# Extremal properties of Synge's world function and discrete geometry

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**Abstract**

Properties of $\sigma$ space [a set $\Omega$ of points $P$ with a real function $\sigma(P, P')$ given on $\Omega$] are investigated. A continuity of the set $\Omega$ is not necessary and, generally, geometry is discrete. The properties of the world function $\sigma$ are investigated. At certain (extremal) world function properties the $\sigma$ space is shown to be a subset of points of Euclidean space or Riemannian space. The presented approach has the pecularity that no operation other than the function $\sigma$ is given on $\sigma$ space. In particular, all such operations as linear operations over vectors, constructing lines and planes, and dimension of the space are expressed through the world function $\sigma$ and only through it (if it is extremal). A violation of the $\sigma$-space extremality leads to going out beyond the scope of Riemannian geomery (lines are substituted by tubes of lines, etc.). The presented approach can be useful in quantum gravitation, string models, and other problems, where the properties of the event space at small distances are important.

## 1 Introduction.

Event space (pseudo-Euclidean or Riemannian) is used to consider a set of two independent structures: (i) linear vector space (or manifold in the case of a Riemannian space) and (ii) metric space. Linear vector space properties are used for constructing straight lines, planes, and for determination of space dimensionality. Metric space properties are used for determining distances, volumes, etc. The concept of continuity is important for describing linear vector space.

In general, the hierarchy of concepts can be represented as follows: (i) the manifold, which includes continuity and affine properties, including space dimensionality; (ii) metric properties; and (iii) the topological type of the space (i.e., whether it is topologically equivalent to a plane, cylinder, sphere, etc.).

Another approach to the description of event space properties is possible. This approach uses only the structure of metric space. Linear space and its properties are considered not as a new additional structure, but as a structure generated by the metric space.

Let us illustrate the structure of metric space in the example of three-dimensional proper Euclidean space $E_3$, where the distance $d(P_1,P_2)$ between the points $P_1$ and $P_2$ with the Cartesian coordinates $\mathbf{x}_1$ and $\mathbf{x}_2$ is defined in the conventional way:

$$d(P_1,P_2)=d(\mathbf{x}_1,\mathbf{x}_2)=\sqrt{(\mathbf{x}_1-\mathbf{x}_2)^2}=|\mathbf{x}_1-\mathbf{x}_2|. \tag{1.1}$$

It is easy to see that

$$d(P_1,P)+d(P,P_2)=d(P_1,P_2), \tag{1.2}$$

considered as an equation for the point $P$ at fixed $P_1$, and $P_2$ ($P_1\neq P_2$), determines an inetrcept $\mathcal{L}_{[P_1P_2]}$ of the straight line $\mathcal{L}_{P_1P_2}$ between the points $P_1,P_2$. Equation (1.2), considered as an equation for $P_2$ and $P_1\neq P$, determines a ray of the straight line, passing through the points $P_1,P\in E_3$. Hence, solving the algebraic equation of the type (1.2), one can construct a straight line passing through two points.

If one can construct straight lines, then one can construct two-dimensional planes, etc. and determine the space dimensionality. Practically, all properties (except continuity) of the Euclidean space, which are usually described in terms of the linear space, can be obtained from properties of the distance function $d$ of the metric space.

Such an approach is possible only in the case when the distance $d(P_1,P_2)$ has specific (extremal) properties. For instance, if the expression (1.1) for the distance $d(\mathbf{x}_1,\mathbf{x}_2)$ is substituted by

$$d(\mathbf{x}_1,\mathbf{x}_2)=\sqrt{(\mathbf{x}_1-\mathbf{x}_2)^2-a|\mathbf{x}_1-\mathbf{x}_2|/\left(1+a|\mathbf{x}_1-\mathbf{x}_2|^2\right)}, \tag{1.3}$$

where $a$ is a nonvanishing constant, then the set of points $P$ satisfying Eq.(1.2) forms a two-dimensional cigar-shaped surface. In the case when $0<a\ll 1$ the transversal size of the "cigar" of unit length is much less than the longitudinal size. Then the cigar distinguishes itself slightly from the intercept of the straight line.

In the conventional approach the geodesic (straight) line is defined as a curve of an extremal (in the given case of the shortest) length. It is natural that the definition of the curve must be given and the space dimensionality concept must be defined.

In our approach the geodesic (straight) line is merely a set of points which is determined by the form of the distance function $d(\mathbf{x}_1,\mathbf{x}_2)$. The circumstance that the line is one-dimensional (but not two-dimensional, for instance) is conditioned by specific properties of the function $d$. There is no necessity to introduce such concepts as continuity, curve, and dimensionality. The distance function can be given on *any set of points.*

Practically, it is more convenient to use another function

$$\sigma(P_1,P_2)=\frac{1}{2}d^2(P_1,P_2) \tag{1.4}$$

instead of the function $d$. The function $\sigma$ will be referred to as the Synge world function or merely the $\sigma$ function. The properties of the world function that provide, in general, the degeneration of the many-dimensional surface (1.2) into the one-dimensional geodesic will be referred to as extremal properties. The present paper is devoted to investigating extremal properties of the $\sigma$ function.

The real event space description is terms of the $\sigma$ function realizes an intuitive conception that the event space is described by and only by an interval between any pair of events. This circumstance, together with the permission of discrete space can conveniently occur at the event space quantization. Finally, the potential violation of the $\sigma$-function extremality at small distances leads to the fact that a pointlike particle is described by a world tube (but not by a one-dimensional world line): It is associated with the string model in elementary particle theory.

Apparently, the world function was first introduced for the Riemannian space description by Ruse [1, 2] and Synge [3]. In the gravitation theory the world function is denoted by different symbols: by $\Omega$ in Synge [4]; by $G$ in Rylov [5, 6]; where the world function is used for the two-metric formalism, by $I$ in Gorelik [7], which is devoted to the introduction of special coordinate systems on the base of the world function. In current quantum gravitation [8, 9] the symbol $\sigma$ is used. This designation will be used further.

In the presentation of the event space properties in terms of the world function the hierarchy of suppositions is as follows: (i) determination of the world function, (ii) world function extremality, and (iii) continuity properties.

The world function extremality is equivalent to the Euclidean axiom: One and only one straight line passes through two different points. If the world function is extremal, then the space dimensionality and topological type of the space (plane,cylinder, sphere, etc.) can be determined in some cases independently of the continuity property, for instance, in the case of a discrete space.

The circumstance that the continuity is nonessential can be useful in quantum gravitation because in a certain sense the quantization is a substitution of continuous variables by discrete ones.

Section II is devoted to the introduction of $\sigma$ space and its extremality. In Section III the Euclidean space properties are described in terms of the $\sigma$ function. In Sec. IV we investigate to what extent the $\sigma$ space extremality determines its properties. In Sec. V curtailed tubes are introduced and the manifold is described in terms of the $\sigma$ function. Section VI is devoted to the description of Riemannian space in terms of the $\sigma$ function. In Sec. VII some violations of the $\sigma$-function extremality are investigated.

# 2 $\sigma$ space

**Definition 1** *The $\sigma$ space $V$ is a set $\Omega$ of points $P$ with a real function $\sigma$ of two points $P$ and $Q$ given on $\Omega$, where the function $\sigma$ has the properties*

$$\sigma(P,Q)=\sigma(Q,P), \qquad \sigma(P,P)=0, \qquad P,Q\in\Omega \tag{2.1}$$

*The function $\sigma$ will be referred to as the world function, or merely as the $\sigma$ function.*

The interval between the points $P$ and $Q$ is defined as

$$S(P,Q)=\sqrt{2\sigma(P,Q)}=\left\{\begin{array}{ll}\left|\sqrt{2\sigma(P,Q)}\right|, & \sigma(P,Q)\geq 0\\ i\left|\sqrt{2\sigma(P,Q)}\right|, & \sigma(P,Q)<0\end{array}\right., \qquad P,Q\in\Omega \quad (2.2)$$

Let us introduce more a real function of three points $P_0, P_1, P_2$:

$$\Gamma(P_0,P_1,P_2)=\sigma(P_0,P_1)+\sigma(P_0,P_2)-\sigma(P_1,P_2), \qquad P_0,P_1,P_2\in\Omega \qquad (2.3)$$

The functions $\sigma(P,Q)$, $S(P,Q)$ are symmetric with respect to arguments $P$ and $Q$: $\Gamma(P_0,P_1,P_2)$ is symmetric only with respect to the arguments $P_1$ and $P_2$.

It is easy to see that any subset $\Omega'\subset\Omega$ of the $\sigma$ space points is a $\sigma$ space.

Let us introduce the designations

$$^k\mathcal{P}^n=\{P_k,P_{k+1}...P_n\}, \qquad ^k\mathcal{P}_l^n=\ ^k\mathcal{P}^n\backslash\{P_l\}, \qquad \mathcal{P}^n\equiv\ ^0\mathcal{P}^n, \qquad \mathcal{P}_l^n=\mathcal{P}^n\backslash\{P_l\} \qquad (2.4)$$

and define a real function $F_n$ of $n+1$ points $\mathcal{P}^n\subset\Omega$:

$$F_n(\mathcal{P}^n)=\det||\Gamma(P_0,P_i,P_k)||. \qquad i,k=1,2,...n \qquad (2.5)$$

One can show that as a result of Eq. (1.2) $F_n(\mathcal{P}^n)$ is symmetric with respect to any pair of points $P_i$ and $P_k$, $\quad i,k=1,2,...n$.

The meaning of the functions $\sigma$, $\Gamma$, $F_n$ can be understood most easily in the example of $D$-dimensional proper Euclidean space, considering it as a $\sigma$ space with the $\sigma$ function

$$\sigma(P,Q)=\sigma(x,y)=\frac{1}{2}\sum_{i,k=1}^{D}g_{ik}(x^i-y^i)(x^k-y^k) \qquad (2.6)$$

where $\{x^i\}$ and $\{y^i\}$ are contravariant coordinates of the points $P$ and $Q$, respectively, in some coordinate system $K$. Here $g_{ik}$ =const, $i,k=1,2,...D$ is the metric tensor in the coordinate system $K$, which is formed by $D+1$ points $P_0,P_1,...P_D$ with the point $P_0$ as an origin of $K$.

The vectors

$$\mathbf{e}_i=\mathbf{P}_0\mathbf{P}_i, \qquad |\mathbf{e}_i|=|\mathbf{P}_0\mathbf{P}_i|=\sqrt{2\sigma(P_0,P_i)}, \qquad i=1,2,...D \qquad (2.7)$$

are directed along coordinate axes of the coordinate system $K$. Then using the cosine theorem and Eqs.(2.7) and (2.3), it is easy to verify that

$$g_{ik}=(\mathbf{e}_i.\mathbf{e}_k)=\frac{1}{2}\left(|\mathbf{P}_0\mathbf{P}_i|^2+|\mathbf{P}_0\mathbf{P}_k|^2-|\mathbf{P}_i\mathbf{P}_i|^2\right)=\Gamma(P_0,P_i,P_k), \qquad i,k=1,2,...D \qquad (2.8)$$

where $(\mathbf{e}_i.\mathbf{e}_k)$ means the scalar product of vectors $\mathbf{e}_i$ and $\mathbf{e}_k$. The function (2.5)

$$F_n(\mathcal{P}^n)=\det||g_{ik}||=\det||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)||=(n!V_n(\mathcal{P}^n))^2 \qquad (2.9)$$

is the Gram determinant, and $V_n(\mathcal{P}^n)$ is the volume of $(n+1)$ edron with vertices at the points $\mathcal{P}^n$. In proper Euclidean space the condition

$$F_n\left(\mathcal{P}^n\right) \neq 0 \tag{2.10}$$

is the necessary and sufficient condition of the linear independence of $n$ vectors $\mathbf{e}_i = \mathbf{P}_0\mathbf{P}_i, \quad i = 1, 2, ...D$ and that of the fact that $n+1$ points $\mathcal{P}^n$ do not lie on one $(n-1)$-dimensional plane.

**Definition 2** *The $n+1$ point basis $\mathcal{P}^n$ is $n+1$ points $P_i \in \Omega, \quad i = 0, 1, ...n$ that satisfy condition (2.10) .*

The $n+1$ point basis $\mathcal{P}^n$ in the Euclidean space is associated with the basis of $n$ vectors $\mathbf{e}_i = \mathbf{P}_0\mathbf{P}_i, \quad i - 1, 2, ...n$ in an $n$-dimensional plane $\mathcal{L}(\mathcal{P}^n)$ containing the points $\mathcal{P}^n$.

**Definition 3** *The line tube (or merely tube) $\mathcal{T}_n$ of nth order formed by the basis $\mathcal{P}^n$ is a set of points $P \in \Omega$:*

$$\mathcal{T}_{\mathcal{P}^n} = \mathcal{T}(\mathcal{P}^n) = \{P|F_{n+1}(P, \mathcal{P}^n) = 0\}, \qquad F_n(\mathcal{P}^n) \neq 0 \tag{2.11}$$

**Definition 4** *Section $\mathcal{S}_{n;P}$ of the tube $\mathcal{T}(\mathcal{P}^n)$ at the point $P \in \mathcal{T}(\mathcal{P}^n)$ is the set of points $P' \in \Omega$:*

$$\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n)) = \left\{ P'|F_{n+1}(P', \mathcal{P}^n) = 0 \bigwedge_{l=o}^{l=n} \sigma(P_l, P') = \sigma(P_l, P) \right\}, \tag{2.12}$$

$$F_{n+1}(P, \mathcal{P}^n) = 0$$

In the Euclidean space the tube $\mathcal{T}_n$ of $n$th order corresponds to the $n$-dimensional plane containing the points $\mathcal{P}^n$ and the section $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n))$ consists of the point $P$.

The tubes of zeroth order are of most interest. One has, for $F_1$

$$F_1(P_0, P_1) = 2\sigma(P_0, P_1)$$

and

$$\mathcal{T}_{P_0} = \mathcal{T}(P_0) = \{P|\sigma(P_0, P) = 0\} \tag{2.13}$$

In the proper Euclidean space the tube $\mathcal{T}_{P_0} = \{P_0\}$ consists of one point $P_0$ and $\mathcal{S}_{0;P_0}(\mathcal{T}_{P_0}) = \{P_0\}$. However, in the pseudo-Euclidean space, which is the space-time in special relativity, $\mathcal{T}_{P_0}$ is the light cone with the vertex at point $P_0$: its section at the point $P_0$,

$$\mathcal{S}_{0;P}(\mathcal{T}_{P_0}) = \left\{ P'|\sigma(P', P_0) = 0 \bigwedge \sigma(P_0, P') = \sigma(P_0, P) \right\} = \mathcal{T}_{P_0}, \tag{2.14}$$

coincides with the light cone.

