Finsler geometry in terms of world function

Yuri A. Rylov

Institute for Problems in Mechanics, Russian Academy of Sciences,
101-1, Vernadskii Ave., Moscow, 119526, Russia.
e-mail: rylov@ipmnet.ru
Web site: $http://rsfq1.physics.sunysb.edu/\tilde{\ }rylov/yrylov.htm$
or mirror Web site:
$http://gasdyn-ipm.ipmnet.ru/\tilde{\ }rylov/yrylov.htm$

Abstract

It is shown that the space-time geometry should be formulated in terms of the world function, because only description in terms of world function admits one to recognize similar geometrical objects in regions of the space-time geometry with different geometries. The Berwald-Moor geometry formulated in terms of the world function appears to be multivariant geometry, which hardly can be used as a space-time geometry, because in this geometry the world lines wobbling of free particles differs from the real wobbling.

Key words: recognition of similar geometric objects; deformation principle; multivariant geometry; world function; linear vector space; vector bundle; role of coordinates

1 Introduction

Finsler geometry is some generaization of the Riemannian geometry, which uses a vector bundle [1, 2]. This bundle is equipped by a metric function. This metric function generates some geometry, which may be not locally Euclidean (or pseudoeuclidean). In means that the Finsler geometry is a generalization of the Riemannian geometry, because the Riemannian geometry is locally Euclidean. But the Finsler geometries as well as the Riemannian geometries are local geometries. They are described by infinitesimal distance ds between the points x and $x + dx$, which in the case of the Riemannian geometry is defined by the relation

$$2\sigma (x, x + dx) = ds^2 = g_{ik}dx^i dx^k \tag{1.1}$$

Here σ is the world function of the Riemannian geometry. For construction of a geometry in some finite region one uses a coordinate system which connects descriptions in different infinitesimal regions. Besides, a geometry $\mathcal{G}_1$ in the region Ω_1

appears to be not connected with the geometry $\mathcal{G}_2$ in the region Ω_2. For instance, in the region Ω_1 with the space-time geometry of Minkowski $\mathcal{G}_{\mathrm{M}}$ there is a geometric object $\mathcal{O}_{\mathrm{M}}$. This geometric object moves in the space-time without a deformation. It appears in other region Ω_2 of the space-time with the space-time geometry $\mathcal{G}$. In the geometry $\mathcal{G}$ this object is described as $\mathcal{O}$. How description of the geometrical object $\mathcal{O}$ in the space-time geometry $\mathcal{G}$ can be expressed via description of the same object $\mathcal{O}_{\mathrm{M}}$ in the geometry $\mathcal{G}_{\mathrm{M}}$? Neither Riemannian geometry, nor Finsler geometry can answer this question, because these geometries do not consider the problem of a geometry deformation. The only exclusion is the segment $\mathcal{T}_{[AB]}$ of straight line between the points A and B. It is supposed that the segment $\mathcal{T}_{[AB]}$ is a one-dimensiomal segment of a curve in the Riemannian geometry and in the Finsler geometry.

One can answer this important question on a connection between $\mathcal{O}$ and $\mathcal{O}_{\mathrm{M}}$ only in the case, when the space-time geometry is described in terms of the world function σ. Such a geometry is called the physical geometry. Any physical geometry is obtained as a result of a deformation of the proper Euclidean geometry which is considered as a standard geometry. As a result a physical geometry can be obtained from other physical geometry by means of some deformation.

The idea, that a geometry is described completely by means of a distance function (or world function) is very old. At first it was a metric space, described by metric (distance). The metric has been restricted by a set of conditions such as the triangle axiom and nonnegativity of the metric. Condition of nonnegativity of the metric does not permit to apply the metric space for description of the space-time. The main defect of the metric geometry and of the distance geometry [3, 4] is impossibility of construction of geometrical objects in terms of the metric ρ , or in terms of the world function $\sigma = \frac{1}{2}\rho^2$. Construction of geometrical objects in terms of the world function is to be possible, because it is supposed that the geometry is described completely by the world function and in terms of the world function. Furthermore, a physical geometry is to admit a coordinateless description.

Such a situation is possible, if one defines concepts of a geometry and those of a geometrical objects correctly [6].

Definition 1.1: The physical geometry $\mathcal{G} = \{\sigma, \Omega\}$ is a point set Ω with the single-valued function σ on $\Omega \times \Omega$

$$\sigma: \qquad \Omega \times \Omega \rightarrow \mathbb{R}, \qquad \sigma(P,P) = 0, \qquad \sigma(P,Q) = \sigma(Q,P), \qquad \forall P,Q \in \Omega \tag{1.2}$$

Definition 1.2: Two physical geometries $\mathcal{G}_1 = \{\sigma_1, \Omega_1\}$ and $\mathcal{G}_2 = \{\sigma_2, \Omega_2\}$ are equivalent ($\mathcal{G}_1 \mathrm{eqv} \mathcal{G}_2$), if the point set $\Omega_1 \subseteq \Omega_2 \wedge \sigma_1 = \sigma_2$, or $\Omega_2 \subseteq \Omega_1 \wedge \sigma_2 = \sigma_1$.

Remark: Coincidence of point sets Ω_1 and Ω_2 is not necessary for equivalence of geometries $\mathcal{G}_1$ and $\mathcal{G}_2$. If one demands coincidence of Ω_1 and Ω_2 in the case of equivalence of $\mathcal{G}_1$ and $\mathcal{G}_2$, then an elimination of one point P from the point set Ω_1 turns the geometry $\mathcal{G}_1 = \{\sigma_1, \Omega_1\}$ into geometry $\mathcal{G}_2 = \{\sigma_1, \Omega_1 \setminus \{P\}\}$, which appears to be not equivalent to the geometry $\mathcal{G}_1$. Such a situation seems to be inadmissible, because a geometry on a part $\omega \subset \Omega_1$ of the point set Ω_1 appears to be not equivalent to the geometry on the whole point set Ω_1.

According to definition the geometries $\mathcal{G}_1$= $\{\sigma,\omega_1\}$ and $\mathcal{G}_2$= $\{\sigma,\omega_2\}$ on parts of Ω, $\omega_1\subset\Omega$ and $\omega_2\subset\Omega$ are equivalent ($\mathcal{G}_1$eqv$\mathcal{G}$), ($\mathcal{G}_2$eqv$\mathcal{G}$) to the geometry $\mathcal{G}=\{\sigma,\Omega\}$, whereas the geometries $\mathcal{G}_1$= $\{\sigma,\omega_1\}$ and $\mathcal{G}_2$= $\{\sigma,\omega_2\}$ are not equivalent, in general, if $\omega_1\nsubseteqq\omega_2$ and $\omega_2\nsubseteqq\omega_1$. Thus, the relation of equivalence is intransitive, in general. The space-time geometry may vary in different regions of the space-time. It means, that a physical body, described as a geometrical object, may evolve in such a way, that it appears in regions with different space-time geometry.

Definition 1.3: A geometrical object $g_{\mathcal{P}_n}$ of the geometry $\mathcal{G}=\{\sigma,\Omega\}$ is a subset $g_{\mathcal{P}_n}\subset\Omega$ of the point set Ω. This geometrical object $g_{\mathcal{P}_n}$ is a set of roots $R\in\Omega$ of the function $F_{\mathcal{P}_n}$, $\mathcal{P}_n=\{P_0,P_1,...P_n\}\subset\Omega$

$$F_{\mathcal{P}_n}:\qquad \Omega\rightarrow\mathbb{R} \tag{1.3}$$

where

$$F_{\mathcal{P}_n}\quad:\qquad F_{\mathcal{P}_n}(R)=G_{\mathcal{P}_n}(u_1,u_2,...u_s)\,,\qquad s=\frac{1}{2}(n+1)(n+2) \tag{1.4}$$

$$u_l\;=\;\sigma(w_i,w_k)\,,\qquad i,k=0,1,...n+1,\qquad l=1,2,...\frac{1}{2}(n+1)(n+2) \tag{1.5}$$

$$w_k\;=\;P_k\in\Omega,\qquad k=0,1,...n,\qquad w_{n+1}=R\in\Omega \tag{1.6}$$

Here $\mathcal{P}_n=\{P_0,P_1,...,P_n\}\subset\Omega$ are $n+1$ points which are parameters, determining the geometrical object $g_{\mathcal{P}_n}$

$$g_{\mathcal{P}_n}=\{R|F_{\mathcal{P}_n}(R)=0\}\,,\qquad R\in\Omega,\qquad \mathcal{P}_n\in\Omega^{n+1} \tag{1.7}$$

$F_{\mathcal{P}_n}(R)=G_{\mathcal{P}_n}(u_1,u_2,...u_s)$ is some function of $\frac{1}{2}(n+1)(n+2)$ arguments u_s and of $n+1$ parameters $\mathcal{P}_n$. The set $\mathcal{P}_n$ of the geometric object parameters will be referred to as the skeleton of the geometrical object. The subset $g_{\mathcal{P}_n}$ will be referred to as the envelope of the skeleton.

