Deformation principle and problem of parallelism in geometry and physics.

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Abstract

The deformation principle admits one to obtain a very broad class of nonuniform geometries as a result of deformation of the proper Euclidean geometry. The Riemannian geometry is also obtained by means of a deformation of the Euclidean geometry. Application of the deformation principle appears to be not consecutive, and the Riemannian geometry appears to be not completely consistent. Two different definitions of two vectors parallelism are investigated and compared. The first definitions is based on the deformation principle. The second definition is the conventional definition of parallelism, which is used in the Riemannian geometry. It is shown, that the second definition is inconsistent. It leads to absence of absolute parallelism in Riemannian geometry and to discrimination of outcome outside the framework of the Riemannian geometry at description of the space-time geometry.

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1 Introduction

There are two different approaches two geometry: mathematical and physical ones. In the mathematical approach a geometry is a construction founded on a system of axioms about points and straights. Practically any system of axioms, containing concepts of a point and a straight, may be called a geometry. Well known mathematician Felix Klein [1] supposed that only such a construction on a point set is a geometry, where all points of the set have the same properties (uniform geometry). For instance, Felix Klein insisted that Euclidean geometry and Lobachevsky geometry are geometries, because they are uniform, whereas the Riemannian geometries are not geometries at all. As a rule the Riemannian geometries are not uniform, and their points have different properties. According to the Felix Klein viewpoint, they should be called as "Riemannian topographies" or as "Riemannian geographies". Thus, at the mathematical approach to geometry the main feature of geometry is existence of some axiomatics. One may say that the mathematical geometry (mathematical approach to geometry) is a system of axioms. Practically one can construct axiomatics only for uniform geometries, and any mathematical geometry is a uniform geometry.

Riemannian geometries are not uniform geometries, in general. Practically one cannot construct axiomatics for each of the continuous set of Riemannian geometries, and any Riemannian geometry is obtained as a result of some deformation of the proper Euclidean geometry, when the infinitesimal Euclidean interval ds_{E}^2 is replaced by the infinitesimal Riemannian interval $ds^2 = g_{ik}dx^i dx^k$. Such a replacement is a deformation of the Euclidean space.

Such an approach to geometry, when a geometry is obtained as a result of deformation of the proper Euclidean geometry will be referred to as the physical approach to geometry. The obtained geometry will be referred to as physical geometry. The physical geometry has not its own axiomatics. It uses "deformed " Euclidean axiomatics. The term "physical geometry" is used, because it is very convenient for application to physics and can be used as a space-time geometry. Felix Klein referred to a physical geometry as a topography, but we think that another name is important not in itself, but only because it describes another method of the geometry construction.

Physical geometry describes mutual disposition of geometric objects in the space, or mutual dispositions of events in the event space (space-time). The mutual dispositions is described by the distance between any two points. It is of no importance, whether the geometry has any axiomatics or not. One may say that the physical geometry (physical approach to geometry) is a conception, describing mutual dispositions of geometric objects and points. Physical geometry may be not uniform, and it is not uniform in many cases. Metric ρ (distance between two points) is a unique characteristic of a physical geometry. World function $\sigma = \frac{1}{2}\rho^2$ [2] is more convenient for description of a physical geometry, because it is real even for the space-time, where $\rho = \sqrt{2\sigma}$ may be imaginary. Besides, usually the term metric is associated with some constraints on metric (triangle axiom, positivity of ρ). The term world function does not associate with these constraints directly.

Construction of any physical geometry is determined by *the deformation principle*. It works as follows. The proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ can be described in terms and only in terms of the world function σ_{E}, provided σ_{E} satisfies some constraints formulated in terms of σ_{E} [3]. In means that all geometric objects $\mathcal{O}_{\mathrm{E}}$ can be described σ-immanently (i.e. in terms of σ_{E} and only of σ_{E}) $\mathcal{O}_{\mathrm{E}} = \mathcal{O}_{\mathrm{E}}(\sigma_{\mathrm{E}})$. Relations between geometric objects are described by some expressions $\mathcal{R}_{\mathrm{E}} = \mathcal{R}_{\mathrm{E}}(\sigma_{\mathrm{E}})$. Any physical geometry $\mathcal{G}_{\mathrm{A}}$ can be obtained from the proper Euclidean geometry by means of deformation, when the Euclidean world function σ_{E} is replaced by some other world function σ_{A} in all definitions of Euclidean geometric objects $\mathcal{O}_{\mathrm{E}} = \mathcal{O}_{\mathrm{E}}(\sigma_{\mathrm{E}})$ and in all Euclidean relations $\mathcal{R}_{\mathrm{E}} = \mathcal{R}_{\mathrm{E}}(\sigma_{\mathrm{E}})$ between them. As a result we have the following change

$$\mathcal{O}_{\mathrm{E}} = \mathcal{O}_{\mathrm{E}}(\sigma_{\mathrm{E}}) \rightarrow \mathcal{O}_{\mathrm{A}} = \mathcal{O}_{\mathrm{E}}(\sigma_{\mathrm{A}}), \qquad \mathcal{R}_{\mathrm{E}} = \mathcal{R}_{\mathrm{E}}(\sigma_{\mathrm{E}}) \rightarrow \mathcal{R}_{\mathrm{A}} = \mathcal{R}_{\mathrm{E}}(\sigma_{\mathrm{A}})$$

