j'}\right]_x = -\left[\Sigma_{,ilkj'}\right]_x - \tilde{g}_{pj}\left(\tilde{\gamma}_{(\mathrm{f})}{}_{il,k}^{p} + \tilde{\gamma}_{(\mathrm{f})}{}_{il}^{s}\tilde{\gamma}_{(\mathrm{f})}{}_{sk}^{p}\right) \tag{9.25}$$

Alternating with respect to indices k, l, one obtains

$$\tilde{f}_{ilkj} - \tilde{f}_{iklj} = \tilde{g}_{pj}\tilde{r}_{(\mathrm{f})}{}_{i.kl}^{\cdot p} \tag{9.26}$$

where $\tilde{r}_{(\mathrm{f})}{}_{i.kl}^{\cdot s}$ is the Riemann-Christoffel curvature tensor, constructed on the base of the Christoffel symbol $\tilde{\gamma}_{(\mathrm{f})}{}_{ik}^{s}$

$$\tilde{r}_{(\mathrm{f})}{}_{i.kl}^{\cdot p} = \tilde{\gamma}_{(\mathrm{f})}{}_{ik,l}^{p} - \tilde{\gamma}_{(\mathrm{f})}{}_{il,k}^{p} + \tilde{\gamma}_{(\mathrm{f})}{}_{ik}^{s}\tilde{\gamma}_{(\mathrm{f})}{}_{sl}^{p} - \tilde{\gamma}_{(\mathrm{f})}{}_{il}^{p}\tilde{\gamma}_{(\mathrm{f})}{}_{sk}^{k} \tag{9.27}$$

In the same way one can express $\tilde{f}_{ilkj} - \tilde{f}_{iklj}$ via the Riemann-Christoffel curvature tensor $\tilde{r}_{(\mathrm{p})}{}_{i.kl}^{\cdot s}$, constructed on the base of the Christoffel symbol $\tilde{\gamma}_{(\mathrm{p})}{}_{ik}^{s}$

$$\tilde{f}_{ilkj} - \tilde{f}_{iklj} = \tilde{g}_{lp}\tilde{r}_{(\mathrm{p})}{}_{k.ij}^{\cdot p} \tag{9.28}$$

where

$$\tilde{r}_{(\mathrm{p})}{}_{i.kl}^{\cdot p} = \tilde{\gamma}_{(\mathrm{p})}{}_{ik,l}^{p} - \tilde{\gamma}_{(\mathrm{p})}{}_{il,k}^{p} + \tilde{\gamma}_{(\mathrm{p})}{}_{ik}^{s}\tilde{\gamma}_{(\mathrm{p})}{}_{sl}^{p} - \tilde{\gamma}_{(\mathrm{p})}{}_{il}^{p}\tilde{\gamma}_{(\mathrm{p})}{}_{sk}^{k} \tag{9.29}$$

To obtain representation (9.28), let us use another representation (9.5)

$$\tilde{F}_{ilk'j'} = \Sigma_{iq'}\tilde{\Gamma}_{k'j',l}^{q'}$$

of the $\tilde{F}$-tensor, which differs from the representation (9.4) by a change $x \leftrightarrow x'$. Producing the same operations (9.23) – (9.25), one obtains (9.28) instead of (9.26).

Note that relations (9.26) and (9.28) are different, because the tensor $\tilde{g}_{ik}$ is not symmetric. In (9.26) summation is produced over the first index, whereas in (9.28) it is produced over the second index. In the symmetric T-geometry, when $\tilde{g}_{ik}$ is symmetric, three expressions (9.17) (9.26) and (9.28) coincide.

There are two essentially different cases of asymmetric T-geometry:

1. Rough antisymmetry, when the field $a_i(x) \neq 0$. In this case the field $a_i(x)$ dominates at small distances $x - x'$, and the world function is determined by the linear form

$$\Sigma(x, x') = a_i(x)\left(x^i - x'^i\right) + ...$$

In this case the antisymmetry is the main phenomenon at small distances.

2. Fine antisymmetry, when the field $a_i(x) \equiv 0$. In this case the antisymmetric effects are described by the field a_{ikl}. At small distances $x - x'$ the symmetric structure dominates, and the world function is determined by the quadratic form

$$\Sigma(x, x') = \frac{1}{2} g_{ik}(x)\left(x^i - x'^i\right)\left(x^k - x'^k\right) + ...$$

as in the symmetric T-geometry. In this case the antisymmetric effects may be considered as corrections to gravitational effects. This corrections may be essential at large distances $\xi^i = x^i - x'^i$, when the form $\frac{1}{6} a_{ikl}\xi^i\xi^k\xi^l$ becomes of the same order as the form $\frac{1}{2} g_{ik}\xi^i\xi^k$,

The asymmetric T-geometry with fine antisymmetry is simpler, because it is rather close to the usual symmetric T-geometry.

