Dynamics and Interpretation in Quantum Theory

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It is shown that the dynamic system $\mathcal{S}_{KG}$, described by the Klein-Gordon equation is a partial (potential) case of a more general dynamic system $\mathcal{S}$ which can be considered as a set (ensemble) of identical classical particles, interacting between themselves via a self-consistent field κ. The κ-field is responsible for quantum effects, can escape from the matter and exist separately in the space-time. The κ-field enables to generate pairs. The κ-field can be interpreted as a force field, describing dynamically a statistical effect of the world lines reconnection. It is shown that main quantum effects can be considered as dynamical effects which are not connected with a usage of the QM principles.

1 Introduction

Conventional scheme (C-scheme) of the quantum mechanics presentation can be written as follows:

$$\text{Single system} \quad \Rightarrow \quad \text{Stat. propositions} \quad \Rightarrow \quad \text{Dynamics}$$

where "Single system" means a real single physical system and "statistical propositions" means a set of rules for calculations of average values of physical quantities. These rules describe the way of calculation of the mean value of a physical quantity R in the pure state ψ and can be written in the form [1]

$$\langle R \rangle_\psi = \langle \psi^* \mid \hat{R} \mid \psi \rangle \tag{1.1}$$

Here $\hat{R}$ means linear operator corresponding to the physical quantity R. Dynamics describe time evolution of mean values of physical quantities (in the Heisenberg picture), or the time evolution of the state vector ψ of the physical system (in the Schrödinger picture). In both cases the dynamics are described by fixing the evolution operator H called Hamiltonian. The arrows show the logical connection between the terms of the scheme. In particular, the second arrow shows that the statistical propositions are formulated before the dynamics, whereas the dynamics and dynamic equations are formulated after and in terms of the statistical propositions. In such a scheme practically all quantum properties [1] are contained in the statistical propositions. In particular, the QM principles can be derived from the statistical propositions [1].

Dynamical equations and statistical propositions generate some associations with the classical mechanics or with some elements of the classical mechanics. These associations are commonly considered as quantum mechanics interpretations. These associations are rather indefinite in the sense that as a rule they are not formulated in a mathematical form. It is a reason why some scientists believe that the quantum mechanics can exist without any interpretations, and only mathematical statements of the statistical propositions and the dynamic equations are of importance. Nevertheless, a lot of papers [2] - [11] are devoted to development of different interpretations of the quantum mechanics. It is commonly supposed that a valid interpretation is useful for a correct development of the quantum theory.

Here another scheme of quantum mechanics presentation is used. It will be refered to as a SET-scheme (SET is an abbreviation of "statistical ensemble technique"). The SET-scheme can be presented in the form:

$$\text{Single system} \quad \Rightarrow \quad \text{Dynamics} \quad \Rightarrow \quad \text{Interpretation}$$

where "Single system" means a single stochastic physical system $\mathcal{S}_{st}$, "Dynamics" denote a dynamical system $\mathcal{E}$, its attributes and its dynamic equations, describing the time evolution. This dynamic system $\mathcal{E}$ associates with the single stochastic system $\mathcal{S}_{st}$. "Interpretation" means a correlation between the dynamic variables of the dynamic system $\mathcal{E}$ and mean values related to the single stochastic system $\mathcal{S}_{st}$. The interpretation means the same statistical propositions and, maybe, some other statements which admit to formulate properties of the stochastic system $\mathcal{S}_{st}$ in terms of dynamic variables of the dynamic system $\mathcal{E}$. The arrows show also the logical connection between the terms of the scheme. In particular, the second arrow shows that the dynamics is formulated before and independently of the interpretation (statistical propositions). It means also that the interpretation (statistical propositions) is formulated after and in terms of the dynamics.

A usage of the SET-scheme can be explained in the following example. Precise experiments with a single electron are irreproducible, in general. It means by definition that a single electron is a stochastic system $\mathcal{S}_{st}$, and there are no dynamic equations for the time evolution of the single electron. At the same time a series of many precise independent experiments with the stochastic system $\mathcal{S}_{st}$ gives distributions of physical quantities. These distributions are reproducible in other series of like experiments with the stochastic system $\mathcal{S}_{st}$. It means that the set of many similar independent stochastic systems $\mathcal{S}_{st}$ can be considered as a deterministic dynamic system $\mathcal{E}$, associated with the stochastic system $\mathcal{S}_{st}$. It is a common practice to call such a dynamic system $\mathcal{E}$ as a statistical ensemble of stochastic systems $\mathcal{S}_{st}$. The time evolution of the statistical ensemble is described by dynamic equations which are determined by properties of stochastic systems $\mathcal{S}_{st}$ constituting the statistical ensemble $\mathcal{E}$.

Interpretation (statistical propositions) determines mean values of physical quantities of $\mathcal{S}_{st}$ in terms of dynamic variables of $\mathcal{E}$. It is an interpretation of the stochastic system in terms of a deterministic dynamic system, i.e. it is an interpretation in a conventional sense of this word. In particular, the interpretation (statistical propositions) determines the time evolution of the mean values of $\mathcal{S}_{st}$.

It should note a conceptual difference between the C-scheme and the SET-scheme. In the C-scheme all quantum properties are contained in the statistical propositions, whereas in the SET-scheme they are contained in dynamics. Indeed, removing the statistical propositions from the C-scheme, one removes the quantum mechanics as a whole. At the same time in the SET-scheme one can derive practically all quantum effects (including the uncertainty principle, interference and diffraction) only from dynamics, i.e. without a usage of the interpretation (statistical propositions). From the viewpoint of the C-scheme it seems rather strange and unexpected. Nevertheless, it will be demonstrated below.

In the second section one considers dynamics of a set of identical classical particles interacting via a self-consistent field κ_l. Properties of this field are investigated in the third section. An interpretation of the κ-field is given in the fourth section. The fifth section is devoted to an interpretation of the quantum effects in dynamical terms.

2 Dynamics of a set of relativistic particles

Let us consider a dynamic system $\mathcal{S}$, which is a set of identical classical particles, interacting via some self-consistent field κ. The action has the form

$$\mathcal{A}_L[x,\kappa] = -\int mcK\sqrt{g_{kl}\dot{x}^k\dot{x}^l}d^4\xi, \qquad \dot{x}^k \equiv dx^k/d\tau, \qquad \tau = \xi_0 \tag{2.1}$$

$$K \equiv \sqrt{1+\lambda^2(\partial_l\kappa^l + \kappa^l\kappa_l)}, \tag{2.2}$$

where c is the speed of the light, g_{kl} =diag$\{c^2,-1,-1,-1\}$ is the metric tensor and $\lambda = \hbar/mc$ is the Compton wave length. $x = \{x^l\}$ and $\kappa = \{\kappa^l\}$, $l = 0,1,2,3$ are dependent dynamic variables. $\xi = \{\xi_0, \boldsymbol{\xi}\} = \{\xi_k\}$, $k = 0,1,2,3$ are independent dynamic variables. κ depends on ξ via x:

$$\kappa^k = \kappa^k(x), \qquad k = 0,1,2,3; \qquad x^k = x^k(\xi), \qquad \xi = \{\xi_k\}, \qquad k = 0,1,2,3. \tag{2.3}$$

Here and further a summation is produced over repeated Latin indices $(0-3)$ and over the Greek ones $(1-3)$. The dynamic system $\mathcal{S}$ is a distributed system which can be considered as a fluid. $\boldsymbol{\xi} = \{\xi_\alpha\}, \alpha = 1,2,3$ are Lagrangian coordinates labelling particles of the fluid, $\xi_0 = \tau$ is the time Lagrangian coordinate along the world line of the fluid particle. The Lagrangian coordinates ξ are independent variables. Dynamic equaitons for the field κ^k are determined as a result of a variation of the action (2.1) with respect κ^k. $\delta\mathcal{A}_L/\delta\kappa^k = 0$, $k = 0,1,2,3$.

By means of a change of variables the action (2.1) can be transformed to the Eulerian coordinates x, when $x = \{x^0, \mathbf{x}\}$, $x^0 = t, \mathbf{x} = \{x^\alpha\}$, $\alpha = 1,2,3$ are independent variables, and the Lagrangian coordinates $\boldsymbol{\xi}$ are functions of the Eulerian coordinates x. In this case the action (2.1) takes the form (see Appendix)

$$\mathcal{A}_E[j,\kappa,\xi] = \int\{-mcK\sqrt{j^k j_k} - \hbar j^k[\partial_k\xi_0 + g^\alpha(\boldsymbol{\xi})\partial_k\xi_\alpha]\}d^4x \tag{2.4}$$

where g^α, $\quad \alpha = 1,2,3$ are arbitrary functions of the argument $\boldsymbol{\xi}$.

The dynamic system $\mathcal{S}$ described by the action (2.1), or (2.4) is rather symmetric. The actions (2.1), (2.4), the corresponding dynamic equtions and the expressions for the current

$$j^k = \frac{\partial J}{\partial\xi_{0,k}} \equiv \frac{\partial(x^k,\xi_1,\xi_2,\xi_3)}{\partial(x^0,x^1,x^2,x^3)} \tag{2.5}$$

and for the energy-momentum tensor

$$T^i{}_k = \frac{mcKj^ij_k}{\sqrt{j^sj_s}} - \lambda^2\frac{mc\sqrt{j^sj_s}}{2K}\partial_k\kappa^i \tag{2.6}$$

are form-invariant with respect to the transformation of Lagrangian coordinates

$$\xi_\alpha \to \tilde{\xi}_\alpha = \tilde{\xi}_\alpha(\boldsymbol{\xi}), \qquad \alpha = 1,2,3, \qquad \frac{\partial(\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\xi}_3)}{\partial(\xi_1,\xi_2,\xi_3)} = 1 \tag{2.7}$$
$$\xi_0 \to \tilde{\xi}_0 = \tilde{\xi}_0(\xi_0,\boldsymbol{\xi}), \qquad \partial\tilde{\xi}_0/\partial\xi_0 > 1$$

The action (2.1) is invariant with respect to the transformation (2.7) in the evident way. The action (2.4) transforms to the form

$$\mathcal{A}_E[j,\kappa,\tilde{\boldsymbol{\xi}}] = \int\{-mcK\sqrt{j^kj_k} - \hbar j^k[\partial_k\tilde{\xi}_0 + \tilde{g}^\alpha(\tilde{\boldsymbol{\xi}})\partial_k\tilde{\xi}_\alpha]\}d^4x \tag{2.8}$$

where

$$\tilde{g}^{\alpha}(\tilde{\boldsymbol{\xi}}) = g^{\beta}(\boldsymbol{\xi})\frac{\partial \xi_{\beta}}{\partial \tilde{\xi}_{\alpha}} \tag{2.9}$$

The actions (2.4) and (2.8) should be considered as having the same form, because the functions $g^{\alpha}(\boldsymbol{\xi})$ and $\tilde{g}^{\alpha}(\tilde{\boldsymbol{\xi}})$ are supposed to be arbitrary.