The set of all geometric objects $\mathcal{O}_{\mathrm{A}}$ and all relations $\mathcal{R}_{\mathrm{A}}$ between them forms a physical geometry, described by the world function σ_{A}. Index E in the relations of physical geometry $\mathcal{G}_{\mathrm{A}}$ means that axiomatics of the proper Euclidean geometry was used for construction of geometric objects $\mathcal{O}_{\mathrm{E}} = \mathcal{O}_{\mathrm{E}}(\sigma_{\mathrm{E}})$ and of relations between them $\mathcal{R}_{\mathrm{E}} = \mathcal{R}_{\mathrm{E}}(\sigma_{\mathrm{E}})$. The same axiomatics is used for all geometric objects $\mathcal{O}_{\mathrm{A}} = \mathcal{O}_{\mathrm{E}}(\sigma_{\mathrm{A}})$ and relations between them $\mathcal{R}_{\mathrm{A}} = \mathcal{R}_{\mathrm{E}}(\sigma_{\mathrm{A}})$ in the geometry $\mathcal{G}_{\mathrm{A}}$. But now this axiomatics has another form, because of deformation $\sigma_{\mathrm{E}} \rightarrow \sigma_{\mathrm{A}}$. It means that the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ is the basic geometry for all physical geometries $\mathcal{G}$ obtained by means of a deformation of the proper Euclidean geometry. If basic geometry is fixed (it is this case that will be considered further), the geometry on the arbitrary set Ω of points is called T-geometry (tubular geometry). The T-geometry is determined [3] by setting the world function σ:

$$\sigma: \quad \Omega \times \Omega \rightarrow \mathbb{R}, \qquad \sigma(P,P) = 0, \qquad \forall P \in \Omega \tag{1.1}$$

In general, no other constraints are imposed, although one can impose any additional constraints to obtain a special class of T-geometries. T-geometry is symmetric, if in addition

$$\sigma(P,Q) = \sigma(Q,P), \qquad \forall P,Q \in \Omega \tag{1.2}$$

Deformation $\mathbb{R}^n \rightarrow \Omega$ of the n-dimensional proper Euclidean space to an arbitrary set Ω of points is a deformation in the broad sense. This deformation can change the dimension of a geometric object and the dimension of the whole space. For instance, the resulting T-geometry does not depend on the dimension n of deformed proper Euclidean space. Only final world function σ is important for the T-geometry properties. This admits one to consider T-geometry as something self-sufficient and to ignore the deformation which produces T-geometry from the Euclidean geometry.

The Riemannian geometry is a physical geometry. It is constructed on the basis of the *deformation principle*, i.e. in the same way as T-geometry. But class of possible

Riemannian deformations is not so general as the class of all possible deformations. It is restricted by the constraint

$$\sigma_{\mathrm{R}}\left(x,x'\right)=\frac{1}{2}\left(\int\limits_{\mathcal{L}_{[xx']}}\sqrt{g_{ik}dx^{i}dx^{k}}\right)^{2} \tag{1.3}$$

where σ_{R} is the world function of Riemannian geometry, and $\mathcal{L}_{[xx']}$ denotes segment of geodesic between the points x and x'. The Riemannian geometry is determined by the dimension n and $n\left(n+1\right)/2$ functions g_{ik} of one point x, whereas the class of possible T-geometries is determined by one function σ of two points x and x'.

A use of the deformation principle is sufficient for construction of any physical geometry. All relations between geometric objects are appear to be as consistent as they are consistent in the proper Euclidean geometry. The deformation principle does not use any logical conclusions and leads to a construction of a consistent physical geometry. Moreover, a use of additional means of the geometry construction is undesirable, because these means may disagree with the deformation principle. In the case of such a disagreement the obtained geometry appears to be inconsistent.

Although the Riemannian geometry is a kind of physical geometry, at its construction one uses additional means of description (dimension, concept of a curve, coordinate system, continuous manifold). Some of them appear to be incompatible with the principle of the geometry deformation, and as a result the Riemannian geometry appears to be inconsistent. Constraint (1.3), imposed on the world function of Riemannian geometry, restricts the class of possible physical geometries and reduces this inconsistency, but it fails to eliminate inconsistency completely. The σ-Riemannian geometry, i.e. the physical geometry, constructed by means of only the deformation principle on n-dimensional manifold and restricted by the constraint [2]

$$\sigma_{i}\left(x,x'\right)g^{ik}\left(x\right)\sigma_{k}\left(x,x'\right)=2\sigma\left(x,x'\right),\qquad\sigma_{i}\left(x,x'\right)\equiv\frac{\partial\sigma\left(x,x'\right)}{\partial x^{i}} \tag{1.4}$$

which is equivalent to (1.3), is rather close to the Riemannian geometry. Nevertheless, the absolute parallelism is absent in the Riemannian geometry, but it takes place in the σ-Riemannian geometry. This difference means that the Riemannian geometry is inconsistent, because the σ-Riemannian geometry cannot be inconsistent.