When a particle is considered as a geometrical object, its motion in the space-time is described mainly by the skeleton $\mathcal{P}_n$. The skeleton is an analog of a frame attached rigidly to a physical body. Following the frame motion, one may follow the motion of a body. One skeleton may have many envelopes and describe different geometric objects.

If two geometric objects $g_{P_{n,\sigma}}$ and $g'_{\mathcal{P}'_n,\sigma'}$ are similar their skeletons $\mathcal{P}_{n,\sigma}=\{P_0,P_1,...P_n\}$, and $\mathcal{P}'_{n.\sigma'}=\{P'_0,P'_1,...P'_n\}$ are to be similar. It means that

$$\sigma(P_i,P_k)=\sigma'(P'_i,P'_k)\,,\quad i,k=0,1,...n \tag{1.8}$$

Remark: Arbitrary subset of the point set Ω is not a geometrical object, in general. It is supposed, that physical bodies may have a shape of a geometrical object only in the case, when it is defined by (1.3) - (1.7), because only in this case one can identify identical physical bodies (geometrical objects) in different space-time geometries.

Example: The straight line segment $\mathcal{T}_{[A,B]}$ in the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}} = \{\sigma_{\mathrm{E}}, \Omega\}$ is defined as a set of points $R \in \Omega$

$$\mathcal{T}_{[A,B]} = \left\{ R | \sqrt{2\sigma_{\mathrm{E}}\left(A,R\right)} + \sqrt{2\sigma_{\mathrm{E}}\left(R,B\right)} = \sqrt{2\sigma_{\mathrm{E}}\left(A,B\right)} \right\} \tag{1.9}$$

This segment $\mathcal{T}_{[A,B]}$ is one-dimensional in $\mathcal{G}_{\mathrm{E}}$. It means by definition that a section $S\left(\mathcal{T}_{[A,B]}, P\right)$ of $\mathcal{T}_{[A,B]}$ at any point $P \in \mathcal{T}_{[A,B]}$ consists of one point P.

$$S\left(\mathcal{T}_{[A,B]}, P\right) = \left\{ R | \bigwedge_{C=A,B} \sqrt{2\sigma_{\mathrm{E}}\left(R,C\right)} = \sqrt{2\sigma_{\mathrm{E}}\left(P,C\right)} \right\} = \{P\} \tag{1.10}$$

In other physical geometry $\mathcal{G} = \{\sigma, \Omega\}$ the straight line segment $\mathcal{T}_{[A,B]}$ is defined by the relation

$$\mathcal{T}_{[A,B]} = \left\{ R | \sqrt{2\sigma\left(A,R\right)} + \sqrt{2\sigma\left(R,B\right)} = \sqrt{2\sigma\left(A,B\right)} \right\} \tag{1.11}$$

Its section has the form

$$S\left(\mathcal{T}_{[A,B]}, P\right) = \left\{ R | \bigwedge_{C=A,B} \sqrt{2\sigma\left(R,C\right)} = \sqrt{2\sigma\left(P,C\right)} \right\} \tag{1.12}$$

The set of points $S\left(\mathcal{T}_{[A,B]}, P\right)$ may contain many points, because one equation (1.11) in n-dimensional space is a $(n-1)$-dimensional surface, in general. The fact, that (1.9) in $\mathcal{G}_{\mathrm{E}}$ is one-dimensional segment is a corollary of special properties of the world function σ_{E}.

Let us stress that the definitions (1.9), (1.11) of the straight line segment $\mathcal{T}_{[A,B]}$ in geometries $\mathcal{G}_{\mathrm{E}}$ and $\mathcal{G}$ do not contain a reference to a coordinate system. It is important, because the coordinateless description deals with the space-time geometry in itself (without influence of the coordinate system, which may appear to be essential). We shall show, that the conventional coordinate system can be introduced not always, because some physical geometries (for instance a discrete space-time geometry) have indefinite metrical dimension (maximal number of linear independent vectors).

Identification of geometrical objects in different regions of the space-time geometry is a very important operation, which can be realized only if the description is produced in terms of world function. Conventional description of the space-time geometry based on a use of the linear space formalism is effective only in the space-time geometry of Minkowski $\mathcal{G}_{\mathrm{M}}$ and partly in the Riemannian geometry. Even description of the straight line segment $\mathcal{T}_{[AB]}$ leads to different results in the physical geometry and in the geometry of Minkowski. In the physical geometry the timelike segment $\mathcal{T}_{[AB]}$ (1.11) is a 3-dimensional tube, in general, whereas in $\mathcal{G}_{\mathrm{M}}$ it is a one-dimensional line. According to conventional axiomatic approach to the space-time geometry the segment $\mathcal{T}_{[AB]}$ is one-dimensional in any space-time geometry. In general, mathematical technique of the conventional space-time geometry is not applicable for simultaneous consideration of several different geometries of different space-time regions.

We can perceive the space-time geometry only via motion of physical bodies in the space-time, or via construction of geometrical objects corresponding to these physical bodies. As it follows from the *definition 1.3* of the geometrical object, the function F as a function of its arguments u_s (of world functions of different points) is the same in all physical geometries. It means, that a geometrical object $\mathcal{O}_1$ in the geometry $\mathcal{G}_1 = \{\sigma_1, \Omega_1\}$ is obtained from the same geometrical object $\mathcal{O}_2$ in the geometry $\mathcal{G}_2 = \{\sigma_2, \Omega_2\}$ by means of the replacement $\sigma_2 \rightarrow \sigma_1$ in the definition of this geometrical object. This method of the geometrical objects comparison in different geometries is simple and effective. It cannot be used at the conventional (axiomatic) approach to geometry. It is a reason, why we try to present the Finsler geometry in terms of the world function. The Finsler space-time geometry is to be described in terms of the world function, in order that one can recognize similar geometrical objects in different regions of space-time.

2 The world function as a generator of vector bundle

Capacity of the world function of the Riemannian geometry to generate vector bundle were investigated in the paper [7]. These properties were used for construction of the relative gravitational field [8]. To construct the Finsler geometry one uses a vector bundle TM of the Riemannian geometry $\mathcal{G}_{\mathrm{M}}$ given on a smooth manifold M, where the coordinate system K is given. The Riemannian geometry $\mathcal{G}_{\mathrm{R}}$ may be considered as a special case of the physical geometry $\mathcal{G} = \{\sigma, M\}$ with the world function $\sigma_{\mathrm{R}} = \sigma(x, x')$, where x and x' are coordinates of points $P, P' \in M$ in the coordinate system K. The metric tensor of Riemannian geometry $\mathcal{G}$ has the form

$$g_{ik}(x) = \left[-\frac{\partial^2 \sigma(x, x')}{\partial x^i \partial x'^k}\right]_{x'=x}, \quad i, k = 0, 1, ...n-1 \tag{2.1}$$

To construct the vector bundle $TM_{x'}$ at the point $x' \in M$, one uses usually a set $S_{\mathrm{L}x'}$ of one-dimensional lines, passing trough the point x'. The vector bundle $TM_{x'}$ is tangent to all lines $L \subset S_{\mathrm{L}x'}$. In the physical geometry the vector bundle TM can be constructed without a reference to the set $S_{\mathrm{L}x'}$. It is important, because in the the physical geometry one-dimensional curves may not exist.

