Discrete space-time geometry and skeleton conception of particle dynamics

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Abstract

It is shown that properties of a discrete space-time geometry distinguish from properties of the Riemannian space-time geometry. The discrete geometry is a physical geometry, which is described completely by the world function. The discrete geometry is nonaxiomatizable and multivariant. The equivalence relation is intransitive in the discrete geometry. The particles are described by world chains (broken lines with finite length of links), because in the discrete space-time geometry there are no infinitesimal lengths. Motion of particles is stochastic, and statistical description of them leads to the Schrödinger equation, if the elementary length of the discrete geometry depends on the quantum constant in a proper way.

Key words: nonaxiomatizable geometry; discrete space-time geometry; geometrization of particle parameters; skeleton conception of particle dynamics; monistic conception

1 Introduction

Is the space-time geometry discrete, or is it continuous, but equipped by quantum properties? The quantum properties associate with some portions, quanta and discreteness. However, it seems intuitively, that the space-time geometry is rather discrete, than it has mystic quantum properties, because the concept of discreteness does not contain anything mysterious. Nevertheless if one consider the scales, which are much more, than the elementary length λ_0 (characteristic length of the discrete geometry), it is of no importance, whether the space-time geometry is discrete or continuous.

Contemporary space-time geometry is a differential geometry. All concepts of the differential geometry are based on the concept of continuity. Such concepts (manifold, dimension, coordinate system, differential equations of dynamics) can be introduced and used only in a continuous space-time geometry. Besides, a discrete geometry is nonaxiomatizable geometry. It means, that a discrete geometry cannot be obtained as a logical construction, because the equivalence relation is intransitive, in general, in a discrete geometry. In its turn the intransitivity of the equivalence relation is a corollary of the following circumstance. For determination of a vector $\mathbf{P}_0\mathbf{P}_1$ at the given point P_0, which is equivalent to a given vector $\mathbf{Q}_0\mathbf{Q}_1$ at the point Q_0, one needs to solve two algebraic equations (condition of vectors parallelism, condition of equality of their lengths). The number of coordinates (four) of the point P_1 is more than the number of equations. As a result there are many solutions, in general. There are many vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0\mathbf{P}_1'$,... at the given point P_0, which are equivalent to the vector $\mathbf{Q}_0\mathbf{Q}_1$ at the point Q_0, but vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0\mathbf{P}_1'$,... are not equivalent between themselves. Such a situation means intransitivity of the equivalence relation for vectors. Hence, a discrete geometry cannot be a logical construction, because in any logical construction the equivalence relation is transitive. It cannot be constructed by the Euclidean method, which is used for construction of the proper Euclidean geometry.

On the other hand, the Euclidean method of a geometry construction is used longer, than two thousands years, and we do not know any other method. The only Euclidean method of a geometry construction is learnt in all schools. As a result we perceive hardly the idea of alternative method of a geometry construction. We cannot imagine, how one can construct a discrete space-time geometry, if it is nonaxiomatizable and one cannot use the conventional Euclidean method. It seems to be simpler to consider continuous space-time geometry, equipping it by mystic quantum properties, which imitate discreteness. This imitation appeared to be very successful. Unfortunately, this imitation is not complete (as well as the imitation of thermal phenomena by means of the thermogen), and we are forced to return to idea of discrete space-time geometry, if we want to describe physical phenomena in microcosm.

A geometry is a science on geometrical objects, on their shape and on their disposition in the space or in the space-time. The proper Euclidean geometry is the first geometry, which has been constructed. Geometrical objects of the Euclidean geometry are usually constructed as combination of fundamental blocks (point, segment of straight, angle). Properties of these fundamental blocks and the rules of their combination at the construction of geometrical objects were formulated as axioms of some logical construction. The proper Euclidean geometry was studied in the form of a logical construction in the last two thousand years. As a result practically all scientists believe, that any geometry is a logical construction, and that any geometry is to be constructed as a logical construction. It is supposed, that any geometry can be deduced from a system of properly chosen axioms, and there are no geometries which cannot be deduced from a proper chosen axiomatics.

This fact is formulated as follows. Only axiomatizable geometries do exist and there are no nonaxiomatizable geometries. In other words, a geometry is identified with a logical construction. The belief in the identity of a geometry and a logical construction was so large, that one uses the term "geometry" with respect to disciplines, which have a logical structure of the Euclidean geometry, but have no relation to the science on geometrical objects and their shape (for instance, symplectic geometry). A geometry, identified with a logical construction (axiomatizable geometry) will be referred to as a mathematical geometry. The proper Euclidean is a mathematical geometry.

A geometry as a science on geometrical objects, on their shape and on their disposition is described completely by a distance function $\rho(P, Q)$ between any two points $P, Q \in \Omega$, where Ω is a point set, where the geometry is given. It is more convenient to use world function $\sigma = \frac{1}{2}\rho^2$ instead of the distance function ρ, because the world function σ is real in any geometry (even in the space-time geometry of Minkowski). The geometry, which is described completely by their world function will be referred to as a physical geometry, because physicists are not interested in the way of the geometry construction. They are not interested in the circumstance, whether or not the physical geometry is axiomatizable.

The proper Euclidean geometry as well as the geometry of Minkowski are continuous geometries. However, there may exist discrete geometries, where the distance between any two points of the space-time is larger, than some elementary length λ_0. If characteristic scale of the problem is much larger, than the elementary length λ_0, one may set $\lambda_0 = 0$ and consider a continuous geometry. However, in microcosm, where characteristic scale is of the order of λ_0, one should consider a discrete space-time geometry.

At the conventional construction of the Euclidean geometry one uses such concepts as manifold, dimension, coordinate system, linear vector space, which might be used only in continuous geometries. Constructing a discrete geometry as a generalization of the proper Euclidean geometry, one may not use these concepts. The only concept, which may be used in both continuous geometry and in the discrete one, is the distance ρ. But the distance ρ is to be introduced as a fundamental quantity. In the Riemannian geometry the distance ρ is introduced as an integral along the geodesic from the infinitesimal distance

$$ds = \sqrt{g_{ik}dx^i dx^k}$$

Such a method of introduction of the distance ρ is inadequate in the discrete geometry, because it uses infinitesimal distance, which does not exist in the discrete geometry. Besides, in the case, when there are several geodesics, connecting two points, one obtains many-valued expressions for the distance or for the world function. Many-valued world function is inadmissible in a geometry.

To construct a discrete geometry, one needs to represent the proper Euclidean geometry in terms of the distance ρ (or in terms of the world function $\sigma = \frac{1}{2}\rho^2$) and to use this representation for generalization of the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ on the case of a discrete geometry $\mathcal{G}_{\mathrm{d}}$. Representation of a geometry in terms of a world function will be referred to as σ-immanent representation. The σ-immanent representation of the proper Euclidean geometry $\mathcal{G}_{\rm E}$ is always possible.

The main geometrical objects and concepts of $\mathcal{G}_{\rm E}$: (1) straight segment $\mathcal{T}_{[PQ]}$ between the points P and Q, (2) scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{Q}_0\mathbf{Q}_1$, (3) linear dependence of n vectors $\mathbf{P}_0\mathbf{P}_1,\mathbf{P}_0\mathbf{P}_2,...\mathbf{P}_0\mathbf{P}_n$, (4) equivalence $(\mathbf{P}_0\mathbf{P}_1\text{eqv}\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{Q}_0\mathbf{Q}_1$ can be expressed in terms of the world function $\sigma_{\rm E}$ of the geometry $\mathcal{G}_{\rm E}$

$$\mathcal{T}_{[PQ]}=\left\{R|\rho\left(P,R\right)+\rho\left(R,Q\right)=\rho\left(P,Q\right)\right\},\qquad \rho\left(P,Q\right)=\sqrt{2\sigma\left(P,Q\right)} \tag{1.1}$$

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)=\sigma\left(P_0,Q_1\right)+\sigma\left(P_1,Q_0\right)-\sigma\left(P_0,Q_0\right)-\sigma\left(P_1,Q_1\right) \tag{1.2}$$

$$F_n\left(\mathcal{P}_n\right)\equiv\det||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)||=0,\qquad i,k=1,2,...n \tag{1.3}$$

where

$$\mathcal{P}_n=\left\{P_0,P_1,...P_n\right\} \tag{1.4}$$

and the scalar product of two vectors is defined by the formula (1.2), which in the case of common origin has the form

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_2)=\sigma\left(P_0,P_2\right)+\sigma\left(P_0,P_1\right)-\sigma\left(P_1,P_2\right) \tag{1.5}$$

Equivalence $(\mathbf{P}_0\mathbf{P}_1\text{eqv}\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors is described by two relations

$$(\mathbf{P}_0\mathbf{P}_1\text{eqv}\mathbf{Q}_0\mathbf{Q}_1):\quad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)=|\mathbf{P}_0\mathbf{P}_1|\cdot|\mathbf{Q}_0\mathbf{Q}_1|\wedge|\mathbf{P}_0\mathbf{P}_1|=|\mathbf{Q}_0\mathbf{Q}_1| \tag{1.6}$$

$$|\mathbf{P}_0\mathbf{P}_1|=\sqrt{2\sigma\left(P_0,P_1\right)} \tag{1.7}$$

where $\sigma=\sigma_{\rm E}$ is the world function of the proper Euclidean geometry $\mathcal{G}_{\rm E}$.