10 Gradient lines on the manifold in the case of fine antisymmetry $a_i \equiv 0$

Let us consider a one-dimensional line $\mathcal{L}_{(\mathrm{f})}$, passing through points x' and x''. This line is defined by the relations

$$\mathcal{L}_{(\mathrm{f})}: \quad \Sigma_{,i'}(x, x') = \tau \Sigma_{,i'}(x'', x') = \tau b_{i'}, \qquad i = 0, 1, ...n \tag{10.1}$$

Let us suppose that $\det ||\Sigma_{,i'k}(x, x')|| \neq 0$. Then $n+1$ equations (10.1) can be resolved with respect to x in the form

$$\mathcal{L}_{(\mathrm{f})}: \quad x^i = x^i(\tau), \qquad i = 0, 1, ...n \tag{10.2}$$

where τ is a parameter along the line $\mathcal{L}_{(\mathrm{f})}$. As it follows from (10.1), this line passes through the point x' at $\tau = 0$ and through the point x'' at $\tau = 1$. Such a line will be referred to as gradient line (curve) from the future. Let us derive differential equation for the gradient curve $\mathcal{L}_{(\mathrm{f})}$.

Differentiating (10.1) with respect to τ, one obtains

$$\Sigma_{,ki'}(x, x')\frac{dx^k}{d\tau} = \Sigma_{,i'}(x'', x') = b_{i'}, \qquad i = 0, 1, ...n \tag{10.3}$$

Differentiating once more, one obtains

$$\Sigma_{,ki'}\left(x,x'\right)\frac{d^2x^k}{d\tau^2}+\Sigma_{,kli'}\left(x,x'\right)\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}=0,\qquad i=0,1,...n \tag{10.4}$$

Using relation (7.3), one can write equations (10.4) in the form

$$\frac{d^2x^i}{d\tau^2}+\tilde{\Gamma}_{kl}^i\left(x,x'\right)\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}=0,\qquad i=0,1,...n \tag{10.5}$$

The equation (10.5) may be interpreted as an equation for a geodesic in some $(n+1)$-dimensional Euclidean space with the Christoffel symbol $\tilde{\Gamma}_{kl}^i\left(x,x'\right)$. This geodesic passes through the points x' and x''.

Let the points x' and x'' be infinitesimally close. Then equation (10.5) can be written in the form

$$\mathcal{L}_{(\mathrm{f})}:\quad \frac{d^2x^i}{d\tau^2}+\tilde{\gamma}_{(\mathrm{f})kl}^{\ i}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}=0,\qquad i=0,1,...n \tag{10.6}$$

where $\tilde{\gamma}_{(\mathrm{f})kl}^{\ i}=\tilde{\gamma}_{(\mathrm{f})kl}^{\ i}\left(x\right)=\left[\tilde{\Gamma}_{kl}^i\right]_x$. Dividing the gradient line $\mathcal{L}_{(\mathrm{f})}$ into infinitesimal segments and writing equations (10.5) in the form (10.6) on each segment, one obtains that the gradient line $\mathcal{L}_{(\mathrm{f})}$ is described by the equations (10.6) everywhere.

The equation (10.6) does not contain a reference to the point x', and any gradient line (10.1), (10.2) is to satisfy this equation.

In the case of fine antisymmetry, when $a_i\equiv 0$, the equation (10.6) can be written in other form. Using relations (8.19), and taking into account that $a_i\equiv 0$, one obtains instead of (10.6)

$$\mathcal{L}_{(\mathrm{f})}:\qquad \frac{d^2x^i}{d\tau^2}+\left(\gamma_{kl}^i+\beta_{kl}^i\right)\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}=0,\qquad a_i\equiv 0 \tag{10.7}$$

where

$$\gamma_{kl}^i=\gamma_{kl}^i\left(x\right)=\frac{1}{2}g^{si}\left(g_{ks,l}+g_{sl,k}-g_{lk,s}\right), \tag{10.8}$$

$$\beta_{kl}^i=\beta_{kl}^i\left(x\right)=g^{si}a_{kls} \tag{10.9}$$

If $a_{ikl}=0$, the equations (10.7) may be considered to be the equations for a geodesic in a Riemannian space with the metric tensor g_{ik}.