In describing first-order tubes it is convenient to use the circumstance that $F_2(\mathcal{P}^2)$ can be represented as a product

$$\begin{aligned} F_2(P_0,P_1,P_2) &= S_+(P_0,P_1,P_2)S_2(P_0,P_1,P_2) \\ &\times S_2(P_1,P_2,P_0)S_2(P_2,P_0,P_1) \end{aligned} \tag{2.15}$$

where

$$S_+(P_0,P_1,P_2) = S(P_0,P_1) + S(P_1,P_2) + S(P_0,P_2) \tag{2.16}$$

$$S_2(P_0,P_1,P_2) = S(P_0,P_1) + S(P_1,P_2) - S(P_0,P_2) \tag{2.17}$$

Since it follows from Eq.(2.2), $S_+$ vanishes in only that case, if all the terms in Eq.(2.16) vanish. Then no two points form a basis, and construction of a tube is not defined. The tube $\mathcal{T}(\mathcal{P}^2)$ can be divided into parts and each of factors (2.17) in Eq.(2.15) is responsible for one part.

Let us set

$$\mathcal{T}_{[P_0P_1]} = \mathcal{T}_{[P_1P_0]} = \{P|S_2(P_0,P,P_1)=0\}\,, \qquad \sigma(P_0,P_1)\neq 0 \tag{2.18}$$

$$\mathcal{T}_{P_0[P_1} = \mathcal{T}_{P_1]P_0} = \{P|S_2(P_0,P_1,P)=0\}\,, \qquad \sigma(P_0,P_1)\neq 0 \tag{2.19}$$

One refers to $\mathcal{T}_{[P_0P_1]}$ as the tube segment between the points $P_0, P_1$ and to $\mathcal{T}_{P_0[P_1}$ as the tube ray outgoing from $P_1$ in the direction from the point $P_0$. The set

$$\mathcal{T}_{[P_0P_1} \equiv \mathcal{T}_{[P_0P_1]} \cup \mathcal{T}_{P_0[P_1} \tag{2.20}$$

will be referred to as the tube ray outgoing from $P_0$ toward point $P_1$. It is evident from Eq.(2.15) that

$$\mathcal{T}_{P_0P_1} = \mathcal{T}_{P_0]P_1} \cup \mathcal{T}_{[P_0P_1]} \cup \mathcal{T}_{P_0[P_1} \tag{2.21}$$

Let us use designations

$$\begin{aligned} \mathcal{T}_{(P_0P_1)} &= \mathcal{T}_{[P_0P_1]}\left[\setminus\bigcup_{l=0}^{l=1}\mathcal{S}_{1;P_l}(\mathcal{T}_{P_0P_1})\right] \\ \mathcal{T}_{(P_0P_1]} &= \mathcal{T}_{[P_0P_1]}\setminus\mathcal{S}_{1;P_0}(\mathcal{T}_{P_0P_1}) \\ \mathcal{T}_{P_0)P_1} &= \mathcal{T}_{P_0]P_1}\setminus\mathcal{S}_{1;P_l}(\mathcal{T}_{P_0P_1}) \end{aligned} \tag{2.22}$$

Here $\mathcal{T}_{(P_0P_1)}$, $\mathcal{T}_{(P_0P_1]}$ will be referred to as the open and half-open tube intercepts, respectively, between the points $P_0$, $P_1$ and $\mathcal{T}_{P_0)P_1}$ will be referred to as the open tube ray outgoing from point $P_0$ in the direction from the point $P_1$.

By definition, the point $P$ is placed between points $P_0$ and $P_1$ if $P\in\mathcal{T}_{(P_0P_1)}$. Let us use designations (2.7), (2.8), and

$$\mathbf{x} = \mathbf{P}_0\mathbf{P}_1. \tag{2.23}$$

Then in the Euclidean space the first order tube $\mathcal{T}_{P_0P_1}$ is described by

$$\begin{vmatrix} (\mathbf{e}_1.\mathbf{e}_1) & (\mathbf{e}_1.\mathbf{x}) \\ (\mathbf{x}.\mathbf{e}_1) & (\mathbf{x}.\mathbf{x}) \end{vmatrix} = 0, \qquad (\mathbf{e}_1.\mathbf{e}_1)\neq 0 \tag{2.24}$$

Its solution has the form

$$\mathbf{P}_0\mathbf{P} = \mathbf{x} = \mathbf{e}_1\tau + \mathbf{q}, \qquad \tau \in \mathbb{R} \tag{2.25}$$

where $\mathbf{q}$ satisfies the conditions

$$(\mathbf{q}.\mathbf{e}_1) = 0, \qquad \mathbf{q}^2 = 0 \tag{2.26}$$

In proper Euclidean space Eqs.(2.26) have unique solution $\mathbf{q}$ =0 and the tube $\mathcal{T}_{P_0P_1}$ coincides with the straight line passing through the points $P_0$, $P_1$. In the pseudo-Euclidean space of index 1 metric signature $(+,-,-,-)$, Eqs.(2.26) have unique solution $\mathbf{q}$ =0 for timelike interval $P_0P_1$, $\mathbf{e}_1^2 = 2\sigma(P_0, P_1) > 0$. For the spacelike interval $P_0P_1$, $[\sigma(P_0,P_1) < 0]$ there are many solutions of Eqs.(2.26) and $\mathcal{T}_{P_0P_1}$ is not reduced to the straight line $\mathcal{L}_{P_0P_1}$.

In the case of the proper Euclidean space $V$ and in the case of the pseudo-Euclidean space $V$ of index 1 the section $\mathcal{S}_{1;P}(\mathcal{T}_{P_0P_1})$ of timelike tube $\mathcal{T}_{P_0P_1}$ has the form

$$\mathcal{S}_{1;P}(\mathcal{T}_{P_0P_1}) = \{P\}, \qquad P \in \mathcal{T}_{P_0P_1} \tag{2.27}$$

If the tube is spacelike $[\sigma(P_0, P_1) < 0]$, then

$$\mathcal{S}_{1;Q}(\mathcal{T}_{P_0P_1}) = \mathcal{T}_{P_0P_1}(\tau_0), \qquad \tau_0 = (\mathbf{P}_0\mathbf{Q}.\mathbf{P}_0\mathbf{P}_1)/(\mathbf{P}_0\mathbf{P}_1)^2 \tag{2.28}$$

where $\mathcal{T}_{P_0P_1}(\tau)$ is a set of points $P$ satisfying Eqs.(2.25) and (2.26).

Thus in the four-dimensional event space the timelike tubes are one-dimensional straight lines; however, the spacelike tube $\mathcal{T}_{P_0P_1}$ is a three-dimensional surface formed as a result of moving the light cone section normal to the vector $\mathbf{P}_0\mathbf{P}_1$: the section moves in the direction of the vector $\mathbf{P}_0\mathbf{P}_1$.

In the conventional approach a geodesic in a $D$-dimensional Riemannian space is considered as a *special type of a curve* having extremal properties, as follows.

1. (i) *Extremality* The distance $(2\sigma)^{1/2}$ between two points measured along a gedesic is the shortest (extremal) compared to distance measured along other curves.

   (ii) *Definiteness.* Any two points of the geodesic determine unambiguously the geodesic passing through these points.

   (iii) *Minimality of section* (one-dimensionality). Any section of a geodesic consists of one point.

Another approach is when the geodesic is considered as a *special type of surface* (or of a line tube) that degenerates into a line. Then properties (ii) and (iii) are supposed to be fulfilled; however, property (i) is not defined because the concept of a line (or curve) is not defined. Let us try to define a geodesic tube having the property of definiteness and minimality of section at the same time.

**Definition 5** *The tube $\mathcal{T}(\mathcal{P}^n)$ has the definiteness property if for any basis $\mathcal{Q}^n$ of $n+1$ points $\mathcal{Q}^n \subset \mathcal{T}(\mathcal{P}^n)$ the condition*

$$\mathcal{T}(\mathcal{Q}^n) \subset \mathcal{T}(\mathcal{P}^n) \tag{2.29}$$

*is fulfilled.*

**Definition 6** *The tube $\mathcal{T}(\mathcal{P}^n)$ has the property of section minimality if* $\forall P \in \mathcal{T}(\mathcal{P}^n)$,

$$\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n)) = \{P\}. \tag{2.30}$$

**Definition 7** *The $\sigma$ space is extremal on the tube $\mathcal{T}(\mathcal{P}^n)$ if the conditions of definiteness and section minimality are fulfilled.*

**Definition 8** *The $\sigma$ space is extremal on the set $\mathcal{T}$ of tubes $\mathcal{T}(\mathcal{P}^n)$ of nth order if it is extremal on each tube of the set $\mathcal{T}$.*

**Definition 9** *The $\sigma$ space is extremal in the nth order if it is extremal on all tubes $\mathcal{T}(\mathcal{P}^n)$ of nth order.*

**Definition 10** *The tube $\mathcal{T}(\mathcal{P}^n)$ is a geodesic tube $\mathcal{L}(\mathcal{P}^n)$ if the $\sigma$ space is extremal on the tube $\mathcal{T}(\mathcal{P}^n)$.*

Introducing the concept of $\sigma$ space and considering a geodesic as a kind of a line tube, one hopes to obtain a more adequate description of the event space. The $\sigma$ space is described in terms of the $\sigma$ function and only in these terms. In this approach the introduction of such concepts as continuity and manifold is not necessary. In this approach the world lines are replaced by world tubes whose section is a surface, but in general not a point. This approaches associated with the string model, which is popular in the contemporary theory of elementary particles. Finally, quantum particles have no definite world lines: They are "spread" over the event space. The tubes are the world lines spread over the event space. Perhaps the tubes are more adequate for describing quantum particles. In other words, perhaps the $\sigma$ space describes the real event space at small distances better than does the Riemannian space.

We are not to be discouraged by the circumstance that in the event space the timelike tubes correspond to timelike straight lines and the spacelike tubes correspond to three-dimensional surfaces. Really, world lines of the real particles are timelike, whereas there are no spacelike world lines. Perhaps the $\sigma$ space makes even this distinction.

In reference to the null tubes $\mathcal{T}^*(\mathcal{P}^1)$ ($\sigma(P_0, P_1) = 0$ ), definition (2.24) or (2.11) does not provide them. For continuous $\sigma$ space the null tubes $\mathcal{T}^*_{P_0P_1}$ can be defined as follows:

$$\mathcal{T}^*_{P_0P_1} = \lim_{P' \to P_1} \mathcal{T}_{P_0P'}, \qquad \sigma(P', P_0) > 0, \qquad \sigma(P_0, P_1) = 0 \tag{2.31}$$

as a limit of the timelike tube $\mathcal{T}_{P_0P'}$ at $P' \to P_1$. In the case of pseudo-Euclidean space the result is a null straight line. If the interval $P_0P'$ is spacelike, then the result of Eq. (2.31) depends on the way in which $P'$ is applied to $P_1$.

*Example 1:* Let points $P$ of $\sigma$ space $V$ be numbered by $n_1, n_2, n_3 \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integer numbers. Let the $\sigma$ function that is between the points $P = (n_1, n_2, n_3)$, $P' = (n'_1, n'_2, n'_3)$; $n_1, n_2, n_3, n'_1, n'_2, n'_3 \in \mathbb{Z}$ be defined by the relation

$$\sigma(P,P') = \frac{a^2}{2}\sum_{i=1}^{3}(n_i - n_i')^2, \qquad a = \text{const}, \qquad a > 0 \tag{2.32}$$

The $\sigma$ function depends only on the difference $n_i - n_i'$, $i = 1,2,3$ and the tube's properties can be investigated without loss of generality in the example of $\mathcal{L}_{P_0P'}$, $P_0 = (0,0,0)$. Solving

$$F_2(P_0, P', P) = 0 \tag{2.33}$$

with the $\sigma$ function (2.32) one obtains

$$\mathcal{L}_{P_0P'}: \qquad n_i = \tau n_i', \qquad i = 1,2,3, \qquad \tau = k/N, \qquad N, k \in \mathbb{Z} \tag{2.34}$$

where $n_i$, $i = 1,2,3$ are coordinates of points of $\mathcal{L}_{P_0P'}$, $k$ is an arbitrary integer number and $N$ is determined by the relation

$$N = \max_{P''\in\mathcal{L}_{P_0P'}} \left\{\sqrt{\sigma(P,P')/\sigma(P,P'')}\right\} \tag{2.35}$$

Thus points of $\mathcal{L}_{P_0P'}$ are points of the straight line $\mathbf{x} = \mathbf{n}\tau$, $[\mathbf{n}' = n_1', n_2', n_3')$, $\mathbf{x} = (x^1, x^2, x^3)]$ with integer coordinates. It is easy to verify that the $\sigma$ space (2.32) is extremal in the first order.

The determination of straight lines permits us to introduce the linear vector space over the ring of integer numbers $\mathbb{Z}$. Let us refer to an ordered pair of points $P_0, P_1$ as the vector $\mathbf{P}_0\mathbf{P}_1$. The modulus of the vector $\mathbf{P}_0\mathbf{P}_1$ is the number

$$|\mathbf{P}_0\mathbf{P}_1| = \sqrt{2\sigma(P_0, P_1)} \tag{2.36}$$

The vector $\mathbf{P}_0\mathbf{P}' = a\mathbf{P}_0\mathbf{P}$ is a result of multiplying the vector $\mathbf{P}_0\mathbf{P}$ by the number $a \in \mathbb{Z}$. The point $P'$ is determined by

$$\sigma(P_0, P') = a^2\sigma(P_0, P) \tag{2.37}$$

with $P' \in \mathcal{L}_{[P_0P}$ if $a \geq 0$ and $P' \in \mathcal{L}_{P_0]P}$ if $a < 0$. The sum of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{P}_2$

$$\mathbf{P}_0\mathbf{P}'' = \mathbf{P}_0\mathbf{P}_1 + \mathbf{P}_0\mathbf{P}_2, \tag{2.38}$$

$$P' \in \mathcal{L}_{[P_1P_2]}, \qquad S(P',P_1) = S(P',P_2), \qquad P'' \in \mathcal{L}_{P_0P'}, \qquad S(P_0,P') = S(P',P'')$$

Thus defined operations of summation and multiplication by a number satisfy all axioms of linear vector space.

The vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0\mathbf{P}_2$, ...$\mathbf{P}_0\mathbf{P}_n$ are referred to as linear independent if the relation

$$\sum_{k=1}^{n}\alpha_k\mathbf{P}_0\mathbf{P}_k = \mathbf{P}_0\mathbf{P}_0, \qquad \alpha_1, \alpha_2, ...\alpha_n \in \mathbb{Z} \tag{2.39}$$

is fulfilled only at $\alpha_i = 0$, $i = 1,2,...n$. The maximal number of the linear-independent vectors is called the dimensionality of the space. In the given case $D = 3$.

