Coordinateless description and deformation principle as foundations of physical geometry

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Abstract

Physical geometry studies mutual disposition of geometrical objects and points in space, or space-time, which is described by the distance function d, or by the world function $\sigma = d^2/2$. One suggests a new general method of the physical geometry construction. The proper Euclidean geometry is described in terms of its world function σ_{E}. Any physical geometry $\mathcal{G}$ is obtained from the Euclidean geometry as a result of replacement of the Euclidean world function σ_{E} by the world function σ of $\mathcal{G}$. This method is very simple and effective. It introduces a new geometric property: nondegeneracy of geometry. Using this method, one can construct deterministic space-time geometries with primordially stochastic motion of free particles and geometrized particle mass. Such a space-time geometry defined properly (with quantum constant as an attribute of geometry) allows one to explain quantum effects as a result of the statistical description of the stochastic particle motion (without a use of quantum principles).

1 Introduction

A geometry lies in the foundation of physics, and a true conception of geometry is very important for the consequent development of physics. It is common practice to think that all problems in foundations of geometry have been solved many years ago. It is valid, but this concerns the geometry considered to be a logical construction. Physicists are interested in the geometry considered as a science on mutual disposition of geometrical objects in the space or in the space-time. The two aspects of geometry are quite different, and one can speak about two different geometries, using for them two different terms. Geometry as a logical construction is a homogeneous geometry, where all points have the same properties. Well known mathematician Felix Klein [1] believed that only the homogeneous geometry deserves to be called a geometry. It is his opinion that the Riemannian geometry (in general, inhomogeneous geometry) should be qualified as a Riemannian geography, or a Riemannian topography. In other words, Felix Klein considered a geometry mainly as a logical construction. We shall refer to such a geometry as the mathematical geometry.

The geometry considered to be a science on mutual disposition of geometric objects will be referred to as a physical geometry, because the physicists are interested mainly in this aspect of a geometry. The physical geometries are inhomogeneous, in general, although they may be homogeneous also. On the one hand, the proper Euclidean geometry is a physical geometry. On the other hand, it is a logical construction, because it is homogeneous and can be constructed of simple elements (points, straights, planes, etc.). All elements of the Euclidean geometry have similar properties, which are described by axioms. Similarity of geometrical elements allows one to construct the mathematical (homogeneous) geometry by means of logical reasonings. The proper Euclidean geometry was constructed many years ago by Euclid. Consistency of this construction was investigated and proved in [2]. Such a construction is very complicated even in the case of the proper Euclidean geometry, because simple geometrical objects are used for construction of more complicated ones, and one cannot construct a complicated geometrical object $\mathcal{O}$ without construction of more simple constituents of this object.

Note that constructing his geometry, Euclid did not use coordinates for labeling of the space points. His description of the homogeneous geometry was coordinateless. It means that the coordinates are not a necessary attribute of the geometry. Coordinate system is a method of the geometry description, which may or may not be used. Application of coordinates and of other means of description poses the problem of separation of the geometry properties from the properties of the means of the description. Usually the separation of the geometry properties from the coordinate system properties is carried out as follows. The geometry is described in all possible coordinate systems. Transformations from one coordinate system to the another one form a group of transformation. Invariants of this transformation group are the same in all coordinate system, and hence, they describe properties of the geometry in question.

At this point we are to make a very important remark. *Any geometry is a totality of all geometric objects $\mathcal{O}$ and of all relations $\mathcal{R}$ between them.* Any geometric object $\mathcal{O}$ is a subset of points of some point set Ω, where the geometry is given. In the Riemannian geometry (and in other inhomogeneous geometries) the set Ω is supposed to be a n-dimensional manifold, whose points P are labelled by n coordinates $x = \{x^1, x^2, ...x^n\}$. This labelling (arithmetization of space) is considered to be a necessary attribute of the Riemannian geometry. Most geometers believe that the Riemannian geometry (and physical geometry, in general) cannot be constructed without introduction of the manifold. In other words, they believe that the manifold is an attribute of the Riemannian geometry (and of any continuous geometry, in general). This belief is founded on the fact, that the Riemannian geometry is always constructed on some manifold. But this belief is a delusion. The fact, that we always construct the physical geometry on some manifold, does not mean that the physical geometry cannot be constructed without a reference to a manifold, or to a coordinate system. Of course, some labelling of the spatial points (coordinate system) is convenient, but this labelling has no relations to the construction of the geometry, and the physical geometry should be constructed without a reference to coordinate system. Application of the coordinate system imposes constraints on properties of the constructed physical geometry. For instance, if we use a continuous coordinate system (manifold) we can construct only continuous physical geometry. To construct a discrete physical geometry, the geometry construction is not to contain a reference to the coordinate system.

Here we present the method of the physical geometry construction, which does not contain a reference to the coordinate system and other means of description. It contains a reference only to the distance function d, which is a real characteristic of physical geometry.

If a geometry is inhomogeneous, and the straights located in different places have different properties, it is impossible to describe properties of straights by means of axioms, because there are no such axioms for the whole geometry. Mutual disposition of points in a physical (inhomogeneous) geometry, which is given on the set Ω of points P, is described by the distance function $d(P,Q)$

$$d: \qquad \Omega\times\Omega\rightarrow\mathbb{R}, \qquad d(P,P)=0, \qquad \forall P\in\Omega \tag{1.1}$$

The distance function d is the main characteristic of the physical geometry. Besides, the distance function d is an *unique characteristic* of any physical geometry. *The distance function d determines completely the physical geometry, and one does not need any additional information for determination of the physical geometry.* This statement is very important for construction of a physical geometry. It will be proved below. Any physical geometry $\mathcal{G}$ is constructed from the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ by means of a deformation, i.e. by a replacement of the Euclidean distance function d_{E} by the distance function of the geometry in question. For instance, constructing the Riemannian geometry, we replace the Euclidean infinitesimal distance $dS_{\mathrm{E}}=\sqrt{g_{\mathrm{E}ik}dx^idx^k}$ by the Riemannian one $dS=\sqrt{g_{ik}dx^idx^k}$. There is no method of the inhomogeneous physical geometry construction other, than the deformation of the Euclidean geometry (or some other homogeneous geometry) which is constructed as a mathematical geometry on the basis of its axiomatics and logic. Unfortunately, conventional method of the Riemannian geometry construction contains a reference to the coordinate system. But this reference can be eliminated, provided that we use finite distances d instead of infinitesimal distances dS.

For description of a physical geometry one uses the world function σ [3], which is connected with the distance function d by means of the relation $\sigma(P,Q)=\frac{1}{2}d^2(P,Q)$. The world function σ of the σ-space $V=\{\sigma,\Omega\}$ is defined by the relation

$$\sigma: \qquad \Omega\times\Omega\rightarrow\mathbb{R}, \qquad \sigma(P,P)=0, \qquad \forall P\in\Omega \tag{1.2}$$

Application of the world function is more convenient in the relation that the world function is real, when the distance function d is imaginary and does not satisfy definition (1.1). It is important at the consideration of the space-time geometry as a physical geometry.

In general, a physical geometry cannot be constructed as a logical building, because any change of the world function should be accompanied by a change of axiomatics. This is practically aerial, because the set of possible physical geometries is continual. Does the world function contain full information which is necessary for construction of the physical geometry? It is a very important question. For instance, can one derive the space dimension from the world function in the case of Euclidean geometry? Slightly below we shall answer this question in the affirmative. Now we formulate the method of the physical geometry construction.