Arbitrary functions $g^{\beta}(\boldsymbol{\xi})$ arise as a result of an integration of equations (A.15) in the process of transformation from the action (2.1) to the action (2.4). Aribtrariness of $g^{\beta}(\boldsymbol{\xi})$ is connected with arbitrary labelling fluid particles in the action (2.1). The transformation (2.7) describes relabelling the particles of the fluid described by the action (2.1). The arbitrary functions $g^{\alpha}(\boldsymbol{\xi})$ transform as a vector in the space $V_{\boldsymbol{\xi}}$ of Lagrangian coordinates $\boldsymbol{\xi}$ and describe the vorticity of the vector field of the momentum

$$p_l = -\frac{mcKj_l}{\sqrt{j^s j_s}} = \hbar[\partial_l \xi_0 + g^{\alpha}(\xi)\partial_l \xi_{\alpha}] \tag{2.10}$$

The curl of the momentum p_l is written in the form

$$\partial_k p_l - \partial_l p_k = \hbar[g^{\alpha,\beta}(\boldsymbol{\xi}) - g^{\beta,\alpha}(\xi)]\xi_{\alpha,l}\xi_{\beta,k}, \tag{2.11}$$
$$g^{\alpha,\beta}(\boldsymbol{\xi}) \equiv \frac{\partial g^{\alpha}(\boldsymbol{\xi})}{\partial \xi_{\beta}}, \qquad \xi_{\alpha,k} \equiv \frac{\partial \xi_{\alpha}}{\partial x^l}.$$

The functions $g^{\alpha}(\boldsymbol{\xi})$ can be written in the form

$$g^{\alpha}(\boldsymbol{\xi}) = g^{\alpha}_{\mathrm{sym}}(\boldsymbol{\xi}) + g^{\alpha}_{\mathrm{asym}}(\boldsymbol{\xi}) \tag{2.12}$$
$$\frac{\partial}{\partial \xi_{\beta}} g^{\alpha}_{\mathrm{sym}}(\boldsymbol{\xi}) = \frac{\partial}{\partial \xi_{\alpha}} g^{\beta}_{\mathrm{sym}}(\boldsymbol{\xi}), \qquad \frac{\partial}{\partial \xi_{\beta}} g^{\alpha}_{\mathrm{asym}}(\boldsymbol{\xi}) = -\frac{\partial}{\partial \xi_{\alpha}} g^{\beta}_{\mathrm{asym}}(\xi) \tag{2.13}$$

As it follows from the first equation (2.13) the symmetric part of g^{α} can be presented in the form

$$g^{\alpha}_{\mathrm{sym}}(\xi) = \frac{\partial}{\partial \xi_{\alpha}}\Phi(\xi) \tag{2.14}$$

and included in the term $\partial_l \xi_0$. One obtains

$$\partial_l \xi_0 + g^{\alpha}(\boldsymbol{\xi})\partial_l \xi_{\alpha} = \partial_l \varphi + g^{\alpha}_{\mathrm{asym}}(\boldsymbol{\xi})\partial_l \xi_{\alpha}, \qquad \varphi = \xi_0 + \Phi \tag{2.15}$$

Thus, without a loss of generality one can consider that the functions $g^{\alpha}(\boldsymbol{\xi})$ satisfy the second condition (2.13). Then the action (2.4) is written in the form

$$\mathcal{A}_E[j,\kappa,\varphi,\xi]=\int\{-mcK\sqrt{j^k j_k}-\hbar j^k[\partial_k\varphi_0+g^{\alpha}(\boldsymbol{\xi})\partial_k\xi_\alpha]\}d^4x \tag{2.16}$$

$$g^{\alpha,\beta}(\boldsymbol{\xi})=-g^{\beta,\alpha}(\boldsymbol{\xi}),\qquad \alpha,\beta=1,2,3 \tag{2.17}$$

Let us introduce designation

$$\rho\equiv\frac{\sqrt{j^s j_s}}{mcK}\equiv\frac{1}{mc}\frac{\sqrt{j^s j_s}}{\sqrt{1+\lambda^2(\partial_k\kappa^k+\kappa_k\kappa^k)}}. \tag{2.18}$$

Then dynamic equations generated by the action (2.16) take the form

$$\frac{\delta\mathcal{A}_E}{\delta j^k}=-j_k/\rho-\hbar[\partial_k\varphi+g^{\alpha}(\boldsymbol{\xi})\partial_k\xi_\alpha]=0 \tag{2.19}$$

$$\delta\mathcal{A}_E/\delta\varphi=\hbar\partial_k j^k=0 \tag{2.20}$$

$$\delta\mathcal{A}_E/\delta\kappa^k=-\hbar^2[\partial_k\rho/2-\rho\kappa_k]=0 \tag{2.21}$$

$$\frac{\delta\mathcal{A}_E}{\delta\xi_\alpha}=\hbar j^k\partial_k\xi_\beta(g^{\alpha,\beta}-g^{\beta,\alpha})=0 \tag{2.22}$$

The equation (2.21) is integrated in the form

$$\kappa_k=\partial_k\kappa=\frac{1}{2}\partial_k\log\rho,\qquad \rho=a^2e^{2\kappa},\qquad a=\text{const}. \tag{2.23}$$

where κ is a potential of the field κ_k. Deminsionality of a : $[a]=[\rho^{1/2}]$. Equations (2.19) and (2.20) take respectively the form

$$j_k=-\hbar\rho[\partial_k\varphi+g^{\alpha}(\boldsymbol{\xi})\partial_k\xi_\alpha],\qquad \kappa=0,1,2,3 \tag{2.24}$$

$$2\kappa^k[\varphi_k+g^{\alpha}(\boldsymbol{\xi})\partial_k\xi_\alpha]+\partial^k\varphi_k+g^{\alpha}(\boldsymbol{\xi})\partial^k\partial_k\xi_\alpha=0 \tag{2.25}$$

$$\varphi^k\equiv\partial^k\varphi,\qquad \varphi_k\equiv\partial_k\varphi,$$

the term $g^{\alpha,\beta}(\boldsymbol{\xi})\partial_k\xi_\alpha\partial_k\xi_\beta$ vanishes in virtue of the condition (2.17). Using Eqs. (2.2) and (2.23) the designation (2.18) takes the form of the dynamic equation

$$\lambda^2\{\partial_k\kappa^k+\kappa_k\kappa^k-[\varphi_k+g^{\alpha}(\boldsymbol{\xi})\partial_k\xi_\alpha][\varphi^k+g^{\beta}(\boldsymbol{\xi})\partial^k\xi_\beta]\}+1=0 \tag{2.26}$$

In virtue of Eq.(2.17) the equation (2.22) can be written in the form

$$j^k\partial_k g^\alpha(\boldsymbol{\xi})=0,\qquad \alpha=1,2,3 \tag{2.27}$$

It means that $g^\alpha(\boldsymbol{\xi})$ is constant along the line tangent to the current j^k. Thus, dynamic equations (2.25), (2.26), (2.27), (2.17) with j^k and ρ determined respectively by Eqs. (2.24) and (2.23) form the complete system of dynamic equations in terms of "potentials" κ, φ, $\boldsymbol{\xi}$ for the dynamic system $\mathcal{S}$.

The same system can be written in the hydrodynamic form (A.19), (A.18), i.e. in the dynamic variables j^k, κ:

$$\partial_k j^k=0 \tag{2.28}$$

$$j^l[\partial_l(e^{-2\kappa}j_k)-\partial_k(e^{-2\kappa}j_l)]=0,\qquad k=0,1,2,3 \tag{2.29}$$

$$\lambda^2\partial^l\partial_l e^\kappa=[\frac{j^s j_s}{a^4m^2c^2e^{4\kappa}}-1]e^\kappa. \tag{2.30}$$

At any fixed $g^\alpha(\xi)$ the system of equations (2.25), (2.26), (2.27) can be considered as a result of a partial integration of the system (2.28)-(2.30).

Let us consider the potential case, when the momentum vector (2.10) has a potential, and $g^\alpha(\boldsymbol{\xi})=0$, $\alpha=1,2,3$. In this case Eqs. (2.27) are satisfied identically. Multiplying Eq.(2.26) by $\psi=\exp(\kappa+i\varphi)$ and Eq.(2.25) by $i\psi=i\exp(\kappa+i\varphi)$ and summing them, one obtains after simplifications

$$\lambda^2\partial_k\partial^k\psi+\psi=0,\qquad \lambda=\hbar/mc,\qquad \psi=e^{\kappa+i\varphi}, \tag{2.31}$$

where ψ is a complex-valued variable, and ψ^* is complex conjugate to ψ. The equation (2.31) is the well known Klein-Gordon equation which can be obtained from the action

$$\mathcal{A}_{KG}[\psi,\psi^*]=\frac{1}{2}\int[\hbar^2(\partial_k\psi^*)(\partial^k\psi)-m^2c^2\psi^*\psi]d^4x \tag{2.32}$$

The current j^k and the canonical energy-momentum tensor T_k^l are determined by the conventional expressions

$$j^k=\frac{i\hbar}{2}(\psi^*\psi^k-\psi^{*k}\psi),\qquad k=0,1,2,3\qquad \psi^k\equiv\partial^k\psi \tag{2.33}$$

$$T_k^l=\frac{\hbar^2}{2}(\psi^{*l}\psi_k+\psi_k^*\psi^l-\delta_k^l\psi^{*s}\psi_s)+\frac{1}{2}\delta_k^l m^2c^2\psi^*\psi,\qquad k,l=0,1,2,3 \tag{2.34}$$

By means of substitution of $\psi = \exp(\kappa + i\varphi)$ and subsequent calculation one can verify that expressions (2.33), (2.34) coincide with expressions (2.24) and (2.6) respectively. It means that the Klein-Gordon equation (2.31) describes the dynamic system $\mathcal{S}_{\mathrm{pot}} = \mathcal{S}_{KG}$ which is determined by the action (2.16) with the additional potentiality constraint

$$\partial_k p_l - \partial_l p_k = 0, \qquad p_l = -\frac{mcKj_l}{\sqrt{j^s j_s}} \tag{2.35}$$

or by the action (2.1) with the same additional constraint (2.35). In the independent Lagrangian coordinates Eq.(2.35) is written in the form

$$\frac{\partial D}{\partial x^{l,s}}\frac{\partial p_k}{\partial \xi_s} - \frac{\partial D}{\partial x^{k,s}}\frac{\partial p_l}{\partial \xi_s} = 0, \qquad p_l = -\frac{mcK\dot{x}_l}{\sqrt{\dot{x}^s\dot{x}_s}}, \qquad k,l = 0,1,2,3 \tag{2.36}$$

where D is the Jacobian

$$D = \frac{\partial(x^0, x^1, x^2, x^3)}{\partial(\xi_0, \xi_1, \xi_2, \xi_3)} = \det \parallel x^{l,k} \parallel, \qquad x^{l,k} \equiv \frac{\partial x^l}{\partial \xi_k}, \qquad l,k = 0,1,2,3 \tag{2.37}$$

and dynamic variables x and κ are considered as functions of ξ_l.