Differentiating world function, one obtains the following quantity

$$\sigma_{ik'}(x, x') \equiv \sigma_{,i,k'}(x, x') \equiv \partial_i \partial_{k'} \sigma(x, x') \equiv \frac{\partial^2 \sigma(x, x')}{\partial x^i \partial x'^k}, \quad i, k = 0, 1, 2, 3 \tag{2.2}$$

This quantity forms a covariant two-point tensor (a vector at the point x and a vector at the point x'). In general, a comma before the index k means differentiation with respect to x^k, and a comma before the index k' means differentiation with respect to x'^k.

For the Riemannian geometry the following property takes place

$$\det ||g_{ik}(x)|| \neq 0 \tag{2.3}$$

Then

$$\det ||\sigma_{ik'}(x, x')|| \neq 0 \tag{2.4}$$

in some finite region $\omega_{x'}$, where

$$|x - x'| \equiv \left|\sqrt{2\sigma(x, x')}\right| < R \tag{2.5}$$

In $\omega_{x'}$ one may determine the contravariant two-tensor $\sigma^{ik'}$ by means of relation

$$\sigma^{ik'}\sigma_{lk'} = \delta_l^i \tag{2.6}$$

Here and farther a summation is produced over like upper and lower indices.

Let us define the quantity

$$\Gamma_{kl}^i = \Gamma_{kl}^i(x, x') = \sigma^{is'}\sigma_{kls'}, \qquad \sigma_{kls'}(x.x') \equiv \frac{\partial^3\sigma(x, x')}{\partial x^k \partial x^l \partial x'^s} \tag{2.7}$$

The quantity $\Gamma_{kl}^i(x, x')$ transforms as a Cristoffel symbol at the point x, and one can define a covariant derivative $\tilde{\nabla}_i$ with respect to x^i

$$\tilde{\nabla}_i T_l^k \equiv T_{l||i}^k = T_{l,i}^k - \Gamma_{is}^k T_l^s + \Gamma_{il}^s T_s^k \tag{2.8}$$

where $T_l^k = T_l^k(x, x')$ is some two-point tensor at the point x.

Connection $\Gamma_{kl}^i(x, x')$ at the point x appears to be a connection of a flat space, because the curvature tensor

$$R_{k,sr}^i = \partial_r\Gamma_{ks}^i - \partial_s\Gamma_{kr}^i + \Gamma_{ks}^m\Gamma_{mr}^i - \Gamma_{kr}^m\Gamma_{ms}^i \equiv 0 \tag{2.9}$$

vanishes identically. Indeed, according to (2.7) and according to relation

$$\partial_s\sigma^{ik'} = -\sigma^{ir'}\partial_s\sigma_{lr'}\sigma^{lk'} = -\sigma^{ir'}\sigma_{slr'}\sigma^{lk'} \tag{2.10}$$

which follows from (2.6), one obtains

$$\begin{aligned}
& \partial_r\Gamma_{ks}^i - \partial_s\Gamma_{kr}^i + \Gamma_{ks}^m\Gamma_{mr}^i - \Gamma_{kr}^m\Gamma_{ms}^i \\
= \; & \partial_r\left(\sigma^{im'}\sigma_{ksm'}\right) - \partial_s\left(\sigma^{im'}\sigma_{krm'}\right) + \left(\sigma^{ml'}\sigma_{ksl'}\sigma^{ip'}\sigma_{mr,p'}\right) - \left(\sigma^{ml'}\sigma_{krl'}\sigma^{ip'}\sigma_{msp'}\right) \\
= \; & \sigma_{ksm'}\partial_r\sigma^{im'} - \sigma_{krm'}\partial_s\sigma^{im'} + \left(\sigma^{ml'}\sigma_{ksl'}\sigma^{ip'}\sigma_{mrp'}\right) - \left(\sigma^{ml'}\sigma_{krl'}\sigma^{ip'}\sigma_{msp'}\right) = 0
\end{aligned}$$

Thus, the Cristoffel symbol (2.7) is a connection of a flat Riemannian space $E_{x'}$. The set of all spaces $E_{x'}$ $x' \in M$ forms a vector bundle TM.

One has

$$\tilde{\nabla}_l\sigma_{ik'} = \sigma_{ik',l} - \Gamma_{li}^s\sigma_{sk'} = \sigma_{lik'} - \sigma^{sm'}\sigma_{lim'}\sigma_{sk'} \equiv 0 \tag{2.11}$$

$$\tilde{\nabla}_l \sigma^{ik'} = -\sigma^{ip'} \tilde{\nabla}_l \sigma_{qp'} \sigma^{qk'} = 0 \tag{2.12}$$

$$\tilde{\nabla}_l u^{\iota' k'}(x') = 0 \tag{2.13}$$

where $u^{\iota' k'}(x')$ is an arbirary one-point tensor at the point x'. Covariant metric tensor $G_{ik} = G_{ik}(x, x')$ of the flat Riemannian space $E_{x'}$ can be presented in the form

$$G_{ik}(x, x') = \sigma_{il'} g_{(0)}^{l's'}(x') \sigma_{k,s'} \tag{2.14}$$

because in this case

$$\tilde{\nabla}_l G_{ik}(x, x') \equiv 0$$

It needs that

$$\det \left|\left| g_{(0)}^{l's'}(x') \right|\right| \neq 0 \tag{2.15}$$

in order that $\det ||G_{ik}|| \neq 0$. It follows from (2.11), (2.13) and (2.14) that

$$\tilde{\nabla}_l G_{ik}(x, x') = \sigma_{ir'} \tilde{\nabla}_l g_{(0)}^{r's'}(x') \sigma_{ks'} = 0 \tag{2.16}$$

Let us define Cristoffel symbol $\tilde{\Gamma}_{kl}^i$ in the flat Riemannian space $E_{x'}$ with metric tensor (2.14)

$$\tilde{\Gamma}_{kl}^i = \frac{1}{2} G^{is} (G_{ks,l} + G_{ls,k} - G_{kl,s}) \tag{2.17}$$

where G^{ik} is the contravariant metric tensor

$$G^{ik} = \sigma^{ip'} g_{(0)p'q'}(x') \sigma^{kp'}, \quad g_{(0)i'k'}(x') g_{(0)}^{i'l'}(x') = \delta_{k'}^{l'} \tag{2.18}$$

Substituting (2.14) in (2.17) and using (2.18), one obtains

$$\tilde{\Gamma}_{kl}^i = \sigma^{is'} \sigma_{kls'} = \Gamma_{kl}^i \tag{2.19}$$

Thus, a physical geometry $\mathcal{G} = \{\sigma, M\}$ on a smooth manifold M, whose world function has the property (2.4), generates a vector bundle TM. The world function σ determines a mapping of the coordinate system K on M into coordinate system $K_{x'}$ on any space $E_{x'}$ of the bundle TM. This mapping determines connection Γ_{kl}^i on any $E_{x'}$ of the vector bundle TM. However, the metric tensor $G_{ik}(x', x')$ in $E_{x'}$, defined by (2.14) may not coincide with $g_{i'k'}(x')$. The flat Riemannian space $E_{x'}$ may have singularity at the point x'. In particular, $E_{x'}$ can be a conical space with the vertex at the point x'.

At the point x' of the space $E_{x'}$

$$G^{i'k'}(x', x') = g^{i'p'}(x') g_{(0)p'q'}(x') g^{k'q'}(x') \tag{2.20}$$

If $g_{(0)i'k'}(x') = g_{i'k'}(x')$, the metric tensor $[G_{ik}(x, x')]_{x=x'}$ in $E_{x'}$ coincides with the metric tensor $g_{i'k'}(x')$ in the geometry $\mathcal{G} = \{\sigma, M\}$, defined by (2.1)

$$[G_{ik}(x, x')]_{x=x'} = g_{i'k'}(x'), \quad \text{if} \quad g_{(0)i'k'}(x') = g_{i'k'}(x') \tag{2.21}$$

It means that the flat Riemannian space $E_{x'}$ is Euclidean. In this case it is tangent to the manifold M at the point x'.