These relations are to use for definition of such quantities as $\mathcal{T}_{[PQ]}$, $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$, $(\mathbf{P}_0\mathbf{P}_1\text{eqv}\mathbf{Q}_0\mathbf{Q}_1)$ and linear dependence, because these definitions do not refer to dimension, coordinate system, linear vector space and other means of description of the proper Euclidean geometry. These definitions contain only such a fundamental geometrical quantity as the world function. These definitions are written in the coordinateless invariant form. As a result these definitions may and must be used as definitions of any physical geometry, i.e. a geometry described completely in terms of the world function. Any geometrical object can be described in terms of the world function. The geometrical object properties are calculated on the basis of its σ-immanent representation (representation in terms of the world function) and of the world function form.

Generalization of these expressions on the case of the discrete geometry $\mathcal{G}_{\rm d}$ is obtained by means of the replacement $\sigma_{\rm E}\rightarrow\sigma_{\rm d}$, where $\sigma_{\rm d}$ is the world function of the discrete geometry $\mathcal{G}_{\rm d}$. We are to be ready, that properties of concepts (1.1) - (1.6) in $\mathcal{G}_{\rm d}$ differ strongly from their properties in $\mathcal{G}_{\rm E}$. However, we have no alternative to relations (1.1) - (1.6) for definition of these geometrical quantities in a discrete geometry $\mathcal{G}_{\rm d}$.

Definition 1.1: The physical geometry $\mathcal{G}=\{\sigma,\Omega\}$ is a point set Ω with the single-valued function σ on it

$$\sigma: \qquad \Omega\times\Omega\rightarrow\mathbb{R}, \qquad \sigma(P,P)=0, \qquad \sigma(P,Q)=\sigma(Q,P), \qquad P,Q\in\Omega \tag{1.8}$$

Definition 1.2: Two physical geometries $\mathcal{G}_1=\{\sigma_1,\Omega_1\}$ and $\mathcal{G}_2=\{\sigma_2,\Omega_2\}$ are equivalent ($\mathcal{G}_1\mathrm{eqv}\mathcal{G}_2$), if the point set $\Omega_1\subseteq\Omega_2\wedge\sigma_1(P,Q)=\sigma_2(P,Q)$, $\forall P,Q\in\Omega_1$, or $\Omega_2\subseteq\Omega_1\wedge\sigma_2(P,Q)=\sigma_1(P,Q)$, $\forall P,Q\in\Omega_2$

Remark: Coincidence of point sets Ω_1 and Ω_2 is not necessary for equivalence of geometries $\mathcal{G}_1$ and $\mathcal{G}_2$. If one demands coincidence of Ω_1 and Ω_2 in the case equivalence of $\mathcal{G}_1$ and $\mathcal{G}_2$, then an elimination of one point P from the point set Ω_1 turns the geometry $\mathcal{G}_1=\{\sigma_1,\Omega_1\}$ into geometry $\mathcal{G}_2=\{\sigma_1,\Omega_1\backslash P\}$, which appears to be not equivalent to the geometry $\mathcal{G}_1$. Such a situation seems to be inadmissible, because a geometry on a part $\omega\subset\Omega_1$ of the point set Ω_1 appears to be not equivalent to the geometry on the whole point set Ω_1.

According to definition the geometries $\mathcal{G}_1=\{\sigma,\omega_1\}$ and $\mathcal{G}_2=\{\sigma,\omega_2\}$ on parts of Ω, $\omega_1\subset\Omega$ and $\omega_2\subset\Omega$ are equivalent $(\mathcal{G}_1\mathrm{eqv}\mathcal{G}),(\mathcal{G}_2\mathrm{eqv}\mathcal{G})$ to the geometry $\mathcal{G}$, whereas the geometries $\mathcal{G}_1=\{\sigma,\omega_1\}$ and $\mathcal{G}_2=\{\sigma,\omega_2\}$ are not equivalent, in general, if $\omega_1\nsubseteqq\omega_2$ and $\omega_2\nsubseteqq\omega_1$. Thus, the relation of equivalence is intransitive, in general. The space-time geometry may vary in different regions of the space-time. It means, that a physical body, described as a geometrical object, may evolve in such a way, that it appears in regions with different space-time geometry.

The space-time geometry of Minkowski as well as the Euclidean geometry are continuous geometries. It is true for usual scales of distances. However, one cannot be sure, that the space-time geometry is continuous in microcosm. The space-time geometry may appear to be discrete in microcosm. We consider a discrete space-time geometry and discuss the corollaries of the suggested discreteness.

The distance function ρ_{d} of a discrete geometry $\mathcal{G}_{\mathrm{d}}$ satisfies the condition

$$|\rho_{\mathrm{d}}(P,Q)|\notin(0,\lambda_0), \qquad \forall P,Q\in\Omega \tag{1.9}$$

which means that in the geometry $\mathcal{G}_{\mathrm{d}}$ there are no distances, which are shorter, than the elementary length λ_0. The distance $\rho_{\mathrm{d}}(P,Q)=0$ is admissible. This condition takes place, if $P=Q$.

Note, that the condition (1.9) is a restriction on the values of the distance function, but not on values of its argument (points of Ω), although one considers usually a discrete geometry as a geometry on a lattice. It is true, that the geometries on a lattice are discrete geometries, but they form a very special case of the discrete geometries. Besides, such a discrete geometry cannot be uniform and isotropic. A general case of a discrete geometry takes place, when restrictions are imposed on the admissible values of the world function (distance function).

The simplest case of a discrete space-time geometry $\mathcal{G}_{\mathrm{d}}$ is obtained, if $\mathcal{G}_{\mathrm{d}}=\{\Omega_{\mathrm{M}},\sigma_{\mathrm{d}}\}$ is given on the manifold Ω_{M}, where the geometry of Minkowski $\mathcal{G}_{\mathrm{M}}=\{\Omega_{\mathrm{M}},\sigma_{\mathrm{M}}\}$ is given. The world function σ_{d} is chosen in the form

$$\sigma_{\mathrm{d}}(P,Q)=\sigma_{\mathrm{M}}(P,Q)+\frac{1}{2}\lambda_0^2\mathrm{sgn}(\sigma_{\mathrm{M}}(P,Q)), \qquad \forall P,Q\in\Omega_{\mathrm{M}} \tag{1.10}$$

where σ_{M} is the world function of the geometry of Minkowski. It is easy to verify, that $\rho_{\mathrm{d}} = \sqrt{2\sigma_{\mathrm{d}}}$, defined by (1.10) satisfies the constraint (1.9). Such a discrete geometry is uniform and isotropic as well as the geometry of Minkowski

Besides, in the discrete space-time geometry (1.10) a pointlike particle cannot be described by a world line, because any world line is a limit of the broken line, when lengths of its links tend to zero. But in the discrete geometry $\mathcal{G}_{\mathrm{d}}$ there are no infinitesimal lengths, and a pointlike particle is described by a world chain (broken line) instead of continuous world line. Description of a pointlike particle state by means of the particle position and its momentum becomes inadequate, because in the continuous space-time geometry the particle 4-momentum p_k is described by the relation

$$p_k = g_{kl}\frac{dx^l}{d\tau} = g_{kl}\lim_{d\tau\rightarrow 0}\frac{x^l(\tau + d\tau) - x^l(\tau)}{d\tau} \tag{1.11}$$

where $x^l = x^l(\tau)$, $l = 0, 1, 2, 3$ is an equation of the world line. The limit in the formula (1.11) does not exist in $\mathcal{G}_{\mathrm{d}}$, and the 4-momentum p_k is not defined (at any rate in such a form). In general, the mathematical formalism, based on the infinitesimal calculus (differential dynamic equations), is inadequate in the discrete space-time geometry.

