Gödel's theorem as a corollary of impossibility of complete axiomatization of geometry

Yuri A.Rylov

Institute for Problems in Mechanics, Russian Academy of Sciences,
101-1, Vernadskii Ave., Moscow, 119526, Russia.
e-mail: rylov@ipmnet.ru
Web site: *http://rsfq1.physics.sunysb.edu/˜rylov/yrylov.htm*
or mirror Web site:
http://gasdyn-ipm.ipmnet.ru/˜rylov/yrylov.htm

Abstract

Not any geometry can be axiomatized. The paradoxical Godel's theorem starts from the supposition that any geometry can be axiomatized and goes to the result, that not any geometry can be axiomatized. One considers example of two close geometries (Riemannian geometry and σ-Riemannian one), which are constructed by different methods and distinguish in some details. The Riemannian geometry reminds such a geometry, which is only a part of the full geometry. Such a possibility is covered by the Godel's theorem.

1 Introduction

Let there be a set Ω of points $P_1, P_2, ...$ A geometrical object O is a subset O of points $P_1, P_2, ...$ of Ω, $(O \subset \Omega)$.

Definition 1 *Geometry $\mathcal{G}$ is an infinite set $\mathcal{S}_\mathcal{G}$ of prepositions $\mathcal{P}_1, \mathcal{P}_2, ...$ on properties of geometrical objects $O_1, O_2, ... \subset \Omega$.*

Definition 2 *If one can choose a finite or countable set $\mathcal{S}_\mathcal{A} \subset \mathcal{S}_\mathcal{G}$ of prepositions in such a way, that the infinite set $\mathcal{S}_\mathcal{G}$ of all prepositions $\mathcal{P}_1, \mathcal{P}_2, ...$ on properties of geometrical objects $O_1, O_2, ... \subset \Omega$ can be obtained as a result of logical reasonings of prepositions $\mathcal{P} \in \mathcal{S}_\mathcal{A}$, the geometry $\mathcal{G}$ may be axiomatized, and the set $\mathcal{S}_\mathcal{A}$ of prepositions $\mathcal{P}$ forms axiomatics (the system of axioms) of the geometry $\mathcal{G}$.*

It is evident, that not any geometry can be axiomatized. However, there are geometries, which can be axiomatized. For instance, the proper Euclidean geometry can be axiomatized.

Let us consider a geometry $\mathcal{G}$, which can be axiomatized partly. It means that a part $\mathcal{S}_{\mathcal{G}_1} \subset \mathcal{S}_{\mathcal{G}}$ of all propositions $\mathcal{P}$ of geometry $\mathcal{G}$ may be obtained as a result of logical reasonings of propositions $\mathcal{P} \in \mathcal{S}_{\mathcal{A}}$, where $\mathcal{S}_{\mathcal{A}}$ is the set of axioms $\mathcal{S}_{\mathcal{A}} \subset \mathcal{S}_{\mathcal{G}}$. The set of axioms $\mathcal{S}_{\mathcal{A}}$ is supposed to be complete in the sense, that any supposition $\mathcal{P} \in \mathcal{S}_{\mathcal{G}_1}$ may be deduced from the set $\mathcal{S}_{\mathcal{A}}$ of axioms. It is supposed that the set $\mathcal{S}_{\mathrm{r}} = \mathcal{S}_{\mathcal{G}} \backslash \mathcal{S}_{\mathcal{G}_1}$ of remaining propositions of geometry $\mathcal{G}$ cannot be deduced from the set $\mathcal{S}_{\mathcal{A}}$ of axioms.

The Godel's theorem proves the same, but Godel starts from the supposition, that the complete axiomatization of geometry is possible, and he deduced essentially that the complete axiomatization is not always possible.

The real problem of the geometry construction lies in the fact, that we know no other method of the geometry construction except for the deduction of the geometry propositions from the geometry axioms. Euclid had deduced his Euclidean geometry from the system of axioms. It was proved [1], that the Euclidean axioms are compatible between themselves, and the proper Euclidean geometry is a consistent geometry.

Mathematicians try to repeat the Euclidean experience for construction of other (non-uniform) geometries. They use the Euclidean method of the geometry construction. As a result one cannot be sure that one deduces the full geometry, but not only its part.

We start from the general supposition that the complete axiomatization of a geometry is not always possible. In this case the result of the Godel's theorem is evident. However, to adduce such a supposition, one is to have an alternative method of the geometry construction, because a formal consideration of the set of the geometrical propositions is not constructive without a method of these propositions obtaining.

In reality there is an alternative method of the geometry construction. It is based on two statements.

1. Geometry is described completely by its metric (distance between any two points of the space).

2. The proper Euclidean geometry is a true geometry, which can be described completely by its metric.

We shall refer to geometry, which is completely described by its metric, as the physical geometry. The geometry, which is deduced from the system of some axioms, will be referred to as an axiomatic geometry, or a mathematical geometry. Then the alternative method of the physical geometry construction is formulated as follows.

Any physical geometry is obtained from the proper Euclidean geometry as a result of its deformation.

It means that all propositions of the proper Euclidean geometry are expressed via Euclidean metric, and the Euclidean metric in all propositions is replaced by the metric of the geometry in question. There is a theorem, which proves that all propositions of the proper Euclidean geometry can be expressed via Euclidean metric [2]. As a result one obtains all propositions of the geometry in question, expressed via its metric.