If the physical geometry $\mathcal{G}=\{\sigma,M\}$ is a Riemannian geometry, the world function $\sigma=\sigma_{\mathrm{R}}$ satisfies the differential equation [5]

$$\sigma_{\mathrm{R}i'}g^{i'k'}\left(x'\right)\sigma_{\mathrm{R}k'}=2\sigma_{\mathrm{R}},\quad \sigma_{\mathrm{R}i'}\equiv\frac{\partial\sigma_{\mathrm{R}}}{\partial x'^{i}} \tag{2.22}$$

Acting to both sides of equation (2.22) by $\tilde{\nabla}_l\tilde{\nabla}_s$ and taking into account relations (2.11), (2.13), one obtains

$$\sigma_{\mathrm{R}li'}g^{i'k'}\left(x'\right)\sigma_{\mathrm{R}sk'}=\tilde{\nabla}_l\sigma_{\mathrm{R}s} \tag{2.23}$$

Comparing (2.23) with (2.14) and (2.21), one concludes that for the Riemannian geometry $\mathcal{G}_{\mathrm{R}}=\{\sigma_{\mathrm{R}},M\}$, the metric tensor in $E_{x'}$ at the point x takes the form

$$G_{ik}\left(x,x'\right)=\tilde{\nabla}_i\tilde{\nabla}_k\sigma_{\mathrm{R}}\left(x,x'\right) \tag{2.24}$$

Thus, let a physical geometry $\mathcal{G}=\{\sigma,M\}$ be given on a smooth manifold M with a coordinate system K on M and satisfy the condition (2.4). Then this geometry generates bundle TM of flat Riemannian spaces $E_{x'}$ at any point $x'\in M$. It generates a mapping of coordinate system $K\rightarrow K_{x'}$ and determines the metric tensor $G_{ik}\left(x,x'\right)$ at any point x of the Euclidean space $E_{x'}$ in the coordinate system $K_{x'}$. The metric tensor $G_{ik}\left(x,x'\right)$ at the point x is determined by some tensor $g_{(0)i'k'}\left(x'\right)$ at the point x' ($\det\left|\left|g_{(0)i'k'}\left(x'\right)\right|\right|\neq 0$). If the geometry $\mathcal{G}=\{\sigma_{\mathrm{R}},M\}$ is a Riemannian geometry, the metric tensor $G_{ik}\left(x,x'\right)$ on $E_{x'}$ at the point x is determined completely by the world function σ_{R} of the Riemannian geometry.

The vector bundle TM is a tangent vector bundle in the case, when $\mathcal{G}=\{\sigma_{\mathrm{R}},M\}$. It means that, if the manifold M is embedded isometrically into an Euclidean manifold M_{E} and form a surface $\mathcal{S}$ in M_{E}, the bundle TM is a set of planes, which are tangent to the surface $\mathcal{S}$. In the case, when $\mathcal{G}=\{\sigma,M\}$ is not a Riemannian geometry, the vector bundle TM is not a tangent vector bundle, in general, because the manifold M cannot be embedded isometrically into an Euclidean manifold M_{E}. In this case the straight lines of manifold M are not one-dimensional lines, and such straight lines of the form of (1.11) cannot be embedded isometrically in the Euclidean manifold M_{E}. Nevertheless a physical geometry $\mathcal{G}=\{\sigma,M\}$ generates a vector bundle TM of flat Riemannian spaces $E_{x'}$ and a mapping of the coordinate system K onto any space $E_{x'}$ of TM.

3 Role of the coordinate system in the space-time geometry description

One assumes usually that the coordinate system is something external with respect to space-time geometry and, in general, with respect to any geometry. But it is not so. To understand the role of the coordinate system, let us consider the proper

Euclidean geometry $\mathcal{G}_{\rm E}$ in the σ-representation. The σ-representation appears at the metric approach to geometry, when all geometrical quantities and relations are described in terms of the world function. The geometry dimension and the coordinate system are expressed in terms of the world function.

In the Cartesian coordinate system K the world function $\sigma_{\rm E}$ of the proper Euclidean geometry has a special form

$$\sigma_{\rm E}\left(P,P'\right)=\sigma_{\rm E}\left(x,x'\right)=\frac{1}{2}\sum_{k=1}^{k=n}\left(x^k-x'^k\right)^2 \tag{3.1}$$

where $P=\{x^1,x^2,...x^n\}$, $P'=\{x'^1,x'^2,...x'^n\}$ are points of the n-dimensional Euclidean space E^n, $P,P'\in E^n$ and $x=\{x^1,x^2,...x^n\}$, $x'=\{x'^1,x'^2,...x'^n\}$ are coordinates in some Cartesian coordinate system K.

The way of generalization of $\mathcal{G}_{\rm E}$ depends essentially on the method of the $\mathcal{G}_{\rm E}$ representation. There are two methods of $\mathcal{G}_{\rm E}$ representation: (1) V-representation and (2) σ-representation [9].

At V-representation one uses axiomatic approach to $\mathcal{G}_{\rm E}$, when the Euclidean geometry is constructed on the basis of linear space $\mathcal{L}_n$. The linear space $\mathcal{L}_n$ is a set Ω_n of elements $u\in\Omega_n$. These elements u will be referred to as linear vectors (linvectors). Multiplication of a linvector $u\in\Omega_n$ by a real number a gives the linvector $au\in\Omega_n$. Sum of two linvectors $u\in\Omega_n$ and $v\in\Omega_n$ gives a new linvector $(u+v)\in\Omega_n$. These operations have linear properties. The term "linvector" (instead of conventional term "vector") is used, because any linvector $u\in\Omega_n$ exists in one copy.

On the contrary, vector $\mathbf{AB}$ in $\mathcal{G}_{\rm E}$ is defined as the ordered set $\mathbf{AB}=\{A,B\}\in\Omega\times\Omega$ of two points $A,B\in\Omega$. Here Ω is the set of points, where the geometry $\mathcal{G}_{\rm E}$ is defined. Among vectors $\mathbf{PQ}\in\Omega\times\Omega$ of the Euclidean space E^n there are equivalent (equal) vectors, and there are many equivalent vectors $\mathbf{PQ}\in\Omega\times\Omega$. We use different terms ("linvector" and " vector") for elements of Ω_n and of $\Omega\times\Omega$, because it is incorrect to use the same term for different objects with different properties.

The set $\Omega_{\mathbf{AB}}$ of vectors $\mathbf{CD}\in\Omega\times\Omega$ which are equivalent to vector $\mathbf{AB}\in\Omega\times\Omega$ is defined as a set of vectors $\mathbf{CD}$ which are in parallel with $\mathbf{AB}$ and their lengths $|\mathbf{CD}|$, $|\mathbf{AB}|$ are equal.

$$\Omega_{\mathbf{AB}}=\left\{\mathbf{CD}|\left(\mathbf{CD}\mathrm{eqv}\mathbf{AB}\right)\right\} \tag{3.2}$$

$$\left(\mathbf{CD}\mathrm{eqv}\mathbf{AB}\right):\quad\left(\mathbf{CD}\upuparrows\mathbf{AB}\right)\wedge|\mathbf{CD}|=|\mathbf{AB}| \tag{3.3}$$

$$\left(\mathbf{CD}\upuparrows\mathbf{AB}\right):\quad\left(\mathbf{CD}.\mathbf{AB}\right)=|\mathbf{CD}|\cdot|\mathbf{AB}| \tag{3.4}$$

Here $(\mathbf{CD}.\mathbf{AB})\in\mathbb{R}$ is the scalar product of two vectors $\mathbf{CD}$ and $\mathbf{AB}$ which is defined by the relation

$$\left(\mathbf{CD}.\mathbf{AB}\right)=\sigma_{\rm E}\left(C,B\right)+\sigma_{\rm E}\left(D,A\right)-\sigma_{\rm E}\left(C,A\right)-\sigma_{\rm E}\left(D,B\right) \tag{3.5}$$

$$|\mathbf{CD}|^2=2\sigma_{\rm E}\left(C,D\right) \tag{3.6}$$

Equivalence (3.3) of two vectors $\mathbf{CD} \in \mathbf{\Omega} \times \mathbf{\Omega}$ and $\mathbf{AB} \in \mathbf{\Omega} \times \mathbf{\Omega}$ is defined in terms of the Euclidean world function σ_{E}. In the Cartesian coordinate system K, where the world function σ_{E} has the form (3.1) and points A, B, C, D have respectively coordinates x_A, x_B, x_C, x_D the scalar product (3.5) and $|\mathbf{CD}|$ take respectively the form

$$(\mathbf{CD}.\mathbf{AB}) = \sum_{k=1}^{k=n} \left(x_D^k - x_C^k\right)\left(x_B^k - x_A^k\right) \tag{3.7}$$

$$|\mathbf{CD}|^2 = \sum_{k=1}^{k=n} \left(x_D^k - x_C^k\right)^2 \tag{3.8}$$

These expressions coincide respectively with the scalar product of two linvectors $(u_{\mathbf{CD}}.u_{AB})$ and with $|u_{\mathbf{CD}}|^2$, provided $u_{\mathbf{CD}}$ and u_{AB} have coordinates respectively $\left(x_D^k - x_C^k\right)$ and $\left(x_B^k - x_A^k\right)$.