From viewpoint of the deformation principle this inconsistency is conditioned by a use of special properties of the world function σ_{E} of n-dimensional proper Euclidean space. It means as follows. Before deformation the geometric objects $\mathcal{O}_{\mathrm{E}}$ and relations $\mathcal{R}_{\mathrm{E}}$ of the proper Euclidean geometry they are to be represented in the σ-immanent form. Representing $\mathcal{O}_{\mathrm{E}}$ and $\mathcal{R}_{\mathrm{E}}$ in terms of σ_{E}, we must not use special properties of Euclidean world function σ_{E}. These special properties of σ_{E} are formulated for n-dimensional Euclidean space and contain a reference to the space dimension n. If these properties are used at the description of $\mathcal{O}_{\mathrm{E}}$, or $\mathcal{R}_{\mathrm{E}}$, the description contains a reference to the dimension n of the space. In this case after deformation we attribute some properties of n-*dimensional* proper Euclidean geometry to the constructed physical geometry. Formal criterion of application of special properties of $\sigma_{\rm E}$ is a reference to the dimension n. Being transformed to σ-immanent form, such a description of $\mathcal{O}_{\rm E}$, or $\mathcal{R}_{\rm E}$ contains additional points which are not characteristic for $\mathcal{O}_{\rm E}$, or $\mathcal{R}_{\rm E}$. Practically, these additional points describe the coordinate system, and number of these points depends on the space dimension n.

Inconsistency of the Riemannian geometry manifests itself in the parallelism problem. The definition of two vectors parallelism in Riemannian geometry has two defects:

1. Definition of parallelism in Riemannian geometry is coordinate dependent, because it contains a reference to the number of coordinates (space dimension).

2. Parallelism is defined only for two infinitesimally close vectors. Parallelism of two remote vectors at points P_1 and P_2 is defined by means of a parallel transport along some curve connecting points P_1 and P_2. In curved space the result of parallel transport depends on the path of transport, and the absolute parallelism is absent, in general.

The problem of definition of two vectors parallelism is very important, because parallelism lies in the foundation of particle dynamics. For instance, in the curved space-time the free particle motion is described by the geodesic equation

$$d\dot{x}^i = -\Gamma^i_{kl}\dot{x}^k dx^l, \qquad dx^l = \dot{x}^i d\tau \tag{1.5}$$

where Γ^i_{kl} is the Christoffel symbol. Equations (1.5) describe parallel transport of the particle velocity vector $\dot{x}^i$ along the direction $dx^i = \dot{x}^i d\tau$ determined by the velocity vector $\dot{x}^i$. If the parallel transport (1.5) appears to be incorrect and needs a modification, the equation of motion of a free particle needs a modification also. For instance, if at the point x a set of vectors u^i, which are parallel to the velocity vector $\dot{x}^i$, appears to be consisting of many mutually noncollinear vectors, the parallel transport of the velocity vector $\dot{x}^i$ stops to be single-valued, and the world line of a free particle becomes to be random.

Definition of the scalar product of two vectors in Riemannian geometry contains special properties of Euclidean world function and attributes to Riemannian geometry some properties of the proper Euclidean geometry, mainly one-dimensionality of straight (geodesics). In general, this definition of scalar product is incompatible with the deformation principle. Restriction (1.3), imposed on the world function $\sigma_{\rm R}$, eliminates geometries admitting non-one-dimensional straight and eliminates some corollaries of this incompatibility, but not all. Creators of the Riemannian geometry tried to conserve one-dimensional straights (geodesics) in the Riemannian geometry. They had achieved this goal, but not completely, because only straight (geodesic) $\mathcal{L}(P_0, \mathbf{P}_0\mathbf{P}_1)$, drawn through the point P_0 parallel to the vector $\mathbf{P}_0\mathbf{P}_1$, is one-dimensional, whereas the straight $\mathcal{L}(Q_0, \mathbf{P}_0\mathbf{P}_1)$, drawn through the point Q_0 ($Q_0 \neq P_0$) parallel to the vector $\mathbf{P}_0\mathbf{P}_1$, is not one-dimensional, in general. Note that the Riemannian geometry denied a possibility of constructing the straight $\mathcal{L}(Q_0, \mathbf{P}_0\mathbf{P}_1)$, referring to lack of absolute parallelism. Lack of one-dimensionality for $\mathcal{L}(Q_0, \mathbf{P}_0\mathbf{P}_1)$ can be seen only in the σ-Riemannian geometry, which is defined as a consistent T-geometry, whose world function is restricted by the relation (1.3). In the present paper we consider and compare definitions of parallelism in Riemannian geometry and in the consistent T-geometry (σ-Riemannian geometry) and discuss corollaries of the Riemannian geometry inconsistency.

2 Definition of parallelism

Vector $\mathbf{P}_0\mathbf{P}_1 \equiv \overrightarrow{P_0P_1}$ in T-geometry is the ordered set of two points $\mathbf{P}_0\mathbf{P}_1 = \{P_0, P_1\}$, $P_0, P_1 \in \Omega$. (The points P_0, P_1 may be similar). The scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is defined by the relation

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \sigma(P_0, Q_1) + \sigma(P_1, Q_0) - \sigma(P_0, Q_0) - \sigma(P_1, Q_1), \tag{2.1}$$

for all $P_0, P_1, Q_0, Q_1 \in \Omega$. As it follows from (1.1), (2.1), in the symmetric T-geometry

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_1), \qquad \forall P_0, P_1, Q_0, Q_1 \in \Omega \tag{2.2}$$

Further we shall consider only symmetric T-geometry and shall not stipulate this. (Asymmetric T-geometry is considered in [4]).