Usually one supposes, that the term "metric" means the distance, satisfying the triangle axiom. In the previous presentation the term "metric" means the distance simply (which is not satisfies the triangle axiom, in general).

To avoid a confusion, one uses the world function $\sigma(P,Q) = \frac{1}{2}\rho^2(P,Q)$ instead of the metric $\rho(P,Q)$. It is more convenient from technical point of view. Besides, in the case of indefinite geometry (for instance, in geometry of Minkowski) the world function σ is real, although $\rho = \sqrt{2\sigma}$ may be imaginary.

2 Deformation principle as a method of a physical geometry construction

If one knows the proper Euclidean geometry, the expression of the proper Euclidean propositions via the Euclidean world function is a purely technical problem. But there are some subtleties in this problem. The fact is that, the proper Euclidean geometry has specific Euclidean properties and general geometric properties. All propositions of the proper Euclidean geometry are to be expressed only via general geometric properties. Only in this case the expression via Euclidean world function may be deformed and used in the deformed geometry. The fact is that, the specific properties of the proper Euclidean geometry are different in the Euclidean spaces of different dimensions. Any specific property of the proper Euclidean geometry contains a reference to the dimension n of the Euclidean space.

The general geometric propositions of the proper Euclidean geometry do not refer to the dimension n of the Euclidean space. Thus, only the Euclidean propositions, which do not contain a reference to the dimension n, may be deformed to obtain corresponding relation of the deformed geometry.

We consider a simple example. Vector $\mathbf{PQ} = \overrightarrow{PQ}$ is an ordered set of two points P and Q. The scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is defined by the relation

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \sigma(P_0,Q_1) + \sigma(P_1,Q_0) - \sigma(P_0,Q_0) - \sigma(P_1,Q_1) \tag{2.1}$$

where σ is the world function

$$\sigma: \qquad \Omega\times\Omega \to \mathbb{R}, \qquad \sigma(P,Q) = \sigma(Q,P), \qquad \sigma(P,P) = 0, \qquad \forall P,Q \in \Omega \tag{2.2}$$

Definition (2.1) of the scalar product of two vectors coincides with the conventional scalar product of vectors in the proper Euclidean space. (One can verify this easily). The relation (2.1) does not contain a reference to the dimension of the Euclidean space and to a coordinate system in it. Hence, the relation (2.1) is a general geometric relation, which may be considered as a definition of the scalar product of two vectors in any physical geometry.

Equivalence (equality) of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is defined by the relations

$$\mathbf{P}_0\mathbf{P}_1\text{eqv}\mathbf{Q}_0\mathbf{Q}_1 : \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = |\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1| \wedge |\mathbf{P}_0\mathbf{P}_1| = |\mathbf{Q}_0\mathbf{Q}_1| \qquad (2.3)$$

where $|\mathbf{P}_0\mathbf{P}_1|$ is the length of the vector $\mathbf{P}_0\mathbf{P}_1$

$$|\mathbf{P}_0\mathbf{P}_1| = \sqrt{(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1)} = \sqrt{2\sigma(P_0, P_1)} \qquad (2.4)$$

In the developed form the condition (2.3) of equivalence of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ has the form

$$\sigma(P_0, Q_1) + \sigma(P_1, Q_0) - \sigma(P_0, Q_0) - \sigma(P_1, Q_1) = 2\sigma(P_0, P_1) \qquad (2.5)$$

$$\sigma(P_0, P_1) = \sigma(Q_0, Q_1) \qquad (2.6)$$

If the points P_0, P_1, determining the vector $\mathbf{P}_0\mathbf{P}_1$, and the origin Q_0 of the vector $\mathbf{Q}_0\mathbf{Q}_1$ are given, we can determine the vector $\mathbf{Q}_0\mathbf{Q}_1$, which is equivalent (equal) to the vector $\mathbf{P}_0\mathbf{P}_1$, solving two equations (2.5), (2.6) with respect to the position of the point Q_1.

In the case of the proper Euclidean space there is one and only one solution of equations (2.5), (2.6) independently of its dimension n. In the case of arbitrary physical geometry one can guarantee neither existence nor uniqueness of the solution of equations (2.5), (2.6) for the point Q_1. Number of solutions depends on the form of the world function σ. This fact means a multivariance of the property of two vectors equivalence in the arbitrary physical geometry. In other words, single-variance of the vector equality in the proper Euclidean space is a specific property of the proper Euclidean geometry, and this property is conditioned by the form of the Euclidean world function. In other physical geometries this property does not take place, in general.

The multivariance is a general property of a physical geometry. It is connected with a necessity of solution of algebraic equations, containing the world function. As far as the world function is different in different physical geometries, the solution of these equations may be not unique, or it may not exist at all.

In the proper Euclidean geometry the equality of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ may be defined also as equality of components of these vectors at some rectilinear coordinate system. (It is the conventional method of definition of two vectors equality).