In the case of rough antisymmetry, when $a_i\neq 0$, equations (10.1), (10.5) also describe a gradient line, but equation (10.6) is not equivalent to (10.5), because the point x' does not belong to $\mathcal{L}_{(\mathrm{f})}$, in general. In this case one cannot choose the points $x'$and x'' infinitesimally close and pass from equation (10.5) to (10.6). Thus, in the case of rough antisymmetry the equation (10.6) does not describe a gradient line, in general.

Now let us consider another type of gradient line $\mathcal{L}_{(\mathrm{p})}$, passing through the points x and x''. Let the gradient line $\mathcal{L}_{(\mathrm{p})}$ be described by the equations (It is supposed again that $a_i\equiv 0$).

$$\mathcal{L}_{(\mathrm{p})}:\qquad \Sigma_{,i}\left(x,x'\right)=\tau\Sigma_{,i}\left(x,x''\right)=\tau b_i,\qquad i=0,1,...n \tag{10.10}$$

which determine

$$\mathcal{L}_{(\mathrm{p})}: \qquad x'^i = x'^i(\tau), \qquad i = 0, 1, ...n \tag{10.11}$$

Equation (10.10) distinguishes from the equation (10.1) only in transposition of the first and second arguments of the world function $\Sigma(x, x')$. The gradient line $\mathcal{L}_{(\mathrm{p})}$, determined by the relation (10.10), may be referred to as the gradient line from the past. Manipulating with the equation (10.10) in the same way as with (10.1), one obtains instead of (10.6)

$$\mathcal{L}_{(\mathrm{p})}: \qquad \frac{d^2x^i}{d\tau^2} + \tilde{\gamma}_{(\mathrm{p})kl}^{\ i} \frac{dx^k}{d\tau}\frac{dx^l}{d\tau} = 0, \qquad i = 0, 1, ...n \tag{10.12}$$

In the case of fine antisymmetry, when $a_i \equiv 0$, the equation (10.12) can be written in other form. Using relation (8.20), and taking into account that $a_i \equiv 0$, one obtains instead of (10.12)

$$\mathcal{L}_{(\mathrm{p})}: \qquad \frac{d^2x^i}{d\tau^2} + \left(\gamma_{kl}^i - \beta_{kl}^i\right) \frac{dx^k}{d\tau}\frac{dx^l}{d\tau} = 0, \qquad a_i \equiv 0 \tag{10.13}$$

where γ_{kl}^i and β_{kl}^i are defined by the relations (10.8), (10.9).

In the case of symmetric T-geometry, when $a_{ikl} = 0$ and $\beta_{kl}^i = 0$, differential equations (10.7) and (10.13) respectively for gradient line $\mathcal{L}_{(\mathrm{f})}$ and for gradient line $\mathcal{L}_{(\mathrm{p})}$ coincide.

In the case of asymmetric T-geometry the quantities $[\Gamma_{kl}^i]_x$ and $\left[\Gamma_{k'l'}^{i'}\right]_x$ do not coincide, in general. In this case the equations (10.6) and (10.13) determine, in general, different gradient curves, passing through the same points x' and x''. Differential equations (10.6) and (10.13) for the gradient curves $\mathcal{L}_{(\mathrm{p})}$ and $\mathcal{L}_{(\mathrm{f})}$ differ in the sign of the "antisymmetric force"

$$\beta_{kl}^i \frac{dx^k}{d\tau}\frac{dx^l}{d\tau} = g^{si}\left(-\frac{1}{2}a_{k,ls} - \frac{1}{2}a_{l,ks} + a_{kls}\right)\frac{dx^k}{d\tau}\frac{dx^l}{d\tau} \tag{10.14}$$

Finally, one can introduce the neutral gradient line $\mathcal{L}_{(\mathrm{n})}$, defining it by the relations

$$\mathcal{L}_{(\mathrm{n})}: \quad G_{,i'}(x, x') = \tau G_{,i'}(x'', x') = \tau b_{i'}, \qquad i = 0, 1, ...n \tag{10.15}$$

which determine

$$\mathcal{L}_{(\mathrm{n})}: \quad x^i = x^i(\tau), \qquad i = 0, 1, ...n \tag{10.16}$$

Equation (10.15) distinguishes from the equation (10.1) only in replacement of the world function $\Sigma(x, x')$ by its symmetric component $G(x, x')$. Manipulating with the equation (10.15) in the same way as with (10.1), one obtains instead of (10.7)

$$\mathcal{L}_{(\mathrm{n})}: \qquad \frac{d^2x^i}{d\tau^2} + \gamma_{kl}^i \frac{dx^k}{d\tau}\frac{dx^l}{d\tau} = 0 \tag{10.17}$$

where the "antisymmetric force" is absent.