The world chain $\mathcal{C}$ of a particle has the form

$$\mathcal{C}: \qquad \bigcup_s \mathbf{P}_s\mathbf{P}_{s+1}, \tag{1.12}$$

where vector $\mathbf{P}_s\mathbf{P}_{s+1} = \{P_s, P_{s+1}\}$ is an ordered set of two points P_s, P_{s+1}. The lengths $|\mathbf{P}_s\mathbf{P}_{s+1}|$ of all vectors are equal

$$|\mathbf{P}_s\mathbf{P}_{s+1}| = \sqrt{2\sigma_{\mathrm{d}}(P_s, P_{s+1})} = \mu, \qquad s = ...0, 1, ... \tag{1.13}$$

A new parameter μ appears in the description of a pointlike particle. This parameter describes the length of links of the world chain $\mathcal{C}$. This new parameter (geometrical mass) may be connected with the particle mass m by means of the relation

$$m = b\mu \tag{1.14}$$

where b is some universal constant.

The world chain $\mathcal{C}$ describes the free motion of the pointlike particle, if in addition to the relation (1.13) the adjacent links of the chain are parallel

$$\mathbf{P}_s\mathbf{P}_{s+1} \upuparrows \mathbf{P}_{s+1}\mathbf{P}_{s+2}: \qquad (\mathbf{P}_s\mathbf{P}_{s+1}.\mathbf{P}_{s+1}\mathbf{P}_{s+2}) = |\mathbf{P}_s\mathbf{P}_{s+1}|\cdot|\mathbf{P}_{s+1}\mathbf{P}_{s+2}| = \mu^2 \tag{1.15}$$

where $(\mathbf{P}_s\mathbf{P}_{s+1}.\mathbf{P}_{s+1}\mathbf{P}_{s+2})$ is the σ-immanent expression for the scalar product of two vectors $\mathbf{P}_s\mathbf{P}_{s+1}$ and $\mathbf{P}_{s+1}\mathbf{P}_{s+2}$.

Relations (1.13) and (1.15) means that adjacent vectors $\mathbf{P}_s\mathbf{P}_{s+1}$ and $\mathbf{P}_{s+1}\mathbf{P}_{s+2}$ are equivalent. According to (1.13) the equation (1.15) may written in the form

$$(\mathbf{P}_s\mathbf{P}_{s+1}.\mathbf{P}_{s+1}\mathbf{P}_{s+2}) = |\mathbf{P}_s\mathbf{P}_{s+1}|^2 = \mu^2, \qquad s = ...0, 1, ... \tag{1.16}$$

The scalar product of vectors in the discrete geometry $\mathcal{G}_\mathrm{d}$ has the same form (1.2), but the Euclidean world function σ_E is replaced by the world function σ_d of the discrete geometry (1.10). In the limit $\lambda_0 \to 0$ the world function $\sigma_\mathrm{d} \to \sigma_\mathrm{M}$ and the scalar product in $\mathcal{G}_\mathrm{d}$ turns to the scalar product in the geometry of Minkowski.

Fixing points P_s and P_{s+1} and solving equations (1.13), (1.16) to determine the point P_{s+2}, one obtains many solutions for the point P_{s+2}. This circumstance is explained by the fact, that one has only two equations for determination of four coordinates of the point P_{s+2}. It means that the links of the world chain (1.12) wobble, and the shape of the world chain appears to be random (stochastic).

Statistical description of random world chains leads to the Schrödinger equation [1], provided the elementary length λ_0 has the form

$$\lambda_0^2 = \frac{\hbar}{bc} \tag{1.17}$$

where $\hbar$ is the quantum constant, c is the speed of the light, and b is the universal constant, defined by the relation (1.14). The Schrödinger equation describes a motion of free stochastic (quantum) particle, and this description contains the particle mass m, although the classical description of a free particle motion does not contain a reference to the particle mass.

Explanation of quantum effects (the Schrödinger equation) as geometrical effects of the discrete space-time geometry is very important, because it shows, that introduction of quantum principles is overabundant. This result suggests to resolve between two paradigms: either discrete space-time geometry, or continuous space-time geometry equipped by quantum principles. It is evident, that discrete space-time geometry is the simplest and more natural solution, than an invention of quantum principles, whose meaning is unclear. Note, that replacing in quantum mechanics the classical momentum by an operator or be a matrix, one imitates a multivariance (indeterminacy) of the particle momentum, described by the relation (1.11) in $\mathcal{G}_\mathrm{d}$.

Besides, the discrete space-time geometry admits one to describe both the indeterministic motion of a single particle and the deterministic motion of an statistically averaged particle, whereas the quantum paradigm admits one to describe only the statistically averaged particle motion. In application to the theory of elementary particles the statistically averaged particle motion does not give a possibility to penetrate in the elementary particle structure, whereas the single particle motion (even indeterministic) gives a hope to understand the elementary particles structure.

Formally it means that the conception of particle dynamics in the discrete space-time geometry distinguishes from the particle dynamics conception in the space-time geometry of Minkowski, because in the space-time geometry of Minkowski the formalism of the particle dynamics uses essentially the continuity of the space-time geometry. This formalism cannot be used in the discrete space-time geometry.

2 Description of geometrical objects in physical geometry

In a physical geometry there is the only basic quantity (world function), which describes completely all geometric objects of the geometry. All geometrical objects are described uniformly in all physical geometries. Such geometrical objects as straight segment and angle are fundamental objects in conventional presentation of the proper Euclidean geometry. Their properties are formulated as axioms and they are used at the construction of more complicate geometric objects. In the physical geometry the straight segment and the angle are derivative objects, which are constructed in terms of the world function.

In the σ-immanent presentation of the proper Euclidean geometry the straight segment $\mathcal{T}_{[PQ]}$ between two points P and Q is described by the relation (1.1). The angle φ between two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is described by the formula

$$\cos\varphi = \frac{(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)}{|\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1|} = \frac{\sigma(P_0,Q_1)+\sigma(P_1,Q_0)-\sigma(P_0,Q_0)-\sigma(P_1,Q_1)}{\sqrt{4\sigma(P_0P_1)\,\sigma(Q_0,Q_1)}} \tag{2.1}$$

where the scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ is defined by the formula (1.2).

If we try to construct the discrete geometry $\mathcal{G}_{\mathrm{d}}$, defined by the relation (1.9), starting from the axioms, which describe properties of the segment and of the angle, we could not determine, what are properties of the segment $\mathcal{T}_{[PQ]}$ in arbitrary geometry (for instance, in a discrete geometry). Starting from the axioms of the proper Euclidean geometry, it is very difficult to imagine, that the straight segment can be a tube (surface), but not a one-dimensional line. Starting from these axioms, one cannot obtain the formula (1.1) for the straight segment in the discrete space-time geometry. As a result the version of the discrete space-time geometry was not developed. One developed the continuous space-time geometry, equipped by quantum principles instead of the discrete space-time geometry. This approach was founded on the supposition, that all geometrical objects can be constructed as combination of fundamental blocks (point, segment, angle). This supposition means, that a geometry is a logical construction (axiomatizable geometry). The axiomatizability condition may be too restrictive for real space-time geometry. At any rate, it was not reasonable to impose this constraint from the beginning.

Almost all basic concepts of differential geometry (manifold, dimension, coordinate system, linear dependence of vectors) are used at the supposition, that the geometry is continuous. Only concept of distance function is an exception, which may be defined in a discrete geometry. Using the distance function as an only basic quantity of a geometry, one may hope to describe both continuous and discrete geometries. The idea to describe a geometry in terms of a distance function is an old idea. The following conditions were imposed on the distance function ϱ

$$\rho(P,Q) \geq 0, \qquad \forall P,Q \in \Omega \tag{2.2}$$

$$\rho(P,Q) = 0, \qquad \text{iff } P = Q \tag{2.3}$$

$$\rho(P,Q)+\rho(Q,R)\geq\rho(P,R), \qquad \forall P,Q,R\in\Omega \tag{2.4}$$

Such a geometry is known as the metric geometry. The condition (2.4) admits only such geometries, where the straight segment (1.1) is a one-dimensional point set. The condition (2.2) forbids an application of the metric geometry to the space-time.

There were attempts to introduce so called distant geometry [2, 3], which is free of restriction (2.4). Note, that the triangle axiom (2.4) means, that the straight is an one-dimensional set of points. However, Blumental [3] could not overcome the concept of the Euclidean geometry, that the straight line has no thickness, although a formal obstruction in the form of the triangle axiom (2.4) has been overcome. From the logical viewpoint the fact, that in the Euclidean geometry a straight is an one-dimensional point set, does not mean, that the straight is a one-dimensional point set in any geometry. Nevertheless, Blumental introduced a curve as a continuous mapping of interval $(0,1)$ onto the space Ω. He has introduced an additional fundamental concept (continuous mapping), which cannot be formulated in terms of a distance. As a result the distant geometry lost its monistic character, when the geometry is described completely in terms of a distance.