Let the dimension of the Euclidean space be equal to n. Let us introduce n linear independent vectors $\mathbf{P}_0\mathbf{P}_k$, $k = 1, 2, ...n$. Linear independence of vectors $\mathbf{P}_0\mathbf{P}_k$ means that the Gram's determinant

$$\det ||\,(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)\,|| \neq 0, \qquad i, k = 1, 2, ...n \qquad (2.7)$$

We construct rectilinear coordinate system with basic vectors $\mathbf{P}_0\mathbf{P}_k$, $k = 1, 2, ...n$ in the n-dimensional Euclidean space. Covariant coordinates $x_k = (\mathbf{P}_0\mathbf{P}_1)_k$ and $y_k = (\mathbf{Q}_0\mathbf{Q}_1)_k$ of vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ in this coordinate system have the form

$$x_k = (\mathbf{P}_0\mathbf{P}_1)_k = (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_k), \qquad y_k = (\mathbf{Q}_0\mathbf{Q}_1)_k = (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_k), \qquad k = 1, 2, ...n \qquad (2.8)$$

Equality of vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is written in the form

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_k) = (\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_k), \qquad k = 1, 2, ...n \tag{2.9}$$

and according to (2.1) it may be written in terms of the world function (metric). The points P_0, P_1, Q_0 are supposed to be given. The point Q_1 is to be determined by equations (2.9). Equations (2.9) determine equality of vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ only in the n-dimensional Euclidean space E_n. Already in the $(n+1)$-dimensional Euclidean space E_{n+1} n equations (2.9) do not determine equality of vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$, because in $(n+1)$-dimensional Euclidean space E_{n+1} one needs $(n+1)$ equations of the form (2.9) to define equality of vectors. There may be no dimension in the physical geometry, or the dimension of the space may be different at different points. In this cases conventional conditions (2.9) of two vectors equality cannot be used also.

From formal point of view the equations (2.9) define some geometrical object, or a set of geometrical objects, whose points are described by means of running point Q_1. This geometrical object depends on parameters $Q_0, P_0, P_1, ...P_n$. How can one interpret this object? It is quite unclear.

From formal viewpoint the relations (2.5), (2.6) describe some geometrical object by means of the running point Q_1. This object depends on parameters Q_0, P_0, P_1, and one interprets this as a set of vectors $\mathbf{Q}_0\mathbf{Q}_1$, which are equivalent to vector $\mathbf{P}_0\mathbf{P}_1$. One may consider, that the relations (2.5), (2.6) describe some geometrical object by means of the running point P_1. This object depends on parameters Q_0, Q_1, P_0, and one interprets this as a set of vectors $\mathbf{P}_0\mathbf{P}_1$, which are equivalent to vector $\mathbf{Q}_0\mathbf{Q}_1$.

It is to note that the proper Euclidean geometry is a degenerate geometry in the sense that the same geometrical object may be described by different ways in terms of the world function. For instance, the circular cylinder $\mathcal{CY}(P_0, P_1, Q)$ is defined by the relation

$$\mathcal{CY}(P_0, P_1, Q) = \{R | S_{P_0P_1R} = S_{P_0P_1Q}\} \tag{2.10}$$

where P_0, P_1 are two different points on the axis of the cylinder and Q is some point on the surface of the cylinder. The quantity $S_{P_0P_1Q}$ is the area of the triangle with the vertices at the points P_0, P_1, Q. The area of triangle may be calculated by means of the Hero's, expressing the triangle area via length of the triangle sides, or by the formula

$$S_{P_0P_1Q} = \frac{1}{2}\sqrt{\left| \begin{array}{cc} (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}) \\ (\mathbf{P}_0\mathbf{Q}.\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{Q}.\mathbf{P}_0\mathbf{Q}) \end{array} \right|} \tag{2.11}$$

which may be expressed in terms of the world function by means of (2.1). In the proper Euclidean geometry the circular cylinder $\mathcal{CY}(P_0, P_1, Q)$ depends only on its axis $\mathcal{T}_{P_0P_1}$, passing through the points P_0, P_1, but not on positions of the points P_0, P_1 on the axis $\mathcal{T}_{P_0P_1}$.

The axis of the cylinder $\mathcal{T}_{P_0P_1}$ is described by the relation

$$\mathcal{T}_{P_0P_1} = \{R | \mathbf{P}_0\mathbf{P}_1 \parallel \mathbf{P}_0\mathbf{R}\} \tag{2.12}$$

where $\mathbf{P}_0\mathbf{P}_1 \parallel \mathbf{P}_0\mathbf{R}$ means that the vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{R}$ are collinear (linear dependent), which means mathematically, that

$$\mathbf{P}_0\mathbf{P}_1 \parallel \mathbf{P}_0\mathbf{R}: \qquad \begin{vmatrix} (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R}) \\ (\mathbf{P}_0\mathbf{R}.\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{R}.\mathbf{P}_0\mathbf{R}) \end{vmatrix} = 0 \tag{2.13}$$

Thus, if the point $P_1' \in \mathcal{T}_{P_0P_1}$, $P' \neq P_1$, $P' \neq P_0$, then the straight line $\mathcal{T}_{P_0P_1'} = \mathcal{T}_{P_0P_1}$ in the Euclidean space, but $\mathcal{T}_{P_0P_1'} \neq \mathcal{T}_{P_0P_1}$ in the physical geometry, in general.