Thus the Klein-Gordon dynamic system (2.31) is a partial case $\mathcal{S}_{\mathrm{pot}}$ of the dynamic system $\mathcal{S}$ described by the action (2.1), or (2.16). The self-consistent field κ^l is responsible for quantum effects in the sense that, setting $\kappa^l \equiv 0, \quad l = 0,1,2,3$ in the action (2.1), or (2.3) one suppreses quantum effects, because the dynamic system $\mathcal{S}$ turns to a statistical ensemble of independent classical particles. The wave function (2.31) is a scalar in the space-time and in the 3-space $V_{\boldsymbol{\xi}}$.

Let a vorticity of the momentum (2.10) do not vanish. In this case one-component complex wave function does not enable to describe four current components $j^k, \quad k = 0,1,2,3$. It is necessary to introduce two-component wave function. Let

$$\psi = \binom{\psi_1}{\psi_2} = ae^{\kappa + i\varphi}\boldsymbol{\eta\sigma}\chi, \qquad \chi = \binom{\chi_1}{\chi_2} \tag{2.38}$$

$$\psi^* = (\psi_1^*, \psi_2^*) = a\chi^*\boldsymbol{\eta\sigma}e^{\kappa - i\varphi}, \qquad \chi^*\chi = 1, \qquad \mathbf{b} = \chi^*\boldsymbol{\sigma}\chi, \tag{2.39}$$

where $\boldsymbol{\sigma} = \{\sigma_\alpha\}$, $\alpha = 1,2,3$ are Pauli matrices, $\boldsymbol{\eta} = \{\eta_\alpha\}$, $\alpha = 1,2,3$ is a unit vector

$$\eta_\alpha = \frac{\xi_\alpha}{\mid \boldsymbol{\xi} \mid} = \frac{\xi_\alpha}{\sqrt{\xi_\beta \xi_\beta}} \tag{2.40}$$

If $g^\alpha(\boldsymbol{\xi})$ in Eq.(2.16) is chosen in the form

$$g^\alpha(\xi) = -\varepsilon_{\alpha\beta\gamma}\xi_\beta b_\gamma/\boldsymbol{\xi}^2, \qquad b_\gamma b_\gamma = 1, \tag{2.41}$$

the action (2.16) can be written in terms of the wave function (2.38) as follows

$$\mathcal{A}[\psi,\psi^*] = \frac{1}{2}\int \{\hbar^2(\partial_k\psi^*)(\partial^k\psi) - m^2c^2\psi^*\psi - (\partial_k s_\alpha)(\partial^k s_\alpha)\psi^*\psi\}d^4x, \tag{2.42}$$

where

$$s_\alpha \equiv \frac{\hbar}{2}\frac{\psi^*\sigma_\alpha\psi}{\psi^*\psi}, \qquad \alpha = 1,2,3 \tag{2.43}$$

and the current j^k is determined by Eq.(2.33). Substituting Eqs.(2.38), (2.39) in Eq. (2.33), one obtains

$$j^k = -\hbar a^2 e^{2\kappa}(\partial^k\varphi + \varepsilon_{\alpha\beta\gamma}\eta_\alpha\partial^k\eta_\beta b_\gamma), \qquad k = 0,1,2,3 \tag{2.44}$$

that coincides with Eq.(2.24) in virtue of Eq.(2.41).

3 Properties of the κ-field

Let us list the properties of the κ-field introduced by the relations (2.1), (2.2).

(1) The κ-field is responsible for quantum effects.

(2) The κ-field can escape from the matter and exist separately in the "empty" space-time.

(3) The κ-field enables to generate pairs.

The first property is rather evident from Eqs.(2.1), (2.2). Setting $\kappa^l \equiv 0$, $l = 0,1,2,3$, the action (2.1) turns to the action for the classical statistical ensemble of independent classical particles.

The second property follows from the dynamic equation (2.30). Setting $j^l \equiv 0$, $l = 0,1,2,3$, Eqs. (2.28), (2.29) are satisfied identically. Eq.(2.30) turns to the Klein-Gordon equation for e^κ.

$$\lambda^2\partial_l\partial^l e^\kappa + e^\kappa = 0 \tag{3.1}$$

In the case $j^l = 0$, $l = 0,1,2,3$ the wave function $\psi = ae^\kappa$. In other words, real wave function ψ, satisfying the Klein-Gordon equation, describes free κ-field without a matter.

According to Eq. (2.34) the energy-momentum tensor of the free κ-field is determined by the relation

$$T_k^l = \frac{1}{2} m^2 c^2 a^2 e^{2\kappa} [\delta_k^l + \lambda^2 (2\kappa^l \kappa_k - \delta_k^l \kappa^s \kappa_s)], \qquad k, l = 0, 1, 2, 3 \tag{3.2}$$

$$T_0^0 = \frac{1}{2} m^2 c^2 a^2 e^{2\kappa} [1 + \lambda^2 \sum_{l=0}^{3} \mid \kappa_l \mid^2] \geq 0 \tag{3.3}$$

If j^l changes rather slightly over the distance of the Compton length λ, the dynamic equation (2.30) for the κ-field is a differential equation with a small parameter at the highest derivative. In this case rhs of Eq. (2.30) vanishes , and its solution has the form

$$\kappa \simeq \frac{1}{2} \log \frac{\sqrt{j^s j_s}}{a^2 mc}, \qquad K \simeq 1, \qquad \lambda \mid \partial_l j^s \mid \ll j^s \tag{3.4}$$

In this case the κ-field is coupled with the matter (like the Coulomb electric field) and has no proper degrees of freedom. In the non-relativistic case the energy difference $mKc^2 - mc^2 = -\frac{1}{2}\lambda^2 mc^2 (j_0)^{-1/2} \nabla^2 \sqrt{j_0}$ associates with the Bohm potential $V_{\mathrm{B}} = [\hbar \nabla \log(\psi^* \psi)]^2 / 8m$ [3].

Before a consideration of the pair production problem let us discuss some conceptual points, connected with a description of particles and antipartiles in classical terms. First, about terminology. The terms "world line of a particle" and "world line of an antiparticle" are essentially non-relativistic. They suppose that a world line is an *attribute* (history) of a pointlike physical object (particle or antiparticle). This physical object is a point in the 3-dimensional space, and its trajectory in the space-time is its world line. From the non-relativistic viewpoint a particle and an antiparticle are different physical objects. From relativistical viewpoint a world line (*WL*) is a real physical object. This object is a one-dimensional line in the 4-dimensional space-time. *WL* has two distinct orienatations which can be considered as one of two possible directions of motion along *WL* [12, 13]. Further the term "*WL*" will be used instead of the term "world line" in those cases, when the world line is considered as a physical object (not as a history of a particle). Particles and antiparticles are derivative pointlike physical objects which arise as intersections of *WL* with the hyperplane t=const at some coordinate system. One *WL* can describe a few particles and antiparticles, placed in different regions.

A particle and an antiparticle are distinguished by the orientation of *WL* at its intersection with the hyperplane t=const. From relativistical viewpoint a particle and an antiparticle are two different states (or attributes) of a WL, whereas from the nonrelativistic viewpoint they are two different physical objects. The term "SWL" (abbreviation of "section of world line") will be used as a common concept with respect to concepts "particle' and "antiparticle". SWL means any point of intersection between the WL and a hyperplane t=const independently of an orientation of the WL at this point.

Describing a particle and an antiparticle, one has to distinguish between the energy-momentum vector $P_l = \{P_0, \mathbf{P}\}$ and the canonical momentum vector $p_l = \{p_0, \mathbf{p}\}$, especially between the energy $E = P_0$ and the time component p_0 of the canonical momentum. These quantities are defined in different way. The energy-momentum vector is defined by the relation

$$P_l = \int\limits_V T_l^0 d\mathbf{x}, \qquad l = 0,1,2,3, \qquad E = P_0 \tag{3.5}$$

where T_k^l is the energy-momentum tensor and V is a 3-volume around SWL. The canonical momentum p_l is defined as a quantity, canonically conjugate to the position x^l, $l = 0,1,2,3$ of SWL. In classical physics it means that $p_l = \partial L/d\dot{x}^l$, $\dot{x}^l \equiv dx^l/d\tau$, $l = 0,1,2,3$, where L is a Lagrangian, and τ is a parameter along WL. In the quantum physics the operator $\hat{p}_l$ of the canonical momentum is defined by the relation

$$[\hat{u}, \hat{p}_l]_- = i\hbar \frac{\partial \hat{u}}{\partial x^l} \tag{3.6}$$

where $[\ldots]_-$ denotes a commutator and $\hat{u}$ means an operator of any dynamic variable in the Heisenberg representation.

The fact that the energy E of a free particle coincide with the time component $-p_0$ of the canonical momentum taken with the opposite sign $E = -p_0$ does not mean that the same relation is valid for a free antiparticle. Indeed, setting for simplicity $\kappa^l \equiv 0$, $l = 0,1,2,3$ in the relations (2.6), (2.10), one obtains

$$T_0^0 = \frac{mcKj^0 j_0}{\sqrt{j^s j_s}} \geq 0, \qquad p_0 = -\frac{mcKj_0}{\sqrt{j^s j_s}} \tag{3.7}$$

$\mathrm{sgn}(K\sqrt{j^s j_s})$ =$\mathrm{sgn}(K/\sqrt{j^s j_s}) = 1$ always. Then it follows from Eqs. (3.5), (3.7) that the energy is positive always, whereas the sign of p_0 depends on the sign of j^0 and is different for a particle and for an antiparticle. If p_0 can be considered as a constant, then it follows from Eqs. (3.5), (3.7)

$$E = P_0 = -p_0 N, \qquad N = \int j^0 d\mathbf{x}, \tag{3.8}$$

where N is the number of SWLs. N is positive for particles and negative for antiparticles For one WL the relation (3.8) reduces to $E = -p_0 \mathrm{sgn}(j^0) =\mid p_0 \mid$.