The gradient lines (10.1) and (10.10) are insensitive with respect to transformation of the world function of the form

$$\Sigma \rightarrow \tilde{\Sigma} = f(\Sigma), \qquad |f'(\Sigma)| > 0 \tag{10.18}$$

where f is an arbitrary function, because for determination of the gradient line only direction of the gradient Σ_i or $\Sigma_{i'}$ is important, but not its module. Indeed, after substitution of $\tilde{\Sigma}$ from (10.18) in (10.1) one obtains the equation

$$\Sigma_{,i'}(x, x') = \tau' \Sigma_{,i'}(x'', x'), \qquad i = 0, 1, ...n, \qquad \tau' = \tau \frac{f'(\Sigma(x'', x'))}{f'(\Sigma(x, x'))} \tag{10.19}$$

which describes the same gradient line, but with another parametrization.

11 Conditions of degeneracy of the neutral first order tube

In general, the asymmetric T-geometry is nondegenerate geometry, even if it is given on n-dimensional manifold $\mathcal{M}_n$. In general case the first order tube $\mathcal{T}_{(\mathrm{n})\mathbf{x}'\mathbf{x}''}$, passing through points x' and x'', does not coincide with gradient line $\mathcal{L}_{(\mathrm{f})}$ or $\mathcal{L}_{(\mathrm{p})}$, passing through the points x' and x'' and defined by the relations (10.1) and (10.10) respectively.

Let us investigate, under what conditions the neutral first order tube $\mathcal{T}_{(\mathrm{n})\mathbf{x}'\mathbf{x}''}$ degenerates into gradient line $\mathcal{L}_{(\mathrm{f})}$ or $\mathcal{L}_{(\mathrm{p})}$. The first order tube $\mathcal{T}_{(\mathrm{n})\mathbf{P}_0\mathbf{P}_1}$, passing through the points $P_0 = \{x\}$, $P_1 = \{x'\}$ is defined by the relation

$$F_2(P_0, P_1, R) = \begin{vmatrix} \left(\overrightarrow{P_0P_1}.\overrightarrow{P_0P_1}\right) & \left(\overrightarrow{P_0P_1}.\overrightarrow{P_0R}\right) \\ \left(\overrightarrow{P_0R}.\overrightarrow{P_0P_1}\right) & \left(\overrightarrow{P_0R}.\overrightarrow{P_0R}\right) \end{vmatrix} = 0 \tag{11.1}$$

where $R = \{x' + dx'\}$ is a running point on the tube $\mathcal{T}_{P_0P_1}$.

One has the following expansion for the scalar Σ-products $\left(\overrightarrow{P_0P_1}.\overrightarrow{P_0R}\right)$ and $\left(\overrightarrow{P_0R}.\overrightarrow{P_0P_1}\right)$

$$\left(\overrightarrow{P_0P_1}.\overrightarrow{P_0R}\right) = \Sigma(P_1, P_0) + \Sigma(P_0, R) - \Sigma(P_1, R)$$

$$\begin{aligned} &= \Sigma(x', x) + \Sigma(x, x' + dx') - \Sigma(x', x' + dx') \\ &= 2G + \left(\Sigma_{,i'} - [\Sigma_{,i'}]_{x'}\right) dx'^{i'} + \frac{1}{2}\left(\Sigma_{,i'k'} - [\Sigma_{,i'k'}]_{x'}\right) dx'^{i'} dx'^{k'} \end{aligned} \tag{11.2}$$

$$\left(\overrightarrow{P_0R}.\overrightarrow{P_0P_1}\right) = \Sigma(P_0, P_1) + \Sigma(R, P_0) - \Sigma(R, P_1)$$

$$\begin{aligned} &= \Sigma(x, x') + \Sigma(x' + dx', x) - \Sigma(x' + dx', x') \\ &= 2G + \left(G_{,i'} - A_{,i'} - [\Sigma_{,i}]_{x'}\right) dx'^{i'} + \frac{1}{2}\left(G_{,i'k'} - A_{,i'k'} - [\Sigma_{,ik}]_{x'}\right) dx'^{i} dx'^{k} \end{aligned} \tag{11.3}$$

where unprimed indices are associated with the first argument and the primed ones with the second argument of the Σ-function.

$$\left(\overrightarrow{P_0P_1}.\overrightarrow{P_0P_1}\right) = 2G \tag{11.4}$$

$$\left(\overrightarrow{P_0R}.\overrightarrow{P_0R}\right) = 2G\left(x, x' + dx'\right) = 2G + 2G_{,i'}dx'^{i'} + G_{,i'k'}dx'^{i'}dx'^{k'} \tag{11.5}$$