Then

$$\mathcal{CY}(P_0, P_1, Q) = \mathcal{CY}(P_0, P_1', Q) \wedge P_1' \in \mathcal{T}_{P_0P_1} \tag{2.14}$$

in the proper Euclidean geometry. But in general,

$$\mathcal{CY}(P_0, P_1, Q) \neq \mathcal{CY}(P_0, P_1', Q) \wedge P_1' \in \mathcal{T}_{P_0P_1} \tag{2.15}$$

In other words, the circular cylinder of the proper Euclidean space is split into many different cylinders after deformation of the Euclidean space.

A more accurate statement is as follows. Cylinders $\mathcal{CY}(P_0, P_1, Q)$ and $\mathcal{CY}(P_0, P_1', Q)$ are different, in general. But in the proper Euclidean geometry they may coincide, even if $P_1' \neq P_1$ but $P_1' \in \mathcal{T}_{P_0P_1}$. It means, that in the proper Euclidean geometry different geometrical objects may coincide, because of very high symmetry of the Euclidean geometry. In other words, a deformation of the Euclidean geometry may violate its symmetry, and coincidence of different geometrical objects ceases.

3 Multivariance of two vectors equivalence

In application to the property of equivalence the multivariance property looks as follows. In general, there are many vectors $\mathbf{Q}_0\mathbf{Q}_1, \mathbf{Q}_0\mathbf{Q}_1', ...$ which are equivalent to vector $\mathbf{P}_0\mathbf{P}_1$ and are not equivalent between themselves. This situation is common for all physical geometries. However, there are possible such geometries, where the set of vectors $\mathbf{Q}_0\mathbf{Q}_1, \mathbf{Q}_0\mathbf{Q}_1', ...$ degenerates into one vector. In the proper Euclidean geometry we have such a degeneration for all vectors $\mathbf{P}_0\mathbf{P}_1$ and any point Q_0, which is an origin of the vector $\mathbf{Q}_0\mathbf{Q}_1$, equivalent to the vector $\mathbf{P}_0\mathbf{P}_1$.

It means that the multivariance of the equivalence property is a general property of a physical geometry, whereas single-variance of the equivalence property in the proper Euclidean geometry is a specific property of the Euclidean geometry, which is conditioned by the form of the Euclidean world function.

The multivariance is a new property of physical geometry. Multivariance properties have not been yet investigated properly. Multivariance of the equivalence property generates intransitivity of the vector equivalence. In other words, if $\mathbf{P}_0\mathbf{P}_1 \text{eqv} \mathbf{Q}_0\mathbf{Q}_1$ and $\mathbf{Q}_0\mathbf{Q}_1 \text{eqv} \mathbf{S}_0\mathbf{S}_1$, then, in general, $\mathbf{P}_0\mathbf{P}_1$ is not equivalent to $\mathbf{S}_0\mathbf{S}_1$. The intransitive equivalence is difficult for investigation. It is reasonable to consider physical geometries, which do not contain multivariance, or contain multivariance in the minimal degree.

The form of the world function is a unique characteristic of a physical geometry. One can change a physical geometry only changing its world function. To obtain the Riemannian geometry we are to impose on the world function the following constraint

$$F_3(P_0,R,P_1) \equiv \sqrt{2\sigma(P_0,R)}+\sqrt{2\sigma(R,P_1)}-\sqrt{2\sigma(P_0,P_1)} \geq 0, \qquad \forall P_0,P_1,R\in\Omega \tag{3.1}$$

It is the triangle axiom. Its meaning is as follows. The segment $\mathcal{T}_{[P_0P_1]}$ of the straight between the points P_0, P_1 is described by the relation

$$\mathcal{T}_{[P_0P_1]} = \{R | F_3(P_0,R,P_1) = 0\} \tag{3.2}$$

In general, the equation

$$F_3(P_0,R,P_1) \equiv \sqrt{2\sigma(P_0,R)}+\sqrt{2\sigma(R,P_1)}-\sqrt{2\sigma(P_0,P_1)} = 0 \tag{3.3}$$

describes some surface S around some volume V in the space Ω. The external points R with respect to the volume V satisfy the relation $F_3(P_0,R,P_1)>0$. The internal points R inside the volume V satisfy the relation $F_3(P_0,R,P_1)<0$. If the triangle axiom (3.1) is fulfilled, the volume V is empty, and the segment $\mathcal{T}_{[P_0P_1]}$ of straight is one-dimensional, because the surface S does not contain any points inside.

On the other hand, the segment (3.2) of the straight can be described by the relation

$$\mathcal{T}_{[P_0P_1]} = \{R | \mathbf{P}_0\mathbf{P}_1 \upuparrows \mathbf{P}_0\mathbf{R} \wedge |\mathbf{P}_0\mathbf{R}| \leq |\mathbf{P}_0\mathbf{P}_1|\} \tag{3.4}$$

where $\mathbf{P}_0\mathbf{P}_1 \upuparrows \mathbf{P}_0\mathbf{R}$ is the condition of parallelism of vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{R}$, which is described by the relation

$$\mathbf{P}_0\mathbf{P}_1 \upuparrows \mathbf{P}_0\mathbf{R}: \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R}) = |\mathbf{P}_0\mathbf{R}|\cdot|\mathbf{P}_0\mathbf{P}_1| \tag{3.5}$$

It easy to verify that two definitions (3.2), (3.3) and (3.4), (3.5) are equivalent because of the identity