A like relation

$$Q = e \ \mathrm{sgn}(j^0) \tag{3.9}$$

is valid for the connection between the electric charge Q of SWL, defined as a source of the electromagnetic field and the constant e, describing interaction of a charged WL with the electromagnetic field [13]. For instance, for a charged WL in a given electromagnetic field the action (2.1) takes the form

$$\mathcal{A}_L[x, \kappa] = \int \{-mcK\sqrt{g_{kl}\dot{x}^k\dot{x}^l} - \frac{e}{c}A_l\dot{x}^l\}d^4\xi, \qquad \dot{x}^k \equiv dx^k/d\tau, \qquad \tau = \xi_0 \tag{3.10}$$

where e is a constant. The electric charge Q is defined by the relation (3.9) which describes the well known fact that a particle and an antiparticle have opposite electric charges.

Identification of P_l with p_l is possible [13, 14], provided particles and antiparticles are considered as different objects, (but not attributes of a whole WL). In the up-to-date quantum field theory such an identification is produced almost always (see, however, [14]) As a result one deals with indefinite number of SWL (instead of a few WLs), and many additional problems arise. In particular, conventionally one claims [15, 16] that some solutions of the Klein-Gordon equation has a negative energy and are not physical. Here one has a confusion of the energy E with the canonical momentum p_0. The energy of any solution is non-negative, as it follows from Eqs.(3.5), (3.7), or (2.34), whereas the p_0 can have both signs, but it does not mean that the solution with $-p_0 < 0$ is not physical. It means only an appearance of antiparticles.

The problem of pair production is connected with a possibilty of changing sign of p_0, or j^0. To change $\mathrm{sgn}(j^0)$, the vector j^0 is to be spacelike $j^k j_k < 0$. It is a necessary condition of changing sign of j^0. The first term of the action (2.16) remains real, provided $j^k j_k < 0$ and

$$K^2 = 1 + \lambda^2(\kappa_k\kappa^k + \partial_k\kappa^k) < 0 \tag{3.11}$$

To satisfy the last condition, the κ-field is to be large enough and to change rather rapidly.

To distinguish between particles and antiparticles, let us introduce the distribution function over position $\mathbf{x}$, canonical momentum $\mathbf{p}$ and ε=sgn$(j^0) = -$sgn(p_0) :

$$F(\mathbf{x},\mathbf{p},\varepsilon) = \frac{\sqrt{j^s j_s}}{mcK}\prod_{\alpha=1}^{3}\delta(p_\alpha + \frac{mcKj^\alpha}{\sqrt{j^s j_s}})\delta_{\varepsilon,\mathrm{sgn}(j^0)} \tag{3.12}$$

where K is defined by the relation (2.2). One distinguishes between the canonical momentum (c-momentum) $\mathbf{p} = \{p_1, p_2, p_3\}$ and the momentum $\mathbf{P} = \{P_\alpha\}, \alpha = 1,2,3$ which are coupled between themselves by the relation

$$\mathbf{P} = \varepsilon\mathbf{p}, \qquad \varepsilon = -\mathrm{sgn}(p_0) = \pm 1 \tag{3.13}$$

$\mathbf{P}$ is the spatial component of the energy-momentum vector (3.5), whereas $\mathbf{p}$ is defined by Eq.(2.10).

The distribution function permits to calculate the mean value $\langle R\rangle$ of any function $R(\mathbf{x},\mathbf{P},\varepsilon)$ by means of the relation

$$\langle R(\mathbf{x},\mathbf{P},\varepsilon)\rangle = A^{-1}\sum_{\varepsilon=\pm1}\int\int R(\mathbf{x},\varepsilon\mathbf{p},\varepsilon)F(\mathbf{x},\mathbf{p},\varepsilon)d\mathbf{x}d\mathbf{p} \tag{3.14}$$

$$A = \sum_{\varepsilon=\pm1}\int\int F(\mathbf{x},\mathbf{p},\varepsilon)d\mathbf{x}d\mathbf{p} = \int(mcK)^{-1}\sqrt{j^s j_s}d\mathbf{x} = \int\rho d\mathbf{x} \tag{3.15}$$

Relations (3.14), (3.15) are some new statistical propositions alternative to Eq.(1.1), which will be referred to as *WL*-interpretation (interpretation in terms of *WL*). They realize in a mathematical form the "classical" associations" arising at the description of the dynamic system $\mathcal{S}$ by means of the action (2.1). Until formulation of the *WL*-interpretation in the form (3.14) all "classical considerations" were only a description of the dynamic system $\mathcal{S}_{KG}$ in terms of other dynamic variables and nothing more.

A possiblity of an alternative interpretation arises only after mathematical formulation of alternative statistical propositions. The *WL*-interpretation is more general, than the conventional interpretation (1.1), because it is applicable in the case, when ψ satisfies nonlinear dynamic equation and the linear superposition principle does not take place. The distribution function (3.12) describe a classical particle of alternating mass

$$\mu = mK = m\sqrt{1+\lambda^2(\kappa_k\kappa^k + \partial_k\kappa^k)} \tag{3.16}$$

with the mass depending on the κ-field magnitude. μ is imaginary for spacelike momenta.

ε=sgn(j^0) is not a relativistic invariant for the spacelike vector j^k. It means that the spacelike WL describes a particle ($\varepsilon = 1$) in one coordinate system and an antiparticle ($\varepsilon = -1$) in other one. Spacelike WL are possible only inside a region with a large κ-field which associates with an existence of pairs. Apparently, spacelike WL should be regarded as the mean WL, describing simultaneously one SWL and a few pairs.

Pairs are not generated in the dynamic system described by the free Klein-Gordon equation. Let us consider the potential case of the dynamic system (3.10), where particles move in a given electromagnetic field. One can show that this case is described by the Klein-Gordon equation

$$\partial_0^2\psi - (-i\nabla + e\mathbf{A})^2\psi + m^2\psi = 0 \tag{3.17}$$

Here and further for simplicity one uses $\hbar = c = 1$. $A_0 = 0$ that can be obtained by a proper gauge transformation.

Let $\tilde{\psi}$, $\tilde{u}$ and $\tilde{\mathbf{A}}$ denote spatial Fourier components of variables ψ, $\partial_0\psi$ and $\mathbf{A}$ respectively.

$$\tilde{\psi}(t,\mathbf{q}) = (2\pi)^{-3/2}\int_{-\infty}^{\infty} e^{-i\mathbf{qx}}\psi(t,\mathbf{x})d\mathbf{x} \tag{3.18}$$

$$\tilde{u}(t,\mathbf{q}) = (2\pi)^{-3/2}\int_{-\infty}^{\infty} e^{-i\mathbf{qx}}\partial_0\psi(t,\mathbf{x})d\mathbf{x} \tag{3.19}$$

$$\tilde{\mathbf{A}}(t,\mathbf{q}) = (2\pi)^{-3/2}\int_{-\infty}^{\infty} e^{-i\mathbf{qx}}\mathbf{A}(t,\mathbf{x})d\mathbf{x} \tag{3.20}$$

Let $b(t,Q)$ be a function of the argument $Q = \{\varepsilon,\mathbf{q}\}$, $\varepsilon = \pm 1$, where ε =sgn$(-p_0)$ describes an orientation of WL [13] ($\varepsilon = 1$ for particles, and $\varepsilon = -1$ for antiparticles). Let us define the variable $b(t,Q)$ by means of the relation

$$b(t,Q) = E^{1/2}(\mathbf{q})\tilde{\psi}(t,\mathbf{q}) + i\varepsilon E^{-1/2}(\mathbf{q})\tilde{u}(t,\mathbf{q}) \tag{3.21}$$

where $E(\mathbf{q}) \equiv \mid \sqrt{m^2+\mathbf{q}^2} \mid$. If ψ satisfies Eq.(3.17), $b(t,Q)$ satisfies the dynamic equation

$$\partial_0 b(t,Q) = -i\varepsilon E(\mathbf{q}) b(t,Q) - \frac{e}{(2\pi)^{3/2}} \int F(\mathbf{q},\mathbf{q}') b(t,Q') dQ', \tag{3.22}$$

$$F(\mathbf{q},\mathbf{q}') = (\mathbf{q}+\mathbf{q}')\tilde{\mathbf{A}}(\mathbf{q}-\mathbf{q}') + e(2\pi)^{-3/2} \int \tilde{\mathbf{A}}(\mathbf{q}-\mathbf{q}'')\tilde{\mathbf{A}}(\mathbf{q}''-\mathbf{q}')d\mathbf{q}'', \tag{3.23}$$

where $\int(.)dQ \equiv \sum_{\varepsilon=\pm1}\int(.)d\mathbf{q}$. At $A_0 = 0$ the energy $\mathcal{E}$ and the number $\mathcal{N}$ of SWLs can be presented as follows

$$\mathcal{E} \equiv \int T^0{}_0 d\mathbf{x} = \int E(\mathbf{q}) n(Q) dQ + \int e\mathbf{j}\mathbf{A} d\mathbf{x} \tag{3.24}$$

$$\mathcal{N} \equiv \int j^0 d\mathbf{x} = \int \varepsilon n(Q) dQ, \tag{3.25}$$

where $n(Q) \equiv b^*(Q)b(Q)$, and b^* is the quantity complex conjugate to b. The non-negative quantity $n(Q)$ can be considered as the number of world lines (WL) having direction described by the vector of the canonical momentum $p_l = \{-\varepsilon E(\mathbf{q}), \mathbf{q}\}$. For $\varepsilon = 1 \quad p_0 < 0$, for $\varepsilon = -1 \quad p_0 > 0$. Nothing prevents from interpretation of $b(1,\mathbf{q})$ and $b(-1,\mathbf{q})$ as spectra of particles and antiparticles respectively (at this point the QM principles are used essentially). The total number $\mathcal{N}$ of SWLs can have any sign and does not depend on time t, but the energy $\mathcal{E}$ depends, in general.

Let at $t = -\infty \quad b(-1,\mathbf{q}) = 0$. After interaction with the strong electromagnetic field the spectrum changes, and it is possible that at $t = +\infty \quad b(-1,\mathbf{q}) \neq 0$. It means that the external electromagnetic field generates pairs particle-antiparticle, so that $\mathcal{N} = \mathcal{N}_p + \mathcal{N}_a = \mid \mathcal{N}_p \mid - \mid \mathcal{N}_a \mid$, where $\mid \mathcal{N}_a \mid$ is the number of generated antiparticles (and pairs). Generally, such a situation is possible, when $\mid \mathcal{N}_a \mid \gg \mathcal{N}$.