The scalar product of vectors $\mathbf{P}_0\mathbf{P}$ and $\mathbf{P}_0\mathbf{P}'$ is the number $(\mathbf{P}_0\mathbf{P}.\mathbf{P}_0\mathbf{P}')$, which is defined by the relation

$$(\mathbf{P}_0\mathbf{P}.\mathbf{P}_0\mathbf{P}') = \Gamma(P_0, P, P') = \sigma(P_0, P) + \sigma(P_0, P') - \sigma(P, P'), \qquad P_0, P, P' \in \Omega \tag{2.40}$$

Thus all operations in the vector space are defined through the world function $\sigma$. The space $V$ is a subset of three-dimensional Euclidean space. Indeed, let us imagine that coordinates $(n_1, n_2, n_3)$, $(n_1', n_2', n_3')$ in expression (2.32) are real numbers. Then relations (2.32)-(2.40) define the three dimensional proper Euclidean space, where $\tau$ in Eq.(2.34) and $a$ in Eq.(2.37) are arbitrary real numbers. The constraint (2.35) is to be omitted.

*Example 2:* Let points of the $\sigma$ space $V_c$ be numbered by three numbers $n_1, n_2, n_3$, $(n_1, n_2, n_3 \in \mathbb{Z},\ 0 \leq n_3 < N)$, where $N$ is some natural number and $\mathbb{Z}$ is the set of all integer numbers. Let us define the $\sigma$ function by the relation

$$\sigma(P, P') = \frac{a^2}{2}\sum_{i=1}^{3}(n_i - n_i')^2, \qquad n_1, n_2, n_3, n_1', n_2', n_3' \in \mathbb{Z} \tag{2.41}$$

$$P = (n_1, n_2, \operatorname{mod}_N n_3), \qquad P' = (n_1', n_2', \operatorname{mod}_N n_3')$$

Here $\sigma$ is a multivalued function of two points $P, P'$ because the same value $\operatorname{mod}_N(n_3 - n_3')$ corresponds to different values of $n_3 - n_3'$. The space $V_c$ is obtained from the space $V$ of Example 1 by means of identifying those points $P$ whose coordinates $n_3$ are distinguished by $sN$, where $s$ is integer. At such identification $P = P_1 = P_2 = ...$ the world function $\sigma(P', P_i)$, $i = 1, 2, ...$ converts into $\sigma(P', P)$ and becomes multivalued, i.e. the world function contains information about the identification. The space $V_c$ is a discrete analog of a cylinder, whereas $V$ from Example 1 is a discrete analog of the three-dimensional plane.

The $\sigma$ space with the world function (2.41) is not extremal in the first order. In particular, the tube $\mathcal{T}_{P_0P''}$, $P_0 = (0,0,0)$, $P'' = (n_1'', n_2'', \operatorname{mod}_N n_3'')$ consists of geodesics of type (2.34):

$$\mathcal{T}_{P_0P''} = \bigcup_{P' \in \mathbb{Z}} \mathcal{L}_{P_0P'}, \qquad n_1' = n_1'',\ \ n_2' = n_2'',\ \ n_3' = \operatorname{mod}_N n_3'' + sN$$

Usually, the difference between a cylinder an a plane is formulated as a distinction of topology of these surfaces. When discussing the distinction between $V$ and $V_c$, it is hardly appropriate to speak about topology because topology is connected with the concept of continuity, which is not used here. Formally, a "cylindricity" of the space $V_c$ manifests itself in "closed" geodesics consisting of $N$ points. In the $\sigma$ space $V$ of the Example 1 any geodesic contains an infinite number of points.

In Example 2 a single-valued world function can be defined by the relation

$$\sigma(P, P') = \frac{a^2}{2}\{(n_1 - n_1')^2 + (n_2 - n_2')^2 + [\operatorname{mod}_N(n_3 - n_3' + q) - q]_E^2\} \tag{2.42}$$

$$q = [N/2]_E$$

where $[...]_E$ denotes the entire part of the number. In this case the space is also no extremal in the first order. Equation (2.42) corresponds to the case when the unique (minimal) value of the function is chosen among many values (2.41). Thus almost all properties of the Euclidean space can be formulated in terms of an extremal $\sigma$ space without using the concept of continuity. A discrete analog of the Euclidean space can be constructed by removing all points except a countable set, with the values of the world function for the remaining points being conserved.

**Definition 11** *The space $V$, the point set of which is a subset of the points of the Euclidean space $E$, is an Euclidean $\sigma$ space.*

# 3 Properties of the $\sigma$ space

Let us consider a Euclidean space $E_n$ of the dimensionality $n > 1$ and introduce coordinates of an arbitrary point $P$ in the basis $\mathcal{P}^n$ using only world function. Let $\mathcal{P}^n$ be $n+1$ points that do not lie on one $(n-1)$-dimensional plane. In this case $F_n(\mathcal{P}^n) \neq 0$ and $\mathcal{P}^n$ is a point basis in $E_n$ connected with the basis

$$\mathbf{e}_i = \mathbf{P}_0\mathbf{P}_i, \qquad i = 1, 2, ...n \tag{3.1}$$

in the linear space of vectors $\mathbf{P}_0\mathbf{P}$. Then in this basis according to Eq.(2.7), the metric tensor $g_{ik}(\mathcal{P}^n)$ has the form (2.8) and, according to (2.9)

$$\det ||g_{ik}(\mathcal{P}^n)|| \neq 0, \qquad i, k = 1, 2, ...n \tag{3.2}$$

The covariant coordinates $x_i$ of the vector $\mathbf{P}_0\mathbf{P}$ in this basis are

$$x_i = (\mathbf{P}_0\mathbf{P}.\mathbf{e}_i) = \Gamma(P_0, P, P_i) = \sigma(P_0, P) + \sigma(P_0, P_i) - \sigma(P, P_i),$$

$$y_i = (\mathbf{P}_0\mathbf{Q}.\mathbf{e}_i) = \Gamma(P_0, Q, P_i), \qquad i = 1, 2, ...n \tag{3.3}$$

The world function of two points $P, Q$ of the Euclidean space $E_n$ has the form

$$\sigma(P, Q) = \frac{1}{2} g^{ik}(\mathcal{P}^n)(x_i - y_i)(x_k - y_k) \tag{3.4}$$

where $g^{ik}$ are contravariant components of the metric tensor, defined by the relation

$$g^{ik}(\mathcal{P}^n) g_{kl}(\mathcal{P}^n) = \delta^i_l, \qquad i, l = 1, 2, ...n \tag{3.5}$$

Here the summation is made over repeated superindices and subindices $(1-n)$.

The following definitions, equivalent to conventional definitions, will be used.

**Definition 12** *: The $n$-dimensional Euclidean space is the set $\mathbb{R}^n$ of all ordered number $x = \{x_1, x_2, ...x_n\}$, where the $\sigma$ function is given by relations (3.4) and (3.2), $\forall x \in \mathbb{R}^n$, $\forall y \in \mathbb{R}^n$.*

**Definition 13** *: The proper $n$-dimensional Euclidean space is the Euclidean space for which the equation $\sigma(x, y) = 0$, $\forall y \in \mathbb{R}^n$ has the unique solution $x = y$.*

**Definition 14** *: The pseudo-Euclidean space is the Euclidean space, which is not proper.*

**Definition 15** *: The flat space is the Riemannian space with an everywhere vanishing Riemannian curvature tensor. A flat space can differ from the Euclidean space in topology.*

Let us consider a $(n+2)\times(n+2)$ matrix

$$\mathcal{A}_{n+2} = ||a_{ik}||, \qquad a_{ik} = \Gamma(P_0, P_i, P_k), \qquad i,k = 1,2,...n+2, \tag{3.6}$$

$$P_{n+1} = P, \qquad P_{n+2} = Q$$

$$F_n(\mathcal{P}^n) \neq 0 \tag{3.7}$$

**Theorem 1** *: Let $\mathcal{P}^n$ be a basis of $n+1$ points in the $\sigma$ space $V$, i.e., Eq.(3.7) is fulfilled. Then for the tube $\mathcal{T}(\mathcal{P}^n)$ to be a Euclidean $\sigma$ space, fulfillment of the following conditions is necessary and sufficient:*

*(i) The* $\mathrm{rank}(\mathcal{A}_{n+2}) = n, \quad \forall P \in \mathcal{T}(\mathcal{P}^n), \ \ \forall Q \in \mathcal{T}(\mathcal{P}^n)$.

*(ii)* $\mathcal{S}_{n;Q}(\mathcal{T}(\mathcal{P}^n)) = \{Q\}, \ \ Q \in \mathcal{T}(\mathcal{P}^n)$.

*Proof:* $\mathcal{T}(\mathcal{P}^n) \subset E_n$ be a subset of a Euclidean space $E_n$ and $P \in \mathcal{T}(\mathcal{P}^n), \ \ Q \in \mathcal{T}(\mathcal{P}^n)$. Then using the designations (2.8), (3.3), (3.7) and property (3.4), one can represent the matrix in the form

$$\mathcal{A}_{n+2} = \left\| \begin{array}{ccc} g_{ik}(\mathcal{P}^n) & x_i & y_i \\ x_k & \sum\limits_{l=1}^{n} x_l x^l & \sum\limits_{l=1}^{n} y_l x^l \\ y_k & \sum\limits_{l=1}^{n} x_l y^l & \sum\limits_{l=1}^{n} y_l y^l \end{array} \right\|, \qquad i,k = 1,2,...n \tag{3.8}$$

where

$$x^i = \sum_{k=1}^{n} g^{ik}(\mathcal{P}^n) x_k, \qquad y^i = \sum_{k=1}^{n} g^{ik}(\mathcal{P}^n) y_k \tag{3.9}$$

are contravariant coordinates of points $P$ and $Q$ in the basis $\mathcal{P}^n$.

According to Eq.(3.7)

$$g \equiv \det ||g_{ik}|| = F_n(\mathcal{P}^n) \neq 0 \tag{3.10}$$

and rank $r$ of the matrix $\mathcal{A}_{n+2}$ is not less than $r$ $(r \geq n)$. On the other hand, the last and next to last columns of the matrix (3.8) are linear combinations of the first $n$ columns with the factors $y^k$ and $x^k$, $k = 1,2,...n$, respectively, i.e. rank of the matrix $r \leq n$. Then one concludes that in the Euclidean space the rank of the matrix $\mathcal{A}_{n+2}$ is equal to $n$ provided that when fulfilling Eq. (3.7),

$$\mathrm{rank}(\mathcal{A}_{n+2}) = n \tag{3.11}$$

It is easy to verify that the section of any plane $\mathcal{T}(\mathcal{P}^n)$ of the $n$-dimensional Euclidean space is minimal in the sense of Eq. (2.30).

Now let $n+1$ points $\mathcal{P}^n$ of a $\sigma$ space $V$ satisfy Eq. (3.7). Let $P \in \mathcal{T}(\mathcal{P}^n)$, $Q \in \mathcal{T}(\mathcal{P}^n)$, rank of matrix (3.6) be equal to $n$, and any section of the tube $\mathcal{T}(\mathcal{P}^n)$ be minimal, i.e. Eq. (2.30) is fulfilled $\forall P \in \mathcal{T}(\mathcal{P}^n)$. According to Eq. (2.11), the conditions $P \in \mathcal{T}(\mathcal{P}^n)$, $Q \in \mathcal{T}(\mathcal{P}^n)$ mean that

$$M_{n+1,n+1} = F_{n+1}(P, \mathcal{P}^n) = \left\| \begin{array}{cc} g_{ik} & x_i \\ x_k & 2\sigma(P_0, P) \end{array} \right\| = 0$$

$$M_{n+2,n+2} = F_{n+1}(Q, \mathcal{P}^n) = 0 \tag{3.12}$$

where $M_{n+1,n+1}$ and $M_{n+2,n+2}$ are two principal minors of the matrix (3.6). It follows from Eq. (3.12) that

$$\sigma(P_0, P) = \frac{1}{2} \sum_{i,k=1}^{n} g^{ik} x_i x_k, \qquad \forall P \in \mathcal{T}(\mathcal{P}^n) \tag{3.13}$$

Using Eq. (3.13), the matrix $\mathcal{A}_{n+2}$ can be written in the form

$$\mathcal{A}_{n+2} = \left\| \begin{array}{ccc} g_{ik}(\mathcal{P}^n) & x_i & y_i \\ x_k & \sum\limits_{l=1}^{n} x_l x^l & \Gamma(P_0, P, Q) \\ y_k & \Gamma(P_0, P, Q) & \sum\limits_{l=1}^{n} y_l y^l \end{array} \right\|, \tag{3.14}$$

In order that the rank of matrix (3.14) be equal to $n$, it is necessary that each of the two last columns be linear combination of the first $n$ columns: This means that

$$\Gamma(P_0, P, Q) = \sum_{l=1}^{n} x_l y^l \tag{3.15}$$

As a result of definition (3.3) and Eq. (3.13), Eq. (3.15) lead to Eq.(3.4).

Now let us map $\mathcal{T}(\mathcal{P}^n) \to \Omega_n \subset \mathbb{R}^n$ where $\mathbb{R}^n$ is a set of all elements consisting of $n$ ordered real numbers $x = \{x_i\}$ $(i = 1, 2, ...n)$. Let us correspond to each point $P$ its covariant coordinates in the basis $\mathcal{P}^n$. Such a mapping is one-to-one as a result of the minimal section condition (2.30), i.e., each point $P \in \mathcal{T}(\mathcal{P}^n)$ corresponds to one point $x \in \Omega_n \subset \mathbb{R}^n$ and each point $x \in \Omega_n$ corresponds to only one point $P \in \mathcal{T}(\mathcal{P}^n)$. Really, if the points $P \in \mathcal{T}(\mathcal{P}^n)$ and $Q \in \mathcal{T}(\mathcal{P}^n)$ have the same coordinates

$$x_i = \Gamma(P_0, P_i, P) = \Gamma(P_0, P_i, Q) = y_i, \qquad i = 1, 2, ...n, \tag{3.16}$$

then it follows from Eq. (3.13) that

$$\sigma(P_0, P) = \sigma(P_0, Q). \tag{3.17}$$

As a corollary of Eqs. (2.3) and (3.16) one has

$$\sigma(P_i, P) = \sigma(P_i, Q), \qquad i = 1, 2, ...n \tag{3.18}$$

Conversely, Eq. (3.16) follows from Eqs. (3.17), (3.18) and (2.3). Then definition (2.12) of the tube $\mathcal{T}(\mathcal{P}^n)$ section can also be represented in the form

$$\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n)) = \left\{ P' | F_{n+1}(P', \mathcal{P}^n) = 0 \bigwedge_{l=o}^{l=n} \Gamma(P_0, P_i, P') = \Gamma(P_0, P_i, P) \right\}, \qquad (3.19)$$

$$F_{n+1}(P, \mathcal{P}^n) = 0$$

It follows from (3.19) that if $Q \in \mathcal{T}(\mathcal{P}^n)$ and has coordinates (3.16), then $Q \in \mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n))$. As a result of the minimal section condition the section consists of one point. Then any coordinates $x$ correspond to the unique point $P$ and the mapping $\mathcal{T}(\mathcal{P}^n) \to \Omega_n$ is one-to-one.