Let us imagine that the proper Euclidean geometry $\mathcal{G}_{\rm E}$ can be described completely in terms and only in terms of the Euclidean world function $\sigma_{\rm E}$. Such a description is called σ-immanent. It means that any geometrical object $\mathcal{O}_{\rm E}$ and any relation $\mathcal{R}_{\rm E}$ between geometrical objects in $\mathcal{G}_{\rm E}$ can be described in terms of $\sigma_{\rm E}$ in the form $\mathcal{O}_{\rm E}(\sigma_{\rm E})$ and $\mathcal{R}_{\rm E}(\sigma_{\rm E})$. To obtain corresponding geometrical object $\mathcal{O}$ and corresponding relation $\mathcal{R}$ between the geometrical objects in other physical geometry $\mathcal{G}$, it is sufficient to replace the Euclidean world function $\sigma_{\rm E}$ by the world function σ of the physical geometry $\mathcal{G}$ in description of $\mathcal{O}_{\rm E}(\sigma_{\rm E})$ and $\mathcal{R}_{\rm E}(\sigma_{\rm E})$.

$$\mathcal{O}_{\rm E}(\sigma_{\rm E}) \rightarrow \mathcal{O}_{\rm E}(\sigma), \qquad \mathcal{R}_{\rm E}(\sigma_{\rm E}) \rightarrow \mathcal{R}_{\rm E}(\sigma)$$

Index 'E' shows that the geometric object is constructed on the basis of the Euclidean axiomatics. Thus, one can obtain another physical geometry $\mathcal{G}$ from the Euclidean geometry $\mathcal{G}_{\rm E}$ by a simple replacement of $\sigma_{\rm E}$ by σ. For such a construction one needs no axiomatics and no reasonings. One needs no means of descriptions (topological structures, continuity, coordinate system, manifold, dimension, etc.). In fact, one uses implicitly the axiomatics of the Euclidean geometry, which is deformed by the replacement $\sigma_{\rm E} \rightarrow \sigma$. This replacement may be interpreted as a deformation of the Euclidean space. Absence of a reference to the means of description is an advantage of the considered method of the geometry construction. Besides, there is no necessity to construct the whole geometry $\mathcal{G}$. We can construct and investigate only that part of the geometry $\mathcal{G}$ which we are interested in. Any physical geometry may be constructed as a result of a deformation of the Euclidean geometry.

The geometric object $\mathcal{O}$ is described by means of the skeleton-envelope method [5]. It means that any geometric object $\mathcal{O}$ is considered to be a set of intersections and joins of elementary geometric objects (EGO).

The finite set $\mathcal{P}^n \equiv \{P_0, P_1, ..., P_n\} \subset \Omega$ of parameters of the envelope function $f_{\mathcal{P}^n}$ is the skeleton of elementary geometric object (EGO) $\mathcal{E} \subset \Omega$. The set $\mathcal{E} \subset \Omega$ of points forming EGO is called the envelope of its skeleton $\mathcal{P}^n$. For continuous physical geometry the envelope $\mathcal{E}$ is usually a continual set of points. The envelope function $f_{\mathcal{P}^n}$, determining EGO is a function of the running point $R \in \Omega$ and of parameters $\mathcal{P}^n \subset \Omega$. The envelope function $f_{\mathcal{P}^n}$ is supposed to be an algebraic function of s arguments $w = \{w_1, w_2, ...w_s\}$, $s = (n+2)(n+1)/2$. Each of arguments

$w_k = \sigma(Q_k, L_k)$ is a σ-function of two arguments $Q_k, L_k \in \{R, \mathcal{P}^n\}$, either belonging to skeleton $\mathcal{P}^n$, or coinciding with the running point R. Thus, any elementary geometric object $\mathcal{E}$ is determined by its skeleton and its envelope function.

For instance, the sphere $\mathcal{S}(P_0, P_1)$ with the center at the point P_0 is determined by the relation

$$\mathcal{S}(P_0, P_1) = \{R | f_{P_0 P_1}(R) = 0\}, \qquad f_{P_0 P_1}(R) = \sqrt{2\sigma(P_0, P_1)} - \sqrt{2\sigma(P_0, R)} \quad (1.3)$$

where P_1 is a point belonging to the sphere. The elementary object $\mathcal{E}$ is determined in all physical geometries at once. In particular, it is determined in the proper Euclidean geometry, where we can obtain its meaning. We interpret the elementary geometrical object $\mathcal{E}$, using our knowledge of the proper Euclidean geometry. Thus, the proper Euclidean geometry is used as a sample geometry for interpretation of any physical geometry.

We do not try to repeat subscriptions of Euclid at construction of the geometry. We take the geometrical objects and relations between them, prepared in the framework of the Euclidean geometry and describe them in terms of the world function. Thereafter we deform them, replacing the Euclidean world function σ_{E} by the world function σ of the geometry in question. In practice the construction of the elementary geometry object is reduced to the representation of the corresponding Euclidean geometrical object in the σ-immanent form, i.e. in terms of the Euclidean world function. The last problem is the problem of the proper Euclidean geometry. The problem of representation of the geometrical object (or relation between objects) in the σ-immanent form is a real problem of the physical geometry construction.

It is very important, that such a construction does not use coordinates and other methods of description, because the application of the means of description imposes constraints on the constructed geometry. Any means of description is a structure St given on the basic Euclidean geometry with the world function σ_{E}. Replacement $\sigma_{\mathrm{E}} \to \sigma$ is sufficient for construction of unique physical geometry $\mathcal{G}_\sigma$. If we use an additional structure St for construction of physical geometry, we obtain, in general, other geometry $\mathcal{G}_{St}$, which coincide with $\mathcal{G}_\sigma$ not for all σ, but only for some of world functions σ. Thus, a use of additional means of description restricts the list of possible physical geometries. For instance, if we use the coordinate description at construction of the physical geometry, the obtained geometry appears to be continuous, because description by means of the coordinates is effective only for continuous geometries, where the number of coordinates coincides with the geometry dimension.

Constructing geometry $\mathcal{G}$ by vtans of a deformation we use essentially the fact that the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ is a mathematical geometry, which has been constucted on the basis of Euclidean axiomatics and logical reasonings.

We shall refer to the described method of the physical geometry construction as the deformation principle and interpret the deformation in the broad sense of the word. In particular, a deformation of the Euclidean space may transform an Euclidean surface into a point, and an Euclidean point into a surface. Such a deformation may remove some points of the Euclidean space, violating its continuity, or decreasing its dimension. Such a deformation may add supplemental points to the Euclidean space, increasing its dimension. In other words, the deformation principle is a very general method of the physical geometry construction.

The deformation principle as a method of the physical geometry construction contains two essential stages:

(i) Representation of geometrical objects $\mathcal{O}$ and relations $\mathcal{R}$ of the Euclidean geometry in the σ-immanent form, i.e. in terms and only in terms of the world function σ_{E}.

(ii) Replacement of the Euclidean world function σ_{E} by the world function σ of the geometry in question.

A physical geometry, constructed by means of the only deformation principle (i.e. without a use of other methods of the geometry construction) is called T-geometry (tubular geometry) [4, 5, 6]. The T-geometry is the most general kind of the physical geometry.

Application of the deformation principle is restricted by two constraints.