4 Interpretation of the κ-field on the dynamic-statistical base

Let us discuss interpretation of the κ-field. For simplicity only the case of two-dimensional space-time is considered. Generalization for the case of four dimensions can be obtained easily. Let us consider a statistical ensemble $\mathcal{E}$ of free classical (deterministic) WLs in the two-dimensional space-time. All WLs have the same mass m and the constant e. The state of the statistical ensemble $\mathcal{E}$ is described by the distribution function F as a function of coordinates $x = \{x^0, x^1\}$ and momentum $p = \{p_0, p_1\}$. As far as variables p_0 and p_1 are not independent because of the relation $p_l p^l = m^2, \quad (c = 1)$, it is convenient to use variables p_1 and $\varepsilon = -\mathrm{sgn}(p_0), \quad \varepsilon = \pm 1$ as independent arguments of the distribution function $F = F(x, p_1, \varepsilon)$. The discrete variable ε is different for particles and antiparticles ($\varepsilon = 1$ for particles and $\varepsilon = -1$ for antiparticles).

The distribution function F satisfies the free Liouville equation

$$\partial_l [p^l F(x, p_1, \varepsilon)] = 0, \tag{4.1}$$

$$p^1 = -p_1, \qquad p_0 = -\varepsilon E(p_1), \qquad E(p) \equiv \mid \sqrt{m^2 + p_1^2} \mid$$

The mean current in the statistical ensemble $\mathcal{E}$ is defined by the relation

$$j^l(x) = \sum_{\varepsilon = \pm 1} \int p^l F(x, p_1, \varepsilon) dp_1, \qquad l = 0, 1, \tag{4.2}$$

Let us consider a very important case of the state of $\mathcal{E}$

$$F(x, p_1, \varepsilon) = f(x, \varepsilon) \delta(p_1 - k) \tag{4.3}$$

where k is a constant, or a slowly varying function of x. If WLs of the state (4.3) of $\mathcal{E}$ do not intersect (have no common points) inside some region $\mathcal{R}$ of the space-time, then by definition the $\mathcal{E}$ is a simple ensemble (more exactly an ensemble simple inside $\mathcal{R}$).

A simple ensemble has the remarkable property that its state $F(x, p_1, \varepsilon)$ is determined by its current $j^l(x)$.

$$F(x, p_1, \varepsilon) = m^{-1} \sqrt{j_s j^s} \delta[p_1 + m j_1 (j^s j_s)^{-1/2}], \qquad \varepsilon = \mathrm{sgn}(j^0), \tag{4.4}$$

Besides it is possible to write the action and dynamic equations for the current of the simple ensemble state

$$\mathcal{A}[j, \varphi] = \int \{-m \sqrt{j_s j^s} - \hbar j^l \partial_l \varphi\} d^2 x \tag{4.5}$$

After elimination of φ the dynamic equations for the action (4.5) reduce to the form

$$\partial_l j^l = 0, \qquad \partial_l T^{lk} = 0, \qquad k = 0, 1 \tag{4.6}$$

$$T^{kl} = \frac{mj^k j^l}{\sqrt{j_s j^s}}, \qquad k, l = 0, 1 \tag{4.7}$$

In such a form the dynamic equations (4.6) describe the conservation law of SWLs and the energy-momentum. One can verify that in virtue of the dynamic equations (4.6) the distribution function (4.4) satisfies the Liouville equation (4.1), provided ε=sgn(j^0).

Let us consider two different simple states F_1 and F_2 of the statistical ensemble $\mathcal{E}$, which will be referred to as simple statistical ensembles $\mathcal{E}_1$ and $\mathcal{E}_2$ of free WLs.

$$\mathcal{E}_1 : \qquad F_1(x, p_1, \varepsilon) = f_1(x^1 p^0 - x^0 p^1)\delta(p_1 - k), \qquad p_0 = -\varepsilon E(p_1) \tag{4.8}$$

$$\mathcal{E}_2 : \qquad F_2(x, p_1, \varepsilon) = f_2(x^1 p^0 - x^0 p^1)\delta(p_1 + k), \qquad p_0 = -\varepsilon E(p_1) \tag{4.9}$$

where k =const, $E(p) \equiv \mid (m^2 + p^2)^{1/2} \mid$ and

$$f_s(x) = \left\{ \begin{array}{ll} A, & \mid x \mid < L_s \\ 0, & \mid x \mid > L_s \end{array} \right. \qquad s = 1, 2, \tag{4.10}$$

$$A, L_1, L_2 = \text{const}, \qquad A > 0.$$

If in Eqs. (4.8), (4.9) $p_0 < 0$, then $\varepsilon = 1$, and WLs of the ensemble describe particles. If $p_0 > 0$, then $\varepsilon = -1$, and WLs describe antiparticles. The ensembles $\mathcal{E}_1$ and $\mathcal{E}_2$ are described respectively by the currents $j^0_{(1)}$ and $j^0_{(2)}$

$$\mathcal{E}_1 : \qquad j^0_{(1)} = k_0 f_1(x^1 k_0 + x^0 k), \qquad j^1_{(1)} = -k f_1(x^1 k_0 + x^0 k), \tag{4.11}$$

$$\mathcal{E}_2 : \qquad j^0_{(2)} = k_0 f_2(x^1 k_0 - x^0 k), \qquad j^1_{(2)} = k f_2(x^1 k_0 - x^0 k), \tag{4.12}$$

$$k_0 = -\varepsilon E(k)$$

As far as the distribution functions F_1 and F_2 satisfy the linear equation (4.1) and describe two different states of the same dynamic system $\mathcal{E}$, the distribution function $F = F_1 + F_2$ describes also a state of $\mathcal{E}$. This state F can be considered as a result of composition of two statistical ensembles $\mathcal{E}_1$ and $\mathcal{E}_2$. Thus, the statistical ensemble $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$ is simple everywhere in the space-time except for the region $\mathcal{R}_c$. The region $\mathcal{R}_c$ is located inside the parallelogram formed by the straight lines $x^1 - kx^0/k_0 = \pm L_1, \qquad x^1 + kx^0/k_0 = \pm L_2$. Inside $\mathcal{R}_c$ any point belongs to two different WLs of the statistical ensemble $\mathcal{E}$. These intercepts of WLs are shown in Fig.1a, 2a. as dashed lines. In Fig.1 $\varepsilon > 0$ in both ensembles $\mathcal{E}_1$ and $\mathcal{E}_2$, and SWLs are particles. Thus, the statistical ensemble $\mathcal{E}$ is not simple, in general.

The total current j^l of $\mathcal{E}$ is determined by Eq.(4.2). One obtains the relation

$$j^l = j^l_{(1)} + j^l_{(2)} \tag{4.13}$$

as a corollary of Eqs. (4.8), (4.9) and constructs the mean WLs as lines tangent to the total current j^l. Then the ensemble $\mathcal{E}$ becomes a simple ensemble $\mathcal{E}_m$ of mean WLs, shown in Fig.1b. But now $\mathcal{E}_m$ is not a statistical ensemble of mean WLs, because by definition a statistical ensemble is a set of independent and, hence, non-interacting elements (WLs).

Outside the region $\mathcal{R}_c$ the mean WLs coincide with WLs of the ensembles $\mathcal{E}_1$ or $\mathcal{E}_2$, but inside $\mathcal{R}_c$ the mean WLs distinguish from WLs of both ensembles $\mathcal{E}_1, \mathcal{E}_2$. Looking at Fig.1*b*, one can see that after the unification (4.13) a reconnection of WLs take place. From the non-relativistic viewpoint this reconnection can be interpreted as an exchange effect between the particles of the ensembles $\mathcal{E}_1$ and $\mathcal{E}_2$. This reconnection (or the exchange effect) is interpreted as some statistical effect. There is no dynamic interaction between WLs here, nevertheless the mean WLs of the ensemble $\mathcal{E}_m$ look, as if they interact very strongly inside the region $\mathcal{R}_c$. In other words, considering non-simple statistical ensemble $\mathcal{E}$ of deterministic WLs as a simple ensemble $\mathcal{E}_m$ of mean WLs, one introduces some *effective dynamic interaction between the mean WLs.* This dynamic interaction imitates the statistical effect of reconnection of WLs (from relativistic viewpoint), or the effect of the particle exchange (from the non-relativistic standpoint).

Now let us look at Fig.2b. In this case ε is different for $\mathcal{E}_1$ and $\mathcal{E}_2$. Here one sees the effect of the WLs reconnection, but this effect can be hardly interpreted from the non-relativistic viewpoint as an exchange effect, because from this point of view particles and antiparticles are not identical physical objects. Besides, the picture of mean world lines in Fig.2b should be interpreted as follows. Particles and antiparticles annihilated in the region $\mathcal{R}_c$, are converted into some field, and thereafter this field produces particle-antiparticle pairs. Such a situation, when particles disappear and arise, can be hardly explained from the non-relativistic standpoint.

The dynamic interaction is self-consistent in the sense that it depends on the state of the whole ensemble $\mathcal{E}$. For instance, the fact that the mean *WL*s are straight lines inside $\mathcal{R}_c$ (Fig.1b,2b) is a corollary of the fact that $f_s(x) = A$=const. If one violates the form of the functions f_s, the shape of the mean *WL*s in $\mathcal{R}_c$ changes.

From the relativistical viewpoint both pictures in Fig.1,2 describe some statistical effect, when non-simple statistical ensemble $\mathcal{E}$ of free *WL*s is described in terms of a simple ensemble $\mathcal{E}_m$ of *mean* interacting *WL*s. This statistical effect is described as a reconnection of *WL*s and can be imitated by means of dynamic interaction between mean *WL*s.

Thus, there are two different ways of statistical description of deterministic *WL*s: (1) description F in terms of the distribution function, (2) description j in terms of currents. The description F is the most detailed one. It is produced in the 3-dimensional phase space and uses dynamic equations of the type (4.1). The description j is less detailed one. It is produced in the 2-dimensional space-time and uses dynamic equations of the type (4.6).

If there is a non-simple ensemble, then it is possible to describe each beam of *WL*s in the space-time by means of dynamic equations of the type (4.6), (4.7). The mean *WL*s are described by the mean current j^l of the type (4.13). But the dynamic equations cannot be written only in terms of j^l. They contain also variables of the type $j^l_{(1)} - j^l_{(2)}$ which imitate some self-consistent field describing the interaction between mean *WL*s. In the case of the ensemble of determinisitc *WL*s one knows definitely that real *WL*s do not coincide with the mean *WL*s. It is valid also in the case of the ensemble of stochastic *WL*s.