Thus for $x \in \Omega_n \subset \mathbb{R}^n$, $y \in \Omega_n \subset \mathbb{R}^n$ the $\sigma$ function has the form (3.4). Let us propagate Eq. (3.4) over all points of the set $\mathbb{R}^n$. Then the $\mathbb{R}^n$ with the $\sigma$ function defined on it is a Euclidean space. The $\Omega_n$ with the $\sigma$ function defined on it is a subset of $\mathbb{R}^n$. Hence, $\Omega_n$ is the Euclidean $\sigma$ space. As a result of the one-to-one correspondence $\mathcal{T}(\mathcal{P}^n) \leftrightarrow \Omega_n$ and the invariance of the $\sigma$ function one concludes that $\mathcal{T}(\mathcal{P}^n)$ is the Euclidean $\sigma$ space. The theorem 1 has been proved.

Theorem 1 permits us to determine the dimensionality $D$ of the proper Euclidean $\sigma$ space $V$ based only upon the $\sigma$ function. Let us use the following procedure. Let $P_0 \in V$. If $\mathcal{T}_{P_0} = V$, then $D = 0$; otherwise, $\exists P_1 \notin \mathcal{T}_{P_0}$, $P_1 \in V$. Then $F_1(\mathcal{P}^1) \neq 0$, $D > 0$. If $\mathcal{T}(\mathcal{P}^1) = V$, $\mathcal{P}^1 = \{P_0, P_1\}$, then $D = 1$; otherwise $\exists P_2 \notin \mathcal{T}(\mathcal{P}^1)$, $P_2 \in V$, and $D > 1$. If $\mathcal{T}(\mathcal{P}^2) = V$, then $D = 2$; otherwise $\exists P_3 \notin \mathcal{T}(\mathcal{P}^2)$, $P_3 \in V$, and $D > 2$, etc. Let us continue the procedure until at some $n$, $\mathcal{T}(\mathcal{P}^n) = V$, and $D = n$. Such a procedure can be produced in any space, but for the single-valued result it is necessary that it does not depend on the choice of the basis $\mathcal{P}^n$: It is provided $V$ is a proper Euclidean space.

# 4 The Euclidean $\sigma$ space and extremality

The proper Euclidean space is extremal in all orders. Let us investigate whether the reverse statement is valid: A $\sigma$ space that is extremal in all orders is proper Euclidean. In general, this statement is not valid, although its violations are rather an exception than a rule.

Let a basis $\mathcal{P}^n \subset \Omega$, where $\Omega$ is the set of points of the extremal $\sigma$ space $V$, $P \in \mathcal{L}(\mathcal{P}^n)$, and $Q \in \mathcal{L}(\mathcal{P}^n)$. The definiteness condition (2.29) containing $n+3$ points $P$, $Q$, $\mathcal{P}^n$ can be written in the form

$$F_{n+1}(P, \mathcal{P}^n) = 0, \qquad F_{n+1}(Q, \mathcal{P}^n) = 0 \qquad (4.1)$$

$$F_{n+1}(P, Q, \mathcal{P}^n_l) = 0, \qquad l = 0, 1, ...n \qquad (4.2)$$

Here the $n+3$ equations (4.1), (4.2) (except for the case $l = 0$ ) represent the condition of vanishing $n+2$ principal minors of the matrix (3.6). Using the designations(3.2) and (3.3), one obtains from Eq. (4.1), relation (3.13) and a similar relation with the substitution $P \to Q, \quad x \to y$. Each of Eqs. (4.2) represents a quadratic equation with respect to the variable

$$z = \Gamma(P_0, P, Q) - \sum_{l=1}^{n} x_l y^l \tag{4.3}$$

Equations (4.2) have a trivial solution

$$z_1 = 0 \tag{4.4}$$

which corresponds to relation (3.15) and leads to expression (3.4) for $\sigma(P,Q)$, i.e. to the Euclidean $\sigma$ space $\mathcal{T}(\mathcal{P}^n)$.

Using the trivial solution (4.4), one can reduce the order of Eqs. (4.2). Then Eqs. (4.2) (except for the case $l = 0$) are reduced to $n$ linear equations with respect to $z$:

$$g^{ll} z + 2x^l y^l = 0, \qquad l = 1, 2, ...n \tag{4.5}$$

One can show that

$$(x^l)^2 = \frac{F_n(P, \mathcal{P}_l^n)}{F_n(\mathcal{P}^n)}, \qquad (y^l)^2 = \frac{F_n(Q, \mathcal{P}_l^n)}{F_n(\mathcal{P}^n)}, \qquad g^{ll} = \frac{F_{n-1}(\mathcal{P}_l^n)}{F_n(\mathcal{P}^n)}, \qquad l = 1, 2, ...n \tag{4.6}$$

Let us substitute Eqs. (4.6) into Eqs. (4.5) and take into account that the first ($l = 0$) of Eqs. (4.2) is obtained from the case $l = k$ as a result of substituting $P_0 \leftrightarrow P_k$. Then one obtains a system of $n$ linear equations for $z$:

$$F_{n-1}(\mathcal{P}_l^n) z + 2\sqrt{F_n(P, \mathcal{P}_l^n) F_n(Q, \mathcal{P}_l^n)} = 0, \qquad l = 0, 1, ...n \tag{4.7}$$

The coefficients of Eqs. (4.7) do not depend on $\sigma(P,Q)$: The condition of a common solution of Eqs. (4.7) imposes constraints upon these coefficients, which are constraints upon coordinates of the points $P$ and $Q$.

Only nontrivial solution of system (4.7) is of interest. The trivial solution $z = 0$ returns us to case (4.4) of the Euclidean space.

If the points $\mathcal{P}^n = \{P_0, P_1, ...P_n\}$ are such that no $n$ points of them lie on a tube of $(n-2)$th order, i.e.,

$$F_{n-1}(\mathcal{P}_l^n) \neq 0, \tag{4.8}$$

then according to (4.6)

$$g^{ll} \neq 0, \qquad l = 1, 2, ...n \tag{4.9}$$

and Eq. (4.5) can be represented in the form

$$\frac{x^l y^l}{g^{ll}} = \frac{\sqrt{F_n(P, {}^1\mathcal{P}^n) F_n(Q, {}^1\mathcal{P}^n)}}{2F_{n-1}({}^1\mathcal{P}^n)}, \qquad l = 1, 2, ...n \tag{4.10}$$

**Definition 16** : *The $(n+1)$ point basis $\mathcal{P}^n$ in $\sigma$ space $V$ is nondegenerate basis provided that any $n$ points of $\mathcal{P}^n$ form a $n$-point basis $\mathcal{P}_l^n$ $(F_{n-1}(\mathcal{P}_l^n) \neq 0, \quad l = 0, 1, ...n)$*

**Definition 17** *: Points $P$ and $Q$ of the $\sigma$ space $V$ are mutually conjugate with respect to the nondegenerate $(n+1)$ point basis $\mathcal{P}^n \subset V$ if $P \notin \mathcal{T}(\mathcal{P}_l^n)$, $Q \notin \mathcal{T}(\mathcal{P}_l^n)$, $l = 0, 1, ...n$, and their contravariant coordinates $x$ and $y$ satisfy Eq. (4.10).*

If one of coordinates $x^l = 0$ [which is equivalent to $P \in \mathcal{T}(\mathcal{P}_l^n)$ or $F_n(P, \mathcal{P}_l^n) = 0$], then according to Eqs. (4.5)–(4.10) one obtains the trivial solution (4.4).

Fixing the coordinates $x^i \neq 0$, $i = 1, 2, ...n$ of point $P$, one concludes from (4.10) that the set of points $Q$ satisfying Eq. (4.10) with an indefinite rhs forms a straight line passing through point $P_0$. From symmetry under consideration one concludes that point $Q$ must lie on the straight line passing through point $P_1$, through $P_2$,... . Thus in this case there is no more than one point $Q$ conjugate to $P$ with respect to the basis $\mathcal{P}^n$.

One can formulate the following theorems.

**Theorem 2** *: Let $\mathcal{P}^n$ be a nondegenerate $(n+1)$ point basis in the $\sigma$ space $V$, which is extremal on $\mathcal{T}(\mathcal{P}^n) = \mathcal{L}(\mathcal{P}^n)$. If $P \in \mathcal{L}(\mathcal{P}^n)$, $P \notin \mathcal{T}(\mathcal{P}_l^n)$, $l = 0, 1, .., n$, then there exists no more than one point $Q$ conjugate to $P$ with respect to $\mathcal{P}^n$.*

**Theorem 3** *: Let $\mathcal{P}^n$ form a nondegenerate $(n+1)$ point basis in the $\sigma$ space $V$ and $V$ be extremal on $\mathcal{T}(\mathcal{P}^n) = \mathcal{L}(\mathcal{P}^n)$. If $P, Q \in \mathcal{L}(\mathcal{P}^n)$, $P, Q \notin \mathcal{T}(\mathcal{P}_l^n)$, $l = 0, 1, .., n$, then either the $\sigma$ space $\mathcal{L}(\mathcal{P}^n)$ is Euclidean or the points $P$ and $Q$ are mutually conjugate with respect to $\mathcal{P}^n$.*

**Theorem 4** *: Let $\mathcal{P}^n$ be nondegenerate $(n+1)$ point basis in the $\sigma$ space $V$, and $V$ be extremal on $\mathcal{T}(\mathcal{P}^n) = \mathcal{L}(\mathcal{P}^n)$. Then the $\sigma$ space $V_1 = \mathcal{L}(\mathcal{P}^n)$ is Euclidean provided that the number $N$ of points in $V_1 = \mathcal{L}(\mathcal{P}^n)$ is distinguished from $n+3$.*

*Proof:* Insofar as $\mathcal{P}^n \subset \mathcal{L}(\mathcal{P}^n)$, then $N \geq n+1$. If $N = n+1$, then $V_1 = \mathcal{P}^n$ and Theorem 4 is evident from Eqs.(3.3), (3.5), and (2.8).

If $N = n+2$, $V_1 = \{\mathcal{P}^n, P\}$. Then it follows from the first of Eqs. (3.12) that the relation (3.13) takes place. Thereafter, one can verify Eq. (3.4) using Eqs. (3.3), (2.8) (3.5), and (3.13).

Now if $N = n+4$, then $V_1 = \{\mathcal{P}^n, P, Q, R\}$. Let $V_1$ be non-Euclidean. Then according to Theorem 2 $Q$ is conjugate to $P$ and $R$ is conjugate to $P$ with respect to the basis $\mathcal{P}^n$. according to Theorem 2 the conjugate point is unique; then $Q \neq R$ cannot exist. Then the $\sigma$ space is Euclidean. The same consideration can be given in the case of $N > n+4$. Theorem 4 has been proved.

Now let us consider an illustration for Theorem 4: when $N = n+3$, the $\sigma$ space is extremal on $\mathcal{T}(\mathcal{P}^n)$; however, $\mathcal{T}(\mathcal{P}^n)$ is non-Euclidean.

*Example 3:* Let the $\sigma$ space $\Omega_4$ consist of the four points $0, 1, P, Q$.

$$S(0,1) = S(P,Q) = a,$$
$$S(0,P) = S(Q,1) = b, \qquad 0 < b < a \tag{4.11}$$
$$S(P,1) = S(0,Q) = a - b,$$

It is easy to verify that

$$\mathcal{T}_{01} = \mathcal{T}_{0P} = \mathcal{T}_{0Q} = \mathcal{T}_{1P} = \mathcal{T}_{1Q} = \mathcal{T}_{PQ} = \Omega_4,$$

i.e. the tube $\mathcal{T}_{01}$ contains all points of the $\sigma$ space $\Omega_4$ and $\Omega_4$ is one-dimensional. The line tube has the definiteness property. Besides, if $a - b \neq b$, then the section of the tube is minimal. Thus at $a \neq 2b$ the $\sigma$ space is extremal in the first order.

In addition, one has the following situation, which is unusual for the proper Euclidean space:

$$P \in \mathcal{L}_{(01)}, \qquad Q \in \mathcal{L}_{(01)}$$

i.e. the points $P$ and $Q$ are placed between the points 0 and 1. At the same time $0 \in \mathcal{L}_{(PQ)}$, $1 \in \mathcal{L}_{(PQ)}$, i.e. the points 0 and 1 lie between the points $P$ and $Q$. Here $\Omega_4$ is not a Euclidean $\sigma$ space.

Relations (4.11) can be understood from the Euclidean point of view if one imagines that the points are placed on closed geodesic in the order $0, P, 1, Q, 0, \ldots$. For the transition to the unclosed geodesic it is sufficient to substitute $S(P,Q) = a$ with $S(P,Q) = a - 2b$.

Another version of interpretation is shown in Fig.1. One-dimensional proper Euclidean $\sigma$ space consists of the five points $Q'_1, 0, P, 1, Q'_0$. The points $Q'_0$ and $Q'_1$ are mirror images of point $Q$ at a reflection with respect to points 1 and 0, respectively. The point $Q$ is conjugate to point $P$ with respect to basis $(0,1)$ provided that $S(P,Q'_1) = S(P,Q'_0)$. Let the points $P$ and $Q$ be conjugate with respect to the basis $(0,1)$. Then it is possible to identify the points $Q'_0$ and $Q'_1$, denoting them by means of $Q$ and conserving all distances except $S(Q'_1,Q'_0)$, $S(0,Q'_0)$, and $S(1,Q'_0)$. As a result of the $\sigma$ space (4.11) arises: It is flat, but non-Euclidean.

*Example 4: Let the* $\sigma$ space $\Omega_5$ consist of the five points $0, 1, 2, P, Q$.

$$\begin{gathered}
S(0,1) = S(0,2) = S(1,2) = a, \\
S(0,P) = a/\sqrt{3} + \varepsilon + O(\varepsilon^2) \\
S(1,P) = S(2,P) = a/\sqrt{3} - \varepsilon/2 + O(\varepsilon^2) \\
S(1,Q) = S(2,Q) = a/\sqrt{3} + \varepsilon/2 + O(\varepsilon^2) \\
S(0,Q) = a/\sqrt{3} - \varepsilon + O(\varepsilon^2) \\
S(P,Q) = a/\sqrt{3} + O(\varepsilon^2), \qquad \varepsilon \ll 1
\end{gathered} \tag{4.12}$$

One can verify that $\mathcal{T}_{012} = \Omega_5$ and $\Omega_5$ is a two-dimensional $\sigma$ space extremal in the second order. However, $\Omega_5$ is not a Euclidean $\sigma$ space. The disposition of points on the proper Euclidean plane is shown in Fig. 2. All distances (4.12) except $S(P,Q)$ correspond to Euclidean distances in Fig. 2. If the distance $S(P,Q)$ were Euclidean, then one would have $S(P,Q) = 2\varepsilon + O(\varepsilon^2)$.