In $\mathcal{G}_{\mathrm{E}}$ the equivalence relation (3.3) is reflexive, symmetric and transitive. Then the set $\Omega_{\mathbf{AB}}$ is the equivalence class of the vector $\mathbf{AB}$. One may identify the linvector $u_{\mathbf{AB}} \in L_n$ with the equivalence class $\Omega_{\mathbf{AB}}$ of the vector $\mathbf{AB} \in \mathbf{\Omega} \times \mathbf{\Omega}$. Axiomatics of the linear space L_n and operations in L_n can be used for construction of geometric relations in $\mathcal{G}_{\mathrm{E}}$. After generalization of $\mathcal{G}_{\mathrm{E}}$, when σ_{E} is replaced by another world function σ, the equivalence relation (3.3) ceases to be transitive, in general. As a result the set $\Omega_{\mathbf{AB}}$ ceases to be an equivalence class of the vector $\mathbf{AB}$. One may not identify the linvector $u_{\mathbf{AB}} \in L_n$ with the set $\Omega_{\mathbf{AB}}$, because not all vectors $\mathbf{CD} \in \Omega_{\mathbf{AB}}$ are equivalent between themselves. The geometry $\mathcal{G} = \{\sigma, \Omega\}$, obtained as a result of the replacement $\sigma_{\mathrm{E}} \to \sigma$, appears to be multivariant.

A physical geometry $\mathcal{G} = \{\sigma, \Omega\}$ is a multivariant geometry, in general. In the multivariant geometry there are many vectors $\mathbf{CD}, \mathbf{CD}', \mathbf{CD}'', ...$at the point C, which are equivalent to vector $\mathbf{AB}$ at the point A, but they are not equivalent between themseves. Equivalence of vectors $\mathbf{CD}$ and $\mathbf{AB}$ is defined by formulas (3.3) -(3.6). The equivalence relation (3.3) -(3.6) becomes intransitive. This intransitivity is a reason of the physical geometry multivariance. Only the proper Euclidean geometry is not multivariant. The space-time geometry of Minkowski is multivariant with respect to spacelike vectors, but it is single-variant with respect to timelike vectors. The discrete space-time geometry is multivariant with respect to timelike vectors, and this circumstance is a reason of quantum effects [10]. Multivariance of the space-time geometry is a very important property [11], but it cannot be described by the formalism of linear space, which is used usually for the space-time description.

At the generalization of the proper Euclidean geometry one obtains a physical geometry $\mathcal{G} = \{\sigma, \Omega\}$, replacing the world function σ_{E} by the world function σ of the geometry $\mathcal{G}$ in all geometric relations of $\mathcal{G}_{\mathrm{E}} = \{\sigma_{\mathrm{E}}, \Omega\}$, which can be expressed in terms of only the Euclidean world function σ_{E}. These relations will be referred to as general geometric relations. Expressions (3.5), (3.6) are examples of general geometric relations.

Another example of such a relation is the definition of linear dependence of n

vectors $\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2, ...\mathbf{P}_0\mathbf{P}_n$. Vectors $\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2, ...\mathbf{P}_0\mathbf{P}_n$ are linear dependent, if the condition

$$F_n\left(\mathcal{P}^n\right) = 0 \tag{3.9}$$

is fulfilled. Here $\mathcal{P}^n = \{P_0, P_1, ...P_n\}$ and $F_n\left(\mathcal{P}^n\right)$ is the Gram determinant

$$F_n\left(\mathcal{P}^n\right) \equiv \det \left|\left|\left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k\right)\right|\right|, \quad i,k = 1,2,...n \tag{3.10}$$

Scalar product in (3.10) is expressed via the world function by means of (3.5).

Let us consider a generalization $\mathcal{G} = \{\sigma, \Omega\}$ of the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}} = \{\sigma_{\mathrm{E}}, \Omega\}$. One replaces the world function σ_{E} by the world function σ in all general geometric relations. But there are special relations of the geometry $\mathcal{G}_{\mathrm{E}}$, which depends on special properties of the world function σ_{E}. One cannot replace world function in the special relations. These special properties determine dimension of the geometry $\mathcal{G}_{\mathrm{E}}$ and properties of the rectilinear coordinate system in $\mathcal{G}_{\mathrm{E}}$.

If σ_{E} is the world function of n-dimensional Euclidean space E^n, it satisfies the following relations.

I. Definition of the dimension and introduction of the rectilinear coordinate system:

$$\exists\mathcal{P}^n \equiv \{P_0, P_1, ...P_n\} \subset \Omega, \qquad F_n\left(\mathcal{P}^n\right) \neq 0, \qquad F_k\left(\Omega^{k+1}\right) = 0, \qquad k > n \tag{3.11}$$

where $F_n\left(\mathcal{P}^n\right)$ is the Gram's determinant (3.10). Vectors $\mathbf{P}_0\mathbf{P}_i$, $i = 1,2,...n$ are basic vectors of the rectilinear coordinate system K_n with the origin at the point P_0. The covariant metric tensor $g_{ik}\left(\mathcal{P}^n\right)$, $i,k = 1,2,...n$ and the contravariant one $g^{ik}\left(\mathcal{P}^n\right)$, $i,k = 1,2,...n$ in a rectilinear coordinate system K_n are defined by the relations

$$\sum_{k=1}^{k=n} g^{ik}\left(\mathcal{P}^n\right) g_{lk}\left(\mathcal{P}^n\right) = \delta_l^i, \qquad g_{il}\left(\mathcal{P}^n\right) = \left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_l\right), \qquad i,l = 1,2,...n \tag{3.12}$$

$$F_n\left(\mathcal{P}^n\right) = \det\left|\left|g_{ik}\left(\mathcal{P}^n\right)\right|\right| \neq 0, \qquad i,k = 1,2,...n \tag{3.13}$$

II. Linear structure of the Euclidean space:

$$\sigma_{\mathrm{E}}\left(P,Q\right) = \frac{1}{2}\sum_{i,k=1}^{i,k=n} g^{ik}\left(\mathcal{P}^n\right)\left(x_i\left(P\right) - x_i\left(Q\right)\right)\left(x_k\left(P\right) - x_k\left(Q\right)\right), \qquad \forall P,Q \in \Omega \tag{3.14}$$

where coordinates $x_i\left(P\right)$, $i = 1,2,...n$ of the point P are covariant coordinates of the vector $\mathbf{P}_0\mathbf{P}$, defined by the relation

$$x_i\left(P\right) = \left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}\right), \qquad i = 1,2,...n \tag{3.15}$$

III: The metric tensor matrix $g_{lk}\left(\mathcal{P}^n\right)$ has only positive (or only negative) eigenvalues

$$g_k > 0, \qquad k = 1,2,...,n \tag{3.16}$$

IV. The continuity condition: the system of equations

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}) = y_i \in \mathbb{R}, \qquad i = 1, 2, ...n \tag{3.17}$$

considered to be equations for determination of the point P as a function of coordinates $y = \{y_i\}$, $i = 1, 2, ...n$ has always one and only one solution.

Not all conditions I – IV are independent, they determine different properties of $\mathcal{G}_{\mathrm{E}}$. For instance, the condition (3.11) determines the dimension n of the Euclidean space E^n. This dimension n is the maximal number of linear independent vectors in $\mathcal{G}_{\mathrm{E}}$. This number is determined by the general geometric expression (3.10) which depends on the form of the world function. If conditions (3.11) are not fulfilled, one cannot introduce a coordinate system in the conventional form, because the metric dimension $n_m = n$ of the geometry $\mathcal{G}$ remains to be not determined.