At the F-description in the phase space the *WL*s of the ensembles $\mathcal{E}_1$ and $\mathcal{E}_2$ do not interact. At the j-description in the space-time they do. Of course, in reality the *WL*s do not interact (at any rate they do not interact dynamically), but describing stochastic *WL*s in quantum systems, one is forced to use only j-description in the space-time (but not the F-description in the phase space). Mathematically an interplay between descriptions of stochastic *SWL*s and deterministic *WL*s can be established only for j-description of mean *WL*s. Although intuitively *WL*s of ensembles $\mathcal{E}_1$ and $\mathcal{E}_2$ seem to be independent, nevertheless the imitation by means of mean interacting *WL*s seems to be more effective mathematically in the case of stochastic WLs.

A possibility of imitating statistical effects by means of dynamic interaction is very important for a practical use. For instance, WLs describing microparticles (electron, proton,...etc) are stochastic, and statistical ensembles of such stochastic WLs cannot be simple, because they cross between themselves many times. But it is possible to imitate such non-simple statistical ensembles of WLs by simple ensembles of mean WLs which do not cross by definition. But in this case the mean WLs of the ensemble interact between themselves via some self-consistent field. Thus, considering an ensemble $\mathcal{E}$ (not a statistical ensemble) of WLs, interacting via some self-consistent field, it is possible to take into account statistical effects connected with the stochasticity and substitute them by dynamic interaction with the some self-consistent field.

One can imagine such a field as a field describing closed WLs, or closed current loops associated with particle-antiparticle pairs. A join of such closed WLs to a WL describing a particle (or an antiparticle) changes effectively a shape of the WL (see Fig.3).

Let us return to consideration of $\mathcal{S}_{KG}$ and consider the special case of a stationary state in the two-dimensional space-time, when the state of $\mathcal{S}_{KG}$ does not depend on the temporal coordinate. It is described by the wave function $\psi_s(t,x,W)$

$$\psi = \psi_s(t,x,W) = (C_1 e^{ik_1x+i\varphi_1} + C_2 e^{-ik_1x+i\varphi_2})e^{-ik_0t} \tag{4.14}$$

$$\psi^* = \psi_s^*(t,x,W) = (C_1 e^{-ik_1x-i\varphi_1} + C_2 e^{ik_1x-i\varphi_2})e^{ik_0t}, \qquad k_0^2 - k_1^2 = m^2 \tag{4.15}$$

where $W = \{k_0, k_1, C_1, C_2, \varphi_1, \varphi_2\}$ are real parameters. Then one obtains for the current and energy-momentum tensor

$$j_0 = k_0[C_1^2 + C_2^2 + 2C_1C_2\cos\nu], \qquad j_1 = -k_1(C_1^2 - C_2^2), \tag{4.16}$$

$$T^{00} = k_0^2(C_1^2 + C_2^2) - 2m^2C_1C_2\cos\nu, \qquad T^{01} = -k_0k_1(C_1^2 - C_2^2),$$

$$T^{11} = k_1^2(C_1^2 + C_2^2), \qquad \nu = 2k_1x + \varphi_1 - \varphi_2 \tag{4.17}$$

Let us consider a state of $\mathcal{S}_{KG}$ in the form of two wave packets of size $L \gg m^{-1}$ which pass one through another in vicinity of the coordinate origin. The state can be described by the wave function

$$\psi(t,x,q_1)=\int\psi_s(t,x,W)\Psi_0(q_1-k_1)dk_1 \tag{4.18}$$

where ψ_s is determined by Eq.(4.14) with

$$W=\{\sqrt{m^2+k_1^2},k_1,1,1+\alpha,0,0\} \tag{4.19}$$

$$\Psi_0(k_1)=A(L^2/2\pi)^{1/2}\exp(-L^2k_1^2/2),\qquad \epsilon=(Lm)^{-1}\ll 1,\qquad \alpha,A=\text{const} \tag{4.20}$$

Let us consider the region

$$\mathcal{R}_s:\qquad |\, t \,|< L\epsilon^{-1/2}=m^{-1}\epsilon^{-3/2},\qquad |\, x \,|< L\epsilon^{-1/2}=m^{-1}\epsilon^{-3/2} \tag{4.21}$$

Calculation of the integral (4.18) inside this region leads to the following result

$$\psi(t,x,q_1)=\psi_s(t,x,W(q_1,T,X))+o(\epsilon^2) \tag{4.22}$$

Now parameters $W=\{k_0,k_1,C_1,C_2,0,0\}$ of the function (4.14) are some functions of $q_1,T=\epsilon t,X=\epsilon$x. The parameter $\epsilon=(Lm)^{-1}\ll 1$ is small, and parameters W are slowly varying functions of x and t. q_1 is a parameter which is used instead of the parameter k_1. Inside $\mathcal{R}_s$ the following approximate relations take place

$$C_1=A\exp\left[-m^2(X+vT)^2/2+O(\epsilon^{1/2})\right], \tag{4.23}$$

$$C_2=A(1+\alpha)\exp[-m^2(X-vT)^2/2+O(\epsilon^{1/2})], \tag{4.24}$$

$$v=q_1(m^2+q_1^2)^{-1/2},\qquad \Gamma=(m^2+q_1^2)^{1/2}m^{-1} \tag{4.25}$$

$$k_0=\sqrt{m^2+k_1^2}=m\Gamma+O(\epsilon^2),\qquad k_1=q_1+O(\epsilon^2) \tag{4.26}$$

The region $\mathcal{R}_s :|\, t \,|< L,|\, x \,|< L$ can be considered approximately as a region, where the wave packets overlap.

The state ψ of the system $\mathcal{S}_{KG}$ can be described by means of world lines tangent to the vector j^l for the state ψ. The schematic picture of world lines associated with the vector j^l for the wave function (4.22) with small enough $\alpha>0$ is shown in Fig.4a. In reality the world lines oscillate rather strongly in the region, where $C_1\simeq C_2$. This oscillations are not shown in Fig.4a.

It is possible to construct schematic picture of world lines in the case, when the stationary wave function ψ_s is substituted in Eq.(4.18) by the wave function ψ_u of an uniform state which is defined by the relation

$$\psi = \psi_u(t, x, W) = (C_1 e^{ik_0 t + i\varphi_1} + C_2 e^{-ik_0 t + i\varphi_2}) e^{ik_1 x} \tag{4.27}$$

where W are the same parameters, as in Eq.(4.14). In this case instead of Eqs.(4.16)-(4.20) one obtains

$$j_0 = -k_0(C_1^2 - C_2^2), \qquad j_1 = -k_1[C_1^2 + C_2^2 + 2C_1C_2\cos\nu], \tag{4.28}$$

$$T^{00} = k_0^2(C_1^2 + C_2^2), \qquad T^{01} = k_0k_1(C_1^2 - C_2^2),$$

$$T^{11} = k_1^2(C_1^2 + C_2^2) - 2m^2C_1C_2\cos\nu, \qquad \nu = 2k_0t + \varphi_1 - \varphi_2 \tag{4.29}$$

Corresponding picture of world lines associated with the current j^l for the wave function (4.27) with parameters (4.23)–(4.26) is shown in Fig. 4b.

According to the definition of the world lines in Fig. 4 are lines tangent to corresponding currents. They can be interpreted as mean *WL*s describing mean motion of *SWL*s. In both cases (4.15) and (4.27) the κ-field can be expressed by the formula (2.23)

$$\kappa_k = \frac{1}{2}\partial_k \log\rho, \qquad \rho = C_1^2 + C_2^2 + 2C_1C_2\cos\nu \tag{4.34}$$

The main contribution is given by the rapidly oscillating last term. κ-field is very small outside the region $\mathcal{R}_s$, where one of C_k, $\quad k = 1, 2$ is very small (of the order ϵ). Inside the region $\mathcal{R}_s$ it is convenient to estimate the κ-field, calculating the mass ratio (2.23)

$$K^2 - 1 = m^{-2}\rho^{-1/2}\partial_l\partial^l\rho^{1/2} \tag{4.35}$$

Estimation of K^2 inside $\mathcal{R}_s$ gives for the stationary state ψ_s and the uniform state ψ_u respectively

$$K_s^2 \simeq 1 + 2(k_1/m)^2 > 1, \qquad K_u^2 \simeq 1 - 2(k_0/m)^2 < 0 \tag{4.36}$$

Let us compare classical *WL*s in Fig. 1b, 2b with the quantum *WL*s in Fig. 4a and Fig. 4b. In Fig. 1b, 2b one can see interacting mean *WL*s. The reason of the interaction is a reconnection effect which cannot be described dynamically. The reason of interaction of quantum *WL* in Fig. 4 is the κ-field. One concludes that the κ-field describes the reconnection, or at any rate imitates it. The independence of deterministic WLs in the ensembles $\mathcal{E}_1$ and $\mathcal{E}_2$ associates with the independence of the wave packets, passing one through another "without interaction". The last expression "without interaction" means that outside the overlapping region each of the wave packets evolves in such a way, as though another wave packet were absent (or a sum of solutions is a solution of the dynamic equation, written in terms of the wave function).

All this can be interpreted in the sense that the κ-field describes the reconnection (exchange) effect, or at least imitates it dynamically.

5 Quantum properties as dynamic effects

Let us try to explain quantum effects from point of view of the SET-scheme, i.e. without a reference to the QM principles. Of course, it is impossible to explain all quantum effects in one paper, and only the main quantum effects (uncertainty principle, interference, diffraction, appearance of discrete quantum numbers) will be considered.

Uncertainty principle. Let a beam of electrons with the momentum $\mathbf{p} = \{0, 0, p_3\}$ drop on a diaphragm placed in the plane of the coordinate axis x^1, x^2. Let the diaphragm have a narrow slit of the width $\Delta x \ll \hbar/p_3$ along the axis x^2, and the width of the beam in the direction x^1 $L_1 \gg \Delta$x. Then only small part of electrons passes through the diaphragm. This passage of electrons through the slit can be considered as a measurement of the beam electron position in the direction x^1. The energy of the κ-field is very small in the electron beam, but it increases inside the slit and becomes of the order $U_\kappa = (2m)^{-1}(\hbar\partial_1\rho/\rho)^2\rho \simeq (2m)^{-1}(\hbar/2\Delta x)^2\rho$. After the passage of the beam through the slit the potential energy U_κ of the κ-field turns to the kinetic energy of electrons which passed through the slit. As it follows from dynamic equations only the momentum component p_1 is changed, and this energy is added only to the component $p_1^2/2m$ of the kinetic energy. The mean value of this energy component corresponds to $\langle p_1^2\rangle \simeq (\hbar/2\Delta x)^2$. It can be interpreted as a result of the measurement of the electron position in the x^1 direction. Conventionally this result is considered as a corollary of the uncertainty principle, but here it is simply a result of dynamic equations. One may put the question. From where do the passing electrons take the additional energy? The formal answer looks as follows. The result follows from the dynamic equations for the quantum fluid. The observed effect reminds of cumulative effect in the usual hydrodynamics. Another answer is as follows. Electrons of the beam move stochastically, and the slit selects only those of them which have the large enough energy. In other words, the electrons of the passed beam take additional energy from the electrons absorbed by the diaphragm. The more narrow slit, the more larger part of electrons is absorbed, and the more larger additional energy is added to the passed electrons.