The properties of $\Omega_5$ can be understood if one considers the $\sigma$ spaces $\Omega_5\backslash\{0\}$, $\Omega_5\backslash\{1\}$, $\Omega_5\backslash\{2\}$, $\Omega_5\backslash\{P\}$, and $\Omega_5\backslash\{Q\}$ consisting of four points. Each of them is Euclidean $\sigma$ space. The disposition of points in these spaces is shown in Fig. 2. The point $Q$ in the first three $\sigma$ spaces is replaced respectively by $Q'_0$, $Q'_1$, and $Q'_2$

in the picture of $\sigma$ spaces on the proper Euclidean plane. Thus one has $\Omega_5\backslash\{0\} = \{1,2,P,Q'_0\}$, $\Omega_5\backslash\{1\} = \{0,2,P,Q'_1\}$, $\Omega_5\backslash\{2\} = \{0,1,P,Q'_2\}$, $\Omega_5\backslash\{P\} = \{0,1,2,Q\}$, and $\Omega_5\backslash\{Q\} = \{0,1,2,P\}$. Then all the distances shown in Fig. 2 coincide with those in Eq. (4.12). The presented examples illustrate Theorem 4 in the case when $N = n+3$ and $\sigma$ space is non-Euclidean.

# 5 The dense $\sigma$ space and curtailed tubes

**Definition 18** *: A non-null tube ray $\mathcal{T}_{[P_0P_1}$ ($\sigma(P_0,P_1)\neq 0$) is dense at point $P_0$ if an infinite-converging-to-$P_0$ sequence $P'_1, P'_2, ...$ of distinguishing in pairs points can be found on the open tube ray $\mathcal{T}_{(P_0P_1}$. This means that for $\forall\varepsilon > 0$ there exists $N_\varepsilon$ such that $|2\sigma(P_0,P'_n)|^{1/2} < \varepsilon$ if $n > N_\varepsilon$.*

**Definition 19** *: The $\sigma$ space $V$ is dense at point $P_0$ in the timelike direction if any timelike tube ray $\mathcal{T}_{[P_0P_1}$ is dense at $P_0$.*

**Definition 20** *: The $\sigma$ space $V$ is dense at point $P_0$ if any non-null tube ray $\mathcal{T}_{[P_0P_1}$ is dense at $P_0$.*

**Definition 21** *: If a non-null tube ray $\mathcal{T}_{[P_0Q} \subset V$ is dense at $P_0$ and the limit .*

$$(\mathbf{u}_{P_0Q}.\mathbf{P}_0\mathbf{P}) = |\sqrt{2\sigma(P_0,Q)}|\Pi(P_0,Q',P) = \lim_{Q'\to P_0,\, Q'\in\mathcal{T}_{[P_0Q}} \left|\sqrt{\frac{\sigma(P_0,Q)}{\sigma(P_0,Q')}}\right| \Gamma(P_0,Q',P) \tag{5.1}$$

*exists $\forall P \in V$ and for any way of tending $Q'$ to $P_0$, then the tube ray $\mathcal{T}_{[P_0Q}$ determines a non-null direction vector $\mathbf{u}_{P_0Q}$ at $P_0$. Here $\Pi(P_0,Q',P)$ is a projection of the vector $\mathbf{P}_0\mathbf{P}$ onto the direction $\mathbf{u}_{P_0Q}$.*

The limit (5.1) is defined in the conventional way. For any $\varepsilon > 0$ such $\delta_\varepsilon > 0$ exists, that the inequality

$$\left|\Pi(P_0,Q',P) - \frac{\Gamma(P_0,Q',P)}{\sqrt{|2\sigma(P_0,Q')|}}\right| < \varepsilon \tag{5.2}$$

is a consequence of the conditions

$$\left|\sqrt{2\sigma(P_0,Q')}\right| < \delta_\varepsilon, \qquad Q' \in \mathcal{T}_{[P_0Q} \tag{5.3}$$

**Definition 22** *: If the $\sigma$ space $V$ is dense at the point $P_0 \in V$ and if the equation*

$$\frac{(\mathbf{u}_{P_0Q'}.\mathbf{P}_0\mathbf{P})}{|\sqrt{2\sigma(P_0,Q')}|} = \frac{(\mathbf{u}_{P_0Q}.\mathbf{P}_0\mathbf{P})}{|\sqrt{2\sigma(P_0,Q)}|}, \qquad \sigma(P_0,Q)\neq 0, \qquad \forall P\in V \tag{5.4}$$

*considered as an equation for the determination of $Q' \in \mathcal{T}_{[P_0Q}$, then the non-null direction vector $\mathbf{u}_{P_0Q}$ at point $P_0$ determines the non-null tube ray $\mathcal{T}_{[P_0Q}$.*

Definition 22 is formulated briefly as follows: $\mathcal{T}_{[P_0Q} = \mathrm{gen}(\mathbf{u}_{P_0Q})$ ($\mathcal{T}_{[P_0Q}$ is generated by $\mathbf{u}_{P_0Q}$). This property is a variety of the definiteness property (Definition 5) when the two points $P_0$ and $Q'$ defining the tube are infinitely close.

In the conventional approach the concept of direction is connected with the concept of a curve. The latter is connected with the possibility of one-to-one continuous mapping into the set $\mathbb{R}$ of all real numbers. In our approach the concept of direction is connected with the extremality of $\mathcal{T}_{[P_0Q}$ in Eq. (5.1) because only in this case the limit does exist as a rule. It is quite natural since the concepts of a curve and an extremal tube that is dense at all points are rather close.

However, there are $\sigma$ spaces that do not contain the extremal tubes $\mathcal{T}_{[P_0P}$. For instance, the $D$-dimensional pseudo-Euclidean space $E_D$ of index $i$ ($g_{kl}$ =diag$(1,1,...1,-1,-1,...-1)$, $1<i<D-1$, $D\geq 4$) is not extremal on any tube $\mathcal{T}_{[P_0P}$. In this case nonextremality in the first order is caused by nonextremality in the zeroth order. The extremal tube ray $\mathcal{T}_{[P_0P}$ can be obtained from the nonextremal tube ray by means of the special curtailing operation $\mathcal{C}$ :

$$\mathcal{T}^c_{[P_0P} = \mathcal{C}\mathcal{T}_{[P_0P} \tag{5.5}$$

There are different ways of defining the curtailing operation. One way is operation of intense definiteness (intense extremality)

$$\mathcal{T}^c_{[P_0P} = \mathcal{I}\mathcal{T}_{[P_0P} = \{P'|\mathcal{T}_{[P_0P'} = \mathcal{T}_{[P_0P}\} \tag{5.6}$$

when the set $\mathcal{T}^c_{[P_0P}$ contains only those points $P'$ of $\mathcal{T}_{[P_0P}$ for which $\mathcal{T}_{[P_0P'}$ coincides with $\mathcal{T}_{[P_0P}$. Here the $\sigma$ space is extremal on $\mathcal{T}^c_{[P_0P_1}$ if

$$\mathcal{T}^c_{[P_0P'} = \mathcal{T}^c_{[P_0P_1}, \qquad P' \in \mathcal{T}^c_{[P_0P_1}, \qquad P' \neq P_0 \tag{5.7}$$

$$\mathcal{S}_{1;Q}(\mathcal{T}^c_{[P_0P_1}) = \{P'|P' \in \mathcal{T}^c_{[P_0P_1} \bigwedge_{l=0}^{l=1} \sigma(P_l,Q) = \sigma(P_l,P')\} = \{Q\}, \qquad Q \in \mathcal{T}^c_{[P_0P_1}. \tag{5.8}$$

*Example 5:* Let us consider the pseudo-Euclidean space $E_4$ of index 2. In the Cartesian coordinates $x=(x^1,x^2,x^3,x^4)$ the $\sigma$ function has the form (3.4), with $g_{ik}$ =diag$(1,1,-1,-1)$. Let $P_0=(0,0,0,0)$, $P_1=(1,0,0,0)$, and $P_2=(0,0,1,0)$. Let $P=(x^1,x^2,x^3,x^4)$ be a running point of the set. The calculation leads to

$$\mathcal{T}_{[P_0P_1}: \quad (x^2)^2-(x^3)^2-(x^4)^2=0 \wedge x^1\geq 0$$

$$\mathcal{T}^c_{[P_0P_1} = \mathcal{I}\mathcal{T}_{[P_0P_1}: \quad x^2=x^3=x^4=0 \wedge x^1\geq 0$$

$$\mathcal{T}_{[P_0P_2}: \quad (x^2)^2+(x^3)^2-(x^4)^2=0 \wedge x^3\geq 0$$

$$\mathcal{T}^c_{[P_0P_2} = \mathcal{I}\mathcal{T}_{[P_0P_2}: \quad x^1=x^2=x^4=0 \wedge x^3\geq 0$$

Thus the curtailing operation transforms the three-dimensional tube ray $\mathcal{T}_{[P_0P_1}$ into the one-dimensional curtailed tube ray $\mathcal{T}^c_{[P_0P_1}$, where the $\sigma$ space is extremal on $\mathcal{T}^c_{[P_0P_1}$ in the sense of Eqs. (5.7) and (5.8), where Eq. (5.7) is fulfilled as a result of Eq. (5.6).

Further, the $\sigma$ space that is extremal on all timelike tubes $\mathcal{T}_{[P_0P}$ will be considered. Spacelike tubes are not in general extremal. Such a $\sigma$ space is important from a physical viewpoint because the real event space is $\sigma$ space of this kind.

**Definition 23** : *The scalar product between two non-null direction vectors* $\mathbf{u}_{P_0P}$ *and* $\mathbf{u}_{P_0Q}$ *at* $P_0$ *(or between two non-null tube rays* $\mathcal{T}_{[P_0P}$ *and* $\mathcal{T}_{[P_0Q}$*) is determined if the following limit exists:*

$$(\mathbf{u}_{P_0P}.\mathbf{u}_{P_0Q}) \equiv \Gamma^*_{P_0}(P_0, P, Q) =$$

$$\lim_{\substack{Q'\to P_0 \\ Q'\in\mathcal{T}_{(P_0Q}}} \lim_{\substack{P'\to P_0, \\ P'\in\mathcal{T}_{(P_0P}}} \left|\sqrt{\frac{\sigma(P_0,P)\sigma(P_0,Q)}{\sigma(P_0,P')\sigma(P_0,Q')}}\right| \Gamma(P_0, P', Q') \tag{5.9}$$

Let $\sigma$ space $V$ be given on the set $\Omega$. On the set

$$\Omega^+_{P_0} = \{P|\sigma(P_0,P) > 0\} \cup \{P_0\} \tag{5.10}$$

there are defined two kinds of objects: vector $\mathbf{P}_0\mathbf{P}$ (the point $P \in \Omega^+_{P_0}$ determines $\mathbf{P}_0\mathbf{P}$) and the direction vector $\mathbf{u}_{P_0P}$, $P \in \Omega^+_{P_0}$. The scalar products between the two vectors are defined by Eqs.(5.1), (5.9) and

$$(\mathbf{P}_0\mathbf{P}.\mathbf{P}_0\mathbf{Q}) = \Gamma(P_0, P, Q) \tag{5.11}$$

Two different $\sigma$ spaces can be defined on $\Omega^+_{P_0}$: the $\sigma$ space $V^+_{P_0}$ with the world function $\sigma$ and the $\sigma$ space $V^{+*}_{P_0}$ with the world function

$$\sigma^*_{P_0}(P_0, Q) = \sigma(P_0, P) + \sigma(P_0, Q) - \Gamma^*_{P_0}(P_0, P, Q), \qquad P, Q \in \Omega^+_{P_0} \tag{5.12}$$

It is supposed that the vectors $\mathbf{u}_{P_0P}$, $P \in \Omega^+_{P_0}$ belong to $V^{+*}_{P_0}$, whereas the vectors $\mathbf{P}_0\mathbf{P}$, $P \in \Omega^+_{P_0}$ $P \in \Omega^+_{P_0}$ belong to $V^+_{P_0}$.

**Definition 24** : *The* $\sigma$ *space* $V$ *given on a set* $\Omega$ *determines a* $\sigma$ *space* $V^{+*}_{P_0}$ *with the* $\sigma$ *function defined by Eq. (5.12) on the set* $\Omega^+_{P_0}$ *if the following conditions are fulfilled.*

*(i) The* $\sigma$ *space* $V$ *is dense at* $P_0$ *and extremal on any timelike tube ray* $T_{[P_0Q} \ni P_0$*,* $(\sigma(P_0,Q) > 0)$*,* $Q \in \Omega$*.*

*(ii) Any timelike tube ray* $\mathcal{T}_{[P_0Q}$*,* $(\sigma(P_0,Q) > 0)$ *determines the timelike direction vector* $\mathbf{u}_{P_0Q}$ $(\mathbf{u}_{P_0Q} = \mathrm{gen}(\mathcal{T}_{[P_0Q}),\quad \mathbf{u}_{P_0Q}$*).*

*(iii) The scalar product of two direction vectors* $\mathbf{u}_{P_0P}$ *and* $\mathbf{u}_{P_0Q}$ $(\sigma(P_0,Q) > 0,\ \sigma(P_0,P) > 0)$ *at* $P_0$ *is determined.*

It is easy to verify that the function $\Gamma^*_{P_0}(P_0, P, Q)$ calculated by means of Eq. (2.3) with the $\sigma$ function (5.12) coincides with expression (5.9).