The physical geometry distinguishes from the conventional presentation of the Euclidean geometry in the relation, that it is a monistic conception, which is described completely in terms of the world function (or distance). There is no necessity to reconcile different fundamental concepts of the geometry, because there is only one fundamental quantity.

A geometrical object is a geometrical image of a physical body. Any geometrical object is some subset of points in the space-time. However, geometrical object is not an arbitrary set of points. Geometrical object is to be defined in the physical geometry in such a way, that similar geometrical objects (which are images of corresponding physical bodies) could be recognized in different space-time geometries.

Definition 2.1: A geometrical object $g_{\mathcal{P}_n,\sigma}$ of the geometry $\mathcal{G}=\{\sigma,\Omega\}$ is a subset $g_{\mathcal{P}_n,\sigma}\subset\Omega$ of the point set Ω. This geometrical object $g_{\mathcal{P}_n,\sigma}$ is a set of roots $R\in\Omega$ of the function $F_{\mathcal{P}_n,\sigma}$

$$F_{\mathcal{P}_n,\sigma}: \qquad \Omega\rightarrow\mathbb{R} \tag{2.5}$$

where

$$F_{\mathcal{P}_n,\sigma} \quad : \quad F_{\mathcal{P}_n,\sigma}(R)=G_{\mathcal{P}_n,\sigma}(u_1,u_2,...u_s), \qquad s=\frac{1}{2}(n+1)(n+2) \tag{2.6}$$

$$u_l = \sigma(w_i,w_k), \qquad i,k=0,1,...n+1, \qquad l=1,2,...\frac{1}{2}(n+1)(n+2) \tag{2.7}$$

$$w_k = P_k\in\Omega, \qquad k=0,1,...n, \qquad w_{n+1}=R\in\Omega \tag{2.8}$$

Here $\mathcal{P}_n=\{P_0,P_1,...,P_n\}\subset\Omega$ are $n+1$ points which are parameters, determining the geometrical object $g_{\mathcal{P}_n,\sigma}$

$$g_{\mathcal{P}_n,\sigma}=\{R|F_{\mathcal{P}_n,\sigma}(R)=0\}, \qquad R\in\Omega, \qquad \mathcal{P}_n\in\Omega^{n+1} \tag{2.9}$$

$F_{\mathcal{P}_n,\sigma}(R)=G_{\mathcal{P}_n,\sigma}(u_1,u_2,...u_s)$ is an arbitrary function of $\frac{1}{2}(n+1)(n+2)$ arguments u_s and of $n+1$ parameters $\mathcal{P}_n$. The set $\mathcal{P}_n$ of the geometric object parameters will be referred to as the skeleton of the geometrical object. The subset

$g_{\mathcal{P}_n,\sigma}$ will be referred to as the envelope of the skeleton. The skeleton is a analog of a frame of reference, attached rigidly to a physical body. Tracing the skeleton motion, one can trace the motion of the physical body. When a particle is considered as a geometrical object, its motion in the space-time is described by the skeleton $\mathcal{P}_n$ motion. At such an approach (the rigid body approximation) the shape of the envelope is of no importance.

Remark: An arbitrary subset Ω' of the point set Ω is not a geometrical object, in general. It is supposed, that physical bodies may have only a shape of a geometrical object, because only in this case one can identify identical physical bodies (geometrical objects) in different space-time geometries.

Existence of the same geometrical objects in different space-time regions, having different geometries, arises the question on equivalence of geometrical objects in different space-time geometries. Such a question was not arisen before, because one does not consider such a situation, when a physical body moves from one space-time region to another space-time region, having another space-time geometry. In general, mathematical technique of the conventional space-time geometry (differential geometry) is not applicable for simultaneous consideration of several different geometries of different space-time regions.

We can perceive the space-time geometry only via motion of physical bodies in the space-time, or via construction of geometrical objects corresponding to these physical bodies. As it follows from the *definition 2.1* of the geometrical object, the function F as a function of its arguments (of world functions of different points) is the same in all physical geometries. It means, that a geometrical object $\mathcal{O}_1$ in the geometry $\mathcal{G}_1 = \{\sigma_1, \Omega_1\}$ is obtained from the same geometrical object $\mathcal{O}_2$ in the geometry $\mathcal{G}_2 = \{\sigma_2, \Omega_2\}$ by means of the replacement $\sigma_2 \to \sigma_1$ in the definition of this geometrical object.

Definition 2.2: Geometrical object $g_{P'_n,\sigma'}$ ($\mathcal{P}'_n = \{P'_0, P'_1, ..P'_n\}$) in the geometry $\mathcal{G}' = \{\sigma', \Omega'\}$ and the geometrical object $g_{P_n,\sigma}$ ($\mathcal{P}_n = \{P_0, P_1, ..P_n\}$) in the geometry $\mathcal{G} = \{\sigma, \Omega\}$ are equivalent (equal), if

$$\sigma'\left(P'_i, P'_k\right) = \sigma\left(P_i, P_k\right), \qquad i,k = 0,1,..n \tag{2.10}$$

and the functions $G'_{\mathcal{P}'_n,\sigma'}$ for $g_{P'_n,\sigma'}$ and $G_{\mathcal{P}_n,\sigma}$ for $g_{P_n,\sigma}$ in the formula (2.6) coincide

$$G'_{\mathcal{P}'_n,\sigma'}\left(u_1, u_2, ...u_s\right) = G_{\mathcal{P}_n,\sigma}\left(u_1, u_2, ...u_s\right) \tag{2.11}$$

In this case

$$u_l \equiv \sigma\left(P_i, P_k\right) = u'_l \equiv \sigma'\left(P'_i, P'_k\right), \qquad i,k = 0,1,...n, \qquad l = 1,2,..n\left(n+1\right)/2 \tag{2.12}$$

As far as the physical geometry is determined by its geometrical objects construction, a physical geometry $\mathcal{G} = \{\sigma, \Omega\}$ can be obtained from some known standard geometry $\mathcal{G}_{\rm st} = \{\sigma_{\rm st}, \Omega\}$ by means of a deformation of the standard geometry $\mathcal{G}_{\rm st}$. Deformation of the standard geometry $\mathcal{G}_{\rm st}$ is realized by the replacement $\sigma_{\rm st} \to \sigma$ in all definitions of the geometrical objects in the standard geometry. The proper

Euclidean geometry is an axiomatizable geometry. It has been constructed by means of the Euclidean method as a logical construction. The proper Euclidean geometry is a physical geometry. It may be used as a standard geometry $\mathcal{G}_{\rm st}$. Construction of a physical geometry as a deformation of the proper Euclidean geometry will be referred to as the deformation principle [4]. The most physical geometries are nonaxiomatizable geometries. They can be constructed only by means of the deformation principle.

Description of the elementary particle motion in the space-time contains only the particle skeleton $\mathcal{P}_n = \{P_0, P_1, ...P_n\}$. The form of the function (2.6) is of no importance at the approach, when the particle is considered as a rigid body. In the elementary particle dynamics only the lengths $|\mathbf{P}_i\mathbf{P}_k| = \sqrt{2\sigma(P_i, P_k)}$ of vectors $\mathbf{P}_i\mathbf{P}_k$, $i, k = 0, 1, ...n$ are essential. These vectors are defined by the particle skeleton $\mathcal{P}_n$.

The equivalence $(\mathbf{P}_0\mathbf{P}_1\text{eqv}\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is defined by the relations

$$(\mathbf{P}_0\mathbf{P}_1\text{eqv}\mathbf{Q}_0\mathbf{Q}_1): \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = |\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1| \wedge |\mathbf{P}_0\mathbf{P}_1| = |\mathbf{Q}_0\mathbf{Q}_1| \tag{2.13}$$

where

$$|\mathbf{P}_0\mathbf{P}_1| = \sqrt{2\sigma(P_0, P_1)} \tag{2.14}$$

and the scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ is defined by the relation (1.2)

Skeletons $\mathcal{P}_n = \{P_0, P_1, ...P_n\}$ and $\mathcal{P}'_n = \{P'_0, P'_1, ...P'_n\}$ may belong to the same geometrical object, if

$$|\mathbf{P}_i\mathbf{P}_k| = |\mathbf{P}'_i\mathbf{P}'_k|, \qquad i, k = 0, 1, ...n \tag{2.15}$$

i.e. lengths of all vectors $\mathbf{P}_i\mathbf{P}_k$ and corresponding vectors $\mathbf{P}'_i\mathbf{P}'_k$ are equal. However, it is not sufficient for equivalence of skeletons $\mathcal{P}_n$ and $\mathcal{P}'_n$, because the mutual orientation of two skeletons (corresponding systems of reference) is important for their equivalence.