When the world function σ is such one [3] that the σ-space $V = \{\sigma, \Omega\}$ is the n-dimensional proper Euclidean space E_n the scalar product (2.1) turns to the scalar product of two vectors in E_n. Besides, it follows from (1.1), (2.1) that in any T-geometry

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = -(\mathbf{P}_1\mathbf{P}_0.\mathbf{Q}_0\mathbf{Q}_1), \qquad \forall P_0, P_1, Q_0, Q_1 \in \Omega \tag{2.3}$$

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) + (\mathbf{P}_1\mathbf{P}_2.\mathbf{Q}_0\mathbf{Q}_1) = (\mathbf{P}_0\mathbf{P}_2.\mathbf{Q}_0\mathbf{Q}_1), \tag{2.4}$$

for all $P_0, P_1, P_2, Q_0, Q_1 \in \Omega$. Two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ are parallel ($\mathbf{P}_0\mathbf{P}_1 \uparrow\uparrow \mathbf{Q}_0\mathbf{Q}_1$), if the angle between the vectors is equal to 0, i.e.

$$\frac{(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)}{|\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1|} = 1, \qquad |\mathbf{P}_0\mathbf{P}_1| \equiv \sqrt{(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1)}, \qquad |\mathbf{Q}_0\mathbf{Q}_1| \equiv \sqrt{(\mathbf{Q}_0\mathbf{Q}_1.\mathbf{Q}_0\mathbf{Q}_1)} \tag{2.5}$$

Definition of parallelism (2.5) does not contain a reference to coordinate system, to a path of parallel transport, or to other means of description. The relation (2.5) determines parallelism of two remote vectors, using only world function σ. Parallelism of two vectors is absolute in the sense that any two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ are either parallel or not.

Vector **u** in n-dimensional Riemannian geometry is defined as a set of n quantities $\mathbf{u}=\{u_i\}$, $i=1,2,...n$, given at some coordinate system K_n with coordinates $x=\{x^i\}$, $i=1,2,...n$. At the coordinate transformation $K_n\rightarrow\tilde{K}_n$

$$x^i\rightarrow\tilde{x}^i=\tilde{x}^i(x), \qquad i=1,2,...n \tag{2.6}$$

covariant components u_i of the vector **u** transforms as follows

$$u_i\rightarrow\tilde{u}_i=\frac{\partial x^k}{\partial\tilde{x}^i}u_k, \qquad i=1,2,...n \tag{2.7}$$

Summation from 1 to n is made over repeating indices.

Let x be coordinates of the point P, and x' be coordinates of the point P'. Then the vector $\mathbf{PP}'$ at the point P is introduced by the relation

$$\mathbf{PP}'=\{-\sigma_i(x,x')\}, \qquad i=1,2,...n \tag{2.8}$$

$$\sigma_i\equiv\partial_i\sigma(x,x')\equiv\frac{\partial\sigma(x,x')}{\partial x^i}, \qquad i=1,2,...n \tag{2.9}$$

where the world function σ is defined by the relation (1.3). Here $\sigma(x,x')=\sigma(P,P')$ is the world function between the points P and P'.

In the n-dimensional proper Euclidean space E_n and rectilinear coordinate system K_n the world function has the form

$$\sigma(x,x')=\frac{1}{2}g_{(\mathrm{E})ik}\left(x^i-x'^i\right)\left(x^k-x'^k\right), \qquad g_{(\mathrm{E})ik}=\mathrm{const} \tag{2.10}$$

and according to (2.8) the vector $\mathbf{PP}'$ has covariant coordinates $\left\{g_{(\mathrm{E})ik}\left(x^k-x'^k\right)\right\}$, $i=1,2,...n$. Scalar product of two vectors $\mathbf{PP}'$ and $\mathbf{PP}''$, having common origin at the point P has the form

$$(\mathbf{PP}'.\mathbf{PP}'')_{\mathrm{R_n}}=g^{ik}(x)\,\sigma_i(x,x')\,\sigma_k(x,x'') \tag{2.11}$$

where index "$\mathrm{R_n}$" means that the scalar product is defined in the Riemannian space R_n according to conventional rules of Riemannian geometry.

According to (2.11) and in virtue of properties (1.4) of the world function of the Riemannian space we obtain

$$|\mathbf{PP}'|^2\equiv(\mathbf{PP}'.\mathbf{PP}')=2\sigma(P,P') \tag{2.12}$$

The definition (2.11) coincide with the general definition (2.1) in the following cases: (1) if the Riemannian space R_n coincide with the Euclidean space E_n, (2) if vectors $\mathbf{PP}'$ and $\mathbf{PP}''$ are infinitesimally small, (3) if $\sigma_i(x,x')=a\sigma_i(x,x'')$, $i=1,2,...n$, $a=$const (as it follows from (1.4), (2.12)). In other cases the scalar products (2.11) and (2.1) do not coincide, in general. Besides, the scalar product (2.11) is defined only for vectors having a common origin. In the case of vectors $\mathbf{PP}'$ and $\mathbf{QQ}'$ with different origins the scalar product $(\mathbf{PP}'.\mathbf{QQ}')$ must be defined in addition. But this scalar product is not defined in Riemannian geometry, because to define $(\mathbf{PP}'.\mathbf{QQ}')$ for $Q\neq P$, the vector $\mathbf{PP}'$ must be transported at the point Q in parallel, and thereafter the definition (2.11) should be used. Result of parallel transport depends on the path of transport, and the scalar product $(\mathbf{PP}'.\mathbf{QQ}')$ for $Q\neq P$ cannot be defined uniquely. If one uses definition (2.1) and relation (1.3) for determination of $(\mathbf{PP}'.\mathbf{QQ}')$ for $\mathbf{Q}\neq\mathbf{P}$ the result is unique, but definition of parallelism on the base of this scalar product leads to a set of many vectors $\mathbf{QQ}'$, which are parallel to $\mathbf{PP}'$, whereas the conventional conception of Riemannian geometry demands that such a vector would be only one. In other words, the Riemannian geometry becomes to be inconsistent at this point.