Using relations (11.2) – (11.5) and (8.14), (8.15), one reduces equation (11.1) to the form

$$\left(\left(G_{,i'} - A_{,i'} - a_i\left(x'\right)\right)dx'^{i'}\right)\left(\left(G_{,l'} + A_{,l'} + a_l\left(x'\right)\right)dx'^{l'}\right) = 2Gg_{l'm'}\left(x'\right)dx'^{l'}dx'^{m'} \tag{11.6}$$

We suppose that the world function is such, that the tube $\mathcal{T}_{\mathbf{x}'\mathbf{x}}$ degenerates to a line. Then the solution of (11.6) has either the form

$$dx'^{k'} = g^{k'l'}\left(x'\right)\left(G_{,i'} - A_{,i'} - a_{i'}\left(x'\right)\right)d\tau \tag{11.7}$$

or the form

$$dx'^{k'} = g^{k'l'}\left(x'\right)\left(G_{,k'} + A_{,k'} + a_{k'}\left(x'\right)\right)d\tau \tag{11.8}$$

where $d\tau$ is an infinitesimal parameter. The relations (11.7), (11.8) describe a one-dimensional lines in vicinity of the point x'. Both expressions (11.7), (11.8) are solutions of the equation (11.6), provided the relation

$$\left(G_{,i'} - A_{,i'} - a_{i'}\left(x'\right)\right)g^{i'k'}\left(x'\right)\left(G_{,k'} + A_{,k'} + a_{k'}\left(x'\right)\right) = 2G \tag{11.9}$$

is fulfilled.

There is only one solution, provided solutions (11.7) and (11.8) coincide. It means that

$$A_{,k'} + a_{k'}\left(x'\right) = 0 \tag{11.10}$$

and the condition (11.9) transforms to the equation

$$G_{,i'}g^{i'k'}\left(x'\right)G_{,k'} = 2G \tag{11.11}$$

This is well known equation for the world function of a Riemannian space [3]. The Riemannian geometry is locally degenerate in the sense of definition 4.8, and the equation (11.11) describes this property of Riemannian geometry. The world function (5.7), (5.8) of the distorted space-time geometry $\mathcal{G}_{\mathrm{D}}$ does satisfy the relation (11.11)

According to expansion (8.3) the condition (11.10) may take place, only if

$$A\left(x, x'\right) = a_i\left(x^i - x'^i\right), \qquad a_i = \mathrm{const} \tag{11.12}$$

In this case the quantities (4.9) vanish, i.e.

$$\eta_{\mathrm{f}} = A\left(x, x'\right) + A\left(x', y\right) + A\left(y, x\right) = 0, \qquad \forall x, x', y \in \mathcal{M}_n \tag{11.13}$$

and the first order tubes are similar in symmetric and nonsymmetric geometries.

For the case of the past first order tube and the future one the conditions of degeneracy are also rather rigid. In this case instead of (11.6) one obtains two conditions

$$4G\left(A_{,i'}+a_{i'}\right)dx'^{i'}=0 \tag{11.14}$$

$$\left(2G\left(A_{,i'k'}-\left[\Sigma_{,i'k'}\right]_{x'}\right)-G_{,i'}G_{,k'}\right)dx'^{i'}dx'^{k'}=0 \tag{11.15}$$

In the case (11.12) the condition (11.14) is fulfilled, and (11.15) reduces to (11.11).

If the first order tube is nondegenerate in symmetric T-geometry, it cannot degenerate after addition of antisymmetric component, because the local degeneracy condition (11.11) remains to be not fulfilled.

Thus, practically any antisymmetric component of the world function destroys degeneracy of the neutral first order tube. If one connects quantum effects with the first order tube degeneracy [1], one concludes that the possible asymmetry of the space-time geometry is connected with quantum effects.

12 Examples of the first order tubes in nonsymmetric T-geometry.

To imagine the possible corollaries of asymmetry in T-geometry, let us construct the first order tube $\mathcal{T}_{P_0P_1}$ in the Σ-space. Let us consider Σ-space on the 4-dimensional manifold with the world function

$$\begin{aligned}\Sigma\left(x,x'\right) &= a_i\xi^i+\frac{1}{2}g_{ki}\xi^i\xi^k, \qquad a_i=b_i\left(1+\alpha f\left(\xi^2\right)\right), \\ \xi^i &= x^i-x'^i, \qquad \xi^2\equiv g_{ki}\xi^i\xi^k, \qquad \alpha,b_i,g_{ik}=\text{const},\end{aligned} \tag{12.1}$$

where f is some function of ξ^2 and summation is made over repeating indices from 0 to 3. One can interpret the relation (12.1) as an Euclidean space with a linear structure $a_i\xi^i$ given on it. Such a Σ-space is not isotropic, because there is a vector a_i, describing some preferable direction in the Σ-space.