1. Describing Euclidean geometric objects $\mathcal{O}(\sigma_{\mathrm{E}})$ and Euclidean relation $\mathcal{R}(\sigma_{\mathrm{E}})$ in terms of σ_{E}, we are not to use special properties of Euclidean world function σ_{E}. In particular, definitions of $\mathcal{O}(\sigma_{\mathrm{E}})$ and $\mathcal{R}(\sigma_{\mathrm{E}})$ are to have similar form in Euclidean geometries of different dimensions. They must not depend on the dimension of the Euclidean space.

2. The deformation principle is to be applied separately from other methods of the geometry construction. In particular, one may not use topological structures in construction of a physical geometry, because for effective application of the deformation principle the obtained physical geometry must be determined only by the world function (metric).

2 Description of the proper Euclidean space in terms of the world function

The crucial point of the T-geometry construction is the description of the proper Euclidean geometry in terms of the Euclidean world function σ_{E}. We shall refer to this method of description as the σ-immanent description. Unfortunately, it was unknown for many years, although all physicists knew that the infinitesimal interval $dS = \sqrt{g_{ik}dx^i dx^k}$ is the unique essential characteristic of the space-time geometry, and changing this expression, we change the space-time geometry. From physical viewpoint the σ-immanent description is very reasonable, because it does not contain any extrinsic information. The σ-immanent description does not refer to the means of description (dimension, manifold, coordinate system). Absence of references to means of description is important in the relation, that there is no necessity to separate the information on the geometry in itself from the information on the means of description. The σ-immanent description contains only essential characteristic of geometry: its world function. At first the σ-immanent description was obtained in 1990 [4].

The first question concerning the σ-immanent description is as follows. Does the world function contain sufficient information for description of a physical geometry? The answer is affirmative, at least, in the case of the proper Euclidean geometry, and this answer is given by the prove of the following theorem.

Let σ-space $V=\{\sigma,\Omega\}$ be a set Ω of points P with the given world function σ

$$\sigma: \qquad \Omega\times\Omega\rightarrow\mathbb{R}, \qquad \sigma\left(P,P\right)=0, \qquad \forall P\in\Omega \tag{2.1}$$

Let the vector $\mathbf{P}_0\mathbf{P}_1=\{P_0,P_1\}$ be the ordered set of two points P_0, P_1, and its length $|\mathbf{P}_0\mathbf{P}_1|$ is defined by the relation $|\mathbf{P}_0\mathbf{P}_1|^2=2\sigma\left(P_0,P_1\right)$.

Theorem

The σ-space $V=\{\sigma,\Omega\}$ is the n-dimensional proper Euclidean space, if and only if the world function σ satisfies the following conditions, written in terms of the world function σ.

I. Condition of symmetry:

$$\sigma\left(P,Q\right)=\sigma\left(Q,P\right), \qquad \forall P,Q\in\Omega \tag{2.2}$$

II. Definition of the dimension:

$$\exists\mathcal{P}^n\equiv\{P_0,P_1,...P_n\}, \qquad F_n\left(\mathcal{P}^n\right)\neq 0, \qquad F_k\left(\Omega^{k+1}\right)=0, \qquad k>n \tag{2.3}$$

where $F_n\left(\mathcal{P}^n\right)$ is the Gram's determinant

$$F_n\left(\mathcal{P}^n\right)=\det||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)||=\det||g_{ik}\left(\mathcal{P}^n\right)||, \qquad i,k=1,2,...n \tag{2.4}$$

The scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is defined by the relation

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)=\sigma\left(P_0,Q_1\right)+\sigma\left(P_1,Q_0\right)-\sigma\left(P_0,Q_0\right)-\sigma\left(P_1,Q_1\right) \tag{2.5}$$

Vectors $\mathbf{P}_0\mathbf{P}_i$, $i=1,2,...n$ are basic vectors of the rectilinear coordinate system K_n with the origin at the point P_0, and the metric tensors $g_{ik}\left(\mathcal{P}^n\right)$, $g^{ik}\left(\mathcal{P}^n\right)$, $i,k=1,2,...n$ in K_n are defined by the relations

$$\sum_{k=1}^{k=n}g^{ik}\left(\mathcal{P}^n\right)g_{lk}\left(\mathcal{P}^n\right)=\delta_l^i, \qquad g_{il}\left(\mathcal{P}^n\right)=\left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_l\right), \qquad i,l=1,2,...n \tag{2.6}$$

III. Linear structure of the Euclidean space:

$$\sigma\left(P,Q\right)=\frac{1}{2}\sum_{i,k=1}^{i,k=n}g^{ik}\left(\mathcal{P}^n\right)\left(x_i\left(P\right)-x_i\left(Q\right)\right)\left(x_k\left(P\right)-x_k\left(Q\right)\right), \qquad \forall P,Q\in\Omega \tag{2.7}$$

where coordinates $x_i\left(P\right)$, $i=1,2,...n$ of the point P are covariant coordinates of the vector $\mathbf{P}_0\mathbf{P}$, defined by the relation

$$x_i\left(P\right)=\left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}\right), \qquad i=1,2,...n \tag{2.8}$$

IV: The metric tensor matrix $g_{lk}(\mathcal{P}^n)$ has only positive eigenvalues

$$g_k > 0, \qquad k = 1, 2, ..., n \tag{2.9}$$

V. The continuity condition: the system of equations

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}) = y_i \in \mathbb{R}, \qquad i = 1, 2, ...n \tag{2.10}$$

considered to be equations for determination of the point P as a function of coordinates $y = \{y_i\}$, $i = 1, 2, ...n$ has always one and only one solution. Conditions II – V contain a reference to the dimension n of the Euclidean space.

As far as the σ-immanent description of the proper Euclidean geometry is possible, it is possible for any T-geometry, because any geometrical object $\mathcal{O}$ and any relation $\mathcal{R}$ in the physical geometry $\mathcal{G}$ is obtained from the corresponding geometrical object $\mathcal{O}_{\mathrm{E}}$ and from the corresponding relation $\mathcal{R}_{\mathrm{E}}$ in the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ by means of the replacement $\sigma_{\mathrm{E}} \to \sigma$ in description of $\mathcal{O}_{\mathrm{E}}$ an $\mathcal{R}_{\mathrm{E}}$. For such a replacement be possible, the description of $\mathcal{O}_{\mathrm{E}}$ and $\mathcal{R}_{\mathrm{E}}$ is not to refer to special properties of σ_{E}, described by conditions II – V. A formal indicator of the conditions II – V application is a reference to the dimension n, because any of conditions II – V contains a reference to the dimension n of the proper Euclidean space.

If nevertheless we use one of special properties II – V of the Euclidean space in the σ-immanent description of a geometrical object $\mathcal{O}$, or relation $\mathcal{R}$, we refer to the dimension n and, ultimately, to the coordinate system, which is only a means of description.