The definition (2.1) does not contain any reference to the means of description, whereas the definition (2.11) does. The definition (2.11) is invariant with respect to coordinate transformation (2.6), but it refers to the dimension n of the space R_n and existence of n-dimensional manifold. It means that the definition (2.1) is more general and perfect, because it does not use special properties of the Euclidean world function σ_{E}.

These special properties of n-dimensional proper Euclidean space are determined as follows [3].

I:

$$\exists\mathcal{P}^n=\{P_0,P_1,...P_n\}\,,\qquad F_n(\mathcal{P}^n)\neq 0,\qquad F_k\left(\Omega^{k+1}\right)=0,\qquad k>n \tag{2.13}$$

where

$$F_n(\mathcal{P}^n)=\det||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)||=\det||g_{ik}(\mathcal{P}^n)||\neq 0,\qquad i,k=1,2,...n \tag{2.14}$$

Vectors $\mathbf{P}_0\mathbf{P}_i,\ i=1,2,...n$ are basic vectors of the rectilinear coordinate system K_n with the origin at the point P_0, and metric tensors $g_{ik}(\mathcal{P}^n)$, $g^{ik}(\mathcal{P}^n)$, $i,k=1,2,...n$ in K_n are defined by relations

$$g^{ik}(\mathcal{P}^n)\,g_{lk}(\mathcal{P}^n)=\delta^i_l,\qquad g_{il}(\mathcal{P}^n)=(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_l)\,,\qquad i,l=1,2,...n \tag{2.15}$$

II:

$$\sigma_{\mathrm{E}}(P,Q)=\frac{1}{2}g^{ik}(\mathcal{P}^n)\,(x_i(P)-x_i(Q))\,(x_k(P)-x_k(Q))\,,\qquad \forall P,Q\in\mathbb{R}^n \tag{2.16}$$

where coordinates $x_i(P)$ of the point P are defined by the relation

$$x_i(P)=(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P})\,,\qquad i=1,2,...n \tag{2.17}$$

III: The metric tensor matrix $g_{lk}(\mathcal{P}^n)$ has only positive eigenvalues

$$g_k>0,\qquad k=1,2,...,n \tag{2.18}$$

IV. Continuity condition: the system of equations

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P})=y_i\in\mathbb{R},\qquad i=1,2,...n \tag{2.19}$$

considered to be equations for determination of the point P as a function of coordinates $y = \{y_i\}$, $i = 1, 2, ...n$ has always one and only one solution.

Conditions I – IV are necessary and sufficient conditions of that the σ-space $V = \{\sigma, \Omega\}$ is the n-dimensional proper Euclidean space [3]. These special properties of E_n are different for different dimension n, and contain a reference to n.

Let us use in Riemannian geometry two different definitions of parallelism, based on application of relations (2.5), (2.1) and (2.5), (2.11) respectively. Although definitions of (2.1) and (2.11) for the scalar product are different, they give the same result for parallelism of to vectors having a common origin.

The relations (2.5), (2.11) define, parallelism only for two vectors, having a common origin. To define parallelism of two remote vectors $\mathbf{u}(x)$ and $\mathbf{u}(x')$ in Riemannian geometry, one defines parallelism of two infinitesimally close vectors $\mathbf{u}(x)$ and $\mathbf{u}(x+dx)$ by means of the relation

$$u_i(x+dx) = u_i(x) - \Gamma_{il}^k(x)\, u_k(x)\, dx^l, \qquad i = 1, 2, ...n \tag{2.20}$$

$$\Gamma_{il}^k = \frac{1}{2} g^{kj}\left(g_{ij,l} + g_{lj,i} - g_{il,j}\right), \qquad g_{ij,l} \equiv \frac{\partial g_{ij}}{\partial x^l} \tag{2.21}$$

The vector $\mathbf{u}(x')$ at the point x' parallel to the vector $\mathbf{u}(x)$ is obtained by subsequent application of the infinitesimally small transport (2.20) along some path $\mathcal{L}$, connecting points x and x'. Note that the vectors $\mathbf{u}(x)$ and $\mathbf{u}(x+dx)$ are parallel, and besides they have the same length. In general, result of the parallel transport along $\mathcal{L}$ depends on $\mathcal{L}$. Such a situation is known as a lack of absolute parallelism. For flat Riemannian spaces there is the absolute parallelism, but for the curved Riemannian spaces the absolute parallelism is absent, in general.

Application of the parallelism definition, based on relations (2.5), (2.1) to vectors $\mathbf{PP}'$ and $\mathbf{P}_1\mathbf{P}''$ in Riemannian geometry with infinitesimally close points P and P_1 gives a result coinciding with (2.20), only if the displacement vector $\mathbf{PP}_1||\mathbf{PP}'$ (and hence $\mathbf{PP}_1||\mathbf{PP}''$). This property provides one-dimensionality of geodesics, obtained as a result of deformation of Euclidean straights. In other cases, the results of two definitions of parallelism appear to be different, in general, because the relation (2.20) gives only one vector $\mathbf{u}(x+dx)$, parallel to $\mathbf{u}(x)$, whereas relations (2.5), (2.1) generate, in general, a set of many vectors $\mathbf{P}_1\mathbf{P}''$, which are parallel to $\mathbf{PP}'$, but which are not parallel, in general, between themselves [3]. The difference is conditioned by the fact that the condition of parallelism (2.5) contains only one relation, whereas the condition of parallelism (2.20) contains n relations.