Let us construct the first order neutral tube $\mathcal{T}_{P_0P_1}$. Coordinates of points $P_0=\{0\}$, $P_1=\{y\}$, $R=\{x\}$, where R is the running point. In the given case the characteristic quantity (4.9) has the form

$$\eta_{\mathrm{f}}=\eta_{\mathrm{f}}\left(P_0,P_1,R\right)=\alpha\left(-b_ix^if\left(x^2\right)+b_iy^if\left(y^2\right)+b_i\left(x^i-y^i\right)f\left(\left(x-y\right)^2\right)\right), \tag{12.2}$$

The quantity η_{f} does not depend on the constant component of the vector a_i. Then according to (4.2) - (4.8) the shape of the tube does not depend on the constant component of the vector a_i. If $\alpha=0$ and a_i =const, shape of all first order tubes is the same, as in the case of symmetric T-geometry, when $a_i=0$. In other words, the shape of the first order tubes is insensitive to the space-time anisotropy, described by the vector field a_i =const. We omit the constant component of the field a_i and consider the cases, when its variable part has the form

$$\begin{aligned} 1 \quad &: \quad f\left(\xi^2\right) = \xi^2, \qquad 2: \quad f\left(\xi^2\right) = \frac{1}{1+\beta\xi^2}, \qquad \beta = \text{const} \\ \xi^2 \quad &\equiv \quad g_{ik}\xi^i\xi^k, \qquad \xi^i \equiv x^i - x'^i \end{aligned} \tag{12.3}$$

In the first case the antisymmetric structure is essential at large distances $\xi = x - x'$. In the second case the antisymmetric structure vanishes at large ξ.

The equation (4.2), determining the shape of the tube $\mathcal{T}_{0y}$ has the form

$$\left| \begin{array}{cc} 2G(0,y) & (\mathbf{0y}.\mathbf{0x}) \\ (\mathbf{0x}.\mathbf{0y}) & 2G(0,x) \end{array} \right| = 4G(0,y)\,G(0,x) - (\mathbf{0y}.\mathbf{0x})(\mathbf{0x}.\mathbf{0y}) = 0 \tag{12.4}$$

In the first case, when $f\left(\xi^2\right) = \xi^2$, calculation gives for (12.4)

$$\left(x_i y^i\right)^2 - x^2 y^2 = \eta_{\mathrm{f}}^2 \tag{12.5}$$

$$\eta_{\mathrm{f}} = \alpha\left[\left(x_i y^i\right)\left(-2\left(b_k y^k\right) + 2\left(b_k x^k\right)\right) - \left(b_i x^i\right) y^2 + \left(b_i y^i\right) x^2\right] \tag{12.6}$$

where $x^2 = x^i x_i$, $y^2 = y_i y^i$. In the case, when the metric tensor g_{ik} is the metric tensor of the proper Euclidean space, $x^2y^2 \geq \left(x_i y^i\right)^2$, the equation (12.5) has an interesting solution, only if $\alpha = 0$. Then

$$x^2 y^2 = \left(x_i y^i\right)^2, \qquad \mathcal{T}_{0y} = \left\{ x \left| \bigwedge_{i=0}^{i=3} x^i = y^i \tau \right. \right\}, \qquad \alpha = 0 \tag{12.7}$$

In the case $\alpha \neq 0$, the first order tube $\mathcal{T}_{0y}$ degenerates to the set of basic points $\{0, y\}$, because substitution of $x^i = y^i\tau$ in the square bracket in (12.5) shows that the bracket vanishes only at $\tau = 0$ or $\tau = 1$. Thus, in the case of proper Euclidean metric tensor g_{ik} the first order tube shape does not depend on a_i, provided a_i =const.

Let us consider a more interesting case, when the metric tensor g_{ik} of Σ-space is the Minkowski one. Then $x^2y^2 \leq \left(x_i y^i\right)^2$, provided the $\mathbf{0y}$ is timelike ($|\mathbf{0y}|^2 = 2G(0,y) > 0$). In this case the equation (12.5) has the solution (12.7), if $\alpha = 0$.