Diffraction. Let there be a screen in the back of the diaphragm. The electron are distributed over the screen in the form of narrow parallel straightlinear zones, but not uniformly. This effect is known as a diffraction effect. It can be calculated on the base of dynamic equations and expression for the current j^k of the particle. As far as j^k is an attribute of dynamics, it means that the diffraction can be calculated *only* on the base of dynamics, i.e. without a usage of statistical propositions. Usually, the diffraction is calculated in terms of the wave function ψ, because dynamic equation in terms of ψ are most simple for solution (linear differential equation with constant coefficients). This is convenient, but it is not conceptual. The dynamic equations can be solved in any variables. The result is to be the same.

Interference. The interference effects are calculated on the base dynamic equations and expressions for the current j^k in the same way, as diffraction effects. They are also explained on the base of only dynamics.

Discrete quantum numbers. Conventionally the discrete quantum numbers are considered as eigenvalues of linear operators associated with physical quantities. It seems that it is impossible to explain their appearance without a reference to the statistical propositions (or QM principles) which describe correspondence between the physical quantities and corresponding linear operators.

However, let us note that the Hamiltonian $\hat{H}$, describing the evolution of the dynamic system, can be obtained directly from dynamic equations written in terms of the wave function ψ. Thus, the linear operator $\hat{H}$ can be derived from dynamics only. Eigenstates of the Hamiltonian are stationary states, that also follows from the dynamics (without a reference to the statistical propositions). Although QM dynamic equations admit existence of atoms at non-stationary states, nevertheless all atoms and molecules exist practically all the time at the stationary states, because only stationary states are stable. Indeed, being at the non-stationary state, an atom, or a molecule emits the electromagnetic radiation until it becomes at a stationary state [17]. As far as for the non-relativistic case the Hamiltonian coincides with the energy, one can measure eigenvalues of the Hamiltonian by spectrometric methods, measuring frequency ω (and the energy $\hbar\omega$) of the emitted electromagnetic radiation. In other words, measuring eigenvalues of the Hamiltonian, one uses only dynamic equations and the energy conservation law, which is also a dynamic relation.

Only position can be measured instantly. Other physical quantities such as the momentum and functions of the momentum need a rather long time for exact measurement. A physical quantity can be measured, provided its operator commutes with the Hamiltonian and its eigenvalues enable to label stationary states [18]. Thus, if some operator $\hat{S}$ commute with the Hamiltonian $\hat{H}$, the eigenvalues of $\hat{S}$ label the stationary states and can be measured by an identification of the stationary state, that can be made on the base of the dynamics only. If $\hat{S}$ does not commute with $\hat{H}$, the eigenvalues of $\hat{S}$ cannot be measured, and $\hat{S}$ cannot be considered as an observable quantity.

Let us analyze the Stern-Gerlach experiment [19]. Passing across a strongly non-uniform magnetic field, a beam of atoms is splitted into two discrete beams. A reason of such a splitting is explained in the following way. Energy of nth stationary atom state depends on an external magnetic field by means of an additional term $\Delta E_n = \Delta E_n(\mathbf{H})$. The magnetic moment $\boldsymbol{\mu}_n$ of the nth stationary state is defined by the relation

$$\boldsymbol{\mu}_n = \frac{\partial \Delta E_n(\mathbf{H})}{\partial \mathbf{H}} \tag{5.1}$$

If $\boldsymbol{\mu}_n$ does not depend on $\mathbf{H}$, the force, acting on the atom has the form

$$\mathbf{F}_n = -\nabla \Delta E_n(\mathbf{H}) = \mu_{n\alpha} \nabla H_\alpha \tag{5.2}$$

Let a beam of atoms is found at such a state, where there are two stationary states, having different projections μ_H of the magnetic moment onto the magnetic field. In the non-uniform magnetic field the force (5.2) is different for different atoms, and the beam is splitted into discrete beams, provided stationary states are discrete. Using the relation (5.2), one can calculate projections $\mu_H = \boldsymbol{\mu}_n \mathbf{H}/H$ of the magnetic moment on the base of the measured split of the beam.

Thus, in the Stern-Gerlach experiment only the projection μ_H of the magnetic moment of the atom stationary state is measured directly (Note that μ_H can be determined also from spectrometric experiments of the atom in the magnetic field). The statement that in the Stern-Gerlach experiment one measures the spin projection σ_H onto the magnetic field is based on the additional operator relation

$$\boldsymbol{\mu} = -\frac{e\hbar}{2mc}\boldsymbol{\sigma} \tag{5.3}$$

One can consider this relation as a definition of the spin, or as a corollary of the statistical propositions. In any case the Stern-Gerlach experiment tests only discreteness of stationary states, but not the discreteness of such quantities as the magnetic moment, or spin. In other words, the Stern-Gerlach experiment tests only dynamics, but not the statistical propositions.

6 Concluding Remarks

Formal results, concerning the properties and interpretation of the κ-field, are presented in the text of the paper and in the abstract. One should like to mention here about some informal results of the above consideration. First, the SET approach demonstrates that the quantum mechanics can be constructed as a dynamic construction, where the statistical propositions play a secondary role.

Second, the SET approach appears to be more physical and reasonable, than the conventional one which describes all quantum effects in terms of wave functions. The conventional approach cannot exist without a reference to a wave function, although from the physical point of view the wave function is something indefinite. In the SET approach the wave function is not a vehicle of quantum properties. It is simply a way of description of a continuous medium. The wave function (for instance, (2.31), or (2.38)) is constructed of potentials κ, $\boldsymbol{\xi}$. The Lagrangian coordinates $\boldsymbol{\xi}$ are determined to within a rather general transformation (2.7) that determines an indefiniteness of the wave function.

Third, the SET approach is a more general approach which permits to develop the quantum theory in an alternative direction. Especially it concerns the pair production problem. The statement of this problem in the up-to-date QFT seems rather artifical as compared with the statement of this problem in the SET approach, where the pair production is an intrinsic property of the κ-field, and one needs only to investigate and to develop properly this property.

APPENDIX A

Transformation from Lagrangian Coordinates to Eulerian Ones

Let us show equivalency of two forms (2.1) and (2.16) of the action for a set (ensemble) of identical classical systems interacting via some self-consistent field κ. This equivalency can be formulated in a more general form. Let $L(t,\mathbf{x},d\mathbf{x}/dt,\kappa)$, $\quad \mathbf{x}=\{x^\alpha\}$, $\alpha=1,2,..n$ be a Lagrangian of a classical system $\mathcal{S}_d$ in some given external field κ. Then the action for the statistical ensemble of the systems $\mathcal{S}_d$ has the form

$$\mathcal{A}_L[\mathbf{x}]=\int L(t,\mathbf{x},d\mathbf{x}/dt,\kappa)dt d\boldsymbol{\xi}, \qquad d\boldsymbol{\xi}=\prod_{\alpha=1}^{\alpha=n} d\xi_\alpha \tag{A.1}$$

where $\boldsymbol{\xi}=\{\xi_\alpha\}$, $\alpha=1,2,\ldots n$ are Lagrangian coordinates labelling the systems of the ensemble, $\mathbf{x}=\mathbf{x}(t,\boldsymbol{\xi})$ is a function of t, $\boldsymbol{\xi}$ and $\kappa=\kappa(t,\boldsymbol{\xi})$ is some given external field. If the field κ is not a given external field, but a self-consistent field which is defined by the set of dynamical systems $\mathcal{S}_d$, κ should be considered as a dynamic variable and include the κ as argument of the functional $\mathcal{A}_L$. It shows that $\mathcal{A}_L$ should be varyied with respect to κ to obtain dynamic equation for κ. In this case one obtains

$$\mathcal{A}_L[\mathbf{x},\kappa]=\int L(t,\mathbf{x},d\mathbf{x}/dt,\kappa)dt d\boldsymbol{\xi}, \qquad d\boldsymbol{\xi}=\prod_{\alpha=1}^{\alpha=n} d\xi_\alpha \tag{A.2}$$

The dynamic system (A.2) stops to be a statistical ensemble, because its elements enteract between themselves and are not independent.

The variational problem with functional (A.2) is equivalent to the variational problem with the functional

$$\mathcal{A}_E[j,\varphi,\boldsymbol{\xi},\kappa]=\int\{\mathcal{L}(x,j,\kappa)-\hbar j^i[\partial_i\varphi+g^\alpha(\boldsymbol{\xi})\partial_i\xi_\alpha]\}d^{n+1}x, \qquad d^{n+1}x=\prod_{i=0}^{n} dx^i, \tag{A.3}$$

where $j=\{j^0,\mathbf{j}\}$, $\mathbf{j}=\{j^\alpha\}$, $\alpha=1,2,\ldots n$, φ, $\boldsymbol{\xi}=\{\xi_\alpha\}$, $\alpha=1,2,\ldots n$ are functions of $x=\{x^0,\mathbf{x}\}$, $x^0=t$, $\mathbf{x}=\{x^\alpha\}$, $\alpha=1,2,\ldots n$, and a summation is made over repeating indices: over Latin ones $(0-n)$ and over Greek ones $(1-n)$. All variables $j=\{\rho,\mathbf{j}\}$,

φ, $\boldsymbol{\xi}$ are functions of x. $g^\alpha(\boldsymbol{\xi})$, $\alpha = 1, 2, \dots n$ are arbitrary functions of $\boldsymbol{\xi}$. The function $\mathcal{L}(x, j, \kappa)$ is defined by the relation

$$\mathcal{L}(x, j, \kappa) = j^0 L(x^0, \mathbf{x}, \mathbf{j}/j^0, \kappa) \tag{A.4}$$

where L is the Lagrangian of a single system $\mathcal{S}_d$. The variables $j = \{j^0, \mathbf{j}\}$ are connected with the variables $\mathbf{x}(t, \boldsymbol{\xi})$, $d\mathbf{x}(t, \boldsymbol{\xi})/dt$ by means of relations

$$\rho = \det \parallel \xi_{\alpha,\beta} \parallel, \qquad \alpha, \beta = 1, 2, \dots n, \qquad \xi_{\beta,\alpha} \equiv \partial_\alpha \xi_\beta, \qquad \mathbf{j} = \rho \frac{d\mathbf{x}}{dt} \tag{A.5}$$

and the functions $\boldsymbol{\xi} = \boldsymbol{\xi}(x)$ are determined as a solution of the system of equations

$$x^\alpha = x^\alpha(t, \boldsymbol{\xi}), \qquad \alpha = 1, 2, \dots n \tag{A.6}$$