**Definition 25** : *The direction vector* $\mathbf{u}_{P_0P}$ $(P \in \Omega)$ *is tangent to the vector* $\mathbf{P}_0\mathbf{P}$ $(P \in \Omega)$ *on* $\Omega^+_{P_0}$ *if*

$$(\mathbf{u}_{P_0Q}.\mathbf{P}_0\mathbf{P}) = (\mathbf{u}_{P_0Q}.\mathbf{u}_{P_0P}) = \Gamma^*_{P_0}(P_0, P, Q), \qquad \forall Q \in \Omega^+_{P_0} \tag{5.13}$$

**Definition 26** *: The $\sigma$ space $V$ given on the set $\Omega$ determines the $D$-dimensional $\sigma$ space $V_{P_0Dt}^{+*}$ of the timelike tangent direction vectors $\mathbf{u}_{P_0Q}$ on the set $\Omega_{P_0}^{+}$ if the following conditions are fulfilled.*

*(i) Here $V$ determines the $\sigma$ space $V_{P_0}^{+*}$ with the $\sigma$ function (5.12) on $\Omega_{P_0}^{+}$*

*(ii) Any direction vector $\mathbf{u}_{P_0P}$ $(P\in\Omega)$ is tangent to the vector $\mathbf{P}_0\mathbf{P}$ $(P\in\Omega)$ on $\Omega_{P_0}^{+}$.*

*(iii) The set $\mathcal{R}=\{r|r=\mathrm{rank}\mathcal{A}_n(\mathcal{P}^n)\}$ of ranks of all matrices*

$$\mathcal{A}_n(\mathcal{P}^n)=||\Gamma_{P_0}^{*}(P_0,P_i,P_k)||,\qquad i,k=1,2,...n\quad n\in\mathbb{N} \tag{5.14}$$

*is restricted above*

$$\mathrm{rank}\mathcal{A}_n(\mathcal{P}^n)\leq D,\qquad \mathcal{P}^n\subset\Omega_{P_0}^{+} \tag{5.15}$$

*and the supremum $D$ is achieved for the matrix $\mathcal{A}_n(\mathcal{P}^n)$ of the same order $D$:*

$$\det\mathcal{A}_D(\mathcal{P}^D)\neq 0 \tag{5.16}$$

The set $\mathcal{P}^D\subset\Omega_{P_0}^{+}$ forms a $(D+1)$ point basis in $V_{P_0Dt}^{+*}$. Let us parametrize the non-null tube $\mathcal{T}_{P_0P}$ by means of a parameter $\tau$ in such a way that any point $P(\tau)$ of the section $\mathcal{S}_{1;P(\tau)}(\mathcal{T}_{P_0P})$ corresponds to some value of the parameter $\tau$ determined by the relations

$$\begin{gathered}\left(\sqrt{\sigma(P_0,P(\tau))}+\frac{\tau(1-\tau)}{|\tau(1-\tau)|}\sqrt{\sigma(P(\tau),P)}\right)^2=\sigma(P_0,P),\\ \sigma(P_0,P(\tau))=\tau^2\sigma(P_0,P),\qquad \sigma(P,P(\tau))=(1-\tau)^2\sigma(P_0,P),\\ \Gamma(P_0,P,P(\tau))=2\tau\sigma(P_0,P),\qquad P(\tau)\in\mathcal{S}_{1;P(\tau)}(\mathcal{T}_{P_0P})\end{gathered} \tag{5.17}$$

Here $P(\tau)$ is one of the points of the section $\mathcal{S}_{1;P(\tau)}(\mathcal{T}_{P_0P})$.

Such a parametrization means that

$$\begin{gathered}P(\tau)\in\mathcal{T}_{P_0)P},\qquad \tau<0,\qquad P_0\in\{P(0)\},\qquad P\in\{P(1)\}\\ P(\tau)\in\mathcal{T}_{[P_0P)},\qquad 0\leq\tau<1,\qquad P(\tau)\in\mathcal{T}_{P_0[P},\qquad 1\leq\tau.\end{gathered} \tag{5.18}$$

The parametrization is continuous and relations (5.17) for $\sigma$ satisfy the equation of the tube:

$$F_2(P_0,P,P(\tau))=\left|\begin{array}{cc}2\sigma(P_0,P) & \Gamma(P_0,P,P(\tau))\\ \Gamma(P_0,P(\tau),P) & 2\sigma(P_0,P(\tau))\end{array}\right|=0 \tag{5.19}$$

Let us consider the $D$-dimensional Euclidean space $E_D$ given on the set $\mathbb{R}^D$. The world function has the form (3.4) and the tube ray $\mathcal{T}_{0x}$ can be represented in the form

$$\mathcal{T}_{0x}=\bigcup_{s=0}^{s=D-1}\bigcup_{\tau\in\mathbb{R}}\left\{y|y=\tau_0x+\sum_{l=1}^{D-1}a_l(x)\tau_l\right\} \tag{5.20}$$

$$x,y\in\mathbb{R}^D,\qquad a_l\in\mathbb{R}^D,\qquad l=1,2,...D-1,\qquad \tau_s\in\mathbb{R},\qquad s=0,1,...,D-1$$

Here $a_l$ are $D-1$ linear independent vectors which satisfy

$$(a_l(x).a_l(x))=0, \qquad (a_l(x).x)=0, \qquad l=1,2,...D-1, \tag{5.21}$$

where

$$(x,y)=\sum_{i,k=1}^{D} g_{ik}x^iy^k, \qquad x=\{x^l\}, \qquad y=\{y^l\}, \qquad l=1,2,...,D$$

If $E_D$ is the proper Euclidean space, then the system (5.21) has only a trivial solution $a_l=0$ $(l=1,2,...,D-1)$, and $E_D$ is extremal on $\mathcal{T}_{0x}$. If $E_D$ is the pseudo-Euclidean space of index 1 $(D>2)$, then $a_l(x)=0$ $(l=1,2,...,D-1)$ for timelike $x$, $\sigma(0,x)>0$, but $a_l(x)\neq 0$ for spacelike $x$ $(\sigma(0,x)<0)$.

Let us use the curtailing procedure (5.6). Then for the non-null tube one obtains

$$\mathcal{T}_{0x}^c=\{y|\mathcal{T}_{0y}=\mathcal{T}_{0x}\}\bigcup_{\tau\in\mathbb{R}}\{y|y=\tau x\}, \qquad x,y\in\mathbb{R}^D, \tag{5.22}$$

where $E_D$ is extremal on $\mathcal{T}_{0x}^c$.

Thus in the case of the pseudo-Euclidean space the curtailing procedure can be defined as an addition of the curtailing equations $y=\tau x$:

$$\mathcal{T}_{0x}^c=\bigcup_{\tau\in\mathbb{R}}\{y|F_2(0,x,y)=0\wedge y=\tau x\}, \qquad x,y\in\mathbb{R}^D, \tag{5.23}$$

It is easy to see that the curtailing procedure does not change the extremal timelike tubes, but it does change the nonextremal spacelike tubes.

**Theorem 5** *: If the $\sigma$ space $V$ given on the set $\Omega$ determines the $D$-dimensional $\sigma$ space $V_{P_0Dt}^{+*}$ of the timelike tangent vectors $\mathbf{u}_{P_0P}$ on the set $\Omega_{P_0}^+$ and any timelike direction vector $\mathbf{u}_{P_0P}$ determines the tube ray $\mathcal{T}_{[P_0P}$, then $V_{P_0Dt}^{+*}$ is a $D$-dimensional Euclidean $\sigma$ space.*

*Proof:* According to Eq. (5.15) there is a $(D+1)$ point basis $\mathcal{P}^D\subset\Omega_{P_0}^+$ in $V_{P_0Dt}^{+*}$. Let us use the designations

$$g_{ik}(\mathcal{P}^D)=\Gamma_{P_0}^*(P_0,P_i,P_k), \qquad i,k=1,2,...,D \tag{5.24}$$

$$x_i(P)\equiv(\mathbf{u}_{P_0P_i},\mathbf{P}_0\mathbf{P})=(\mathbf{u}_{P_0P_i},\mathbf{u}_{P_0P})=\Gamma_{P_0}^*(P_0,P_i,P) \tag{5.25}$$

where $g_{ik}(\mathcal{P}^D)$ is the metric tensor and $x_i(P)$ are covariant coordinates of the point $P$ in the basis $\mathcal{P}^D$ with the basis vectors $\mathbf{e}_i=\mathbf{u}_{P_0P_i}$ $(i=1,2,...,D)$. According to Eq. (5.13) the vector $\mathbf{u}_{P_0P}$ is supposed to be tangent to $\mathbf{P}_0\mathbf{P}$.

Let us write Eq. (5.15) with $n=D+2$, $P_{D+1}=P\in\Omega_{P_0}^+$, $P_{D+2}=Q\in\Omega_{P_0}^+$. Using designations (5.24) and (5.25) one obtains

$$\operatorname{rank}\left\|\begin{array}{ccc} g_{ik}(\mathcal{P}^D) & x_i(P) & x_i(Q) \\ x_k(P) & 2\sigma(P_0,P) & \Gamma^*(P_0,P,Q) \\ x_k(Q) & \Gamma^*(P_0,Q,P) & 2\sigma(P_0,Q) \end{array}\right\|=D, \qquad i,k=1,2,...D \tag{5.26}$$

Insofar as the last two columns are linear combinations of the first $D$ columns one obtains

$$\sigma(P_0,P)=\frac{1}{2}\sum_{i=1}^{D}x_i(P)x^i(P),\qquad P\in\Omega^+_{P_0} \tag{5.27}$$

$$\Gamma^*(P_0,P,Q)=\Gamma^*(P_0,Q,P)=\sum_{i=1}^{D}x_i(P)x^i(Q),\qquad P,Q\in\Omega^+_{P_0} \tag{5.28}$$

where

$$x^i(P)=\frac{1}{2}\sum_{r=1}^{D}g^{ik}(\mathcal{P}^D)x_k(P),\qquad i=1,2,...D \tag{5.29}$$

Substituting Eqs. (5.27) and (5.28) into (5.12) one obtains

$$\sigma^*_{P_0}(P,Q)=\frac{1}{2}\sum_{i,k=1}^{D}g_{ik}(\mathcal{P}^D)[x^i(P)-x^i(Q)][x^k(P)-x^k(Q)],\qquad P,Q\in\Omega^+_{P_0} \tag{5.30}$$

The above means that the $\sigma$ space $V^{+*}_{P_0Dt}$ is Euclidean if there is a one-to-one correspondence between the direction vectors $\mathbf{u}_{P_0P}$ and their coordinates $x(P)=\{x_i(P)\},\quad i=1,2,...D,\quad x\in\mathbb{R}^D$.

Let $P'=P(\tau)\in\mathcal{T}_{P_0P}$, $Q'=P(\tau')\in\mathcal{T}_{P_0P}$, and $P\in\Omega^+_{P_0}$. Then Eqs. (5.17) and (5.27) lead to

$$\sigma(P',Q')=\frac{1}{2}\sum_{i,k=1}^{D}g_{ik}(\mathcal{P}^D)[x^i(P')-x^i(Q')][x^k(P')-x^k(Q')],\qquad P',Q'\in\mathcal{T}_{P_0P}\subset\Omega^+_{P_0} \tag{5.31}$$

which means that timelike tubes $\mathcal{T}_{P_0P}$ and $\mathcal{T}^*_{P_0P}$ constructed in the $\sigma$ spaces $V$ and $V^{+*}_{P_0Dt}$ coincide, respectively with

$$\mathcal{T}_{P_0P}=\mathcal{T}^*_{P_0P},\qquad \sigma(P,Q)=\sigma^*_{P_0}(P,Q),\qquad Q\in\mathcal{T}_{P_0P} \tag{5.32}$$

Then for $\mathcal{T}_{[P_0P}$ let us take expression (5.23), which is valid for pseudo-Euclidean space,

$$\mathcal{T}^c_{[P_0P}=\bigcup_{\tau>0}\{P'|x(P')=\tau x(P)\wedge 2\tau\sigma(P_0,P)=\Gamma(P_0,P,P')\},\qquad P,P',P_0\in\Omega^+_{P_0} \tag{5.33}$$

For a timelike tube each of conditions (5.33) is a corollary of the other condition. However, conditions (5.33) will be considered as independent, keeping in mind that we will further use them in general case.

According to Eq. (2.12) the section of $\mathcal{T}^c_{[P_0P}$ at the point $Q\in\mathcal{T}^c_{[P_0P}$ has the form

$$\mathcal{S}_{1;Q}(\mathcal{T}^c_{[P_0P})=\bigcup_{\tau>0}\{P'|\sigma(P_0,P')=\sigma(P_0,Q)\wedge\sigma(P,P')=\sigma(P,Q)\wedge x(P')=x(Q)$$

$$\wedge x(Q)=\tau x(P)\wedge 2\tau\sigma(P_0,P)=\Gamma(P_0,P,Q)\wedge\Gamma(P_0,P,P')=\Gamma(P_0,P,Q)\} \tag{5.34}$$

As a result of Eqs. (5.19) and (5.27) the first and last of conditions (5.34) are corollaries of the remaining condition and may be omitted. Then

$$\mathcal{S}_{1;Q}(\mathcal{T}^{c}_{[P_0P}) = \bigcup_{\tau>0}\{P'|x(P') = x(Q) \wedge \sigma(P, P') = \sigma(P, Q)\}, \qquad Q \in \mathcal{T}^{c}_{[P_0P} \quad (5.35)$$

Thus any section corresponds to some value $\tau$. The section is described by

$$x(P') = x(Q) = \tau x(P), \qquad x = \{x_i\}, \qquad i = 1, 2, ..., D \quad (5.36)$$

$$\sigma(P, P') = \sigma(P, Q) \quad (5.37)$$

As a result of the extremality of $V$ on $\mathcal{T}_{[P_0P}$ system (5.36) and (5.37) has the unique solution $P' = Q$. If Eq. (5.36) has another solution $P' = Q' \neq Q$, then $\sigma(P, Q') \neq \sigma(P, Q)$ and $Q' \notin \mathcal{S}_{1;Q}(\mathcal{T}^{c}_{[P_0P})$. Besides, $Q'$ does not belong to the other sections, which correspond to other values $\tau'$. Hence, $Q' \notin \mathcal{T}^{c}_{[P_0P}$. On the other hand, by the supposition of Theorem 5 any direction vector $\mathbf{u}_{P_0P}$ determines $\mathcal{T}^{c}_{[P_0P}$: This means that any solution $Q'$ of Eq. (5.4) belongs to $\mathcal{T}^{c}_{[P_0P}$. As a result of Eqs. (5.17), (5.27), and the arbitrariness of $P$, Eq. (5.4) is reduced to Eq. (5.36). Insofar as $Q' \notin \mathcal{T}^{c}_{[P_0P}$, $P' = Q'$ cannot be a solution of Eq. (5.36). The contradiction obtained shows that there is the unique solution $P' = Q$ of Eq. (5.36) and a one-to-one correspondence between the point $P$ and its coordinates $x(P)$. Thus Theorem 5 has been proved.

In the case of $\sigma$ space, which is extremal in the zeroth order and does not contain spacelike and null vectors $\Omega^{+}_{P_0} = \Omega$ and Theorem 5 can be formulated as follows.

*Corollary 1:* Let the $\sigma$ space $V$ given on the set $\Omega$ be extremal in the zeroth order and $\sigma(P, Q) \geq 0$ for any $P, Q, \in \Omega$. Let $V$ determine the $D$-dimensional $\sigma$ space $V^{+*}_{P_0Dt}$ of the tangent direction vectors $\mathbf{u}_{P_0P}$ on $\Omega$ at $P_0 \in \Omega$. If any direction vector $\mathbf{u}_{P_0P}$, $P \in \Omega$ determines the tube ray $\mathcal{T}_{[P_0P}$, then $V^{+*}_{P_0Dt}$ is the proper Euclidean $\sigma$ space.