Let us show this in the example of the determination of the straight in the n-dimensional Euclidean space. The straight $\mathcal{T}_{P_0Q}$ in the proper Euclidean space is defined by two its points P_0 and Q $(P_0 \neq Q)$ as the set of points R

$$\mathcal{T}_{P_0Q} = \{R \mid \mathbf{P}_0\mathbf{Q} || \mathbf{P}_0\mathbf{R}\} \tag{2.11}$$

where condition $\mathbf{P}_0\mathbf{Q}||\mathbf{P}_0\mathbf{R}$ means that vectors $\mathbf{P}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$ are collinear, i.e. the scalar product $(\mathbf{P}_0\mathbf{Q}.\mathbf{P}_0\mathbf{R})$ of these two vectors satisfies the relation

$$\mathbf{P}_0\mathbf{Q}||\mathbf{P}_0\mathbf{R}: \qquad (\mathbf{P}_0\mathbf{Q}.\mathbf{P}_0\mathbf{R})^2 = (\mathbf{P}_0\mathbf{Q}.\mathbf{P}_0\mathbf{Q})(\mathbf{P}_0\mathbf{R}.\mathbf{P}_0\mathbf{R}) \tag{2.12}$$

where the scalar product is defined by the relation (2.5). Thus, the straight line $\mathcal{T}_{P_0Q}$ is defined σ-immanently, i.e. in terms of the world function σ. We shall use two different names (straight and tube) for the geometric object $\mathcal{T}_{P_0Q}$. We shall use the term "straight", when we want to stress that $\mathcal{T}_{P_0Q}$ is a result of deformation of the Euclidean straight. We shall use the term "tube", when we want to stress that $\mathcal{T}_{P_0Q}$ may be a many-dimensional surface.

In the Euclidean geometry one can use another definition of collinearity. Vectors $\mathbf{P}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$ are collinear, if components of vectors $\mathbf{P}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$ in some coordinate system are proportional. For instance, in the n-dimensional Euclidean space one can introduce rectangular coordinate system, choosing $n+1$ points $\mathcal{P}^n =$

$\{P_0, P_1, ...P_n\}$ and forming n basic vectors $\mathbf{P}_0\mathbf{P}_i$, $i=1,2,...n$. Then the collinearity condition can be written in the form of n equations

$$\mathbf{P}_0\mathbf{Q}||\mathbf{P}_0\mathbf{R}: \qquad (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{Q}) = a\,(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{R}), \qquad i=1,2,...n, \qquad a\in\mathbb{R} \tag{2.13}$$

where a is some constant. Relations (2.13) are relations for covariant components of vectors $\mathbf{P}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$ in the considered coordinate system with basic vectors $\mathbf{P}_0\mathbf{P}_i$, $i=1,2,...n$. Let points $\mathcal{P}^n$ be chosen in such a way, that $(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q})\neq 0$. Then eliminating the parameter a from relations (2.13), we obtain $n-1$ independent relations, and the geometrical object

$$\mathcal{T}_{Q\mathcal{P}^n} = \{R \mid \mathbf{P}_0\mathbf{Q}||\mathbf{P}_0\mathbf{R}\} = \bigcap_{i=2}^{i=n}\mathcal{S}_i, \tag{2.14}$$

$$\mathcal{S}_i = \left\{R\left|\frac{(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{Q})}{(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q})} = \frac{(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{R})}{(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R})}\right.\right\}, \qquad i=2,3,...n \tag{2.15}$$

defined according to (2.13), depends on $n+2$ points $Q,\mathcal{P}^n$. This geometrical object $\mathcal{T}_{Q\mathcal{P}^n}$ is defined σ-immanently. It is a complex, consisting of the straight line and the coordinate system, represented by $n+1$ points $\mathcal{P}^n=\{P_0,P_1,...P_n\}$. In the Euclidean space the dependence on the choice of the coordinate system and on $n+1$ points $\mathcal{P}^n$ determining this system, is fictitious. The geometrical object $\mathcal{T}_{Q\mathcal{P}^n}$ depends only on two points P_0,Q and coincides with the straight line $\mathcal{T}_{P_0Q}$. But at deformations of the Euclidean space the geometrical objects $\mathcal{T}_{Q\mathcal{P}^n}$ and $\mathcal{T}_{P_0Q}$ are deformed differently. The points $P_1,P_2,...P_n$ cease to be fictitious in definition of $\mathcal{T}_{Q\mathcal{P}^n}$, and geometrical objects $\mathcal{T}_{Q\mathcal{P}^n}$ and $\mathcal{T}_{P_0Q}$ become to be different geometric objects, in general. But being different, in general, they may coincide in some special cases.

What of the two geometrical objects in the deformed geometry should be interpreted as the straight line, passing through the points P_0 and Q in the geometry $\mathcal{G}$? Of course, it is $\mathcal{T}_{P_0Q}$, because its definition does not contain a reference to a coordinate system, whereas definition of $\mathcal{T}_{Q\mathcal{P}^n}$ depends on the choice of the coordinate system, represented by points $\mathcal{P}^n$. In general, definitions of geometric objects and relations between them are not to refer to the means of description.

But in the given case the geometrical object $\mathcal{T}_{P_0Q}$ is, in general, $(n-1)$-dimensional surface, whereas $\mathcal{T}_{Q\mathcal{P}^n}$ is an intersection of $(n-1)$ $(n-1)$-dimensional surfaces, i.e. $\mathcal{T}_{Q\mathcal{P}^n}$ is, in general, a one-dimensional curve. The one-dimensional curve $\mathcal{T}_{Q\mathcal{P}^n}$ corresponds better to our ideas on the straight line, than the $(n-1)$-dimensional surface $\mathcal{T}_{P_0Q}$. Nevertheless, in physical geometry $\mathcal{G}$ it is $\mathcal{T}_{P_0Q}$, that is an analog of the Euclidean straight line.

It is very difficult to overcome our conventional idea that the Euclidean straight line cannot be deformed into many-dimensional surface, and *this idea has been prevent for years from construction of T-geometries.* Practically one uses such physical geometries, where deformation of the Euclidean space transforms the Euclidean straight lines into one-dimensional lines. It means that one chooses such geometries, where geometrical objects $\mathcal{T}_{P_0Q}$ and $\mathcal{T}_{Q\mathcal{P}^n}$ coincide.

$$\mathcal{T}_{P_0Q} = \mathcal{T}_{Q\mathcal{P}^n} \tag{2.16}$$

Condition (2.16) of coincidence of the objects $\mathcal{T}_{P_0Q}$ and $\mathcal{T}_{Q\mathcal{P}^n}$, imposed on the T-geometry, restricts list of possible T-geometries.

Let us consider the metric geometry, given on the set Ω of points. The metric space $M=\{\rho,\Omega\}$ is given by the metric (distance) ρ.

$$\rho \quad : \quad \Omega\times\Omega\rightarrow[0,\infty)\subset\mathbb{R} \tag{2.17}$$

$$\rho(P,P) \;=\; 0, \qquad \rho(P,Q)=\rho(Q,P), \qquad \forall P,Q\in\Omega \tag{2.18}$$

$$\rho(P,Q) \;\geq\; 0, \qquad \rho(P,Q)=0, \quad \text{iff } P=Q, \qquad \forall P,Q\in\Omega \tag{2.19}$$

$$0 \;\leq\; \rho(P,R)+\rho(R,Q)-\rho(P,Q), \qquad \forall P,Q,R\in\Omega \tag{2.20}$$

At first sight the metric space is a special case of the σ-space (2.1), and the metric geometry is a special case of the T-geometry with additional constraints (2.19), (2.20) imposed on the world function $\sigma=\frac{1}{2}\rho^2$. However it is not so, because the metric geometry does not use the deformation principle. The fact, that the Euclidean geometry can be described σ-immanently, as well as the conditions (2.3) - (2.10), were not known until 1990. Additional (with respect to the σ-space) constraints (2.19), (2.20) are imposed to eliminate the situation, when the straight line is not a one-dimensional line. The fact is that, in the metric geometry the shortest (straight) line can be constructed only in the case, when it is one-dimensional.