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R})^2 - |\mathbf{P}_0\mathbf{R}|^2|\mathbf{P}_0\mathbf{P}_1|^2 \equiv \frac{1}{4}F_0(P_0,R,P_1)F_1(P_0,R,P_1)F_2(P_0,R,P_1)F_3(P_0,R,P_1) \tag{3.6}$$

where

$$\begin{aligned}
F_0(P_0,R,P_1) &= \sqrt{2\sigma(P_0,R)}+\sqrt{2\sigma(R,P_1)}+\sqrt{2\sigma(P_0,P_1)}\\
F_1(P_0,R,P_1) &= \sqrt{2\sigma(P_0,R)}-\sqrt{2\sigma(R,P_1)}+\sqrt{2\sigma(P_0,P_1)}\\
F_2(P_0,R,P_1) &= -\sqrt{2\sigma(P_0,R)}+\sqrt{2\sigma(R,P_1)}+\sqrt{2\sigma(P_0,P_1)}\\
F_3(P_0,R,P_1) &= \sqrt{2\sigma(P_0,R)}+\sqrt{2\sigma(R,P_1)}-\sqrt{2\sigma(P_0,P_1)}
\end{aligned}$$

Thus, if the world function satisfies the triangle axiom (3.1), any segment (3.4) is single-variant (one-dimensional), and equivalence (2.3) of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{R}$ is single-variant, provided they have a common origin P_0. It means that it follows from the relation $\mathbf{P}_0\mathbf{P}_1\text{eqv}\mathbf{P}_0\mathbf{R}$ and (3.1), that $R=P_1$. Note, that the single-variance of the two vectors equivalence takes place only for the proper Riemannian geometry, when the world function is nonnegative and all terms in the relation (3.1) are real for any points $P_0, P_1, R \in \Omega$. For the pseudo-Riemannian geometry, when the world function σ may have any sign, the equivalence of two vectors is multivariant, in general, even if the vectors have a common origin.

Investigation shows , that the equivalence of two vectors is single-variant only in the flat proper Riemannian space, i.e. in the proper Euclidean space. Note, that in [2] one investigated multivariance of parallelism of two directions, the concept of the vector equivalence had not yet been introduced. However, there is a single-valued connection between the multivariance of two vectors equivalence and multivariance of the two direction parallelism. There are some special cases of the geometry and of vectors, when the equivalence property is single-variant. For instance, in the proper Riemannian space, when the world function is nonnegative and the triangle axiom (3.1) takes place, equivalence of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{R}$ is single-variant, provided that $\mathbf{P}_0\mathbf{P}_1 \parallel \mathbf{P}_0\mathbf{Q}_0$.

4 Riemannian and σ-Riemannian geometries

The physical geometry, whose world function satisfies the triangle axiom (3.1) will be referred to as the σ-Riemannian geometry. The Riemannian geometry distinguishes from the σ-Riemannian geometry by the method of its construction. The σ-Riemannian geometry is constructed by means of the deformation principle, and it is defined completely by its world function. The σ-Riemannian geometry may be discrete, or continuous. This circumstance is of no importance, because the construction of physical geometry by means of the deformation principle does not need introduction of a coordinate system.

The n-dimensional Riemannian geometry is constructed as an internal geometry of the n-dimensional smooth surface S_n in m-dimensional proper Euclidean space E_m $(m > n)$. One introduces a curvilinear coordinate system K_m in E_m with coordinates $\xi = \{\mathbf{x}, \mathbf{y}\} = \{\xi^i\}$, $i = 1, 2, ...m$, where $\mathbf{x} = \{x^1, x^2, ...x^n\}$, $\mathbf{y} = \{y^{n+1}, y^{n+2}, ...y^m\}$. Coordinates are introduced in such a way, that coordinates $\{\mathbf{x}, \mathbf{0}\}$ describe points of the n-dimensional surface S_n. Besides, coordinates ξ are supposed to be chosen in such a way, that any vector $\mathbf{U}(P) = \{0, 0, ...0, U^{n+1}, U^{n+2}, ...U^m\}$ at a point $P \in S_n$ is orthogonal to any tangent to the surface S_n vector $\mathbf{V}(P) = \{V^1, V^2, ...V^n, 0, 0, ...0\}$, taken at the same point $P \in S_n$. Let g_{ik}, $i, k = 1, 2, ...m$ be the metric tensor in the proper Euclidean space E_m in the coordinate system K_m. Then

$$g_{ik}(\xi) = g_{ik}(\mathbf{x}, \mathbf{y}) = \sum_{l=1}^{l=m} \frac{\partial X^l(\xi)}{\partial \xi^i} \frac{\partial X^l(\xi)}{\partial \xi^k}, \qquad i, k = 1, 2, ...m, \tag{4.1}$$

where $X^l(\xi)$, $l = 1, 2, ...m$ are Cartesian coordinates of the point ξ in E_m. Curvilinear coordinates are chosen in such a way, that on the surface S_n they satisfy the conditions

$$\sum_{l=1}^{l=m} \frac{\partial X^l(\mathbf{x},\mathbf{0})}{\partial \xi^i} \frac{\partial X^l(\mathbf{x},\mathbf{0})}{\partial \xi^k} = 0, \qquad i = 1,2,...n, \qquad k = n+1, n+2, ...m \qquad (4.2)$$