If $\alpha \neq 0$, let us consider the special case, when vector b_i is the unit timelike vector, and the basic vector $\mathbf{0y}$ is chosen in such a way, that

$$b_i = \frac{y_i}{|y|}, \qquad y = \{|y|, \mathbf{0}\}, \qquad |y| = \sqrt{y^i y_i} \tag{12.8}$$

Vector $\mathbf{0x}$ is presented in the form

$$x = \left\{x^0, \mathbf{x}\right\} = \{t\,|y|, \mathbf{r}\,|y|\}, \qquad r = \sqrt{\mathbf{r}^2} \tag{12.9}$$

Using relations (12.8), (12.9), one obtains from (12.6)

$$\eta_{\mathrm{f}} = \alpha\,|y|^3\left(3t\,(t-1) - r^2\right) \tag{12.10}$$

and equation (12.5) is reduced to the form

$$-r^2+\alpha^2\left|y\right|^2\left(3t\left(t-1\right)-r^2\right)^2=0, \tag{12.11}$$

Its solution has the form

$$r=\pm\frac{1}{2\alpha\left|y\right|}\left(-1\pm\sqrt{\left(1+12\alpha^2\left|y\right|^2 t\left(t-1\right)\right)}\right) \tag{12.12}$$

Any section $t=$const of the three-dimensional surface $\mathcal{T}_{0y}$ form two (or zero) spheres, whose radii $r=r\left(t\right)$ are determined by the relation (12.12). Equation (12.12) gives four values of r for any value of t, but only two of them are essential, because radii r and $-r$ describe the same surface.

It follows from (12.12) that

$$\lim_{t\to\infty}\frac{r}{t}=\pm\sqrt{3},$$

It means the tube $\mathcal{T}_{0y}$ is infinite only in spacelike directions. In the timelike directions the tube size is bounded.

In the vicinity of the vector $\mathbf{0y}$, generating the tube, the shape of the tube depends on interrelation between the intensity of the antisymmetry, described by the constant α, and the length of the vector $\mathbf{0y}$. The quantity α appears in the equation (12.12) only in the combination $g=\alpha\left|y\right|$. In any case, when $\alpha\neq 0$, the tube $\mathcal{T}_{0y}$ does not degenerate into a one-dimensional curve.

If the antisymmetric structure is strong enough, and $\alpha\left|y\right|>1/\sqrt{3}$, the tube $\mathcal{T}_{0y}$ is empty in its center in the sense that intersection of $\mathcal{T}_{0y}$ with the plane $t=0.5$ is empty. If $\alpha\left|y\right|<1/\sqrt{3}$ intersection of $\mathcal{T}_{0y}$ with the plane $t=0.5$ forms two concentric spheres of radii

$$r_1=\frac{3\alpha\left|y\right|}{2\left(\sqrt{\left(1-3\alpha^2\left|y\right|^2\right)}+1\right)},\qquad r_2=\frac{3\alpha\left|y\right|}{2\left(1-\sqrt{\left(1-3\alpha^2\left|y\right|^2\right)}\right)} \tag{12.13}$$

If $\alpha\left|y\right|\ll 1$, one of radii is small $r_1=0.75\alpha\left|y\right|$ and another one is large $r_2=1/\left(\alpha\left|y\right|\right)$.

The shape of the tube $\mathcal{T}_{0y}$ is symmetric with respect to the reflection $t\to 1-t$. (See figures 1,2). In the same time a separation of the tube into internal and external segments is not symmetric with respect to the reflection $t\to 1-t$. This is shown schematically in the figures 3,4, where internal segment is drawn by a thick line, whereas external segments are drawn by thin line. The shape of internal segment, as well as that of external ones looks rather unexpected. The internal segment $\mathcal{T}_{[0y]}$ is strongly deformed with respect to the case of symmetric geometry. A part of the internal segment $\mathcal{T}_{[0y]}$ spreads to spatial infinity. Both external segments $\mathcal{T}_{0]y}$ and $\mathcal{T}_{0[y}$ are restricted in time direction. The external segment $\mathcal{T}_{0[y}$ is placed in a finite region. The segment $\mathcal{T}_{0]y}$ spreads to the spatial infinity, but it is bounded in any timelike direction.

We have seen that in the symmetric T-geometry the thickness of the internal segment is responsible for non-relativistic quantum effects [1]. At the strong antisymmetric field a_i the internal segment thickness becomes to be infinite. It increases quantum effects and may lead to unexpected phenomena.