The sum of two vectors is defined as follows. If one adds vectors **AB** and **BC,** when the end of one vector is the origin of the other, then one obtains

$$\mathbf{AB} + \mathbf{BC} = \mathbf{AC} \tag{3.18}$$

If one adds arbitrary vectors **AB** and **CD**, one obtains

$$\mathbf{AB} + \mathbf{CD} = \mathbf{AB} + \mathbf{BR} = \mathbf{AR} \tag{3.19}$$

where the point R is defined from the relation

$$(\mathbf{CD}\mathrm{eqv}\mathbf{BR}) \tag{3.20}$$

According to (3.3) - (3.5) the relation (3.20) represents two equations of the type (3.3). If these equations have always one and only one solution for the point R (as in $\mathcal{G}_{\mathrm{E}}$), the operation of addition is defined univalently. However, if the solution is multivariant, one cannot define the addition as a single-valued operation in the form, that is used in linear space for addition of linvectors.

Multiplication of a vector **AB** by a real number a is defined as follows

$$a\mathbf{AB} = \mathbf{AR} \tag{3.21}$$

where the point R is determined from the relations

$$(\mathbf{AB}.\mathbf{AR}) = a\,|\mathbf{AB}|^2, \quad |\mathbf{AR}| = a\,|\mathbf{AB}| \tag{3.22}$$

If solution of equations (3.22) is multivariant, the multiplication operation is multivariant also.

Summarizing, one can say, that the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ can be reduced to linear algebra. However, generalizations of $\mathcal{G}_{\mathrm{E}}$ cannot be reduced, in general, to linear algebra. They are multivariant, in general, and this multivariance is a corollary of the vector directivity which is absent in algebra. Generally speaking, *geometry cannot be reduced to algebra.*

Most restrictions on world function σ_{E} of $\mathcal{G}_{\mathrm{E}}$ arise from restrictions (3.11), which consist of many equations. These restrictions have a global character. One may reduce these restriction to a local form

$$\exists\mathcal{P}^n \equiv \{P_0, P_1, ...P_n\} \subset \Omega_\varepsilon, \qquad F_n\left(\mathcal{P}^n\right) \neq 0, \qquad F_k\left(\Omega_\varepsilon^{k+1}\right) = 0, \qquad k > n \tag{3.23}$$

where Ω_ε is an infinitesimal vicinity of the point P_0, defined by the relation

$$\left|\sqrt{2\sigma\left(P_0, P\right)}\right| < \varepsilon, \quad \varepsilon \to +0 \tag{3.24}$$

If conditions (3.23) take place, one can use formalism of the linear space locally. The Riemannian geometry is locally Euclidean. The Riemannian geometry is obtained at application of restrictions (3.11) in the form (3.23). A use of restriction (3.23) admits one to suppress multivariance of the vector equivalence for vectors having common origin. But multivariance of the vector equality remains for vectors having different origin. Consideration of equality of vectors with different origin is forbidden in the Riemannian geometry or it is connected with the way of the vector transport. It is necessary for a use of the linear space formalism, which can be used only, if the metric dimension exists at least locally, and one can introduce a rectilinear coordinate system locally.

According to consideration of the second section any physical geometry $\mathcal{G} = \{\sigma, M\}$ generates a vector bundle TM with a flat Riemannian geometry on any space $E_{x'}$ of the bundle. In the case, when the geometry $\mathcal{G} = \{\sigma, M\}$ is a Riemannian geometry, according to (2.21) the metric tensor $\left[G_{ik}\left(x, x'\right)\right]_{x'=x}$ on $E_{x'}$ at the point x' in the coordinate system $K_{x'}$ coincides with the metric tensor $g_{i'k'}\left(x'\right)$ of the Riemannian geometry $\mathcal{G} = \{\sigma, M\}$ at the point x'. The metric tensor $G_{ik}\left(x, x'\right)$ at the point x on $E_{x'}$ is determined by the formulae (2.23), (2.24) in the form

$$G_{ik}\left(x, x'\right) = \sigma_{il'} g^{l's'}\left(x'\right) \sigma_{ks'} \tag{3.25}$$

It means that the Riemannian geometry may be described as an Euclidean geometry on a bundle TM of Euclidean spaces $E_{x'}$. It is natural, because the Riemannian geometry is a set of many Euclidean geometries on connected infinitesimal manifolds dM. Transition from the Euclidean geometries on the bundle TdM of infinitesimal manifolds dM to the Riemannian geometry is rather simple. It is described by the formula (3.25).

If the physical geometry $\mathcal{G} = \{\sigma, M\}$ is not a Riemannian geometry, the geometry on any space $E_{x'}$ of the vector bundle TM is a flat Riemannian geometry, but the relation (3.25) does not take place, in general. The Riemannian geometry on $E_{x'}$ may have a singuarity at the point x'. Instead one has the relation (2.18)

$$G_{ik}\left(x, x'\right) = \sigma_{il'} g_{(0)}^{l's'}\left(x'\right) \sigma_{ks'} \tag{3.26}$$

where the tensor $g_{(0)}^{l's'}\left(x'\right)$ is determined by the physical geometry $\mathcal{G} = \{\sigma, M\}$ by some unknown way. But we may hope that the set of flat Riemannian geometry on manifolds $T_{x'}M$ describes the physical geometry $\mathcal{G} = \{\sigma, M\}$. We may hope that the world function σ can be obtained from the set of flat Riemannian geometries on manifolds $T_{x'}M$ of the bundle TM. We cannot prove this statement, but we may hope that the physical geometry $\mathcal{G} = \{\sigma, M\}$ can be described as a set of flat Riemannian geometries. In other words, a single physical geometry $\mathcal{G} = \{\sigma, M\}$ is reduced to a set of many flat Riemannian geometries. This set of flat Riemannian geometries associates with the Finsler geometry, which is given on a vector bundle TM. Note that description on the vector bundle TM is unsufficient in application to the space-time geometry, because one needs a world function for identification of a geometrical object in different regions of the manifold M, as we have seen in introduction.

Conventional presentation of the Finsler geometry is based on a use of the linear space formalism. We try to replace presentation on the basis of linear space by a presentation on the basis of the world function. Presentation in terms of the world function is interesting in the relation that the world function of a Riemannian (or a metric) manifold describes the vector bundle of this manifold.

4 Finsler geometry in terms of world function

Finsler geometry is a generalization of the Riemannian geometry, which may be locally non-Euclidean [1]. The Finsler geometry $\mathcal{G}_{\mathrm{F}}$ is given on a tangent bungle TM of a smooth Riemannian manifold M. We are interested in application of the Finsler geometry for the space-time description.

The Finsler manifold is a differentiable manifold together with the structure of an intrinsic quasimetric space in which the length of any rectifiable curve $\gamma : [a, b] \rightarrow M$ is given by the length functional

$$L(\gamma) = \int_a^b F(\gamma(t), \dot{\gamma}(t))\, dt \tag{4.1}$$

where $F(x, \cdot\ )$ is some asymmetric norm on each tangent space T_xM. Finsler manifolds generalize non-trivially Riemannian manifolds in the sense that they are not necessarily infinitesimally Euclidean. This means that the (asymmetric) norm on each tangent space is not necessarily induced by an inner product (metric tensor).

A Finsler manifold is a differentiable manifold M together with a Finsler function F defined on the tangent bundle of M so that for all tangent vectors v,

1. $F(x, v) \geq 0$ with equality, if and only if $v = 0$ (positivedefiniteness).

2. $F(x, \gamma v) = \lambda F(x, v)$ for all $\lambda \geq 0$ (but not necessarily for $\lambda < 0$) (homogeneity).

3. $F(x, v + w) \leq F(x, v) + F(x, w)$ for all w at the same tangent space with v (subadditivity).

In other words, F is an asymmetric norm on each tangent space. Typically one replaces the subadditivity with the following strong convexity condition: For each tangent vector v, the hessian of F^2 at v is positive definite. Here the hessian of F^2 at v is the symmetric bilinear form

$$g_v\left(X,Y\right)=\frac{1}{2}\left[\frac{\partial^2}{\partial s\partial t}F^2\left(x,v+sX+tY\right)\right]_{s=t=0},\quad X,Y\in T_xM$$

also known as the fundamental tensor of F at v.