To explain the reason of this difference, let us rewrite relations (2.5), (2.11) in the form of scalar product of two unit vectors $\sigma_i(x,x')\left(2\sigma(x,x')\right)^{-1/2}$ and $\sigma_i(x,x'')\left(2\sigma(x,x'')\right)^{-1/2}$

$$g^{ik}(x)\frac{\sigma_i(x,x')}{\sqrt{2\sigma(x,x')}}\frac{\sigma_k(x,x'')}{\sqrt{2\sigma(x,x'')}} = 1, \tag{2.22}$$

Let the matrix of metric tensor $g^{ik}(x)$ has eigenvalues of the same sign. Then both vectors $\sigma_i(x,x')\left(2\sigma(x,x')\right)^{-1/2}$ and $\sigma_i(x,x'')\left(2\sigma(x,x'')\right)^{-1/2}$ are equal, and one relation (2.22) is equivalent to n relations

$$\sigma_i(x,x') = a\sigma_i(x,x''), \qquad i=1,2,...n, \qquad a>0 \tag{2.23}$$

where a is some constant. Conditions (2.23) with arbitrary $a \neq 0$ mean that vectors $\mathbf{PP}'$ and $\mathbf{PP}''$, having a common origin, are collinear (parallel or antiparallel), provided their components are proportional.

In the n-dimensional proper Euclidean space E_n this condition can be written σ-immanently. Let vector $\mathbf{P}_0\mathbf{R}$ be collinear to the vector $\mathbf{P}_0\mathbf{P}_1$. Let us choose $n-1$ points $\{P_2, P_3, ...P_n\}$ in such a way, that n vectors $\mathbf{P}_0\mathbf{P}_i$, $i=1,2,...n$ form a basis. Then the collinearity condition (2.23) of vectors $\mathbf{P}_0\mathbf{R}$ and $\mathbf{P}_0\mathbf{P}_1$ takes the form of n relations

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{R}) = a(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_1), \qquad i=1,2,...n \tag{2.24}$$

Eliminating a from n relations (2.24) we obtain $n-1$ relations, which are written in the form

$$\mathbf{P}_0\mathbf{P}_1||\mathbf{P}_0\mathbf{R}: \qquad \left| \begin{array}{cc} (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R}) & (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{R}) \\ (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_1) \end{array} \right| = 0, \qquad i=2,3,...n \tag{2.25}$$

Thus, we have two different formulation of the collinearity conditions of vectors $\mathbf{P}_0\mathbf{R}$ and $\mathbf{P}_0\mathbf{P}_1$: (2.25) and the relation

$$\mathbf{P}_0\mathbf{P}_1||\mathbf{P}_0\mathbf{R}: \qquad \left| \begin{array}{cc} (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R}) \\ (\mathbf{P}_0\mathbf{R}.\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{R}.\mathbf{P}_0\mathbf{R}) \end{array} \right| = 0 \tag{2.26}$$

which follows from (2.5). In E_n conditions (2.25), and (2.26) are equivalent, because choice of $n-1$ points $\{P_2, P_3, ...P_n\}$ is arbitrary, and they are fictitious in (2.25). The collinearity conditions (2.25) and (2.26) are equivalent due to special properties (2.16) of E_n. In the n-dimensional proper Riemannian geometry the conditions (2.25), and (2.26) are also equivalent, and points $\{P_2, P_3, ...P_n\}$ are also fictitious in (2.25). This is connected with the special choice of the world function (1.3) of n-dimensional Riemannian space. At another choice of the world function the points $\{P_2, P_3, ...P_n\}$ stop to be fictitious.

To manifest difference between the conditions (2.25) and (2.26), let us construct the straight $\mathcal{T}_{P_0P_1}$, passing through points P_0, P_1, defining it as set of such points R, that $\mathbf{P}_0\mathbf{R}||\mathbf{P}_0\mathbf{P}_1$. Using two variants of the collinearity conditions (2.25), and (2.26) we obtain two different geometric objects

$$\mathcal{T}_{P_0P_1} = \{R|\ \mathbf{P}_0\mathbf{P}_1||\mathbf{P}_0\mathbf{R}\} = \left\{R|\,(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R})^2 = |\mathbf{P}_0\mathbf{P}_1|^2\,|\mathbf{P}_0\mathbf{R}|^2\right\} \tag{2.27}$$

and

$$\mathcal{L} = \left\{ R \left| \bigwedge_{k=2}^{k=n} f(P_0, P_1, P_k, R) = 0 \right. \right\} = \bigcap_{k=2}^{k=n} \{R\,|f(P_0, P_1, P_k, R) = 0\} \tag{2.28}$$

where

$$f(P_0, P_1, P_i, R) = \left| \begin{array}{cc} (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R}) & (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{R}) \\ (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_1) \end{array} \right| = 0, \qquad i=2,3,...n \tag{2.29}$$

In the n-dimensional proper Euclidean space and in the n-dimensional proper Riemannian space geometric objects $\mathcal{L}$ and $\mathcal{T}_{P_0P_1}$ coincide, but at a more general form of the world function geometric objects $\mathcal{L}$ and $\mathcal{T}_{P_0P_1}$ are different, in general.