Let us introduce the time Lagrangian coordinate ξ_0, supposing that $t = t(\xi_0)$, $x = \{t, \mathbf{x}\} = x(\xi)$, $\xi = \{\xi_0, \boldsymbol{\xi}\}$. Then the action (A.2) can be rewritten in the form

$$\mathcal{A}_L[x, \kappa] = \int \mathcal{L}(x, \dot{x}, \kappa) d^{n+1}\xi, \qquad \dot{x} \equiv \partial x / \partial \xi_0, \qquad d^{n+1}\xi = d\xi_0 d\boldsymbol{\xi}, \tag{A.7}$$

$$\mathcal{L}(x, \dot{x}, \kappa) = L(x^0, \mathbf{x}, \dot{\mathbf{x}}/\dot{x}^0, \kappa)\dot{x}^0. \tag{A.8}$$

According to Eq.(A.8) the Lagrangian $\mathcal{L}(x, \dot{x}, \kappa)$ is a first order homogeneous function of $\dot{\mathbf{x}}$, i.e.

$$\mathcal{L}(x, a\dot{x}, \kappa) = a\mathcal{L}(x, \dot{x}, \kappa), \tag{A.9}$$

where a is an arbitrary parameter.

Let the variational problem with the functional (A.7) take place. Let us show that it is equivalent to the variational problem with the functional (A.3).

Considering $\xi = \{\xi_i\}$, $i = 0, 1, ..n$ as functions of $x = \{x^i\}$, $i = 0, 1, \dots n$, the variational problem with the action (A.7) can be formulated as a variational problem with the action

$$\mathcal{A}[\xi, \kappa] = \int \mathcal{L}(x, \frac{\partial J}{\partial \xi_{0,i}}, \kappa) d^{n+1}x, \qquad \xi = \xi(x) \tag{A.10}$$

where J is a Jacobian

$$J \equiv \frac{\partial(\xi_0, \xi_1, \ldots \xi_n)}{\partial(x^0, x^1, \ldots x^n)} \equiv \det \parallel \xi_{i,k} \parallel,$$

$$\xi_{i,k} \equiv \partial \xi_i / \partial x^k, \qquad i, k = 0, 1, \ldots n \tag{A.11}$$

and

$$\frac{\partial J}{\partial \xi_{0,i}} \equiv \frac{\partial(x^i, \xi_1, \xi_2, \ldots \xi_n)}{\partial(x^0, x^1, \ldots x^n)} \equiv \frac{\partial x^i}{\partial \xi_0} J, \qquad i = 0, 1, \ldots n \tag{A.12}$$

Deriving Eq.(A.10) from Eq.(A.7), the relations (A.9), (A.12) were used.

In the variational problem with the action (A.10)) the Lagrangian coordinates $\xi = \{\xi_i\}$, $i = 0, 1, \ldots n$ are dynamic variables considered as functions of coordinates $x = \{x^i\}$, $i = 0, 1, \ldots n$.

Let us use designations

$$j^i = \frac{\partial J}{\partial \xi_{0,i}}, \qquad i = 0, 1, \ldots n \tag{A.13}$$

Adding constraints (A.13) to the action (A.10), one does not change the variational problem, because Eq.(A.10) does not contain variables j^i, and Eq. (A.13) is not a constraint in reality. Let us introduce Eq.(A.13) into the action (A.10) by means of Lagrange multpliers $p = \{p_i\}$, $i = 0, 1, \ldots n$. j^i and p_i, $i = 0, 1, \ldots n$ mean respectively a current density and a momentum density in the space $\{t, \mathbf{x}\}$.

Then one obtains

$$\mathcal{A}[j, p, \xi, \kappa] = \int [\mathcal{L}(x, j, \kappa) - p_i (j^i - \frac{\partial J}{\partial \xi_{0,i}})] dx^{n+1}. \tag{A.14}$$

Dynamic equations generated by the action (A.14) have the form

$$\frac{\delta \mathcal{A}}{\delta \xi_\alpha} = -\partial_k (p_l \frac{\partial^2 J}{\partial \xi_{0,l} \partial \xi_{\alpha,k}}) = 0, \qquad \alpha = 1, 2, \ldots n; \tag{A.15}$$

$$\frac{\delta \mathcal{A}}{\delta j^i} = \frac{\partial \mathcal{L}}{\partial j^i} - p_i = 0, \qquad i = 0, 1, \ldots n \tag{A.16}$$

Variation with respect to p_i leads to Eq.(A.13).

Using identities

$$\partial_k(\frac{\partial^2 J}{\partial\xi_{0,l}\partial\xi_{i,k}}) \equiv 0, \qquad i,l = 0,1,\dots n, \qquad \partial_k j^k = \partial_k \frac{\partial J}{\partial\xi_{0,i}} \equiv 0, \tag{A.17}$$

$$\frac{\partial^2 J}{\partial\xi_{0,i}\partial\xi_{j,k}}\xi_{j,l} \equiv \delta_l^k \frac{\partial J}{\partial\xi_{0,i}} - \delta_l^i \frac{\partial J}{\partial\xi_{0,k}}, \qquad i,k,l = 0,1,\dots n, \tag{A.18}$$

and eliminating the variables ξ, one can reduce equatons (A.13), (A.15), (A.16) to the conventional hydrodynamic form

$$\partial_i j^i = 0, \tag{A.19}$$

$$j^i[\partial_i \frac{\partial\mathcal{L}(x,j,\kappa)}{\partial j^k} - \partial_k \frac{\partial\mathcal{L}(x,j,\kappa)}{\partial j^i}] = 0, \qquad k = 0,1,\dots n. \tag{A.20}$$

Instead of elimination of variables ξ Eqs. (A.15), (A.16) can be integrated in the form

$$p_k = \frac{\partial\mathcal{L}(x,j,\kappa)}{\partial j^k} = \hbar\partial_k\varphi + \hbar g^\beta(\boldsymbol{\xi})\partial_k\xi_\beta; \qquad k = 0,1,\dots n, \tag{A.21}$$

where $g^\beta(\boldsymbol{\xi})$, $\beta = 1,2,\dots n$ are arbitrary functions of the argument $\boldsymbol{\xi} = \{\xi_\alpha\}$, $\alpha = 1,2,\dots$n. φ is some new variable. $\hbar$ is the Planck constant which is introduced, for variables φ, $\boldsymbol{\xi} = \{\xi_\alpha\}$, $\alpha = 1,2,\dots n$ and the functions g^β were dimensionless. In this context $\hbar$ has no quantum meaning.

Indeed, let us substitute relation (A.21) into Eq.(A.15) and use the first Eq.(A.17), antisymmetry of $\partial^2 J/\partial\xi_{0,l}\partial\xi_{\alpha,k}$ and symmetry of $\partial_k\partial_l\varphi$ with respect to indices l,k. Then one obtains

$$\partial_k[\frac{\partial^2 J}{\partial\xi_{0,l}\partial\xi_{\alpha,k}} g^\beta(\boldsymbol{\xi})\xi_{\beta,l}] = 0, \qquad \alpha = 1,2,\dots n. \tag{A.22}$$

Now using the identity

$$\frac{\partial^2 J}{\partial\xi_{0,l}\partial\xi_{\alpha,k}}\xi_{\beta,l} \equiv -\delta_\beta^\alpha \frac{\partial J}{\partial\xi_{0,k}}, \qquad \alpha,\beta = 1,2\dots n, \qquad k = 0,1,\dots n, \tag{A.23}$$

and the second relation (A.17), one can reduce Eq.(A.22) to the form

$$-\delta_\beta^\alpha \frac{\partial J}{\partial\xi_{0,k}}\xi_{\gamma,k}\frac{\partial g^\beta(\boldsymbol{\xi})}{\partial\xi_\gamma} = 0, \qquad \alpha = 1,2,\dots n. \tag{A.24}$$

Due to the identity

$$\frac{\partial J}{\partial \xi_{0,k}}\xi_{l,k} \equiv \delta_l^0 J, \qquad l = 0, 1, \dots n \tag{A.25}$$

the equation (A.24) is satisfied for any functions $g^\beta(\boldsymbol{\xi})$, $\beta = 1, 2, \dots$n. Thus, the relation (A.21) is a general solution of the system of equations (A.15).

Let us substitute Eq.(A.21) into the action (A.14) and use that due to Eq.(A.25)

$$\frac{\partial J}{\partial \xi_{0,k}}p_k = \hbar\frac{\partial(\varphi, \xi_1, \xi_2, \dots \xi_n)}{\partial(x^0, x^1, \dots x^n)}. \tag{A.26}$$

Then the action (A.14) turns to Eq.(A.3). The term (A.26) is omitted, because it does not give any contribution into dynamic equations. Thus, Eq.(A.3) is a corollary of Eq.(A.2).

Now let the variational problem with the functional (A.3) take place. Corresponding dynamic equations have the form

$$\frac{\delta \mathcal{A}}{\delta \varphi} = \hbar\partial_i j^i = 0, \qquad i = 0, 1, \dots n; \tag{A.27}$$

$$\frac{\delta \mathcal{A}}{\delta j^i} = \frac{\partial \mathcal{L}}{\partial j^i} - \hbar\partial_i\varphi - \hbar g^\alpha(\boldsymbol{\xi})\xi_{\alpha,i} = 0, \qquad i = 0, 1, \dots n; \tag{A.28}$$

$$\frac{\delta \mathcal{A}}{\delta \xi_\alpha} = -\hbar[\frac{\partial g^\beta(\boldsymbol{\xi})}{\partial \xi_\alpha} - \frac{\partial g^\alpha(\boldsymbol{\xi})}{\partial \xi_\beta}]j^i\partial_i\xi_\beta = 0, \qquad \alpha = 1, 2, \dots n; \tag{A.29}$$

The relations (A.13) satisfy Eqs. (A.27), (A.29) identically, and usage of them as additional constraints does not change the variational problem. Substituting (A.13) into Eq.(A.3) and using identities (A.25), (A.26), one obtains Eq.(A.10) which is equivalent to Eq.(A.2).

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