Distorted Riemannian space and technique of differential geometry

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Abstract

A differential-geometric technique of non-Riemannian space description is considered. This n-dimensional distorted Riemannian space distinguishes from a Riemannian one by a substitution of one-dimensional geodesics by $(n-1)$-dimensional tubes. Such a space is of interest as a model of space-time, taking into account quantum effects.

1 Introduction

Recently it was discovered [1] a new form of the space-time deformation, which converts one-dimensional world lines of particles into three-dimensional tubes. This new form of space-time deformation (called distortion) appears to be associated with quantum effects. Global magnitude of distortion is determined by quantum constant $\hbar$, which describes the thickness of world tubes. This form of space-time deformation can be displayed as follows.

The Minkowski space-time E_{1+3} is described most exhaustively by means of the world function [2],

$$\sigma_E\left(P,P'\right)=\sigma_E\left(x,x'\right)=\frac{1}{2}g_{ik}\left(x^i-x'^i\right)\left(x^k-x'^k\right), \qquad (1.1)$$

where x and x' are Galilean coordinates of points P and P', $g_{ik}=$diag$(c^2,-1,-1,-1,)$ is the metric tensor.

A segment $\mathcal{L}_{[P'P'']}$ between points P' and P'' of the world line of a free particle is described by the equation

$$S\left(P',P\right)+S\left(P,P''\right)=S\left(P',P''\right), \qquad S\left(P,P'\right)=\sqrt{2\sigma\left(P,P'\right)}, \qquad (1.2)$$

where $\sigma=\sigma_E$, $S\left(P',P''\right)$ is an interval between points P',P''. Indeed, if interval $S\left(P',P''\right)$ is time-like [i.e., $\sigma\left(P',P''\right)>0$], the segment $\mathcal{L}_{[P'P'']}$ of the straight line

$\mathcal{L}_{P'P''}$ between points P', P'' is a set of points P satisfying Eq. (1.2). Generally, *one* equation describes a three-dimensional surface in the four-dimensional space.

The circumstance that *one* equation (1.2) describes a *one-dimensional* line [for $\sigma(P', P'') > 0$] associates with some kind of degeneration of the Minkowski space-time, or with a special property (extremality) of its world function. For the non-time-like interval $[\sigma(P', P'') \leq 0]$, Eq. (1.2) describes the infinite three-dimensional surface. Apparently, it is a reason, why world lines of particles are time-like.

Deformation of E_{1+3} into curved space-time R_{1+3} violates uniformity and isotropy (generates gravitation), but the extremality is not violated. Equation (1.2) describes the *one-dimensional* world line of a free particle both in E_{1+3} and in R_{1+3}.

Let us consider some deformation of E_{1+3}, which leads to a distorted space-time D_{1+3} described by a world function σ_D. This deformation (uniform distortion) of E_{1+3} is determined by a transformation,

$$\sigma_E(P, P') \rightarrow \sigma_D(P, P') = \sigma_E(P, P') + D[\sigma_E(P, P')] \tag{1.3}$$

where D is some restricted function of σ_E. The distortion does not violate uniformity and isotropy of the space-time, but it violates extremality and destroys degeneration of E_{1+3}. In D_{1+3}, Eq. (1.2) (with $\sigma = \sigma_D$) describes a segment $\mathcal{T}_{[P'P'']}$ of a three-dimensional tube $\mathcal{T}_{P'P''}$, even if $\sigma(P', P'') > 0$. Now a free particle is described by means of a world tube, not by a world line. The space-time D_{1+3} is not a Riemannian space. One can show [1] that the broken tube that consists of segments $\mathcal{T}_{[P'P'']}$ and describes a free particle is random. Statistical description of such random tubes is equivalent to a quantum description. This circumstance shows that phenomenon of space-time distortion is of interest from a physical point of view.

Thus, the distorted n-dimensional Riemannian space $V = \{\tilde{\sigma}, \mathcal{M}\}$ is a world function $\tilde{\sigma}$ given on n-dimensional manifold $\mathcal{M}$. The distorted Riemannian space is non-Riemannian space, generally.

The use of differential geometry for describing distorted space means as follows. As one will show, any world function σ on $\mathcal{M}$ generates a set of Euclidean spaces $E^*_{x'}$ on $\mathcal{M}$. There is a correspondence between this set and σ. Describing the set of Euclidean spaces by means of the differential-geometric technique, one describes the distorted space V. This technique is applied to describing Riemannian space [3, 4]. It can be used in the more general case.