Let us consider the set $\mathcal{EL}\left(P,Q,a\right)$ of points R

$$\mathcal{EL}\left(P,Q,a\right)=\left\{R|f_{P,Q,a}\left(R\right)=0\right\}, \qquad f_{P,Q,a}\left(R\right)=\rho(P,R)+\rho(R,Q)-2a \tag{2.21}$$

If the metric space coincides with the proper Euclidean space, this set of points is an ellipsoid with focuses at the points P,Q and the large semiaxis a. The relations $f_{P,Q,a}\left(R\right)>0$, $f_{P,Q,a}\left(R\right)=0$, $f_{P,Q,a}\left(R\right)<0$ determine respectively external points, boundary points and internal points of the ellipsoid. If $\rho\left(P,Q\right)=2a$, we obtain the degenerate ellipsoid, which coincides with the segment $\mathcal{T}_{[PQ]}$ of the straight line, passing through the points P, Q. In the proper Euclidean geometry, the degenerate ellipsoid is one-dimensional segment of the straight line, but it is not evident that it is one-dimensional in the case of arbitrary metric geometry. For such a degenerate ellipsoid be one-dimensional in the arbitrary metric space, it is necessary that any degenerate ellipsoid $\mathcal{EL}\left(P,Q,\rho\left(P,Q\right)/2\right)$ have no internal points. This constraint is written in the form

$$f_{P,Q,\rho(P,Q)/2}\left(R\right)=\rho(P,R)+\rho(R,Q)-\rho(P,Q)\geq 0 \tag{2.22}$$

Comparing relation (2.22) with (2.20), we see that the constraint (2.20) is introduced to make the straight (shortest) line to be one-dimensional (absence of internal points in the geometrical object determined by two points).

As far as the metric geometry does not use the deformation principle, it is a poor geometry, because in the framework of this geometry one cannot construct the scalar product of two vectors, define linear independence of vectors and construct such geometrical objects as planes. All these objects as well as other are constructed on the basis of the deformation of the proper Euclidean geometry.

Generalizing the metric geometry, Menger [7] and Blumenthal [8] removed the triangle axiom (2.20). They tried to construct the distance geometry, which would be a more general geometry, than the metric one. As far as they did not use the deformation principle, they could not determine the shortest (straight) line without a reference to the topological concept of the curve $\mathcal{L}$, defined as a continuous mapping

$$\mathcal{L}: \qquad [0,1] \rightarrow \Omega \tag{2.23}$$

which cannot be expressed only via the distance. As a result the distance geometry appeared to be not a pure metric geometry, what the T-geometry is.

3 Conditions of the deformation principle application

Riemannian geometries satisfy the condition (2.16). The Riemannian geometry is a kind of inhomogeneous physical geometry, and, hence, it uses the deformation principle. Constructing the Riemannian geometry, the infinitesimal Euclidean distance is deformed into the Riemannian distance. The deformation is chosen in such a way that any Euclidean straight line $\mathcal{T}_{\mathrm{E}P_0Q}$, passing through the point P_0, collinear to the vector $\mathbf{P}_0\mathbf{Q}$, transforms into the geodesic $\mathcal{T}_{P_0Q}$, passing through the point P_0, collinear to the vector $\mathbf{P}_0\mathbf{Q}$ in the Riemannian space.

Note that in T-geometries, satisfying the condition (2.16) for all points $Q, \mathcal{P}^n$, the straight line

$$\mathcal{T}_{Q_0;P_0Q} = \{R \mid \mathbf{P}_0\mathbf{Q} || \mathbf{Q}_0\mathbf{R}\} \tag{3.1}$$

passing through the point Q_0 collinear to the vector $\mathbf{P}_0\mathbf{Q}$, is not a one-dimensional line, in general. If the Riemannian geometries be T-geometries, they would contain non-one-dimensional geodesics (straight lines). But the Riemannian geometries are not T-geometries, because at their construction one uses not only the deformation principle, but some other methods, containing a reference to the means of description. In particular, in the Riemannian geometries the absolute parallelism is absent, and one cannot to define a straight line (3.1), because the relation $\mathbf{P}_0\mathbf{Q}||\mathbf{Q}_0\mathbf{R}$ is not defined, if points P_0 and Q_0 do not coincide. On one hand, a lack of absolute parallelism allows one to go around the problem of non-one-dimensional straight lines. On the other hand, it makes the Riemannian geometries to be inconsistent, because they cease to be T-geometries, which are consistent by the construction (see for details [9]).

The fact is that the application of *only deformation principle* is sufficient for construction of a physical geometry. Besides, such a construction is consistent, because the original Euclidean geometry is consistent and, deforming it, we do not use any reasonings. If we introduce additional structure (for instance, a topological structure) we obtain a fortified physical geometry, i.e. a physical geometry with additional structure on it. The physical geometry with additional structure on it is a more pithy construction, than the physical geometry simply. But it is valid only in the case, when we consider the additional structure as an addition to the physical geometry. If we use an additional structure in construction of the geometry, we identify the additional structure with one of structures of the physical geometry. If we demand that the additional structure to be a structure of physical geometry, we restrict an application of the deformation principle and reduce the list of possible physical geometries, because coincidence of the additional structure with some structure of a physical geometry is possible not for all physical geometries, but only for some of them.

Let, for instance, we use concept of a curve $\mathcal{L}$ (2.23) for construction of a physical geometry. The concept of curve $\mathcal{L}$, considered as a continuous mapping is a topological structure, which cannot be expressed only via the distance or via the world function. A use of the mapping (2.23) needs an introduction of topological space and, in particular, the concept of continuity. If we identify the topological curve (2.23) with the "metrical" curve, defined as a broken line

$$\mathcal{T}_{\mathrm{br}} = \bigcup_i \mathcal{T}_{[P_i P_{i+1}]}, \qquad \mathcal{T}_{[P_i P_{i+1}]} = \left\{ R | \sqrt{2\sigma\left(P_i, P_{i+1}\right)} - \sqrt{2\sigma\left(P_i, R\right)} - \sqrt{2\sigma\left(R, P_{i+1}\right)} \right\} \tag{3.2}$$

consisting of the straight line segments $\mathcal{T}_{[P_i P_{i+1}]}$ between the points P_i, P_{i+1}, we truncate the list of possible geometries, because such an identification is possible only in some physical geometries. Identifying (2.23) and (3.2), we eliminate all discrete physical geometries and those continuous physical geometries, where the segment $\mathcal{T}_{[P_i P_{i+1}]}$ of straight line is a surface, but not a one-dimensional set of points. Thus, additional structures may lead to (i) a fortified physical geometry, (ii) a restricted physical geometry and (iii) a restricted fortified physical geometry. The result depends on the method of the additional structure application.

Note that some constraints (continuity, convexity, lack of absolute parallelism), imposed on physical geometries are a result of a disagreement of the applied means of the geometry construction. In the T-geometry, which uses only the deformation principle, there is no such restrictions. Besides, the T-geometry accepts some new property of a physical geometry, which is not accepted by conventional versions of physical geometry. This property, called the geometry nondegeneracy, follows directly from the application of arbitrary deformations to the proper Euclidean geometry.