Conventional presentation of the Finsler geometry is based on a use of the linear space formalism. We try to replace presentation on the basis of linear space by a presentation on the basis of the world function. Presentation in terms of the world function is interesting in the relation that the world function of a Riemannian (or a metric) manifold describes the vector bundle of this manifold.

There is an idea that the Finsler geometry may be used as a space-time geometry [14, 15]. It is assumed that the Berwald-Moor geometry is the most adequate Finsler geometry for description of the space-time. We are interested mainly in geometries, which could be used as a space-time geometry. We present the Berwald-Moor geometry in terms of the world function and try to investigate to what extent it can serve as a space-time geometry.

5 Geometry of Berwald - Moor as a possible space-time geometry

We are interested in the Finsler geometry in its application to the space-time. Let us consider the Berwald-Moor space-time geometry. In the isotropic coordinates its line element has the form

$$ds^2=\sqrt[4]{dx^1dx^2dx^3dx^4} \tag{5.1}$$

The corresponding world function has the form

$$\sigma\left(x,x'\right)=\frac{1}{2}\sqrt{(x^1-x'^1)(x^2-x'^2)(x^3-x'^3)(x^4-x'^4)} \tag{5.2}$$

Although the line element (5.1) does not determine the world function uniquely, but the relation (5.1) together with properties 2 and 3 of the previous section leads to expression (5.2) for the world function. Besides, only those values of coordinates x are admissible for which the world function is real.

Instead of isotropic coordinates x^i, $i=1,2,3,4$ we use coordinates

$$t_1=x^1+x^2,\quad t_2=x^3+x^4,\quad y_1=x^1-x^2,\quad y_2=x^3-x^4 \tag{5.3}$$

$$x^1=\frac{t_1+y_1}{2},\quad x^2=\frac{t_1-y_1}{2},\quad x^3=\frac{t_2+y_2}{2},\quad x^4=\frac{t_2-y_2}{2} \tag{5.4}$$

Then the world function has the form

$$\begin{aligned}
& \sigma\left(t_1,t_2,y_1,y_2;t_1',t_2',y_1',y_2'\right) \\
= & \frac{1}{2}\left|\sqrt{\left(\left(t_1-t_1'\right)^2-\left(y_1-y_1'\right)^2\right)\left(\left(t_2-t_2'\right)^2-\left(y_2-y_2'\right)^2\right)}\right| \\
& \times\theta\left(\left(t_1-t_1'\right)^2-\left(y_1-y_1'\right)^2\right)\theta\left(\left(t_2-t_2'\right)^2-\left(y_2-y_2'\right)^2\right) \\
& -\frac{1}{2}\left|\sqrt{\left(\left(t_1-t_1'\right)^2-\left(y_1-y_1'\right)^2\right)\left(\left(t_2-t_2'\right)^2-\left(y_2-y_2'\right)^2\right)}\right| \\
& \times\theta\left(\left(y_1-y_1'\right)^2-\left(t_1-t_1'\right)^2\right)\theta\left(\left(y_2-y_2'\right)^2-\left(t_2-t_2'\right)^2\right)
\end{aligned} \tag{5.5}$$

Here

$$\theta\left(x\right)=\begin{cases}1 & \text{if} \quad x\geq 0\\ 0 & \text{if} \quad x<0\end{cases} \tag{5.6}$$

Vector $\mathbf{P}_0\mathbf{P}_1=\{t_1,t_2,y_1y_2\}$ is timelike, if $(t_1>y_1>0)\wedge(t_2>y_2>0)$, or if $(t_1<y_1<0)\wedge(t_2<y_2<0)$. It is spacelike, if $(y_1>t_1>0)\wedge(y_2>t_2>0)$, or if $(y_1<t_1<0)\wedge(y_2<t_2<0)$ Vector $\mathbf{P}_0\mathbf{P}_1=\{t_1,t_2,y_1y_2\}$ is isotropic, if $t_1=y_1\vee t_2=y_2$. Domains of coordinate values $t_1^2<y_1^2\wedge t_2^2>y_2^2$ and $t_1^2>y_1^2\wedge t_2^2<y_2^2$ should be excluded, because in these domains the world function is imaginary.

Let us consider segment of a world chain in the space-time geometry of Berwald-Moor. We consider three adjacent points P_0,P_1,P_2 of this world chain, describing motion of a free particle. The world chain of a free particle contains two adjacent vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_1\mathbf{P}_2$ which are equivalent. It means that

$$\left|\mathbf{P}_0\mathbf{P}_1\right|=\left|\mathbf{P}_1\mathbf{P}_2\right|,\quad \left(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_1\mathbf{P}_2\right)=\left|\mathbf{P}_0\mathbf{P}_1\right|\cdot\left|\mathbf{P}_1\mathbf{P}_2\right| \tag{5.7}$$

where the scalar product $\left(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_1\mathbf{P}_2\right)$ has the form

$$\left(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_1\mathbf{P}_2\right)=\sigma\left(P_0,P_2\right)-\sigma\left(P_1,P_2\right)-\sigma\left(P_0,P_1\right) \tag{5.8}$$

and

$$\left|\mathbf{P}_0\mathbf{P}_1\right|=2\sigma\left(P_0,P_1\right) \tag{5.9}$$

Equations (5.7) describe both timelike and spacelike world lines.

Using (5.8), (5.9), equations (5.7) are written in the form

$$\sigma\left(P_0,P_1\right)=\sigma\left(P_1,P_2\right),\quad \sigma\left(P_0,P_2\right)=4\sigma\left(P_0,P_1\right) \tag{5.10}$$

Let three points P_0,P_1,P_2 have coordinates

$$P_0=\{0,0,0,\},\quad P_1=\{t_1,t_2,y_1,y_2\},\quad P_2=\{2t_1+\tau_1,2t_2+\tau_2,2y_1+\xi_1,2y_2+\xi_2\} \tag{5.11}$$

Here the Greek variables $\tau_1,\tau_2,\xi_1,\xi_2$ describe wobbling of the world chain. If $\tau_1=\tau_2=\xi_1=\xi_2=0$, the world chain does not wobble. Four variables $\tau_1,\tau_2,\xi_1,\xi_2$ are to be determined from two equations (5.10).

According to (5.5) dynamic equations (5.10) are transformed to the form

$$\left((t_1+\tau_1)^2-(y_1+\xi_1)^2\right)\left((t_2+\tau_2)^2-(y_2+\xi_2)^2\right)=\left(t_1^2-y_1^2\right)\left(t_2^2-y_2^2\right) \tag{5.12}$$

$$\left(\left(t_1+\frac{\tau_1}{2}\right)^2-\left(y_1+\frac{\xi_1}{2}\right)^2\right)\left(\left(t_2+\frac{\tau_2}{2}\right)^2-\left(y_2+\frac{\xi_2}{2}\right)^2\right)=\left(t_1^2-y_1^2\right)\left(t_2^2-y_2^2\right) \tag{5.13}$$

Let us introduce designations

$$f_1(\tau_2,\xi_2)=\frac{\left(t_1^2-y_1^2\right)\left(t_2^2-x_2^2\right)}{(t_2+\tau_2)^2-(y_2+\xi_2)^2} \tag{5.14}$$

$$f_2(\tau_2,\xi_2)=\frac{16\left(t_1^2-y_1^2\right)\left(t_2^2-y_2^2\right)}{(2t_2+\tau_2)^2-(2y_2+\xi_2)^2} \tag{5.15}$$

Then equations (5.12), (5.13) are written in the form

$$\left((t_1+\tau_1)^2-(y_1+\xi_1)^2\right)=f_1(\tau_2,\xi_2) \tag{5.16}$$

$$\left((2t_1+\tau_1)^2-(2y_1+\xi_1)^2\right)=f_2(\tau_2,\xi_2) \tag{5.17}$$

where lhs of equations do not depend on τ_2, ξ_2.