The idea of using a world function can be formulated as follows. Conventionally an n-dimensional Riemannian space is defined by means of the quadratic form, describing the infinitesimal interval dS on an n-dimensional manifold $\mathcal{M}$,

$$dS^2 = g_{ik} dx^i dx^k \tag{1.4}$$

where the metric tensor g_{ik}, $(i, k = 1, 2, ..., n)$.

The quadratic form (1.4) also can be treated as a set of infinitesimal regions of Euclidean space, each region being associated with a point of the Riemannian space. Thus, the Riemannian space is a set of infinitesimal regions of Euclidean spaces united by a manifold (or by a coordinate system).

Infinitesimal regions can be substituted by finite ones. Then quadratic form (1.4) is converted into

$$2\sigma\left(x,x'\right)=S^{2}\left(x,x'\right)=g_{ik}\left(x\right)\Delta^{i}\Delta^{k},\qquad \Delta^{i}=-g^{ik}\left(x\right)\partial_{k}\sigma\left(x,x'\right) \tag{1.5}$$

The form (1.5) is obtained from the form (1.4) by means of a substitution of infinitesimal quantities by finite ones $dS\rightarrow S\left(x,x'\right)$, $dx^{i}\rightarrow\Delta^{i}$. Formally, Eq. (1.5) is an equation with respect to world function σ of a Riemannian space, and the world function of any Riemannian space does satisfy (1.5).

Thus, a *Riemannian space* can be considered as a *set of Euclidean spaces.*

If σ is a world function of non-Riemannian space, then, generally, it does not satisfy Eq. (1.5). Nevertheless, the world function generates a set of Euclidean spaces, although connection between these spaces is not described by the quadratic form (1.5). This connection can be more complicated.

In a sense, one can declare that non-Riemannian space $V=\{\sigma,\mathcal{M}\}$ given on n-dimensional manifold $\mathcal{M}$ is a set of Euclidean spaces. Studying properties of this set, one can investigate properties of the non-Riemannian space.

In the second section one considers the way of Euclidean space generation by a world function. In the third section a uniformly distorted Riemannian space is considered. The fourth section is devoted to calculation of the shape of time-like tubes in the distorted space-time.

2 World function as a generator of Euclidean spaces

Let a world function σ (some symmetric function of two points) be given on n-dimensional manifold $\mathcal{M}$, and have the properties

$$\sigma\left(x,x\right)=0,\qquad \sigma\left(x,x'\right)=\sigma\left(x',x\right) \tag{2.1}$$

$$\det\left\Vert \sigma_{ik'}\right\Vert =\det\left\Vert \frac{\partial^{2}\sigma}{\partial x^{i}\partial x^{k}}\right\Vert \neq0 \tag{2.2}$$

Then the function σ determines a gradient line $\mathcal{G}_{[P'P'']}$, passing through any points P' and P''. Really, let us consider a system of n equations,

$$\sigma_{i'}\left(x,x'\right)=\tau\sigma_{i'}\left(x'',x'\right)=\tau b_{i'},\qquad \sigma_{i'}\equiv\frac{\partial}{\partial x'^{i}}\sigma\left(x,x'\right),\qquad i=1,2,...,n \tag{2.3}$$

$$x'=x\left(P'\right),\qquad x=x\left(P\right),\qquad x''=x\left(P''\right),$$

with respect to variables x. Here τ is a parameter, $b_{i'}$ is a vector at the point x' at some coordinate system K. Primed indices concern to the point x', the unprimed ones concern to the point x. According to (2.2), the system (2.3) considered as a system equations with respect to variables $x'=x'\left(P\right)$ has a sole solution,

$$\mathcal{G}_{P'P''}\colon\ x^{i}=x^{i}\left(\tau,b_{k}\right),\qquad i=1,2,...,n \tag{2.4}$$

Let us differentiate Eq. (2.3) with respect to τ, keeping in mind (2.4),

$$\sigma_{i'k}\left(x,x'\right)\frac{dx^k}{d\tau}=\sigma_{i'}(x'',x')=b_{i'},\qquad \sigma_{i'k}\equiv\sigma_{ki'}\equiv\frac{\partial^2\sigma\left(x,x'\right)}{\partial x'^i\partial x^k} \tag{2.5}$$