The geometry is degenerate at the point P_0 in the direction of the vector $\mathbf{Q}_0\mathbf{Q}$, $|\mathbf{Q}_0\mathbf{Q}| \neq 0$, if the relations

$$\mathbf{Q}_0\mathbf{Q} \uparrow\uparrow \mathbf{P}_0\mathbf{R}: \qquad (\mathbf{Q}_0\mathbf{Q}.\mathbf{P}_0\mathbf{R}) = \sqrt{|\mathbf{Q}_0\mathbf{Q}| \cdot |\mathbf{P}_0\mathbf{R}|}, \qquad |\mathbf{P}_0\mathbf{R}| = a \neq 0 \tag{3.3}$$

considered as equations for determination of the point R, have not more, than one solution for any $a \neq 0$. Otherwise, the geometry is nondegenerate at the point P_0 in the direction of the vector $\mathbf{Q}_0\mathbf{Q}$. Note that the first equation (3.3) is the condition of the parallelism of vectors $\mathbf{Q}_0\mathbf{Q}$ and $\mathbf{P}_0\mathbf{R}$.

The proper Euclidean geometry is degenerate, i.e. it is degenerate at all points in directions of all vectors. Considering the Minkowski geometry, one should distinguish between the Minkowski T-geometry and Minkowski geometry. The two geometries are described by the same world function and differ in the definition of the parallelism. In the Minkowski T-geometry the parallelism of two vectors $\mathbf{Q_0Q}$ and $\mathbf{P_0R}$ is defined by the first equation (3.3). This definition is based on the deformation principle. In Minkowski geometry the parallelism is defined by the relation of the type of (2.13)

$$\mathbf{Q}_0\mathbf{Q}\uparrow\uparrow\mathbf{P}_0\mathbf{R}: \qquad (\mathbf{P}_0\mathbf{P}_i.\mathbf{Q}_0\mathbf{Q}) = a\,(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{R}), \qquad i=1,2,...n, \qquad a>0 \tag{3.4}$$

where points $\mathcal{P}^n = \{P_0, P_1, ...P_n\}$ determine a rectilinear coordinate system with basic vectors $\mathbf{P}_0\mathbf{P}_i$, $i = 1,2,..n$ in the n-dimensional Minkowski geometry (n-dimensional pseudo-Euclidean geometry of index 1). Dependence of the definition (3.4) on the points $(P_1, P_2, ...P_n)$ is fictitious, but dependence on the number $n+1$ of points $\mathcal{P}^n$ is essential. Thus, definition (3.4) depends on the method of the geometry description.

The Minkowski T-geometry is degenerate at all points in direction of all timelike vectors, and it is nondegenerate at all points in direction of all spacelike vectors. The Minkowski geometry is degenerate at all points in direction of all vectors. Conventionally one uses the Minkowski geometry, ignoring the nondegeneracy in spacelike directions.

Considering the proper Riemannian geometry, one should distinguish between the Riemannian T-geometry and the Riemannian geometry. The two geometries are described by the same world function. They differ in the definition of the parallelism. In the Riemannian T-geometry the parallelism of two vectors $\mathbf{Q_0Q}$ and $\mathbf{P_0R}$ is defined by the first equation (3.3). In the Riemannian geometry the parallelism of two vectors $\mathbf{Q_0Q}$ and $\mathbf{P_0R}$ is defined only in the case, when the points P_0 and Q_0 coincide. Parallelism of remote vectors $\mathbf{Q_0Q}$ and $\mathbf{P_0R}$ is not defined, in general. This fact is known as absence of absolute parallelism.

The proper Riemannian T-geometry is locally degenerate, i.e. it is degenerate at all points P_0 in direction of vectors $\mathbf{P}_0\mathbf{Q}$. In the general case, when $P_0 \neq Q_0$, the proper Riemannian T-geometry is nondegenerate, in general. The proper Riemannian geometry is degenerate, because it is degenerate locally, whereas the nonlocal degeneracy is not defined in the Riemannian geometry, because of the lack of absolute parallelism. Conventionally one uses the Riemannian geometry (not Rienannian T-geometry) and ignores the property of the nondegenracy completely.

From the viewpoint of the conventional approach to the physical geometry the nondegeneracy is an undesirable property of a physical geometry, although from the logical viewpoint and from viewpoint of the deformation principle the nondegenracy is an inherent property of a physical geometry. The nonlocal nondegeneracy is ejected from the proper Riemannian geometry by denial of existence of the remote vector parallelism. Nondegeneracy in the spacelike directions is ejected from the Minkowski geometry by means of the redefinition of the two vectors parallelism. But the nondegeneracy is an important property of the real space-time geometry. To appreciate this, let us consider an example.

4 Simple example of nondegenerate space-time geometry

The T-geometry [5] is defined on the σ-space $V = \{\sigma, \Omega\}$, where Ω is an arbitrary set of points and the world function σ is defined by the relations

$$\sigma: \qquad \Omega \times \Omega \to \mathbb{R}, \qquad \sigma(P,Q) = \sigma(Q,P), \qquad \sigma(P,P) = 0, \qquad \forall P,Q \in \Omega \tag{4.1}$$

Geometrical objects (vector $\mathbf{PQ}$, scalar product of vectors $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$, collinearity of vectors $\mathbf{P}_0\mathbf{P}_1 || \mathbf{Q}_0\mathbf{Q}_1$, segment of straight line $\mathcal{T}_{[P_0P_1]}$, etc.) are defined on the σ-space in the same way, as they are defined σ-immanently in the proper Euclidean space. Practically one uses the deformation principle, although it is not mentioned in all definitions.

Let us consider a simple example of the space-time geometry $\mathcal{G}_{\rm d}$, described by the T-geometry on 4-dimensional manifold $\mathcal{M}_{1+3}$. The world function $\sigma_{\rm d}$ is described by the relation

$$\sigma_{\rm d} = \sigma_{\rm M} + D(\sigma_{\rm M}) = \begin{cases} \sigma_{\rm M} + d & \text{if } \sigma_0 < \sigma_{\rm M} \\ \left(1 + \frac{d}{\sigma_0}\right)\sigma_{\rm M} & \text{if } 0 \leq \sigma_{\rm M} \leq \sigma_0 \\ \sigma_{\rm M} & \text{if } \sigma_{\rm M} < 0 \end{cases} \tag{4.2}$$

where $d \geq 0$ and $\sigma_0 > 0$ are some constants. The quantity $\sigma_{\rm M}$ is the world function in the Minkowski space-time geometry $\mathcal{G}_{\rm M}$. In the orthogonal rectilinear (inertial) coordinate system $x = (t, \mathbf{x})$ the world function $\sigma_{\rm M}$ has the form

$$\sigma_{\rm M}(x, x') = \frac{1}{2}\left(c^2(t-t')^2 - (\mathbf{x} - \mathbf{x}')^2\right) \tag{4.3}$$

where c is the speed of the light.