Solutions of equations (5.16), (5.17) have the form

$$\xi_1=\frac{t_1}{y_1}\tau_1-\frac{f_2-f_1-3\left(t_1^2-y_1^2\right)}{2y_1}$$

$$\tau_1=t_1\left(\frac{f_2-f_1}{2\left(t_1^2-y_1^2\right)}-\frac{3}{2}\right)\pm\sqrt{\frac{y_1^2f_1}{\left(t_1^2-y_1^2\right)}-\frac{y_1^2}{4}\left(\frac{f_2-f_1}{\left(t_1^2-y_1^2\right)}-1\right)^2} \tag{5.18}$$

$$\begin{aligned}\xi_1 &= \frac{t_1}{y_1}\left(t_1\left(\frac{f_2-f_1}{2\left(t_1^2-y_1^2\right)}-\frac{3}{2}\right)\pm\sqrt{\frac{y_1^2f_1}{\left(t_1^2-y_1^2\right)}-\frac{y_1^2}{4}\left(\frac{f_2-f_1}{\left(t_1^2-y_1^2\right)}-1\right)^2}\right)\\ &\quad -\frac{f_2-f_1-3\left(t_1^2-y_1^2\right)}{2y_1}\end{aligned} \tag{5.19}$$

Only real solutions are essential. They take place, if and only if

$$F_2(\tau_2,\xi_2)\equiv\frac{f_1(\tau_2,\xi_2)}{\left(t_1^2-y_1^2\right)}-\frac{1}{4}\left(\frac{f_2(\tau_2,\xi_2)-f_1(\tau_2,\xi_2)}{\left(t_1^2-y_1^2\right)}-1\right)^2\geq 0 \tag{5.20}$$

Let us introduce variables

$$a_1=\frac{\tau_1}{t_1},\quad a_2=\frac{\tau_2}{t_2},\quad b_1=\frac{\xi_1}{y_1},\quad b_2=\frac{\xi_2}{y_2} \tag{5.21}$$

and expand (5.20) over powers of a_2, b_2, supposing that a_2^2, $b_2^2 \ll 1$. One obtains

$$F_2(\tau_2,\xi_2) = \frac{4y_2^2}{(t_2^2-y_2^2)}\left(\frac{\tau_2}{t_2}\right)\left(\frac{\xi_2}{y_2}\right) + \frac{3t_2^4}{(t_2^2-y_2^2)^2}\left(\frac{\tau_2}{t_2}\right)^2 + \frac{y_2^2(2y_2^2+t_2^2)}{(t_2^2-y_2^2)^2}\left(\frac{\xi_2}{y_2}\right)^2 \tag{5.22}$$

Thus, in the approximation

$$\frac{\tau_1}{t_1}, \frac{\tau_2}{t_2}, \frac{\xi_1}{y_1}, \frac{\xi_2}{y_2} \ll 1 \tag{5.23}$$

one obtains the following result. In the case of timelike vector $\mathbf{P}_0\mathbf{P}_1$, when $t_1^2 > y_1^2 > 0$ and $t_2^2 > y_2^2 > 0$, $F_2(\tau_2,\xi_2) > 0$, if $\xi_2\tau_2\mathrm{sgn}(t_2y_2) > 0$, and dynamic equations (5.12), (5.13) have many solutions, because τ_2, ξ_2 are arbitrary parameters satisfying inequalities (5.23). In the approximation (5.23) the dynamic equations (5.12), (5.13) take the form

$$\xi_1 = \frac{t_1}{y_1}\tau_1 + \frac{\tau_2^2(t_1^2-y_1^2)}{y_1(t_2^2-y_2^2)} \tag{5.24}$$

$$\left(\tau_1 + t_1\frac{\tau_2^2}{(t_2^2-y_2^2)}\right)^2 = \frac{1}{(t_2^2-y_2^2)^2}\left(4y_2(t_2^2-y_2^2)\frac{\tau_2\xi_2}{t_2} + 3t_2^2\tau_2^2 + \xi_2^2(2y_2^2+t_2^2)\right) \tag{5.25}$$

In the case of spacelike vector $\mathbf{P}_0\mathbf{P}_1$, when $y_2^2 > t_2^2 > 0$ and $y_1^2 > t_1^2 > 0$, $F_2(\tau_2,\xi_2) > 0$, if $\xi_2\tau_2\mathrm{sgn}(t_2y_2) < 0$, and dynamic equations (5.12), (5.13) have many solutions also, because τ_2, ξ_2 are arbitrary parameters satisfying inequalities (5.23).

One can see that world lines of free particles wobble in the space-time, having Berwald-Moor geometry. In the space-time geometry there are two types of world line wobbling: (1) quantum wobbling and (2) tachyon wobbling. The quantum wobbling takes place for tardions (particles having timelike world line). This wobbling is conditioned by the elementary length λ_0 of the discrete space-time geometry. The elementary length is connected with the quantum constant $\hbar$ by the relation $\lambda_0^2 = \hbar/bc$, where c is the speed of the light and b is some universal constant. It may be interpreted in the sense, that wobbling of the tardions world line is connected with quantum effects. World line of a tachion (a particle having a spcelike world line) wobbles with infinite amplitude. This wobbling is not restricted by the quantum constant. A single tachyon cannot be detected, because of unrestricted wobbling of its world line. It is used to think that tachyons do not exist. In reality tachyons exist, but a single tachyon cannot be detected. However, the tachyon gas can be detected by its gravitational field. Tachyon gas is the best candidate for the dark matter [16].

In the space-time geometry of Berwald-Moor the wobbling of world lines differs from the quantum wobbling for tardions and from tachyon wobbling for tachyons. It means that the Berwald-Moor geometry can be used hardly as a real space-time geometry.

References

[1] H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, 1959.

[2] G.I. Garas'ko, Elements of Finsler Geometry for Physicists, Moscow, Tetru, 2009. (in Russian)

[3] K. Menger, Untersuchen über allgemeine Metrik, *Mathematische Annalen,* **100**, 75-113, (1928)

[4] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Oxford, Clarendon Press, 1953

[5] J.L.Synge, *Relativity: the General Theory.* Amsterdam, North-Holland Publishing Company, 1960.

[6] Yu.A.Rylov, Geometry without topology as a new conception of geometry. *Int. Jour. Mat. & Mat. Sci.* **30**, *iss. 12, 733-760, (2002)*

[7] Yu.A.Rylov, On a possibility of the Riemannian space description in terms of a finite interval. *Izvestiya Vysshikh Uchebnych Zavedenii, Matematika. No.3(28), 131-142. (1962*) (in Russian)

[8] Yu.A.Rylov, Relative gravitational field and conservation laws in general relativity. *Ann. Phys.(Leipzig)* **12**, *329-353, (1964)*

[9] Yu.A.Rylov, Different representations of Euclidean geometry and their application to the space-time geometry *e-print :0709.2755*

[10] Yu.A.Rylov, Non-Riemannian model of the space-time responsible for quantum effects. *Journ. Math. Phys.* **32(8)**, *2092-2098, (1991).*

[11] Yu. A.Rylov, Multivariance as immanent property of the space-time geometry (submitted to *IJTP,* now can be found: http://rsfq1.physics.sunysb.edu/~rylov/mimst1e.pdf)

[12] J.L.Synge, A characteristic function in Riemannian space and its applications to the solution of geodesic triangles. *Proc. London Math Soc.* **32**,*241 (1932).*

[13] H.S.Ruse, Some theorem in tensor calculus. *Proc.London Math.Soc.* **31,***225 (1930).*

[14] D.G.Pavlov, Philosophical and mathematical reasons for Finsler extensions of Relativity Theory .140 -147 in *Space-Time Structure. Algebra and Geometry Moscow: Lilia-Print 2007,* editors D. G. Pavlov, Gh. Atanasiu, V. Balan

[15] G. I. Garas'ko, D.G.Pavlov, Construction of the Pseudo-Riemannian geometry on the Base of the Berwald-Moor Geometry. 376 - 385.in *Space-Time Structure. Algebra and Geometry Moscow: Lilia-Print 2007,* editors D. G. Pavlov, Gh. Atanasiu, V. Balan

[16] Yu.A. Rylov, Dynamic equations for tachyon gas. Submitted to *Int. J. Theor. Phys* (now can be found in http://rsfq1.physics.sunysb.edu/~rylov/detg2e.pdf)