Differentiating once more with respect to τ, one obtains

$$\sigma_{i'k}\frac{d^2x^k}{d\tau^2}+\sigma_{i'kl}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}=0,\qquad \sigma_{kli'}\equiv\sigma_{i'kl}\equiv\partial_l\sigma_{i'k}\left(x,x'\right) \tag{2.6}$$

Introducing designation

$$\Gamma_{kl}^i\equiv\Gamma_{kl}^i\left(x,x'\right)=\sigma_{kls'}\sigma^{is'},\qquad \sigma_{kl'}\sigma^{il'}=\delta_k^i \tag{2.7}$$

one can write Eq. (2.6) in the form

$$\frac{d^2x^i}{d\tau^2}+\Gamma_{kl}^i\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}=0,\qquad i=0,1,...,n \tag{2.8}$$

One can show that Γ_{kl}^i defined by Eq. (2.7) transforms as a Christoffel symbol at x and as a scalar at x'. (Function σ is supposed to be a scalar). Moreover, Γ_{kl}^i is a Christoffel symbol of a flat space, because the Riemannian curvature tensor vanishes identically,

$$R_{i.lm}^s=\Gamma_{il,m}^s-\Gamma_{im,l}^s+\Gamma_{il}^j\Gamma_{jm}^s+\Gamma_{im}^j\Gamma_{jl}^s\equiv 0 \tag{2.9}$$

Here a comma denotes differentiation. In the region where Eq. (2.2) takes place, the flat space can be considered as a Euclidean space $E_{x'}^*$.

It follows from Eqs. (2.5),

$$\frac{dx^i}{d\tau}G_{ik}\frac{dx^k}{d\tau}=b_{i'}g^{i'k'}b_{k'}=\text{const} \tag{2.10}$$

where $g^{i'k'}$ and G_{ik} are defined by relations

$$g^{i'k'}g_{k'l'}=\delta_{l'}^{i'},\qquad g_{i'k'}=-\left[\sigma_{ik'}\right]_{x=x'} \tag{2.11}$$

$$G_{ik}=G_{ik}\left(x,x'\right)=\sigma_{il'}g^{l'j'}\sigma_{kj'} \tag{2.12}$$

G_{ik} is the metric tensor in $E_{x'}^*$, where the Christoffel symbol is given by Eq.(2.7), because the relation

$$\Gamma_{kl}^i\left(x\right)=\frac{1}{2}G^{ij}\left(G_{lj,k}+G_{jk,l}-G_{kl,j}\right),\qquad G^{ij}G_{kj}=\delta_k^i \tag{2.13}$$

is satisfied identically by Eqs. (2.12) and (2.7).

Relations (2.8) and (2.10) mean that the line described by Eq. (2.4) is a straight line in the Euclidean space $E_{x'}^*$ defined on manifold $\mathcal{M}$ by means of metric tensor (2.12). Thus, any world function σ generates a Euclidean space and a system of straight lines passing through any two points on $\mathcal{M}$.

If σ is the world function of a Riemannian space, then it realizes geodesic mapping of the Riemannian space onto Euclidean space $E^*_{x'}$. If σ is a world function of non-Riemannian space, then, nevertheless, it generates Euclidean spaces $E^*_{x'}$. This property can be used for investigation of distorted spaces.

Let us introduce a covariant derivative $\tilde{\nabla}_i^{x'}$ with Christoffel symbol Γ^i_{kl} defined by Eq. (2.7) or (2.13), which is the same,

$$\tilde{\nabla}_i^{x'} T^{j\ldots m'\ldots}_{k\ldots l'\ldots} \equiv T^{j\ldots m'\ldots}_{k\ldots l'\ldots||i} - \Gamma^s_{ik} T^{j\ldots m'\ldots}_{s\ldots l'\ldots} - \ldots + \Gamma^j_{is} T^{s\ldots m'\ldots}_{k\ldots l'\ldots} + \ldots, \tag{2.14}$$

where $T^{j\ldots m'\ldots}_{k\ldots l'\ldots}$ is a two-point tensor. Unprimed indices are associated with the point x, and primed ones with x'.