Let us consider now the second case (12.3), when the antisymmetric structure is essential only at small distances. In this case one obtains instead of equation (12.11).

$$r^2 = g^2 \left(-\frac{(t-1)}{1+g_1\left((t-1)^2 - r^2\right)} + \frac{t}{1+g_1\left(t^2-r^2\right)} - \frac{1}{1+g_1} \right)^2 \tag{12.14}$$

where the same designations (12.8) - (12.9) are used, and $g = \alpha\,|y|\,,\ \ g_1 = \beta\,|y|^2$. At large t the equation (12.14) transforms to the equation

$$r^2 = \frac{g^2}{\left(1+\beta\,|y|^2\right)^2}, \qquad t\to\infty \tag{12.15}$$

It means that the tube is unbounded in the timelike direction $\mathbf{0y}$ and has a finite radius at $t\to\infty$. The tube is bounded in any spatial direction. The tube shape is rather fancy, and the section $\mathcal{T}_{0y}\cap S_t$ forms several concentric circles (S_t is the surface $t =$const).

Thus, the local antisymmetric structure produces only local perturbation of the tube shape. At the timelike infinity this perturbation reduces to a nonvanishing radius of the tube. As we have seen in the fifth section, geometrical stochasticity depends on the thickness of tube internal segment. Any asymmetry of the world function increases this thickness and increases stochasticity. It generates additional nondegeneracy of T-geometry, which is connected with the particle mass geometrization and with quantum effects [1].

13 Concluding remarks.

The main goal of the nonsymmetric T-geometry development is its possible application as a space-time geometry, especially as a space-time geometry of microcosm. Approach and methods of T-geometry distinguish from those of the Riemannian (pseudo-Riemannian) geometry, which is used now as a space-time geometry. The Riemannian geometry imposes on the space-time geometry a series of unfounded constraints. These restrictions are generated by methods used at the description of the Riemannian geometry. Let us list some of them.

1. The continuity of space-time. This is a very fine property which cannot be tested by a direct experiment. T-geometry is insensitive to continuity, and it is free of this constraint. For application of T-geometry is unessential, whether the space-time geometry is continuous or only fine-grained.

2. The Riemannian geometry is a geometry with fixed dimension. It is very difficult to imagine a geometry with variable dimension in the scope of the Riemannian geometry. Such a problem is absent in T-geometry.

3. For the Riemannian geometry construction, one uses a coordinate system and the concept of a curve, which are essentially methods of the Riemannian geometry description. The curve is considered conventionally to be a geometrical object (but not as a method of the geometry description), and separation of properties of geometry from properties imported by a use of the description in terms of curves is not considered usually. In particular, in the Riemannian geometry the absolute parallelism is absent, in general. Parallelism of two vectors at remote points is established by means of a reference to a curve, along which the parallel transport of the vector is produced. In other words, geometrical property of parallelism of two vectors is formulated in terms of the method of description, and it is not known, how to remove this dependence on the methods of description. T-geometry is free of this defect. The concept of a curve is not used at the T-geometry construction. There is an absolute parallelism in T-geometry.

4. T-geometry uses a special geometrical language, which contains only concepts immanent to the geometry in itself (Σ-function and finite subspaces). One does not need to eliminate the means of the geometry description.

5. The geometrical language admits one to consider and to investigate effectively such a situation, when the future and the past are not geometrically equivalent.

6. The means of the Riemannian geometry description suppress such an important property of geometry as nondegeneracy. As a corollary the particle mass geometrization appears to be impossible in the framework of Riemannian geometry. Geometrization of the particle mass is important, when the mass of a particle is unknown and must be determined from some geometrical or physical relations. It may appear to be important for determination of the mass spectrum of elementary particles. T-geometry admits geometrization of the particle mass.

7. Consideration of nondegeneracy and geometrization of the particle mass have admitted one to make the important step in understanding of the microcosm space-time geometry. One succeeded in explanation of non-relativistic quantum effects as geometrical effects, generated by nondegeneracy of the space-time geometry. There is a hope that asymmetry of the space-time geometry will admit one to explain important characteristics of elementary particles geometrically.

Capacities of T-geometry as a space-time geometry are far in excess of the Riemannian geometry capacities.

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Captions to figures.

Figure 1. Timelike first order neutral tube for .

Figure 2. Timelike first order neutral tube for $\alpha\,|y| = 0.7$

Figure 3. Schematic division of the timelike first order tube ($\alpha\,|y| = 0.4$) into internal and external segments. Internal segment is drawn by thick line, external ones are drawn by thin line.

Figure 4. Schematic division of the timelike first order tube ($\alpha\,|y| = 0.7$) into internal and external segments. Internal segment is drawn by thick line, external ones are drawn by thin line.