Let us compare the broken line (3.2) in Minkowski space-time geometry $\mathcal{G}_{\rm M}$ and in the distorted geometry $\mathcal{G}_{\rm d}$. We suppose that $\mathcal{T}_{\rm br}$ is timelike broken line, and all links $\mathcal{T}_{[P_iP_{i+1}]}$ of $\mathcal{T}_{\rm br}$ are timelike and have the same length

$$|\mathbf{P}_i\mathbf{P}_{i+1}|_{\rm d} = \sqrt{2\sigma_{\rm d}(P_i, P_{i+1})} = \mu_{\rm d} > 0, \qquad i = 0, \pm1, \pm2, ... \tag{4.4}$$

$$|\mathbf{P}_i\mathbf{P}_{i+1}|_{\rm M} = \sqrt{2\sigma_{\rm M}(P_i, P_{i+1})} = \mu_{\rm M} > 0, \qquad i = 0, \pm1, \pm2, ... \tag{4.5}$$

where indices "d" and "M" mean that the quantity is calculated by means of $\sigma_{\rm d}$ and $\sigma_{\rm M}$ respectively. Vector $\mathbf{P}_i\mathbf{P}_{i+1}$ is regarded as the momentum of the particle at the segment $\mathcal{T}_{[P_iP_{i+1}]}$, and the quantity $|\mathbf{P}_i\mathbf{P}_{i+1}| = \mu$ is interpreted as its (geometric) mass. It follows from definition (2.5) and relation (4.2), that for timelike vectors $\mathbf{P}_i\mathbf{P}_{i+1}$ with $\mu > \sqrt{2\sigma_0}$

$$|\mathbf{P}_i\mathbf{P}_{i+1}|_{\rm d}^2 = \mu_{\rm d}^2 = \mu_{\rm M}^2 + 2d, \qquad \mu_{\rm M}^2 > 2\sigma_0 \tag{4.6}$$

$$(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1})_{\rm d} = (\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1})_{\rm M} + d \tag{4.7}$$

Calculation of the shape of the segment $\mathcal{T}_{[P_0P_1]}(\sigma_\mathrm{d})$ in $\mathcal{G}_\mathrm{d}$ gives the relation

$$r^2(\tau) = \begin{cases} \tau^2\mu_\mathrm{d}^2 \frac{\left(1-\frac{\tau d}{2(\sigma_0+d)}\right)^2}{\left(1-\frac{2d}{\mu_\mathrm{d}^2}\right)} - \frac{\tau^2\mu_\mathrm{d}^2\sigma_0}{(\sigma_0+d)}, & 0<\tau<\frac{\sqrt{2(\sigma_0+d)}}{\mu_\mathrm{d}} \\ \frac{3d}{2}+2d\left(\tau-1/2\right)^2\left(1-\frac{2d}{\mu_\mathrm{d}^2}\right)^{-1}, & \frac{\sqrt{2(\sigma_0+d)}}{\mu_\mathrm{d}}<\tau<1-\frac{\sqrt{2(\sigma_0+d)}}{\mu_\mathrm{d}} \\ (1-\tau)^2\mu_\mathrm{d}^2\left[\frac{\left(1-\frac{(1-\tau)d}{2(\sigma_0+d)}\right)^2}{\left(1-\frac{2d}{\mu_\mathrm{d}^2}\right)} - \frac{\sigma_0}{(\sigma_0+d)}\right], & 1-\frac{\sqrt{2(\sigma_0+d)}}{\mu_\mathrm{d}}<\tau<1 \end{cases}, \tag{4.8}$$

where $r(\tau)$ is the spatial radius of the segment $\mathcal{T}_{[P_0P_1]}(\sigma_\mathrm{d})$ in the coordinate system, where points P_0 and P_1 have coordinates $P_0=\{0,0,0,0\}$, $P_1=\{\mu_\mathrm{d},0,0,0\}$ and τ is a parameter along the segment $\mathcal{T}_{[P_0P_1]}(\sigma_\mathrm{d})$ ($\tau(P_0)=0$, $\tau(P_1)=1$). One can see from (4.8) that the characteristic value of the segment radius is $\sqrt{d}$.

Let the broken tube $\mathcal{T}_\mathrm{br}$ describe the "world line" of a free particle. It means by definition that any link $\mathbf{P}_{i-1}\mathbf{P}_i$ is parallel to the adjacent link $\mathbf{P}_i\mathbf{P}_{i+1}$

$$\mathbf{P}_{i-1}\mathbf{P}_i\uparrow\uparrow\mathbf{P}_i\mathbf{P}_{i+1}: \qquad (\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1}) - |\mathbf{P}_{i-1}\mathbf{P}_i|\cdot|\mathbf{P}_i\mathbf{P}_{i+1}| = 0 \tag{4.9}$$

Definition of parallelism is different in geometries $\mathcal{G}_\mathrm{M}$ and $\mathcal{G}_\mathrm{d}$. As a result links, which are parallel in the geometry $\mathcal{G}_\mathrm{M}$, are not parallel in $\mathcal{G}_\mathrm{d}$ and vice versa.

Let $\mathcal{T}_\mathrm{br}(\sigma_\mathrm{M})$ describe the world line of a free particle in the geometry $\mathcal{G}_\mathrm{M}$. The angle ϑ_M between the adjacent links in $\mathcal{G}_\mathrm{M}$ is defined by the relation

$$\cosh\vartheta_\mathrm{M} = \frac{(\mathbf{P}_{-1}\mathbf{P}_0.\mathbf{P}_0\mathbf{P}_1)_\mathrm{M}}{|\mathbf{P}_0\mathbf{P}_1|_\mathrm{M}\cdot|\mathbf{P}_{-1}\mathbf{P}_0|_\mathrm{M}} = 1 \tag{4.10}$$

The angle $\vartheta_\mathrm{M}=0$, and the geometrical object $\mathcal{T}_\mathrm{br}(\sigma_\mathrm{M})$ is a timelike straight line on the manifold $\mathcal{M}_{1+3}$.

Let now $\mathcal{T}_\mathrm{br}(\sigma_\mathrm{d})$ describe the world line of a free particle in the geometry $\mathcal{G}_\mathrm{d}$. The angle ϑ_d between the adjacent links in $\mathcal{G}_\mathrm{d}$ is defined by the relation

$$\cosh\vartheta_\mathrm{d} = \frac{(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1})_\mathrm{d}}{|\mathbf{P}_i\mathbf{P}_{i+1}|_\mathrm{d}\cdot|\mathbf{P}_{i-1}\mathbf{P}_i|_\mathrm{d}} = 1 \tag{4.11}$$

The angle $\vartheta_\mathrm{d}=0$ also. If we draw the broken tube $\mathcal{T}_\mathrm{br}(\sigma_\mathrm{d})$ on the manifold $\mathcal{M}_{1+3}$, using coordinates of basic points P_i and measure the angle ϑ_dM between the adjacent links in the Minkowski geometry $\mathcal{G}_\mathrm{M}$, we obtain for the angle ϑ_dM the following relation

$$\cosh\vartheta_\mathrm{dM} = \frac{(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1})_\mathrm{M}}{|\mathbf{P}_i\mathbf{P}_{i+1}|_\mathrm{M}\cdot|\mathbf{P}_{i-1}\mathbf{P}_i|_\mathrm{M}} = \frac{(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1})_\mathrm{d}-d}{|\mathbf{P}_i\mathbf{P}_{i+1}|_\mathrm{d}^2-2d} \tag{4.12}$$

Substituting the value of $(\mathbf{P}_{i-1}\mathbf{P}_i.\mathbf{P}_i\mathbf{P}_{i+1})_\mathrm{d}$, taken from (4.11), we obtain

$$\cosh\vartheta_\mathrm{dM} = \frac{\mu_\mathrm{d}^d-d}{\mu_\mathrm{d}^2-2d} \approx 1+\frac{d}{\mu_\mathrm{d}^2}, \qquad d\ll\mu_\mathrm{d}^2 \tag{4.13}$$

Hence, $\vartheta_{\mathrm{dM}} \approx \sqrt{2d}/\mu_{\mathrm{d}}$. It means, that the adjacent link is located on the cone of angle $\sqrt{2d}/\mu_{\mathrm{d}}$, and the whole line $\mathcal{T}_{\mathrm{br}}(\sigma_{\mathrm{d}})$ has a random shape, because any link wabbles with the characteristic angle $\sqrt{2d}/\mu_{\mathrm{d}}$. The wabble angle depends on the space-time distortion d and on the particle mass μ_{d}. The wabble angle is small for the large mass of a particle. The random displacement of the segment end is of the order $\mu_{\mathrm{d}}\vartheta_{\mathrm{dM}} = \sqrt{2d}$, i.e. of the same order as the segment width. It is reasonable, because these two phenomena have the common source: the space-time distortion D.