Derivative $\tilde{\nabla}_i^{x'}$ has the following properties

$$\left[\tilde{\nabla}_i^{x'}, \tilde{\nabla}_k^{x'}\right]_- T^{j\ldots m'\ldots}_{k\ldots l'\ldots} \equiv \left(\tilde{\nabla}_i^{x'}\tilde{\nabla}_k^{x'} - \tilde{\nabla}_k^{x'}\tilde{\nabla}_i^{x'}\right) T^{j\ldots m'\ldots}_{k\ldots l'\ldots} = 0, \tag{2.15}$$

where $[\ldots]_-$ denotes a commutator,

$$g_{k'l'||i} = 0, \qquad g^{k'l'}{}_{||i} = 0, \qquad \delta^k_{l||i} = 0, \tag{2.16}$$

$$\sigma_{kl'||i} \equiv \sigma_{kl',i} - \sigma_{kis'}\sigma^{js'}\sigma_{jl} \equiv 0, \qquad \sigma^{kl'}{}_{||i} \equiv 0. \tag{2.17}$$

Then relation (2.12) can be written in the form

$$G_{ik} = \frac{1}{2}\tilde{\nabla}_i^{x'}\tilde{\nabla}_k^{x'}\left(\sigma_{l'}g^{l'j'}\sigma_{j'}\right). \tag{2.18}$$

Relations (2.14)–(2.18) are valid for any world function, not only for the world function of a Riemannian space. In the case of a Riemannian space, the world function satisfies Eq. (1.5). Then (2.18) can be written in the form

$$G_{ik} = \tilde{\nabla}_i^{x'}\tilde{\nabla}_k^{x'}\sigma \equiv \sigma_{i||k}. \tag{2.19}$$

3 Uniform distortion of a Riemannian space

The system (2.3) of gradient lines is determined by the world function σ. Let us investigate whether the system of gradient lines determines the function σ.

Let $\tilde{\sigma}$ be another world function on $\mathcal{M}$. Let the system of equations

$$\tilde{\sigma}_{i'}(x, x') = \tilde{\tau}\tilde{\sigma}_{i'}(x'', x') = \tau b_{i'}, \tag{3.1}$$

have the same gradient lines (2.4) at all values x'' and x'. These lines can be distinguished only by the way of parametrization. Then

$$\tilde{\sigma}_{i'}(x'', x') = A\sigma_{i'}(x'', x'), \qquad A = A(x', x''), \qquad x', x'' \in \mathcal{M} \tag{3.2}$$

This relation is possible only in the case where $\tilde{\sigma}$ is a function of σ,

$$A = 1 + D'(\sigma), \qquad \tilde{\sigma} = \sigma + D(\sigma), \qquad D'(\sigma) \equiv \frac{dD}{d\sigma} \tag{3.3}$$

Let σ be a world function of a Riemannian space. Then it satisfies [3] the equation (1.5),

$$\sigma_{i'} g^{i'k'} \sigma_{k'} = 2\sigma \tag{3.4}$$

The function $\tilde{\sigma}$ satisfies the equation

$$\tilde{\sigma}_{i'} \tilde{g}^{i'k'} \tilde{\sigma}_{k'} = q(\sigma), \qquad \tilde{g}_{i'k'} = -[\tilde{\sigma}_{ik'}]_{x=x'} = [1 + D'(0)]\, g_{i'k'}, \tag{3.5}$$

where the function $q(\tilde{\sigma})$ is determined by the equation

$$q(\tilde{\sigma}) = q(\sigma + D(\sigma)) = 2\sigma \frac{[1 + D'(\sigma)]^2}{1 + D'(0)}. \tag{3.6}$$

If $D'(\sigma) \neq 0$, then $\tilde{\sigma}$ is not a world function of a Riemannian space. Let us refer to $D(\sigma)$ as a distortion function of a Riemannian space. The space described by the world function $\tilde{\sigma}$ with $D \neq 0$ will be referred to as a Riemannian space with distortion.

Distortion of the type (3.3) should be called uniform distortion, because this type of distortion does not violate any group of motion of a Riemannian space $V = (\sigma, \mathcal{M})$. For instance, if the space is Euclidean, the corresponding distorted space is isotropic and uniform.

Thus, the system of gradient lines determines a set of distorted Riemannian spaces. Each of these spaces can be obtained from one of them by transformation (1.3).