The line element ds on the surface S_n is described by the relation

$$ds^2 = \sum_{l=1}^{l=n} g_{ik}(\mathbf{x},\mathbf{0})\, dx^i dx^k \equiv g_{ik}(\mathbf{x})\, dx^i dx^k \qquad (4.3)$$

where the sign of sum is omitted and the summation over repeated indices is produced from 1 to n. This rule is used further. One can determine geodesics on the surface S_n by means of the relations

$$\frac{d^2 x^k}{d\tau^2} + \gamma_{ls}^k(\mathbf{x}) \frac{dx^l}{d\tau}\frac{dx^s}{d\tau} = 0 \qquad (4.4)$$

where γ_{ls}^k is the Cristoffel symbol in the coordinate system K_n on the surface S_n

$$\gamma_{ls}^k(\mathbf{x}) = \frac{1}{2} g^{ki}(\mathbf{x}) \left(\frac{\partial g_{is}(\mathbf{x})}{\partial x^l} + \frac{\partial g_{li}(\mathbf{x})}{\partial x^s} - \frac{\partial g_{ls}(\mathbf{x})}{\partial x^i} \right), \qquad k,l,s = 1,2,...n \qquad (4.5)$$

$$g^{ki}(\mathbf{x})\, g_{li}(\mathbf{x}) = \delta_l^k, \qquad k,l = 1,2,...n$$

Let for simplicity there be only one geodesic segment $\mathcal{L}_{[P_0P_1]} \subset S_n$, connecting any two points $P_0, P_1 \in S_n$. We can define the world function σ_{R} on S_n by means of the relation

$$\sigma_{\mathrm{R}}(P_0,P_1) = \frac{1}{2} \left(\int\limits_{\mathcal{L}_{[P_0P_1]}} \sqrt{g_{ik}(\mathbf{x})\, dx^i dx^k} \right)^2, \qquad P_0, P_1 \in S_n \qquad (4.6)$$

According to definition (4.6) the world function σ_{R} satisfies the triangle axiom (3.1). It means, that the world function σ_{R} of the Riemannian geometry may coincide with the world function of the σ-Riemannian geometry, if the set Ω is identified with the surface S_n. Then we may repeat construction of the Riemannian geometry on the surface S_n by means of the deformation principle. In this case we obtain the above obtained results. We obtain the line element in the form (4.3) and equation for the straight (geodesic) in the form (4.4). All obtained single-variant results of the Riemannian geometry may be obtained by means of the deformation principle from the world function (4.6).

Two vectors $\mathbf{U}(\mathbf{x}_1,\mathbf{0})$ and $\mathbf{V}(\mathbf{x}_2,\mathbf{0})$ at two different points $P_1 = \{\mathbf{x}_1,\mathbf{0}\} \in S_n$ and $P_2 = \{\mathbf{x}_2,\mathbf{0}\} \in S_n$ are equal in the Euclidean space E_m, if their Cartesian components coincide

$$\sum_{k=1}^{k=m} \frac{\partial X^l}{\partial \xi^k}(\mathbf{x}_1,\mathbf{0})\, U^k(\mathbf{x}_1,\mathbf{0}) = \sum_{k=1}^{k=m} \frac{\partial X^l}{\partial \xi^k}(\mathbf{x}_2,\mathbf{0})\, V^k(\mathbf{x}_2,\mathbf{0}), \qquad l = 1,2,...m \qquad (4.7)$$

where $U^k(\mathbf{x}_1,\mathbf{0})$ and $V^k(\mathbf{x}_2,\mathbf{0})$ are components of vectors $\mathbf{U}(\mathbf{x}_1,\mathbf{0})$ and $\mathbf{V}(\mathbf{x}_2,\mathbf{0})$ respectively in the curvilinear coordinate system K_m. If vectors $\mathbf{U}(\mathbf{x}_1,\mathbf{0})$ and $\mathbf{V}(\mathbf{x}_2,\mathbf{0})$ are vectors of the internal geometry in S_n, they are tangent to the surface S_n, i.e.

$$U^k(\mathbf{x}_1,\mathbf{0})=0, \qquad V^k(\mathbf{x}_2,\mathbf{0})=0, \qquad k=n+1,n+2,...m \tag{4.8}$$

Let the vector $\mathbf{U}(\mathbf{x}_1,\mathbf{0})$, satisfying the first relation (4.8) be fixed. Then the vector $\mathbf{V}(\mathbf{x}_2,\mathbf{0})$, satisfying conditions (4.7), (4.8) does not exist, in general, if the points P_1 and P_2 are different, and

$$\frac{\partial X^l}{\partial \xi^k}(\mathbf{x}_1,\mathbf{0})\neq\frac{\partial X^l}{\partial \xi^k}(\mathbf{x}_2,\mathbf{0}), \qquad l,k=1,2,...m \tag{4.9}$$

But the multivariant relations of the σ-Riemannian geometry cannot be obtained in the Riemannian geometry, because the Riemannian geometry does not contain multivariant relations in principle. Thus, in general, we cannot take from the proper Euclidean geometry the concept of the remote vectors equality in the internal geometry of the surface S_n.