Using property (5.32), let us try to spread the $\sigma$ space $V^{+*}_{P_0Dt}$ on the whole set $\Omega$. In the $D$-dimensional $\sigma$ space $V^{+*}_{P_0Dt}$ the coordinates (5.25) are defined $\forall P \in \Omega$. Then Eq. (5.25) realizes a mapping $\Omega \to \Omega'_{P_0} \subset \mathbb{R}^D$.

Let us consider $D$-dimensional Euclidean $\sigma$ space $V^{+*}_{P_0Dt}$ given on $\Omega'_{P_0}$ by means of the world function (5.30):

$$\sigma^{*}_{P_0}(x, y) = \frac{1}{2}\sum_{i,k=1}^{D} g^{ik}(\mathcal{P}^D)(x_i - y_i)(x_k - y_k), \qquad x, y \in \Omega'_{P_0} \quad (5.38)$$

where the dimensionality $D$ is specified by properties of the $\sigma$ space $V$ mentioned in definitions 24 – 26. Using expressions (5.33) and (5.35) for the curtailed tube rays in the Euclidean $\sigma$ space $V^{+*}_{P_0Dt}$ one can use these expressions in the general case. Equations (5.33) and (5.35) are also well defined in the case $\sigma(P_0, P) = 0$: They can be considered as the definition of the null curtailed tube $\mathcal{T}^{c}_{P_0P}$, $\sigma(P_0, P) = 0$ and its section at the point $Q \in \mathcal{T}^{c}_{P_0P}$. In this case the section of $\mathcal{T}^{c}_{P_0P}$ is defined as a set of points $Q(\tau) \subset \mathcal{T}^{c}_{P_0P}$ with the fixed value $\tau$. Here the null curtailed tube $\mathcal{T}^{c}_{P_0P}$

is considered as a complex of non-null curtailed tubes $\mathcal{T}^c_{P'_iP}$ with $P'_i \to P_0$, $P'_i \in \mathcal{T}^c_{P_0P_i}$ $(i = 1, 2, ..., D)$, if

$$\mathcal{S}_{1;Q}(\mathcal{T}^c_{P_0P}) = \{P'|x(P') = x(Q) \wedge \sigma(P,P') = \sigma(P,Q)\} = \{Q\}, \qquad Q \in \mathcal{T}^c_{P_0P} \quad (5.39)$$

The curtailed tube rays can be used for the calculation of the limits (5.1) and (5.9), which determine the quantities $(\mathbf{u}_{P_0P}, \mathbf{P}_0\mathbf{Q}) = \Gamma^*_{P_0}(P_0, Q, P)$ for the timelike direction vectors $\mathbf{u}_{P_0P}$. Use of the curtailed tube rays enables us to spread definitions (5.4) and (5.6) on the arbitrary direction vectors by replacing $\mathcal{T}_{(P_0Q}$, $\mathcal{T}_{(P_0P}$ by $\mathcal{T}^c_{(P_0Q}$, $\mathcal{T}^c_{(P_0P}$ in Eqs. (5.1) and (5.9).

One should bear in mind that form (5.33) of the curtailed tubes supposes a determination of the $D$-dimensional $\sigma$ space $V^{+*}_{P_0Dt}$ by the $\sigma$ space $V$, although the curtailing operation (5.6) can be used, in principle, in any $\sigma$ space. If the $D$-dimensional $\sigma$ space $V^{+*}_{P_0Dt}$ is determined, then the limits (5.1) and (5.9) take the form

$$(\mathbf{u}_{P_0Q}.\mathbf{P}_0\mathbf{P}) = \lim_{\tau' \to +0} \left| \sqrt{\frac{\sigma(P_0,Q)}{\sigma(P_0,Q(\tau))}} \right| \Gamma(P_0, Q(\tau), P), \qquad x(Q(\tau)) = \tau x(Q) \quad (5.40)$$

$$\Gamma^*_{P_0}(P_0, P, Q) = \lim_{\tau \to +0} \lim_{\tau' \to +0} \left| \sqrt{\frac{\sigma(P_0,Q)\sigma(P_0,P)}{\sigma(P_0,Q(\tau))\sigma(P_0,P(\tau'))}} \right| \Gamma(P_0, Q(\tau), P(\tau')), \quad (5.41)$$

$$x(Q(\tau)) = \tau x(Q), \qquad x(P(\tau')) = \tau' x(P), \qquad P, Q, P_0 \in \Omega$$

For non-null curtailed tubes, when $\mathcal{T}^c_{P_0P} \subseteq \mathcal{T}_{P_0P}$, the limits (5.40) and (5.41) coincide with the limits (5.1) and (5.9). For null curtailed tubes the definition of the limit of Eqs. (5.2) and (5.3) cannot be used. A parametrization of the null curtailed tube ray $\mathcal{T}^c_{P_0P}$, $\sigma(P_0, P) = 0$, which is used in definitions (5.33) and (5.39), can be obtained as a limit of the parametrization (5.17) of the non-null ray $\mathcal{T}^c_{[P_0P'}$ with $P' \to P$, $P' \in \mathcal{T}_{[PP''}$, $\sigma(P, P'') \neq 0$. Such a limit is possible in this case only if $V$ is dense on $\mathcal{T}_{P_0}$.

For determination of the $\sigma$ space $V^{+*}_{P_0Dt}$ on $\Omega$ one can use the following procedure, which does not need a density of $V$ at $\forall P \in \mathcal{T}_{P_0}$. Let $\Omega^-_{P_o} = \Omega \backslash \Omega^+_{P_o}$.

1. (i) $(\mathbf{u}_{P_0Q}, \mathbf{P}_0\mathbf{P})$, $P \in \Omega$, $Q \in \Omega^+_{P_o}$ is determined by Eqs. (5.1)–(5.3).
   (ii) $(\mathbf{u}_{P_0Q}, \mathbf{u}_{P_0P})$, $P \in \Omega^+_{P_o}$, $Q \in \Omega^+_{P_o}$ is determined by Eqs. (5.9).
   (iii) $(\mathbf{u}_{P_0Q}, \mathbf{u}_{P_0P})$, $P \in \Omega^-_{P_o}$, $Q \in \Omega^+_{P_o}$ is determined by Eqs. (5.13) through $(\mathbf{u}_{P_0Q}, \mathbf{P}_0\mathbf{P})$, $P \in \Omega^-_{P_o}$, $Q \in \Omega^+_{P_o}$.
   (iv) $(\mathbf{u}_{P_0Q}, \mathbf{u}_{P_0P})$, $P \in \Omega$, $Q \in \Omega^+_{P_o}$ is determined by Eqs. (5.15) through $(\mathbf{u}_{P_0Q}, \mathbf{u}_{P_0P})$, $P \in \Omega^+_{P_o}$, $Q \in \Omega^+_{P_o}$.
   (v) $(\mathbf{u}_{P_0Q}, \mathbf{P}_0\mathbf{P})$, $P \in \Omega$, $Q \in \Omega^+_{P_o}$ is determined by Eqs. (5.13) through $(\mathbf{u}_{P_0Q}, \mathbf{u}_{P_0P})$, $P \in \Omega$, $Q \in \Omega^-_{P_o}$.

Thus all $(\mathbf{u}_{P_0Q}, \mathbf{P}_0\mathbf{P})$ and $(\mathbf{u}_{P_0Q}, \mathbf{u}_{P_0P})$, $P \in \Omega$, $Q \in \Omega$ are determined through $(\mathbf{u}_{P_0Q}, \mathbf{P}_0\mathbf{P})$, $P \in \Omega$, $Q \in \Omega^+_{P_o}$ and $(\mathbf{u}_{P_0Q}, \mathbf{u}_{P_0P})$, $P \in \Omega^+_{P_o}$, $Q \in \Omega^+_{P_o}$ without using the density of $V$ at $\forall P \in \mathcal{T}_{P_0}$. Practically, it is this procedure that is used for the parametrization (5.33) of null curtailed tube ray.

**Definition 27** *: The $\sigma$ space $V$ given on the set $\Omega$ determines the D-dimensional $\sigma$ space $V^{+*}_{P_0Dt}$ of tangent vectors $\mathbf{u}_{P_0P}$ on $\Omega$ if the following conditions are fulfilled.*

*(i) The $\sigma$ space $V$ is dense at $P_0$ on any curtailed tube ray $\mathcal{T}^c_{[P_0P}$, containing $P_0$ and extremal on it.*

*(ii) Any tube ray $\mathcal{T}^c_{[P_0Q}$ determines the direction vector $\mathbf{u}_{P_0Q}$ =gen($\mathcal{T}^c_{[P_0Q}$).*

*(iii) The scalar product $(\mathbf{u}_{P_0Q}, \mathbf{u}_{P_0P})$ is determined and any direction vector $\mathbf{u}_{P_0P}$, $P \in \Omega$ is tangent on $\Omega$ to the vector $\mathbf{P_0P}$, $P \in \Omega$ at point $P_0$.*

*(iv) The matrix $\mathcal{A}_n(\mathcal{P}^n)$ of any $n+1$ points $\mathcal{P}^n$ satisfies conditions (5.14) and (5.15) with $\mathcal{P}^n \subset \Omega$ instead of $\mathcal{P}^n \subset \Omega^+_{P_0}$.*

**Theorem 6** *: If the $\sigma$ space $V$ given on the set $\Omega$ determines the D-dimensional $\sigma$ space $V^{+*}_{P_0Dt}$ of tangent vectors $\mathbf{u}_{P_0P}$ on $\Omega$ at $P_0$ and any direction vector $\mathbf{u}_{P_0Q}$ determines the curtailed tube ray $\mathcal{T}^c_{[P_0Q}$ ($\mathcal{T}^c_{[P_0Q}$ =gen($\mathbf{u}_{P_0Q}$), then the $\sigma$ space $V^{+*}_{P_0Dt}$ is Euclidean.*

The proof of Theorem 6 is like that of Theorem 5.

**Theorem 7** *: Let the $\sigma$ space $V$ given on the set $\Omega$ determine the D-dimensional $\sigma$ space $V^{+*}_{P_0Dt}$ of the tangent direction vectors $\mathbf{u}_{P_0P}$ on $\Omega$ and let any direction vector $\mathbf{u}_{P_0Q}$ determine the curtailed tube ray $\mathcal{T}^c_{[P_0Q}$. If $V$ is dense at any point $P \in \Omega$, then $V$ determines the D-dimensional manifold on the set $\Omega_0 = \bar{\Omega}\backslash B$, where $\bar{\Omega}$ is the closure of $\Omega$ and $B$ is the boundary of $\bar{\Omega}$.*

*Proof:* According to Theorem 6 there is a one-to-one mapping $\Omega \to \mathbb{M} \subset \mathbb{R}^D$, where $\mathbb{R}^D$ is the space of all coordinates $x = \{x_i(P)\}$, $(i = 1, 2, ..., D)$. Let us define the $\delta$ vicinity of the point $x \in \mathbb{M}$ as a set of $y \in \mathbb{M}$ satisfying the condition

$$|x-y|^2 = \sum_{i=1}^{D}(x_i - y_i)^2 < \delta, \qquad \delta > 0.$$

Then the set $\mathbb{M}$ is dense at any point (i.e., any $\delta$ vicinity of $x \in \mathbb{M}$ contains at least one point $y \neq x$) because the $\sigma$ space $V$ is dense at any point $P$ on any curtailed tube ray $\mathcal{T}^c_{[P_0Q}$. Let $\bar{\mathbb{M}}$ be the closure of $\mathbb{M}$. Here $\bar{\mathbb{M}}$ does not contain isolated points because $\mathbb{M}$ is dense at any point. Let us remove all boundary points of $\bar{\mathbb{M}}$. Then $\bar{\mathbb{M}}$ transforms into an open region $\mathbb{M}_0$ of $\mathbb{R}^D : \mathbb{M} \subset \mathbb{M}_0 \subset \mathbb{R}^D$.

Let us use one-to-one correspondence between $\Omega$ and $\mathbb{M}$ and construct the mappings $\bar{\Omega} \leftrightarrow \bar{\mathbb{M}}$ and $\Omega_0 \leftrightarrow \mathbb{M}_0$ with $\Omega \subset \Omega_0$. The $\sigma$ function on the $\Omega$ can be defined from the world function on $\Omega$ by means of the proper limiting process. Then the $\sigma$ space $V_0$ on $\Omega_0$ arises. The set $\Omega_0$ with the coordinate system defined by Eq. (5.25) is a manifold. Theorem 7 has been proved.

# 6 The Riemannian space

**Definition 28** *: The D-dimensional Riemannian space $V$ is a D-dimensional manifold $\mathbb{M}$ with quadratic form*

$$(dS)^2 = g_{ik}(x)dx^i dx^k, \tag{6.1}$$

*given at any point $P$ of $\mathbb{M}$ in some coordinate system $K$ on $\mathbb{M}$. Here $x = \{x^i\}$, $(i = 1, 2, ..., D)$ are contravariant coordinates of point $P$ in the coordinate system $K$; $g_{ik}$, $(i, k = 1, 2, ..., D)$ is the metric tensor. Here and below the summation is made over repeated superscripts and subscripts $(1 - D)$.*

In the thus-defined Riemannian space $V$ the world function $\sigma(P, P')$ is defined through the interval

$$S(P, P') = S(x, x') = \int_{P'}^{P} \sqrt{g_{ik}(x)dx^i dx^k} \tag{6.2}$$

by means of the relation

$$\sigma(x, x') = \sigma(P, P') = \frac{1}{2}S(P, P'). \tag{6.3}$$

Integration in Eq. (6.2) is produced along the geodesic $\mathcal{L}_{P'P}$, which is an extremal of the integral (6.2) considered as a functional of $x^i = x^i(\tau)$, $(i = 1, 2, ..., D)$. The functions $x^i(\tau)$ satisfy

$$\frac{d^2x^i}{d\tau^2} + \gamma^i_{kl}(x)\frac{dx^k}{d\tau}\frac{dx^l}{d\tau} = 0, \qquad i = 1, 2, ..., D \tag{6.4}$$

$$\gamma^i_{kl}(x) = \frac{1}{2}\left(\frac{\partial g_{sk}}{\partial x^l} + \frac{\partial g_{sl}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^s}\right), \qquad i, k, l = 1, 2, ..., D \tag{6.5}$$

$$g = \det ||g_{ik}||, \qquad g_{ik}\frac{dx^i}{d\tau}\frac{dx^k}{d\tau} = \text{const} \tag{6.6}$$

The rather small region $\Omega$ of the Riemannian space $V$ is considered, so that one and only one geodesic passes through two different points $P, P' \in \Omega$, $P \neq P'$.

If the metric (6.1) is definite, i.e., if

$$g_{ik}x^i x^k = 0$$

has the unique solution $x^i = 0$ $(i = 1, 2, ..., D)$, then the Riemannian $\sigma$ space $V$ with the $\sigma$ function (6.2) and (6.3) is extremal in the zeroth and the first orders. For the opposite case the $\sigma$ space $V$ is nonextremal in the zeroth order, but it can be extremal in the first order (on the curtailed tubes).