In application to space-time models, the case when $D'(\sigma)$ does not vanish only inside small region is most important,

$$\begin{array}{ll} D'(\sigma) \neq 0, & 0 < \sigma < \sigma_0 \\ D'(\sigma) = 0, & \sigma < 0,\ \sigma > \sigma_0 \end{array} \tag{3.7}$$

or

$$\begin{array}{ll} \tilde{\sigma} = \sigma + d, \quad \sigma > \sigma_0, & d = D(\sigma_0) = \text{const.} \\ \tilde{\sigma} = \sigma, & \sigma < 0 \end{array} \tag{3.8}$$

The constant d (integral distortion) describes influence of distortion at large σ. Contribution of integral distortion into interval $(2\sigma)^{1/2}$ reduces at a large interval. If $d \ll \sigma$, then

$$(2\tilde{\sigma})^{1/2} = [2(\sigma + d)]^{1/2} \cong (2\sigma)^{1/2} + d(2\sigma)^{-1/2}, \tag{3.9}$$

distinguishes slightly from the interval of the Riemannian space.

Let us consider uniformly distorted Riemannian space. It is described by the world function $\tilde{\sigma}$ defined by Eq. (3.3), where σ is the world function of a Riemannian space. One obtains

$$\tilde{\sigma}_i = (1 + D')\,\sigma_i, \tag{3.10}$$

$$\tilde{\sigma}_{ik'} = (1 + D')\,\sigma_{ik'} + D''\sigma_i\sigma_{k'} = (1 + D')\,P_{ik'} + (1 + D' + 2D''\sigma)\,\frac{\sigma_i\sigma_{k'}}{2\sigma}, \tag{3.11}$$

$$\tilde{\sigma}^{ik'} = \frac{P^{ik'}}{(1+D')} + \frac{\sigma^i \sigma^{k'}}{2\sigma(1+D'+2D''\sigma)}, \tag{3.12}$$

where

$$P_{ik'} = \sigma_{ik'} - \frac{\sigma_i \sigma_{k'}}{2\sigma}, \qquad P^{ik'} = \sigma^{ik'} - \frac{\sigma^i \sigma^{k'}}{2\sigma} \tag{3.13}$$

Calculating $\tilde{\Gamma}^i_{kl}$ by means of relation (2.7), one obtains

$$\begin{aligned}\tilde{\Gamma}^s_{il} &= \Gamma^s_{il} + \frac{\sigma^s}{(1+D'+2D''\sigma)}\left[D''\left(\frac{\sigma_l\sigma_i}{\sigma} + G_{il}\right) + D'''\sigma_l\sigma_i\right] \\ &+ \frac{D''}{(1+D')}\left(\sigma_l P^s_i + \sigma_i P^s_l\right)\end{aligned} \tag{3.14}$$

where G_{ik} is the metric tensor in $E^*_{x'}$, generated by the world function σ, and

$$P^s_i = \delta^s_i - \frac{\sigma^s \sigma_i}{2\sigma}$$

Calculation of the metric tensor $\tilde{G}_{ik}$ generated by $\tilde{\sigma}$, leads to the result[1]

$$\begin{aligned}\tilde{G}_{ik} &= \frac{(1+D')^2}{(1+D'+2D''\sigma)} G_{ik} \\ &- \frac{(1+D')\sigma_i\sigma_k}{(1+D'+2D''\sigma)}\left[D''(1-D'-2D''\sigma) + 2D'''\sigma\right]\end{aligned} \tag{3.15}$$

The uniform distortion has a local character, i.e., in the region where distortion function D =const., local quantities $\tilde{G}_{ik}$, $\tilde{\Gamma}^i_{kl}$ coincide with G_{ik}, Γ^i_{kl}.

4 Shape of the tube in uniformly distorted Minkowski space-time

From point of view of metric approach [5], the tube is a more simple and fundamental object than the gradient line, because the tube is constructed only on the base of the world function. This construction does not need the use of such concepts as dimensionality, manifold, coordinate system, etc. For this reason the world tubes of particles should be identified with such fundamental objects as tubes (not with gradient lines). In a Riemannian space the tubes degenerate into lines and coincide with gradient lines. It is of no importance whether world lines of particles are tubes or gradient lines.