Skeletons $\mathcal{P}_n = \{P_0, P_1, ...P_n\}$ and $\mathcal{P}'_n = \{P'_0, P'_1, ...P'_n\}$ are equivalent, if

$$(\mathcal{P}_n\text{eqv}\mathcal{P}'_n): \qquad \text{if} \quad \mathbf{P}_i\mathbf{P}_k\text{eqv}\mathbf{P}'_i\mathbf{P}'_k, \qquad i, k = 0, 1, , ...n \tag{2.16}$$

In other words, the equivalence of skeletons needs equality of the lengths of vectors $\mathbf{P}_i\mathbf{P}_k$ and $\mathbf{P}'_i\mathbf{P}'_k$ and equality of their mutual orientations. For identification of two geometrical objects one needs only equality of the lengths of vectors $\mathbf{P}_i\mathbf{P}_k$ and $\mathbf{P}'_i\mathbf{P}'_k$

When we start from the world function as an unique fundamental quantity of the geometry, such geometrical objects as straight segment and the angle appear as some processing of the fundamental quantity (world function). In general, one may use another ways of processing of the world function. However, we use straight segments and skeletons constructed of them, because we know, how skeletons (systems of reference) are connected with the particle dynamics. At the free motion of physical body, the skeleton associated with this body realizes a translational motion. Rotational motion of a rigid body is not a free motion, because internal forces appear inside the physical body, and some parts of the body move with an acceleration.

3 Fluidity of boundary between geometry and dynamics

Motion of rigid bodies is described by dynamic equations, written in a space-time geometry. The boundary between the geometry and dynamics is flexible. One may choose a simple space-time geometry and a complicated dynamics. On the contrary, one may choose a simple dynamics and a complicated space-time geometry.

For instance, a motion of a charged particle in a given electromagnetic field A_l is described in the uniform space-time geometry of Minkowski as a motion under the action of the force field. Corresponding Hamilton-Jacobi equation has the form

$$\left(\frac{\partial S}{\partial x^i}-\frac{e}{c}A_i\right)g^{ik}\left(\frac{\partial S}{\partial x^k}-\frac{e}{c}A_k\right)=m^2c^2,\quad S=S\left(x\right),\quad g^{ik}=\operatorname{diag}\left\{c^{-2},-1,-1,-1\right\}$$

where m is the particle mass and e is its charge. Such a description contains two basic essences: space-time geometry and the electromagnetic field.

The same motion of the charged particle may be described as a free motion of the particle in the five-dimensional space-time of Kaluza- Klein. In this case the electromagnetic field is included in the space-time geometry. Corresponding Hamilton-Jacobi equation has the form

$$\frac{\partial S}{\partial X^A}\gamma^{AB}\frac{\partial S}{\partial X^B}=m_5^2c^2,\quad S=S\left(X\right),\quad X=\left\{x^0,x^1,x^2,x^3,x^5\right\},\quad \frac{\partial S}{\partial x^5}=\frac{e}{ac}=\text{const}$$

$$\gamma^{AB}=\left\Vert \begin{array}{cc} g^{ik} & -g^{il}a_l \\ -g^{kl}a_l & -1+g^{ls}a_la_s \end{array}\right\Vert ,\qquad a_l=aA_l,\qquad m_5^2=m^2-\left(\frac{e}{ac}\right)^2$$

where a is some constant.

Such a description of the space-time geometry "absorbs" dynamics. Dynamics becomes simpler, but geometry becomes more complicated. Geometry ceases to be uniform, because it contains information on the electromagnetic field.

If the space-time geometry is described by the only fundamental quantity (world function), and the particle motion is supposed to be free, such "absorption" of dynamics by the space-time geometry leads to a simplification of the particle dynamics. As a result a monistic description of the particle motion in terms of the world function arises.

In the microcosm, where several different force fields may exist, such a reduction of dynamics to the space-time geometry may strongly simplify the particle motion description. If there are several force fields, their reconciliation is very complicated, because the number of reconciling variants increases rapidly with the number of force fields. In the case, when all fields are absorbed by the space-time geometry, one has a monistic conception, which does not need any reconciling. Of course, the world function becomes more complicated, however it is only one invariant quantity, which is a function of two space-time points. It is an attractive circumstance, that the particle dynamics may be formulated in coordinateless form and in invariant terms.

As we shall see, a skeleton conception of particle dynamics is formulated very simply in the form of difference (not differential) equations. It is quite natural, because the space-time geometry may appear to be discrete. In the discrete space-time geometry, where there are no infinitesimal distances, the differential equation may not be applicable.

However, in the case, when the characteristic scale is much more, than the elementary length, one may use differential equations instead of difference ones.

4 Discreteness and its manifestations

The simplest discrete space-time geometry $\mathcal{G}_{\rm d}$ is described by the world function (1.10). Density of points in $\mathcal{G}_{\rm d}$ with respect to point density in $\mathcal{G}_{\rm M}$ is described by the relation

$$\frac{d\sigma_{\rm M}}{d\sigma_{\rm d}}=\left\{\begin{array}{ccc} 0 & \text{if} & |\sigma_{\rm d}|<\frac{1}{2}\lambda_0^2 \\ 1 & \text{if} & |\sigma_{\rm d}|>\frac{1}{2}\lambda_0^2 \end{array}\right. \tag{4.1}$$

If the world function has the form

$$\sigma_{\rm g}=\sigma_{\rm M}+\frac{\lambda_0^2}{2}\left\{\begin{array}{ccc} \operatorname{sgn}(\sigma_{\rm M}) & \text{if} & |\sigma_{\rm M}|>\sigma_0 \\ \frac{\sigma_{\rm M}}{\sigma_0} & \text{if} & |\sigma_{\rm M}|\leq\sigma_0 \end{array}\right. \tag{4.2}$$

where σ_0 =const, $\sigma_0\geq 0$, the relative density of points has the form

$$\frac{d\sigma_{\rm M}}{d\sigma_{\rm g}}=\left\{\begin{array}{ccc} \frac{2\sigma_0}{2\sigma_0+\lambda_0^2} & \text{if} & |\sigma_{\rm g}|<\sigma_0+\frac{1}{2}\lambda_0^2 \\ 1 & \text{if} & |\sigma_{\rm g}|>\sigma_0+\frac{1}{2}\lambda_0^2 \end{array}\right. \tag{4.3}$$

If the parameter $\sigma_0\rightarrow 0$, the world function $\sigma_{\rm g}\rightarrow\sigma_{\rm d}$ and the point density (4.3) tends to the point density (4.1). The space-time geometry $\mathcal{G}_{\rm g}$, described by the world function $\sigma_{\rm g}$ is a geometry, which is a partly discrete geometry, because it is intermediate between the discrete geometry $\mathcal{G}_{\rm d}$ and the continuous geometry $\mathcal{G}_{\rm M}$. We shall refer to the geometry $\mathcal{G}_{\rm g}$ as a granular geometry.

Deflection of the discrete space-time geometry from the continuous geometry of Minkowski generates special properties of the geometry, which are corollaries of impossibility of the linear vector space introduction.

Let $\mathbf{Q}_0\mathbf{Q}_1$ be a timelike vector in $\mathcal{G}_{\rm d}$ $(\sigma_{\rm d}(Q_0,Q_1)>0)$. We try to determine a vector $\mathbf{P}_0\mathbf{P}_1$ at the point P_0, which is equivalent to vector $\mathbf{Q}_0\mathbf{Q}_1$. Let for simplicity coordinates has the form

$$P_0=Q_0=\{0,0,0,0\}, \qquad Q_1=\{1,0,0,0\}, \qquad P_1=\left\{x^0,\mathbf{x}\right\}=\left\{x^0,x^1,x^2,x^3\right\} \tag{4.4}$$

In this coordinate system the world function of geometry Minkowski has the form

$$\sigma_{\rm M}(x,x')=\frac{1}{2}\left(\left(x^0-x^{0\prime}\right)^2-(\mathbf{x}-\mathbf{x}')^2\right) \tag{4.5}$$

and $\sigma_{\rm d}$ is determined by the relation (1.10). We are to determine coordinates x of the point P_1 from two equations (2.13), which have the form

$$\frac{1}{2}\left(\left(\left(x^0\right)^2-\mathbf{x}^2\right)+\frac{\lambda_0^2}{2}\right)=\frac{1}{2}\left(1+\frac{\lambda_0^2}{2}\right) \tag{4.6}$$

$$\frac{1}{2}\left(1+\frac{\lambda_0^2}{2}\right)+\frac{1}{2}\left(\left(\left(x^0\right)^2-\mathbf{x}^2\right)+\frac{\lambda_0^2}{2}\right)-\frac{1}{2}\left(\left(\left(x^0-1\right)^2-\mathbf{x}^2\right)+\frac{\lambda_0^2}{2}\right)=\left(1+\frac{\lambda_0^2}{2}\right) \tag{4.7}$$

After simplification one obtains

$$x^0=1+\frac{\lambda_0^2}{4},\qquad \mathbf{x}^2=\left(1+\frac{\lambda_0^2}{4}\right)^2-1 \tag{4.8}$$

If $\lambda_0=0$, then the discrete geometry turns to the geometry of Minkowski, and $\mathbf{x}^2=0$. Three equations

$$x^1=0,\quad x^2=0,\quad x^3=0 \tag{4.9}$$

follow from one equation $\mathbf{x}^2=0$. It means, that the geometry of Minkowski is a degenerate geometry, because different solutions of the discrete geometry merge into one solution of the geometry of Minkowski.