If the Riemannian geometry is considered as an abstract logical construction, one may at all not introduce equality of two remote vectors and their parallelism. In this case we obtain a geometry, which has only some concepts of the proper Euclidean geometry, but not all of them. The obtained geometry appears to be more poor in concepts, than the proper Euclidean geometry.

However, if the (pseudo-)Riemannian geometry pretends to be used for description of the space-time geometry, one is forced to introduce these concepts, because a construction of particle dynamics is impossible without a conception of equality of remote vectors. The parallel transport is introduced in the Riemannian geometry as follows.

Let $\mathbf{u}(\mathbf{x})=\left\{u^k(\mathbf{x})\right\}$, $k=1,2,...n$ be a vector on the surface S_n. Simultaneously the vector $\mathbf{U}(\mathbf{x},\mathbf{0})=\{\mathbf{u}(\mathbf{x}),\mathbf{0}\}$ is a vector in the proper Euclidean space E_m. Its coordinates in the coordinate system K_m have the form $U^k=u^k(\mathbf{x})$, $k=1,2,...n$, $U^k=u^k=0$, $k=n+1,n+2,...m$. Let $d\xi$ be an infinitesimal vector of displacement on the surface S_n, $d\xi=\{d\mathbf{x},\mathbf{0}\}$.

Let us transport the vector $\mathbf{U}(\mathbf{x},\mathbf{0})$ in E_m from the point $\xi=\{\mathbf{x},\mathbf{0}\}$ into the point $\xi+d\xi=\{\mathbf{x}+d\mathbf{x},\mathbf{0}\}$. The transport is produced in such a way, that $\mathbf{U}(\mathbf{x},\mathbf{0})=\mathbf{U}(\mathbf{x}+\mathbf{dx},\mathbf{0})$. It is always possible in the proper Euclidean space E_m. As far as $d\mathbf{x}$ is infinitesimal quantity, in the coordinate system K_m the vector $\mathbf{U}(\mathbf{x}+\mathbf{dx},\mathbf{0})$ has the form

$$U^k(\mathbf{x}+\mathbf{dx},\mathbf{0})=U^k(\mathbf{x},\mathbf{0})+\delta U^k(\mathbf{x},\mathbf{0}), \qquad k=1,2,...m \tag{4.10}$$

where $\delta U^k(\mathbf{x},\mathbf{0})$ is an infinitesimal quantity of the order $O(|d\mathbf{x}|)$. As far as $U^k(\mathbf{x},\mathbf{0})=0$, $k=n+1,n+2,...m$, we obtain

$$U^k(\mathbf{x}+\mathbf{dx},\mathbf{0})=\delta U^k(\mathbf{x},\mathbf{0}), \qquad k=n+1,n+2,...m \tag{4.11}$$

and $|\mathbf{U}(\mathbf{x}+\mathbf{dx},\mathbf{0})|$ coincide with $|\mathbf{U}(\mathbf{x},\mathbf{0})|$ to within $O\left(|d\mathbf{x}|^2\right)$.

Let us project the vector $\mathbf{U}(\mathbf{x}+\mathbf{dx},\mathbf{0})$ onto the surface S_n. It means that we set $\delta U^k(\mathbf{x},\mathbf{0})=0, \qquad k=n+1,n+2,...m$. Calculation of $\delta U^k(\mathbf{x},\mathbf{0}), \qquad k=1,2,...n$ gives

$$\delta U^k(\mathbf{x},\mathbf{0})=\sum_{l,s=1}^{l,s=n}\gamma_{ls}^k(\mathbf{x})\,U^l(\mathbf{x},\mathbf{0})\,dx^s+O\left(|d\mathbf{x}|^2\right), \qquad k=1,2,...n \tag{4.12}$$

where the Christoffel symbol $\gamma_{ls}^k(\mathbf{x})$ is given on the surface S_n by the relation (4.5).

The relation (4.12) contains only tangential components $\mathbf{u}$ of the vector $\mathbf{U}\in\mathbf{E}_n$. It means that the relation (4.12) is a relation of the internal geometry on the surface S_n. This relation may be described in the form, containing only quantities of the internal geometry. The vector

$$u^k(\mathbf{x}+d\mathbf{x})=u^k(\mathbf{x})+\delta u^k(\mathbf{x})=u^k(\mathbf{x})+\gamma_{ls}^k(\mathbf{x})\,u^l(\mathbf{x})\,dx^s, \qquad k=1,2,...n \tag{4.13}$$

is considered to be in parallel with the vector $u^k(\mathbf{x})$, $k=1,2,...n$.

It is well known relation for the parallel transport of a vector in the Riemannian geometry. The parallel transport from point $\mathbf{x}$ to the point $\mathbf{x}'$ is produced as follows. The points $\mathbf{x}$ and $\mathbf{x}'$ are connected by some line $\mathcal{L}_{xx'}$. The line is divided into infinitesimal segments. The parallel transport is produced step by step along all segments by means of the formula (4.13). Result of the parallel transport depends on the path $\mathcal{L}_{xx'}$. Essentially the result of the parallel transport is multivariant, but each version of the parallel transport is connected with some path $\mathcal{L}_{xx'}$ of the transport.