The relation (2.28) determines the straight $\mathcal{L}$ in the n-dimensional proper Euclidean space as an intersection of $n-1$ $(n-1)$-dimensional surfaces

$$\mathcal{S}(P_0, P_1, P_k) = \{R\,|f(P_0, P_1, P_k, R) = 0\}, \qquad k = 2, 3, ...n \tag{2.30}$$

In general, such an intersection is a one-dimensional line, but this line is determined by $n+1$ points $\mathcal{P}^n \equiv \{P_0, P_1, ..., P_n\}$, whereas the straight $\mathcal{T}_{P_0P_1}$, defined by the relation (2.27), depends only on two points P_0, P_1.

In general case, when the special properties of the Euclidean space disappear, the relation (2.28) describes one-dimensional object depending on more than two points. Thus, one can eliminate dependence of the collinearity definition (2.23) on the coordinate system, but instead of this dependence a dependence on additional points appears. These additional points $P_2, P_3, ...$ represent the coordinate system in the σ-immanent form. The number of additional points which are necessary for determination of the straight (2.23) as a one-dimensional line depends on the dimension of the Euclidean space. From formal viewpoint the geometric object $\mathcal{L}$, determined σ-immanently by (2.28), is not a straight, but some other geometric object, coinciding with the straight in the n-dimensional proper Euclidean space.

The straight in the n-dimensional proper Euclidean space has two properties: (1) the straight is determined by two points P_0, P_1 independently of the dimension of the Euclidean space, (2) the straight is a one-dimensional line. In general, both properties are not retained at deformation of the Euclidean space. If we use the definition (2.27), we retain the first property, but violate, in general, the second one. If we use the definition (2.28), depending on the Euclidean space dimension and on the way of description (in the form of coordinate system, or in the form of additional arbitrary points), we retain the second property and violate, in general, the first one. Which of the two definitions of the straight should be used?

The answer is evident. Firstly, the definition (2.27) does not refer to any means of description, whereas the definition (2.28) does. Secondly, the property of the straight of being determined by two points is the more natural property of geometry, than the property of being a one-dimensional line. Use of the definition (2.27) is a logical necessity, but not a hypothesis, which can be confirmed or rejected in experiment. Consideration of the straight as a one-dimensional geometric object *in any geometry* is simply a preconception, based on the fact, that in the proper Euclidean geometry the straight is a one-dimensional geometric object. The statement that there is only one vector $\mathbf{Q}_0\mathbf{Q}_1$ of fixed length which is parallel to the vector $\mathbf{P}_0\mathbf{P}_1$ is another formulation of the preconception mentioned above.

3 Consequence of inconsistent definition of parallelism

Abstracting from the history of the Riemannian geometry creation and motives of its creation, let us evaluate what is the Riemannian geometry as a kind of physical geometry. The conventional Riemannian geometry is to be a special case of a physical geometry, constructed on the basis of the principle of geometry deformation. The Riemannian geometry uses definition of the scalar product (2.11), which is completely compatible with the principle of geometry deformation only for several geometries. To compensate inconsistencies, generated by incorrectness of definition (2.11), the Riemannian geometry uses the constraint (1.3), tending to eliminate geometries, for which the definition (2.11) is inconsistent. The constraint (1.3) removes most of possible inconsistencies, but not all, and the Riemannian geometry appears to be inconsistent geometry.

In contemporary geometry and physics the definition (2.23) or (2.28) is used, and this circumstance is a reason for many problems, because this definition lies in the foundation of geometry, and geometry in turn lies in the foundation of physics.

Let us list some consequences of the statement that the straight is a one-dimensional geometric object in any space-time geometry.

1. Lack of absolute parallelism in the space-time geometry (i.e. in Riemannian geometry used for description of the space-time).

2. Discrimination of any space-time geometry, where the timelike straight is not a one-dimensional object, and (as a corollary) discrimination of stochastic motion of microparticles.

3. Consideration of spacelike straights, describing superlight particles (taxyons), in the Minkowski space-time geometry, as one-dimensional geometric objects.

Let us discuss the first point. The world function of the Riemannian geometry is chosen in such a way that the tube $\mathcal{T}_{P_0P_1}$ (we use this term instead of the term "straight "), passing through the points P_0, P_1 and defined by the relation (2.27), is a one-dimensional geometric object in the Riemannian space-time geometry, provided interval between the points P_0, P_1 is timelike ($\sigma(P_0, P_1) > 0$). But the timelike tube

$$\mathcal{T}(P_0, P_1; Q_0) = \{R| \ \mathbf{P}_0\mathbf{P}_1 || \mathbf{Q}_0\mathbf{R}\} = \left\{R| \left(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{R}\right)^2 = |\mathbf{P}_0\mathbf{P}_1|^2 |\mathbf{Q}_0\mathbf{R}|^2\right\} \quad (3.1)$$