One obtains for coordinates of point P_1:

$$P_1=\left\{\sqrt{r+1},r\sin\theta,r\sin\theta\sin\varphi,r\sin\theta\cos\varphi\right\},\qquad r=\frac{\lambda_0}{\sqrt{2}}\sqrt{1+\frac{\lambda_0^2}{4}} \tag{4.10}$$

where the angles θ, φ are arbitrary. Such a situation is formulated as multivariance of the equivalency relation. All solutions lie on the surface of two-dimensional sphere of radius $r\simeq\lambda_0/\sqrt{2}$ ($\lambda_0^2\ll 1$).

Let now $\mathbf{Q}_0\mathbf{Q}_1$ be a spacelike vector in $\mathcal{G}_{\mathrm{d}}$ ($\sigma_{\mathrm{d}}(Q_0,Q_1)<0$). We try to determine a vector $\mathbf{P}_0\mathbf{P}_1$ at the point P_0, which is equivalent to vector $\mathbf{Q}_0\mathbf{Q}_1$. Let

$$P_0=Q_0=\{0,0,0,0\},\qquad Q_1=\{0,1,0,0\},\qquad P_1=\left\{x^0,\mathbf{x}\right\}=\left\{x^0,x^1,x^2,x^3\right\} \tag{4.11}$$

Two equations (2.13) have the form

$$\frac{1}{2}\left(\left(x^0\right)^2-\mathbf{x}^2\right)-\frac{\lambda_0^2}{2}=-\frac{1}{2}-\frac{\lambda_0^2}{2} \tag{4.12}$$

$$\begin{aligned}&\left(-\frac{1}{2}-\frac{\lambda_0^2}{2}\right)+\frac{1}{2}\left(\left(x^0\right)^2-\mathbf{x}^2\right)-\frac{\lambda_0^2}{2}-\frac{1}{2}\left(\left(x^0\right)^2-\left(x^1-1\right)^2-\left(x^2\right)^2-\left(x^3\right)^2\right)\\ =\ &-\frac{\lambda_0^2}{2}-\left(1+\lambda_0^2\right)\end{aligned} \tag{4.13}$$

After simplification one obtains

$$x^1=1+\frac{\lambda_0^2}{2},\qquad \left(x^0\right)^2-\mathbf{x}^2=-1 \tag{4.14}$$

The solution for the point P_1 has the form

$$P_1=\left\{\sqrt{a_2^2+a_3^2+\frac{\lambda_0^2}{2}\left(2+\frac{\lambda_0^2}{2}\right)},1+\frac{\lambda_0^2}{2},a_2,a_3\right\} \tag{4.15}$$

where a_2 and a_3 are arbitrary real numbers. In the case of spacelike vectors $\mathbf{Q}_0\mathbf{Q}_1$, $\mathbf{P}_0\mathbf{P}_1$ the solution is not unique, even if $\lambda_0 = 0$, and geometry is the geometry of Minkowski. All solutions lie on the surface of two-dimensional sphere of arbitrary radius $\sqrt{a_2^2 + a_3^2}$.

Thus, both the discrete geometry and the geometry of Minkowski are multivariant with respect to spacelike vectors. However, this circumstance remains to be unnoticed in the conventional relativistic particle dynamics, because the spacelike vectors do not used there.

Multivariance of the discrete geometry leads to intransitivity of the equivalence relation of two vectors. Indeed, if $(\mathbf{Q}_0\mathbf{Q}_1\text{eqv}\mathbf{P}_0\mathbf{P}_1)$ and $(\mathbf{Q}_0\mathbf{Q}_1\text{eqv}\mathbf{P}_0\mathbf{P}_1')$, but vector $(\mathbf{P}_0\mathbf{P}_1\overline{\text{eqv}}\mathbf{P}_0\mathbf{P}_1')$, it means intransitivity of the equivalence relation. Besides, it means that the discrete geometry is nonaxiomatizable, because in any logical construction the equivalence relation is transitive.

Transport of a vector $\mathbf{P}_0\mathbf{P}_1$ to some point Q_0 leads to some indeterminacy of the result of this transport, because at the point Q_0 there are many vectors $\mathbf{Q}_0\mathbf{Q}_1$, $\mathbf{Q}_0\mathbf{Q}_1'$,..., which are equivalent to the vector $\mathbf{P}_0\mathbf{P}_1$.

Sum $\mathbf{Q}_0\mathbf{S}$ of two vectors $\mathbf{Q}_0\mathbf{Q}_1$ and $\mathbf{Q}_1\mathbf{S}$, when the end of one vector is an origin of the other, is defined by points Q_0 and S

$$\mathbf{Q}_0\mathbf{S} = \mathbf{Q}_0\mathbf{Q}_1 + \mathbf{Q}_1\mathbf{S} \tag{4.16}$$

Sum $\mathbf{Q}_0\mathbf{S}$ of two vectors $\mathbf{Q}_0\mathbf{Q}_1$ and $\mathbf{P}_0\mathbf{P}_1$ at the point Q_0 is defined by the relations

$$\mathbf{Q}_0\mathbf{S} = \mathbf{Q}_0\mathbf{Q}_1 + \mathbf{Q}_1\mathbf{S}, \qquad (\mathbf{Q}_1\mathbf{S}\text{eqv}\mathbf{P}_0\mathbf{P}_1) \tag{4.17}$$

In the discrete geometry the sum of two vectors is not unique, in general.

Result of multiplication of a vector $\mathbf{P}_0\mathbf{P}_1$ by a real number a is not unique also. The result $\mathbf{P}_0\mathbf{S}$ of such a multiplication by a number a is defined by relations

$$\mathbf{P}_0\mathbf{P}_1 \upuparrows \mathbf{P}_0\mathbf{S} \wedge |\mathbf{P}_0\mathbf{S}| = a\,|\mathbf{P}_0\mathbf{P}_1| \tag{4.18}$$

or in terms of algebraic relations

$$((\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{S}) = a\,|\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{P}_0\mathbf{P}_1|) \wedge |\mathbf{P}_0\mathbf{S}| = a\,|\mathbf{P}_0\mathbf{P}_1| \tag{4.19}$$

Thus, results of vectors summation and of a multiplication of a vector by a real number are not unique, in general, in the discrete geometry. It means, that one cannot introduce a linear vector space in the discrete geometry.

Let the discrete geometry is described by n coordinates. Let the skeleton $\mathcal{P}_n = \{P_0, P_1, ...P_n\}$ determine n vectors $\mathbf{P}_0\mathbf{P}_k$, $k = 1, 2, ...n$, which are linear independent in the sense

$$F_n(\mathcal{P}_n) = \det \|(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)\| \neq 0 \qquad i, k = 1, 2, ...n \tag{4.20}$$

One can determine uniquely projections of a vector $\mathbf{Q}_0\mathbf{Q}_1$ onto vectors $\mathbf{P}_0\mathbf{P}_k$, $k = 1, 2, ...n$ by means of relations

$$\text{Pr}\,(\mathbf{Q}_0\mathbf{Q}_1)_{\mathbf{P}_0\mathbf{P}_k} = \frac{(\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_k)}{|\mathbf{P}_0\mathbf{P}_k|} \tag{4.21}$$

However, one cannot reestablish the vector $\mathbf{Q}_0\mathbf{Q}_1$, using its projections onto vectors $\mathbf{P}_0\mathbf{P}_k$. Thus, all operations of the linear vector space are not unique in the discrete geometry.