One should note that the space-time geometry influences the stochasticity of particle motion nonlocally in the sense, that the form of the world function (4.2) for values of $\sigma_{\mathrm{M}} < \frac{1}{2}\mu_{\mathrm{d}}^2$ is unessential for the motion stochasticity of the particle of the mass μ_{d}.

Such a situation, when the world line of a free particle is stochastic in the deterministic geometry, and this stochasticity depends on the particle mass, seems to be rather exotic and incredible. But experiments show that the motion of real particles of small mass is stochastic indeed, and this stochasticity increases, when the particle mass decreases. From physical viewpoint a theoretical foundation of the stochasticity is desirable, and some researchers invent stochastic geometries, noncommutative geometries and other exotic geometrical constructions, to obtain the quantum stochasticity. But in the Riemannian space-time geometry the particle motion does not depend on the particle mass, and in the framework of the Riemannian space-time geometry it is difficult to explain the quantum stochasticity by the space-time geometry properties. Distorted geometry $\mathcal{G}_{\mathrm{d}}$ explains the stochasticity and its dependence on the particle mass freely. Besides, at proper choice of the distortion d the statistical description of stochastic $\mathcal{T}_{\mathrm{br}}$ leads to the quantum description (Schrödinger equation) [10]. It is sufficient to set $d = 0.5\hbar\,(bc)^{-1}$, where $\hbar$ is the quantum constant, c is the speed of the light, and b is some universal constant, connecting the geometrical mass μ with the usual particle mass m by means of the relation $m = b\mu$. In other words, the distorted space-time geometry (4.2) is closer to the real space-time geometry, than the Minkowski geometry $\mathcal{G}_{\mathrm{M}}$.

Further development of the statistical description of geometrical stochasticity leads to a creation of the model conception of quantum phenomena (MCQP), which relates to the conventional quantum theory approximately in the same way as the statistical physics relates to the axiomatic thermodynamics. MCQP is the well defined relativistic conception with effective methods of investigation [11], whereas the conventional quantum theory is not well defined, because it uses incorrect space-time geometry, whose incorrectness is compensated by additional hypotheses (quantum principles). Besides, it has problems with application of the nonrelativistic quantum mechanical technique to the description of relativistic phenomena.

The geometry $\mathcal{G}_{\mathrm{d}}$ is a homogeneous geometry as well as the Minkowski geometry, because the world function σ_{d} is invariant with respect to all coordinate transformations, with respect to which the world function σ_{M} is invariant. In this connection the question arises, whether one could invent some axiomatics for $\mathcal{G}_{\mathrm{d}}$ and derive the geometry $\mathcal{G}_{\mathrm{d}}$ from this axiomatics by means of proper reasonings. Note that such an axiomatics is to depend on the parameter d, because the world function σ_{d} depends on this parameter. If $d = 0$, this axiomatics is to coincide with the axiomatics of the Minkowski geometry $\mathcal{G}_{\mathrm{M}}$. If $d \neq 0$, this axiomatics cannot coincide with the axiomatics of $\mathcal{G}_{\mathrm{M}}$, because some axioms of $\mathcal{G}_{\mathrm{M}}$ are not satisfied in this case. In general, the invention of axiomatics, depending on the parameter d and in the general case on the distortion function D, seems to be a very difficult problem. Besides, why invent the axiomatics? We had derived the axiomatics for the proper Euclidean geometry, when we constructed it before. There is no necessity to repeat this process any time, when we construct a new geometry. It is sufficient to apply the deformation principle to the constructed Euclidean geometry written σ-immanently. Application of the deformation principle to the Euclidean geometry is a very simple and general procedue, which is not restricted by continuity, convexity and other artificial constraints, generated by our preconceived approach to the physical geometry. (Bias of the approach is displayed in the antecedent supposition on the one-dimensionality of any straight line in any physical geometry).

Thus, we have seen that the nondegeneracy of the physical geometry as well as non-one-dimensionality of the straight line are properties of the real physical geometries. The proper Euclidean geometry is a ground for all physical geometries, and it is a degenerate geometry. Nevertheless, it is beyond reason to deny an existence of nondegenerate physical geometries.

Thus, the deformation principle together with the σ-immanent description appears to be a very effective mathematical tool for construction of physical geometries.

1. The deformation principle uses results obtained at construction of the proper Euclidean geometry and does not add any additional supposition on properties of geometrical objects.

2. The deformation principle uses only the real characteristic of the physical geometry – its world function and does not use any additional means of description.

3. The deformation principle is very simple and allows one to investigate only that part of geometry which one is interested in.

4. Application of the deformation principle allows one to obtain the true space-time geometry, whose unexpected properties cannot be obtained at the conventional approach to physical geometry.

References

[1] F. Klein, *Vorlesungen über die Entwicklung die Mathematik im 19. Jahrhundert* teil 1, Berlin, Springer 1926.

[2] D. Hilbert, *Grundlagen der Geometrie.* 7 Auflage, B.G.Teubner, , Leipzig, Berlin, 1930.

[3] J. L. Synge, *Relativity: The General Theory*, North-Holland, Amsterdam, 1960.

[4] Yu. A. Rylov, Extremal properties of Synge's world function and discrete geometry. *J. Math. Phys.* **31**, 2876-2890, (1990).

[5] Yu.A. Rylov, Geometry without topology as a new conception of geometry. *Int. Jour. Mat. & Mat. Sci.* **30**, iss. 12, 733-760, (2002), (available at http://arXiv.org/abs/math.MG/0103002).

[6] Yu. A. Rylov, Asymmetric nondegenerate geometry. (In preparation, available at http://arXiv.org/abs/math.MG/0205061).

[7] K. Menger, Untersuchen über allgemeine Metrik, *Mathematische Annalen,* **100**, 75-113, (1928).

[8] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Oxford, Clarendon Press, 1953.

[9] Yu. A. Rylov, Deformation principle and problem of parallelism in geometry and physics. (In preparation, available at http://arXiv.org/abs/ math.GM /0210413)

[10] Yu. A. Rylov, Non-Riemannian model of space-time responsible for quantum effects. *J. Math. Phys.* **32**, 2092-2098, (1991).

[11] Yu. A. Rylov," Model conception of quantum phenomena: logical structure and investigation methods ". (In preparation, available at http://arXiv.org/abs /physics/0310050, v2).