Let us consider uniformly distorted Minkowski space-time $V = (\tilde{\sigma}, \mathcal{M})$. Let $\mathcal{T}_{[P'P'']}$ be a segment of a time-like world tube defined by Eq. (1.2). Let us calculate

[1] In the text of this paper published in *J. Phys. Math.* **33**, 4220, (1992) formulas (3.14) and (3.15) are written incorrectly.

its shape. Let τ be a parameter along $\mathcal{T}_{[P'P'']}$, taking values 0 and 1 correspondingly at points P' and P''. Then

$$S\left(P',P''\right)=S,\qquad S\left(P',P\right)=\tau S,\qquad S\left(P,P''\right)=\left(1-\tau\right)S, \tag{4.1}$$

where $S=\left(2\sigma\right)^{1/2}$ is the length of $\mathcal{T}_{[P'P'']}$, i.e., interval between P'' and P' measured in V. Let $S_E=g\left(S\right)$ be the interval measured in the associated Minkowski space E, which is connected with S by the relation (3.3). The function g is connected with the distortion function D by means of the relation

$$\frac{1}{2}g^2\left(S\right)+D\left[\frac{1}{2}g^2\left(S\right)\right]=\frac{1}{2}S^2 \tag{4.2}$$

Function g determines the shape of $\mathcal{T}_{[P'P'']}$. Let $R_E\left(\tau\right)$ be the radius of the tube $\mathcal{T}_{[P'P'']}$, measured in the Minkowski space E at the point P that is described by the parameter τ. Then Eq. (1.2) is converted into the equation

$$\sqrt{g^2\left(S\tau\right)+R_E^2\left(\tau\right)}+\sqrt{g^2\left[S\left(1-\tau\right)\right]+R_E^2\left(\tau\right)}=g\left(S\right).$$

The solution of this equation has the form

$$\begin{aligned} R_E^2\left(\tau\right) &= \frac{1}{2}g^{-2}\left(S\right)\left\{\left[g\left(S\tau\right)+g\left(S\left(1-\tau\right)\right)\right]^2-g^2\left(S\tau\right)\right\} \\ &\quad\times\left\{\left[g\left(S\tau\right)-g\left(S\left(1-\tau\right)\right)\right]^2-g^2\left(S\tau\right)\right\} \end{aligned} \tag{4.3}$$

Expression (4.3) describes the shape of the tube segment $\mathcal{T}_{[P'P'']}$ in the Minkowski space E. It has a more simple form for the middle segment section, i.e., for $\tau=\frac{1}{2}$. In this case

$$R_{\rm mid}=R_E\left(\frac{1}{2}\right)=\sqrt{\frac{1}{4}g^2\left(S\right)-g^2\left(S/2\right)} \tag{4.4}$$

In the most interesting case (3.8), the function $g\left(S\right)$ has the form

$$S_E=g\left(S\right)=\sqrt{S^2-2d},\qquad \text{for}\quad S>S_0=\sqrt{2\left(\sigma_0+d\right)}, \tag{4.5}$$

where $d=$const. is the integral distortion. Here σ_0 and S_0 are some constants. In this case $R_{\rm mid}$ does not depend on S.

In the case (4.5) the radius of the tube segment $\mathcal{T}_{[P'P'']}$ is determined by the relation

$$\begin{aligned} R_E^2\left(\tau\right) &= \frac{3}{2}d+\frac{2S^2d}{S^2-2d}\left(\tau-\frac{1}{2}\right)^2, \\ \frac{\sqrt{2d}}{S} &< \tau<1-\frac{\sqrt{2d}}{S},\qquad S>\sqrt{2\left(\sigma_0+d\right)} \end{aligned} \tag{4.6}$$

This expression describes the shape of the segment $\mathcal{T}_{[P'P'']}$ near its middle. Near the ends the form of the distortion function becomes essential for the calculation of $R_E\left(\tau\right)$. The function $R_E\left(\tau\right)$ has a local minimum at $\tau=\frac{1}{2}$,

$$R_E=R_E\left(\tau\right)=\sqrt{3d/2} \tag{4.7}$$

At $S^2 \gg d$ the tube shape does not depend on S,

$$R_E^2(\tau) = 2d \left[\frac{3}{4} + \left(\tau - \frac{1}{2} \right)^2 \right], \tag{4.8}$$

The segment radius $R_E(\tau)$ changes within $\left[(3d/2)^{1/2}, (2d)^{1/2} \right]$, i.e., the variation is rather slight (15%).

The quantity $d^{1/2}$ can be considered as some elementary length that is characteristic for distorted space-time.

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