Mathematical technique of continuous geometry is not adequate for application in a discrete geometry, because it is too special and adapted for continuous geometry.

5 Skeleton conception of particle dynamics

Let us suppose, that the particle motion is free, and all force fields are included in the space-time geometry. The particle motion is described by the motion of the particle skeleton $\mathcal{P}_n = \{P_0, P_1, ...P_n\}$, which is a set of $n+1$ space-time points. The skeleton is an analog of the frame of reference attached rigidly to the particle (physical body). Tracing the skeleton motion, one traces the particle motion. The skeleton motion is described by a world chain $\mathcal{C}$ of connected skeletons

$$\mathcal{C} = \bigcup_{s=-\infty}^{s=+\infty} \mathcal{P}_n^{(s)} \tag{5.1}$$

Skeletons $\mathcal{P}_n^{(s)}$ of the world chain are connected in the sense, that

$$P_1^{(s)} = P_0^{(s+1)}, \qquad s = ...0, 1, ... \tag{5.2}$$

The vector $\mathbf{P}_0^{(s)}\mathbf{P}_1^{(s)} = \mathbf{P}_0^{(s)}\mathbf{P}_0^{(s+1)}$ is the leading vector, which determines the direction of the world chain. If the particle motion is free, the adjacent skeletons are equivalent

$$\mathcal{P}_n^{(s)}\mathrm{eqv}\mathcal{P}_n^{(s+1)}: \qquad \mathbf{P}_i^{(s)}\mathbf{P}_k^{(s)}\mathrm{eqv}\mathbf{P}_i^{(s+1)}\mathbf{P}_k^{(s+1)}, \qquad i,k = 0,1,...n, \qquad s = ..0,1,.. \tag{5.3}$$

If the particle is described by the skeleton $\mathcal{P}_n^{(s)}$, the world chain (5.1) has $n(n+1)/2$ invariants

$$\mu_{ik} = \left|\mathbf{P}_i^{(s)}\mathbf{P}_k^{(s)}\right|^2 = 2\sigma\left(P_i^{(s)}, P_k^{s}\right), \qquad i,k = 0,1,...n, \qquad s = ...0,1,... \tag{5.4}$$

which are constant along the whole world chain.

Equations (5.3) form a system of $n\,(n+1)$ difference equations for determination of nD coordinates of n skeleton points $\{P_1, P_2, ..P_n\}$, where D is the dimension of the space-time.

In the case of pointlike particle, when $n = 1$, $D = 4$, the number of equations $n_e = 2$, whereas the number of variables $n_v = 4$. The number of equations is less, than the number of dynamic variables. In the discrete space-time geometry (1.10) position of the adjacent skeleton is not uniquely determined. As a result the world chain wobbles. In the nonrelativistic approximation a statistical description of the stochastic world chains leads to the Schrödinger equations [1], if the elementary length λ_0 has the form (1.17).

Dynamic equations (5.4) are difference equations. At the large scale, when one may go to the limit $\lambda_0 = 0$, the dynamic equations (5.4) turn to the differential dynamic equations. In the case of pointlike particle ($n = 1$) and of the Kaluza-Klein five-dimensional space-time geometry these equation describe the motion of a charged particle in the given electromagnetic field.

Dynamic equations (5.3) realize the skeleton conception of particle dynamics in the microcosm. The skeleton conception of dynamics distinguishes from the conventional conception of particle dynamics in the relation that the number of dynamic equations may differ from the number of dynamic variables, which are to be determined. In the conventional conception of particle dynamics the number of dynamic equations (first order) coincide always with the number of dynamic variables, which are to be determined. As a result the motion of a particle (or of a averaged particle) appears to be deterministic. In the case of quantum particles, whose motion is stochastic (indeterministic), the dynamical equations are written for a statistical ensemble of indeterministic particles (or for the statistically averaged particle).

Statistical ensemble of indeterministic particles is a continuous dynamic system. It may be considered as a fluid, whose state is described by functions of spatial coordinates. These functions may be hydrodynamic variables: density $\rho(t, \mathbf{x})$ and velocity $\mathbf{v}(t, \mathbf{x})$, or wave function $\psi(t, \mathbf{x})$, which is also a method of the fluid description [5]. In any case the statistically averaged particle (or statistical ensemble) has infinite number of the freedom degrees. Nevertheless the dynamic equations are deterministic. They can be derived from a variational principle.

Dynamic equations (5.3) cannot be derived from a variational principle, because evolution of the particle skeleton may be indeterministic. Let us illustrate the difference between the skeleton conception of particle dynamics and the conventional conception of particle dynamics in the example of stochastic particles $\mathcal{S}_{\mathrm{st}}$ [6]

The action for the statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ is written in the form

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[\mathbf{x}, \mathbf{u}] = \int \int_{V_\xi} \left\{ \frac{m}{2}\dot{\mathbf{x}}^2 + \frac{m}{2}\mathbf{u}^2 - \frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u} \right\} \rho_0(\boldsymbol{\xi})\, dt d\boldsymbol{\xi}, \qquad \dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} \tag{5.5}$$

The variable $\mathbf{x} = \mathbf{x}(t, \boldsymbol{\xi})$ describes the regular component of the particle motion. The variable $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ describes the mean value of the stochastic velocity component, $\hbar$ is the quantum constant. The second term in (5.5) describes the kinetic energy of the stochastic velocity component. The third term describes interaction between the stochastic component $\mathbf{u}(t, \mathbf{x})$ and the regular component $\dot{\mathbf{x}}(t, \boldsymbol{\xi})$. The variable $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}$ labels the elements of the statistical ensemble. The operator

$$\boldsymbol{\nabla} = \left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right\} \tag{5.6}$$

is defined in the space of coordinates $\mathbf{x}$. Dynamic equations for the dynamic system $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ are obtained as a result of variation of the action (5.5) with respect to dynamic variables $\mathbf{x}$ and $\mathbf{u}$.

To obtain the action functional for $\mathcal{S}_{\rm st}$ from the action (5.5) for $\mathcal{E}\left[\mathcal{S}_{\rm st}\right]$, we should omit integration over $\boldsymbol{\xi}$ in (5.5). We obtain

$$\mathcal{A}_{\mathcal{S}_{\rm st}}\left[\mathbf{x},\mathbf{u}\right]=\int\left\{\frac{m}{2}\dot{\mathbf{x}}^{2}+\frac{m}{2}\mathbf{u}^{2}-\frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}\right\}dt,\qquad\dot{\mathbf{x}}\equiv\frac{d\mathbf{x}}{dt}\tag{5.7}$$

where $\mathbf{x}=\mathbf{x}(t)$ and $\mathbf{u}=\mathbf{u}(t,\mathbf{x})$ are dependent dynamic variables. The action functional (5.7) is not well defined (for $\hbar\neq0$), because the operator $\boldsymbol{\nabla}$ is defined in some 3-dimensional vicinity of point $\mathbf{x}$, but not at the point $\mathbf{x}$ itself. As far as the action functional (5.7) is not well defined, one cannot obtain dynamic equations for $\mathcal{S}_{\rm st}$. By definition it means that the particle $\mathcal{S}_{\rm st}$ is stochastic. Setting $\hbar=0$ in (5.7), we transform the action (5.7) into the action for deterministic particle $\mathcal{S}_{\rm d}$

$$\mathcal{A}_{\mathcal{S}_{\rm d}}\left[\mathbf{x}\right]=\int\frac{m}{2}\dot{\mathbf{x}}^{2}dt,\qquad\dot{\mathbf{x}}\equiv\frac{d\mathbf{x}}{dt}\tag{5.8}$$

because in this case $\mathbf{u}=0$ in virtue of dynamic equations.