We shall consider only the cases when the Riemannian space is locally either proper Euclidean or pseudo-Euclidean of index 1. In the latter case the spacelike geodesics are curtailed geodesic tubes of the form (5.33). In both cases the $\sigma$ space $V$ is thought to be extremal in the first order.

The $\sigma$ function defined by Eqs. (6.2) and (6.3) satisfies [4]

$$\sigma_i g^{ik}\sigma_k = 2\sigma, \qquad \sigma(x,x') = \sigma(x',x), \qquad \sigma(x,x) = 0 \tag{6.7}$$

where $g^{ik}(x)$ is the metric tensor at point $P$ with the coordinates $x$ and

$$\sigma_i \equiv \frac{\partial\sigma}{\partial x^i}, \qquad \sigma_{i'} \equiv \frac{\partial\sigma}{\partial x'^{i'}}, \qquad i,i' = 1,2,...,D \tag{6.8}$$

The prime at the index shows that the differentiation is produced with respect to the coordinates $x'$ of point $P'$. The absence of the prime shows that the differentiation is produced with respect to the coordinates $x$ of point $P$. Essentially, Eq. (6.7) is a corollary of the fact that integration in Eq.(6.2) is produced along the extremal.

There is another formulation of the extremal property (6.7) which does not contain the metric tensor explicitly: It is valid for any Riemannian space. This formulation has the form of a system of differential equations containing only the $\sigma$ function and its derivatives [5, 6]

$$\sigma_i\sigma^{i,k'}\sigma_{k'} = 2\sigma, \qquad \sigma(x,x') = \sigma(x',x), \qquad \sigma(x,x) = 0 \tag{6.9}$$

$$G_{ik||l} = 0, \qquad i,k,l = 1,2,...,D \tag{6.10}$$

where $\sigma^{i,k'}$ is determined by

$$\sigma^{i,k'}\sigma_{l,k'} = \delta^i_l, \qquad i,l = 1,2,...,D \tag{6.11}$$

$$\sigma_{l,k'} \equiv \frac{\partial^2\sigma(x,x')}{\partial x^l \partial x'^{k'}}, \qquad k',l = 1,2,...,D \tag{6.12}$$

The symbol $(.)_{||l}$ denotes the tangent derivative with respect to $x^l$, i.e., the covariant derivative with respect to $x^l$ with the Christoffel symbol

$$\Gamma^i_{kl}(x,x') = \sigma^{i,j'}\sigma_{klj'}, \qquad \sigma_{klj'} \equiv \frac{\partial^3\sigma(x,x')}{\partial x^k \partial x^l \partial x'^{j'}}, \qquad i,k,l,j' = 1,2,...,D \tag{6.13}$$

and $G_{ik}$ is defined by the relation

$$G_{ik} = G_{ik}(x,x') \equiv \sigma_{i||k} = \frac{\partial\sigma_i}{\partial x^k} - \Gamma^l_{ik}\sigma_l, \qquad i,k = 1,2,...,D \tag{6.14}$$

Equations (6.9) and (6.10) are corollaries of Eq. (6.7), but Eq. (6.7) can be obtained as a corollary of Eqs. (6.9), (6.10) and the "boundary" conditions [5, 6]

$$[\sigma_{i,k'}] = \sigma_{i,k'}(x,x')\big|_{x'=x} = -g_{ik}(x), \qquad i,k = 1,2,...,D \tag{6.15}$$

The tensor $G_{ik}$ is the metric tensor at point $x$ of the Euclidean space $E_{x'}$, which is tangent to the Riemannian space $V$ at point $x'$. The geodesic mapping $V \to E_{x'}$, is produced in such a way that any geodesic $\mathcal{L}_{P'P}$ of $V$ passing through point $P'$ is mapped into a straight line of $E_{x'}$ tangent to $\mathcal{L}_{P'P}$ at point $P'$. At such a mapping the length of any intercept of the geodesic $\mathcal{L}_{P'P}$ and angles between the geodesics at point $P'$ are conserved. The coordinate system $K$ in $V$ is mapped into the coordinate system $K_{x'}$ in $E_{x'}$. Here $G_{ik}$ is the metric tensor of $E_{x'}$ in the coordinate system $K_{x'}$. In $E_{x'}$ the $\sigma$ function of arguments $P$ and $P''$ has the form [5, 6]

$$\sigma_{P'}^*(P,P'') = \sigma_{x'}^*(x,x'') = \sigma(x',x) + \sigma(x',x'') - \sigma_{i'}(x',x)g^{i'k'}(x')\sigma_{k'}(x',x'') \quad (6.16)$$

Relation (6.16) sets in correspondence the world function $\sigma_{P'}^*$ of the Euclidean space $E_{x'}$ to a point $P'$ and the world function $\sigma$ of the Riemannian space $V$.

**Theorem 8** : *Let the $\sigma$ space $V$ given on the $D$-dimensional manifold $\mathbb{M}$ have the world function $\sigma$ which is the twice-differentiable function of coordinates. Then the $\sigma$ space determines a $D$-dimensional Riemannian space $R$ on $\mathbb{M}$.*

*Proof:* Expanding $\sigma(x,x+dx)$ into a series over powers of $dx^i$, one obtains, as a result of properties (2.1) of the $\sigma$ function,

$$\frac{1}{2}(dS)^2 = \sigma(x,x+dx) - \sigma(x,x) = \frac{1}{2}g_{ik}(x)dx^i dx^k + o(|dx|^2),$$

$$g_{ik}(x) = \left[\frac{\partial\sigma(x,y)}{\partial y^i \partial y^k}\right]_{y=x}, \qquad i,k = 1,2,...,D \quad (6.17)$$

Theorem 8 has been proved.

*Remark:* In general, the $\sigma$ space $V$ given on a set $\Omega_0$ is a Riemannian $\sigma$ space if $\Omega_0$ is a subset of points of a Riemannian space $R$ and the $\sigma$ function $\sigma_R$ of $R$, defined by Eqs. (6.2)–(6.5) and (6.17), coincides with the world function $\sigma$ of $V$:

$$\sigma(P_0,P) = \sigma_R(P_0,P), \qquad P,P_0 \in \Omega_0, \quad (6.18)$$

**Theorem 9** : *Let the $\sigma$ space $V$ be given on the set $\Omega$ and let the following conditions be fulfilled.*

*(i) Here $V$ determines the $D$-dimensional $\sigma$ space $V_{P_0Dt}^{+*}$ of the tangent direction vectors $\mathbf{u}_{P_0P}$ at any point $P_0 \in \Omega$ on $\Omega$.*

*(ii) Each direction vector $\mathbf{u}_{P_0P}$ determines the curtailed tube ray $\mathcal{T}_{[P_0P}^c$.*

*If $V$ is dense at any point $P \in \Omega$, then the $\sigma$ space $V_0$ given on $\Omega_0 = \Omega \backslash B$ is the Riemannian space and the world function defined by Eqs. (6.2)–(6.5) and (6.17) coincides with the $\sigma$ function of $V_0$.*

*Proof:* According to Theorem 7 the $\sigma$ space $V$ determines $D$-dimensional manifold on the set $\Omega_0 = \Omega\backslash B$, where $B$ is the boundary of $\bar{\Omega}$, and the $\sigma$ space $V_0$ on $\Omega_0$ arises. Let $P_0, P \in \Omega_0$ be arbitrary points of $\Omega_0$. According to Eqs. (5.27) and (5.29) the $\sigma$ function is twice-differentiable function of coordinates (5.25). As a result of Theorem 8 the $\sigma$ space $V_0$ determines a $D$-dimensional Riemannian space $R$ on $\Omega_0$ with the metric tensor (5.24) at $P_0 \in \Omega_0$ in the coordinate system $K_{P_0}$ defined by Eq. (5.25). Four $\sigma$ spaces arise on $\Omega_0$ : $V_0$, $V_{P_0Dt}^*$, $R$, and $E_{x'}$, with the corresponding $\sigma$ functions $\sigma$, $\sigma^*_{P_0}$, $\sigma_R$, $\sigma^*_{RP_0}$. According to (5.32) and (6.16) one obtains

$$\sigma(P_0,P)=\sigma^*_{P_0}(P_0,P), \qquad \sigma_R(P_0,P)=\sigma^*_{RP_0}(P_0,P), \qquad P_0,P\in\Omega_0, \tag{6.19}$$

$$x'_i=x_i(P_0), \qquad x_i=x_i(P_0), \qquad i=1,2,...,D.$$

According to Eq. (6.17) two Euclidean $\sigma$ spaces $V^*_{P_0Dt}$, and $E_{x'}$ coincide in the infinitesimal vicinity of point $P_0$; hence, they coincide everywhere:

$$\sigma_R(P_0,P)=\sigma^*_{RP_0}(P_0,P), \qquad P_0,P\in\Omega_0 \tag{6.20}$$

Equations (6.20) and (6.19) lead to Eq. (6.18). The $\sigma$ functions of $V_0$ and $R$ coincide. The $\sigma$ space given on $\Omega_0$ is a Riemannian space. Curtailed tubes in $V_0$ coincide with geodesics in $R$ $\mathcal{T}^c_{[P_0P}=\mathcal{L}_{[P_0P}$.

**Theorem 10** *: The $\sigma$ space $V$ given on a $D$-dimensional manifold $\mathbb{M}$ is a Riemannian $\sigma$ space if $V$ is extremal on any curtailed tube $\mathcal{T}^c_{[P_0P}$, $P_0$, $P\in\mathbb{M}$ and the $\sigma$ function is the twice-differentiable function of coordinates.*

*Proof:* One can verify that all suppositions of Theorem 9 are fulfilled as a result of the suppositions of Theorem 10. Then Theorem 10 is valid as a result of the Theorem 9.

# 7 Violation of extremality in the first order.

A $\sigma$ space defined on a manifold and extremal in all orders $n\geq 1$ is a Euclidean $\sigma$ space. According to the Theorem 10 a $\sigma$ space defined on a manifold and extremal in the first order is a Riemannian space. Which are properties of a $\sigma$ space defined on a manifold, but nonextremal in the first order? Can such a $\sigma$ space have a bearing on a real event space?

The real event space is usually considered as a four-dimensional pseudo-Euclidean space of index 1 or as a four-dimensional Riemannian space. In both cases the event space considered as $\sigma$ space is extremal in the first order (on curtailed tubes) and timelike tubes coincide with timelike curtailed tubes. The world line of a free particle placed at point $x'$ and having the four-velocity $u^i$ is described by an algebraic equation with respect to $x(\tau)$:

$$\sigma_{i'}(x,x')=u_{i'}\tau=g_{i'k'}(x')u^{k'}\tau, \qquad i'=0,1,2,3, \tag{7.1}$$

where $\tau$ is a parameter along the world line. If $\det||\sigma_{i,k'}||\neq 0$, then Eq. (7.1) can be solved with respect to $x'$ : It describes a one-dimensional world line

$$x^i=x^i(\tau), \qquad i=0,1,2,3. \tag{7.2}$$

Equation (7.1) always describes the one-dimensional line; however, Eq. (7.1) does that only in the case when the extremality conditions are fulfilled.

The circumstance that the $\sigma$ function is both the transformation function describing the classical particle motion and the world function describing the event space properties permits us to see test particle observations for the determination and the description of event space properties. In a certain sense the motion of the free classical particles and event space properties are identical because each is described through other.

The drawing of geodesic is a way of describing the space properties, but it is equivalent to the observation of free classical particle motion. Pointlike particles are necessary for a test of space properties at small distances. However, pointlike particles are simultaneously the particles of a small mass (electrons, protons, etc.) which move according to quantum mechanics laws.

If one describes the quantal motion of microparticles in terms of Feynman path integrals, then the particle moves along arbitrary trajectories, but not only along extremal ones, although the motion along them is most probable. This means a violation of extremality in the sense of property (i) of Sec. II and does not permit us to use quantal particles for testing the space-time properties, as one can do by means of classical particles.

Another approach is possible using the following hypothesis: *The space-time considered as $\sigma$ space is not extremal in the first order, i.e., a tube of lines (not geodesic) passes through any two points of the space-time. The microparticle motion is described by this tube.* (i.e., by the world tube and not by a world line).

The above hypothesis permits one to use microparticle motion for testing of the space-time properties and, in particular, for determination of the extremality violation. The section ($t$ =const) of a world tube is in general a surface (string), but not a point. This circumstance is associated with the string model of elementary particles, which is currently popular.

*Example 6*: Let there be a coordinate system $K$ on the four-dimensional manifold. Let the $\sigma$ function have in this coordinate system $K$ the form

$$\sigma(P,P') = \sigma(x.x') = \frac{1}{2}q(1+\varepsilon q/l^2)^2, \qquad q=(x-x')^2 = (t-t')^2 - (\mathbf{x}-\mathbf{x}')^2, \tag{7.3}$$

where $x=(t,\mathbf{x})$, $x'=(t',\mathbf{x}')$ are coordinates of the points $P$ and $P'$, respectively. Here $l$ is some characteristic length and $\varepsilon$ is a small parameter $|\varepsilon| \ll 1$ describing a small violation of extremality.

Let $P_0=(0,\mathbf{0})$, $P_1=(a,\mathbf{0})$, and $\varepsilon a^2/l^2 \ll 1$. The equation for the timelike tube $\mathcal{T}_{P_0P_1}$ has the approximate form

$$\mathbf{x}^2 = -6\frac{\varepsilon t^2(t-a)^2}{l^2} + O(\varepsilon^2), \qquad \frac{\varepsilon t^2}{l^2} \ll 1, \qquad \frac{\varepsilon(t-a)^2}{l^2} \ll 1. \tag{7.4}$$

If $\varepsilon=0$, then the tube is a geodesic $\mathbf{x}=0$. If $\varepsilon>0$, then the timelike tube $\mathcal{T}_{P_0P_1}$ degenerates into two points $\mathcal{T}_{P_0P_1} = \{P_0,P_1\}$, but formally the $\sigma$ space remains extremal on the tube $\mathcal{T}_{P_0P_1}$ because extremal properties degenerate into the trivial form. If $\varepsilon<0$, then the timelike straight line $\mathcal{L}_{P_0P_1}$ transforms into a three-dimensional surface. The extremal properties of definiteness and minimal section are violated. The tube is more close to the geodesic the less $\sqrt{\varepsilon}t/l$.

If, indeed, real space-time is distinguished from Riemannian space at a small distance, then one should expect that attempts to describe the particle motion in terms of world lines leads to contradictions and difficulties. In the nonrelativistic approximation these difficulties have been successfully handled in terms of a probabilistic description of quantum mechanics; however, one cannot be sure that it is the best way of overcoming these difficulties.

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