passing through the point Q_0 parallel to the remote timelike vector $\mathbf{P}_0\mathbf{P}_1$, is not a one-dimensional object, in general, in the σ-Riemannian geometry (in Riemannian geometry $\mathcal{T}(P_0, P_1; Q_0)$ is not defined). One cannot achieve that any timelike tube (2.27) to be a one-dimensional geometric object. In other words, one cannot suppress globally nondegeneracy of all collinearity cones of timelike vectors $\mathbf{Q}_0\mathbf{R}$, parallel to the timelike vector $\mathbf{P}_0\mathbf{P}_1$, although locally the collinearity cone nondegeneracy of timelike vectors $\mathbf{P}_0\mathbf{R}$, parallel to the timelike vector $\mathbf{P}_0\mathbf{P}_1$, can be suppressed, if the world function is restricted by the constraint (1.3). In fact, according to the correct definition (2.26) in the σ-Riemannian geometry there are many timelike vectors $\mathbf{Q}_0\mathbf{R}$ of fixed length, which are parallel to the remote timelike vector $\mathbf{P}_0\mathbf{P}_1$. As far as according to the Riemannian conception of geometry there is to be only one timelike vector $\mathbf{Q}_0\mathbf{R}$ of fixed length, which is parallel to the remote timelike vector $\mathbf{P}_0\mathbf{P}_1$, one cannot choose one vector among the set of equivalent vectors $\mathbf{Q}_0\mathbf{R}$, and one is forced to deny the absolute parallelism.

The point two. The Minkowski space-time geometry T_{M} with the σ-space $\{\sigma_{\mathrm{M}}, \mathbb{R}^4\}$ is the unique uniform isotropic flat geometry in the class of Riemannian geometries. The class of uniform isotropic T-geometries on the set $\mathbb{R}^4$ of points is described by the world function $\sigma = \sigma_{\mathrm{M}} + D(\sigma_{\mathrm{M}})$, where the arbitrary distortion function D describes character of nondegeneracy of timelike tubes $\mathcal{T}_{P_0P_1}$. In the Minkowski space-time geometry a motion of free particles is deterministic. If $D > 0$ the world line of a free particle appears to be stochastic, because the running point moves along the world line in the direction of vector tangent to the world line. There are many vectors tangent to the world line. The particle can move along any of them, and its motion becomes stochastic, (see details in [5]). In fact, motion of microparticles (electrons, protons, etc.) is stochastic. It means that the Minkowski geometry is not a true space-time geometry. One should choose such a space-time geometry, which could explain stochastic motion of microparticles. Such a space-time geometry is possible. In this space-time geometry the distortion function $D(\sigma_{\mathrm{M}}) = \hbar/(2bc)$ for $\sigma_{\mathrm{M}} > \sigma_0 \approx \hbar/(2bc)$, where $\hbar$ is the quantum constant, c is the speed of the light, and b is a new universal constant. In such a space-time geometry the world function contains the quantum constant $\hbar$, and nonrelativistic quantum effects are explained as geometric effects [5]. Insisting on the definition (2.23) of the parallelism, we discriminate space-time geometries with $D \neq 0$. As a result we are forced to use incorrect space-time geometry and to explain quantum effects by additional hypotheses (quantum principles).

Let us consider straights in the Minkowski geometry. Let us define straight by the relation (2.27). Let $\mathbf{e} = \mathbf{P}_0\mathbf{P}_1$ and $\mathbf{x} = \mathbf{P}_0\mathbf{R}$ be the running vector. Then the relation determining the straight $\mathcal{T}_{P_0P_1}$ has the form

$$\mathcal{T}_{P_0P_1}: \qquad \begin{vmatrix} (\mathbf{e}.\mathbf{e}) & (\mathbf{e}.\mathbf{x}) \\ (\mathbf{x}.\mathbf{e}) & (\mathbf{x}.\mathbf{x}) \end{vmatrix} = 0 \tag{3.2}$$

Looking for its solution in the form

$$\mathbf{x} = \mathbf{e}\tau + \mathbf{y} \tag{3.3}$$

and substituting this expression in (3.2), we obtain the equation of the same form.

$$\begin{vmatrix} (\mathbf{e}.\mathbf{e}) & (\mathbf{e}.\mathbf{y}) \\ (\mathbf{y}.\mathbf{e}) & (\mathbf{y}.\mathbf{y}) \end{vmatrix} = 0 \tag{3.4}$$

Evident solution $\mathbf{y} = \alpha\mathbf{e}$ is not interesting, because it has been taken into account in (3.3). Imposing constraint $(\mathbf{e}.\mathbf{y}) = 0$, one obtains from (3.4)

$$(\mathbf{e}.\mathbf{y}) = 0, \qquad \mathbf{y}^2 = 0$$

If the vector $\mathbf{e}$ is timelike, for instance, $\mathbf{e} = \{1, 0, 0, 0\}$, then $\mathbf{y} = 0$. If the vector $\mathbf{e}$ is spacelike, for instance, $\mathbf{e} = \{0, 1, 0, 0\}$, then the solution has the form $\mathbf{y} = \{a, 0, a\cos\psi, a\sin\psi\}$, where a and ψ are arbitrary parameters. Thus, in the Minkowski space the timelike straight is a one-dimensional object, whereas the spacelike "straight" is a three-dimensional surface, containing the one-dimensional spacelike straight $\mathbf{x} = \mathbf{e}\tau$. In other words, timelike directions are degenerate, and free particles, moving with the speed $v < c$, are described by one-dimensional timelike straights. The spacelike directions are nondegenerate, and free particles, moving with the speed $v > c$ (taxyons) are described by three-dimensional surfaces. It is difficult to say, what does it mean practically. But, maybe, taxyons were not discovered, because they were searched in the form of one-dimensional spacelike lines.

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