After proper change of dynamic variables the action (5.5) is transformed to the form (see details of transformation in [6])

$$\mathcal{A}[\psi,\psi^{*}]=\int\left\{\frac{i\hbar}{2}(\psi^{*}\partial_{0}\psi-\partial_{0}\psi^{*}\cdot\psi)-\frac{\hbar^{2}}{2m}\boldsymbol{\nabla}\psi^{*}\cdot\boldsymbol{\nabla}\psi+\frac{\hbar^{2}}{8m}\rho\boldsymbol{\nabla}s_{\alpha}\boldsymbol{\nabla}s_{\alpha}\right\}\mathrm{d}^{4}x\tag{5.9}$$

where ψ is a two-component complex wave function

$$\rho=\psi^{*}\psi,\qquad s_{\alpha}=\frac{\psi^{*}\sigma_{\alpha}\psi}{\rho},\qquad\alpha=1,2,3\tag{5.10}$$

σ_{α} are 2×2 Pauli matrices

$$\sigma_{1}=\left(\begin{array}{cc}0&1\\1&0\end{array}\right),\qquad\sigma_{2}=\left(\begin{array}{cc}0&-i\\i&0\end{array}\right),\qquad\sigma_{3}=\left(\begin{array}{cc}1&0\\0&-1\end{array}\right),\tag{5.11}$$

In the case, when the wave function ψ is one-component, for instance $\psi=\left\{\begin{smallmatrix}\psi_{1}\\0\end{smallmatrix}\right\}$, the quantities $\mathbf{s}=\{s_{1},s_{2},s_{3}\}$ are constant ($s_{1}=0,\quad s_{2}=0,\quad s_{3}=1$), the action (5.9) turns into

$$\mathcal{A}[\psi,\psi^{*}]=\int\left\{\frac{i\hbar}{2}(\psi^{*}\partial_{0}\psi-\partial_{0}\psi^{*}\cdot\psi)-\frac{\hbar^{2}}{2m}\boldsymbol{\nabla}\psi^{*}\cdot\boldsymbol{\nabla}\psi\right\}\mathrm{d}^{4}x\tag{5.12}$$

The dynamic equation, generated by the action (5.12), is the Schrödinger equation

$$i\hbar\partial_{0}\psi+\frac{\hbar^{2}}{2m}\boldsymbol{\nabla}^{2}\psi=0\tag{5.13}$$

This dynamic equation describes the irrotational flow of the fluid.

In the general case one obtains from the action (5.9)

$$()i\hbar\partial_{0}\psi+\frac{\hbar^{2}}{2m}\boldsymbol{\nabla}^{2}\psi+\frac{\hbar^{2}}{8m}\boldsymbol{\nabla}^{2}s_{\alpha}\cdot\left(s_{\alpha}-2\sigma_{\alpha}\right)\psi-\frac{\hbar^{2}}{4m}\frac{\boldsymbol{\nabla}\rho}{\rho}\boldsymbol{\nabla}s_{\alpha}\sigma_{\alpha}\psi=0\tag{5.14}$$

where two last terms describe the vorticity of the fluid flow.

In the conventional conception of dynamics one can obtain dynamic equation for the statistically averaged particle (i.e. statistical ensemble normalized to one particle), but there are no dynamic equations for a single stochastic particle. In the skeleton conception of dynamics there are dynamic equations for a single particle. These equations are many-valued (multivariant), but they do exist. One can derive dynamic equations for the statistically averaged particle which is a kind of a fluid (continuous medium). A flow of this fluid is deterministic, and dynamic equations for this flow can be obtained from a variational principle.

Thus, a difference between the conventional conception of dynamics and the skeleton conception of dynamics lies in the description of a single particle. Conventional conception cannot obtain dynamic equations for a single stochastic particle, whereas skeleton conception can obtain the dynamic equations for a single particle, although these equations are multivariant. This difference is conditioned by the fact, that in the case of the skeleton conception the stochasticity is conditioned by the space-time geometry, whereas in the conventional conception of dynamics the stochasticity is introduced axiomatically, and there is no model of the stochasticity.

One may compare this situation with the situation in description of thermal phenomena. If we use the axiomatic thermodynamics, we cannot say nothing on the structure of gas molecules. We cannot say anything even on the existence of molecules. If we use the statistical physics we obtain some information on behavior and structure of gas molecules, although their motion is chaotic and multivariant.

The skeleton conception of the particle dynamics realizes a more detailed description of elementary particle, one may hope to obtain some information on the elementary particle structure.

We have now only two examples of the skeleton conception application. Considering compactification in the 5-dimensional discrete space-time geometry of Kaluza-Klein, and imposing condition of uniqueness of the world function, one obtains that the value of the electric charge of a stable elementary particle is restricted by the elementary charge [7]. This result has been known experimentally, but it could not be explained theoretically, because in the continuous space-time geometry nobody considers the world function as a fundamental quantity and demand its uniqueness.

Another example concerns structure of Dirac particles (fermions). Consideration in framework of skeleton conception [8] shows, that world chain of a fermion is a spacelike helix with timelike axis. The averaged world chain of a free fermion is a timelike straight line. The helical motion of a skeleton generates an angular moment (spin) and magnetic moment. Such a result looks rather reasonable. In the conventional conception of the particle dynamics the spin and magnetic moment of a fermion are postulated without a reference to its structure.

6 Concluding remarks

Thus, the supposition on the space-time geometry discreteness seems to be more natural and reasonable, than the supposition on quantum nature of physical phenomena in microcosm. Discreteness is simply a property of the space-time, whereas quantum principles assume introduction of new essences.

Formalism of the discrete geometry is very simple. It does not contain theorems with complicated proofs. Nevertheless the discrete geometry and its formalism is perceived hardly. The discrete geometry was not developed in the twentieth century, although the discrete space-time was necessary for description of physical phenomena in microcosm. It was rather probably, that the space-time is discrete in microcosm. What is a reason of the discrete geometry disregard? We try to answer this important question.

The discrete geometry was not developed, because it could be obtained only as a generalization of the proper Euclidean geometry. But almost all concepts and quantities of the proper Euclidean geometry use essentially concepts of the continuous geometry. They could not be used for construction of a discrete geometry. Only world function (or distance) does not use a reference to the geometry continuity. Only coordinateless expressions (1.1) - (1.6) of basic quantities of the Euclidean geometry in terms of world function admit one to construct a discrete geometry and other physical geometries.

Supposition, that any geometry is to be axiomatizable, was the second obstacle on the way of the discrete geometry construction. In general, the fact, that the proper Euclidean geometry is a degenerate geometry, was another obstacle. In particular, being a physical geometry, the proper Euclidean geometry is an axiomatizable geometry, and this circumstance is an evidence of its degeneracy. It is very difficult to obtain a general conception as a generalization of a degenerate conception, because many quantities of the general conception coincide in the degenerate conception. It is rather difficult to disjoint them. For instance, a physical geometry is multivariant, in general. Single-variant physical geometry is a degenerate geometry. In the physical geometry the straight segment (1.1) is a surface (tube), in general. In the degenerate physical geometry (the proper Euclidean geometry) the straight segment is a one-dimensional line. How can one guess, that a straight segment is a surface, in general? Besides, multivariance of the equivalence relation leads to nonaxiomatizability of geometry. But we learn only axiomatizable geometries in the last two thousand years. How can we guess, that nonaxiomatizable geometries exist? The straight way from the Euclidean geometry to physical geometries was very difficult, and the physical geometry has been obtained on an oblique way.

J.L.Synge [9] has introduced the world function for description of the Riemannian geometry. I did not know the papers of Synge and introduced the world function for description of the Riemannian space-time in general relativity. My approach differed slightly from the approach of Synge. In particular, I had obtained an equation for the world function of Riemannian geometry [10].

$$\frac{\partial\sigma\left(x,x'\right)}{\partial x^{i}}G^{ik'}\frac{\partial\sigma\left(x,x'\right)}{\partial x'^{k}}=2\sigma\left(x,x'\right),\qquad G^{ik'}G_{lk'}=\delta_{l}^{i},\qquad G_{lk'}\equiv\frac{\partial^{2}\sigma\left(x,x'\right)}{\partial x^{l}\partial x'^{k}} \tag{6.1}$$

This equation was obtained as a corollary of definition of the world function of the Riemannian geometry as an integral along the geodesic, connecting points x and x'. This equation contains only world function and its derivatives.

This equation put the question. Let a world function does not satisfy the equation. Does this world function describe a nonRiemannian geometry or it does describe no geometry? It was very difficult to answer this question. On one hand, the formalism, based on the world function, is a more developed formalism, than formalism based on a use of metric tensor, because a geodesic is described in terms of the world function by algebraic equation (1.1), whereas the same geodesic is described by differential equations in terms the metric tensor.

On the other hand, the geodesic, described by (1.1) is one-dimensional only in the Riemannian geometry. In general, one equation (1.1) in n-dimensional space describes a $(n-1)$-dimensional surface. I did not know, whether the surface is a generalization of a geodesic in any geometry. I was not sure, because in the Euclidean geometry a straight segment is one-dimensional be definition. I left this question unsolved and returned to it almost thirty years later, in the beginning of ninetieth.

When the string theory of elementary particles appeared, it becomes clear for me, that the particle may be described by means of a world surface (tube) but not only by a world line. As the particle world line associates with a geodesic, I decided, that a world tube may describe a particle. It meant that there exist space-time geometries, where straights (geodesics) are described by world tubes. The question on possibility of the physical space-time geometry has been solved finally, when the quantum description appeared to be a corollary of the space-time multivariance [1].

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