The parallel transport in σ-Riemannian geometry was investigated in sec.6 of paper [2]. Here we present only the result of this investigation. The parallel transport (parallelism of two vectors) is defined by the relation (3.5). In the coordinate form it is written as follows

$$\left(\sigma_{i,l'}(x,x')\,\sigma_{k,s'}(x,x')-g_{ik}(x)\,g_{l's'}(x')\right)u^i(x)\,u^k(x)\,v^{l'}(x')\,v^{k'}(x')=0 \tag{4.14}$$

where $\sigma(x,x')$ is the world function of the σ-Riemannian geometry between the points with coordinates $x=\{x^i\}$, $i=1,2,...n$ and $x'=\{x'^i\}$, $i=1,2,...n$

$$\sigma_{il'}(x,x')\equiv\sigma_{i,l'}(x,x')\equiv\frac{\partial^2\sigma((x,x'))}{\partial x^i\partial x'^l} \tag{4.15}$$

Prime at the index means that this index corresponds to the point x'. $u^i(x)$ is a contravariant vector at the point x. $v^{k'}(x')$ is a contravariant vector at the point x'. If vectors $u^k(x)$ and $v^k(x')$ are in parallel, they satisfy the relation (4.14). The relation (4.14) has been obtained for infinitesimal vectors $u^i(x)$ and $v^{k'}(x')$. But the relation (4.14) is invariant with respect to a change of lengths of vectors $u^i(x)$ and $v^{k'}(x')$, and it appears to be valid also for finite vectors $u^i(x)$, $v^{k'}(x')$. At the deduction of the relation (4.14) one uses the fact that the σ is the world function, defined by the relation (4.6), i.e. it is a world function of the Riemannian space.

Let the vector $\mathbf{x}'-\mathbf{x}$ be infinitesimal. Then

$$d\xi^k = x'^k - x^k, \qquad v^k(\mathbf{x}') = v^k(\mathbf{x}) + \delta v^k(\mathbf{x}), \qquad k = 1,2,...n \tag{4.16}$$

where $d\xi^k$ and δv^k are infinitesimal quantities, and the relation (4.14) may be transformed to the form

$$\begin{aligned}&\left(u_k v^k\right)^2 - u_k u^k v_l v^l + \left(\gamma_{j;is} g_{kr} + g_{ij}\gamma_{r;ks} - g_{ik} g_{rj,s}\right) u^j u^r v^i v^k d\xi^s \\ &+ \left(g_{ij} g_{kr} - g_{ik} g_{rj}\right) u^j u^r \delta v^i v^k + \left(g_{ij} g_{kr} - g_{ik} g_{rj}\right) u^j u^r v^i \delta v^k = 0\end{aligned} \tag{4.17}$$

where

$$\gamma_{j;is} = g_{jl}\gamma_{is}^l, \qquad g_{rj,s} \equiv \frac{\partial g_{rj}}{\partial x^s} \tag{4.18}$$

It is easy to verify, that the conventional parallel transport (4.13)

$$v^k(\mathbf{x}) = u^k(\mathbf{x}), \qquad \delta v^k(\mathbf{x}) = \gamma_{ls}^k(\mathbf{x})\, v^l(\mathbf{x})\, d\xi^s \tag{4.19}$$

satisfies the relation (4.17). However, the solution (4.19) is unique, if direction $d\xi$ of transport coincides with the direction of the vector $\mathbf{u}(\mathbf{x})$. In the general case, when the curvature of the surface S_n does not vanish, the set of solutions $v+\delta v^k$ of equation (4.17) forms a cone (the collinearity cone). This cone degenerates into one-dimensional line, if $\mathbf{u}(\mathbf{x})$ is in parallel with $d\xi$, or if the curvature of the surface S_n vanishes. In these cases the conventional parallel transport (4.19) is the unique solution of (4.17).

One should expect, that the world function determines geometry uniquely. At any rate the Euclidean world function determines the Euclidean geometry uniquely. If the world function satisfies the triangle axiom (3.1), it determines the σ-Riemannian geometry uniquely. However, there is in addition the Riemannian geometry which does not coincide with the σ-Riemannian geometry in some details. The Riemannian geometry pretends to be a true geometry, describing the real space-time. At any rate, most of contemporary mathematicians consider the Riemannian geometry as a true geometry and ignore the σ-Riemannian geometry. We compare the σ-Riemannian geometry and the Riemannian geometry, firstly, as logical constructions and, secondly, as possible space-time geometries.

The Riemannian geometry is rather special geometry. It is described only in terms of coordinates. The Riemannian space is isometrically embeddable in the Euclidean space of rather large dimension. The Riemannian geometry uses the triangle axiom (3.1), taken in the form (4.6), as internal constraint of a geometry. It is internal in the sense, that some basic concepts of the Riemannian geometry (concept of a curve) cannot formulated at all without the triangle axiom. The concept of the world function is a secondary concept in the Riemannian geometry. The concept of the world function, defined by the relation (4.6), cannot be formulated without a reference to the concept of a curve (geodesic). It is impossible to obtain a generalization of the Riemannian geometry in terms of its basic concepts.

The σ-Riemannian geometry is a special case of the physical geometry, when the world function is restricted by the triangle axiom (3.1). The world function is a primary concept of the σ-Riemannian geometry. The σ-Riemannian geometry is not constrained by such conditions as a use of coordinates and isometric embeddability in the Euclidean space. The σ-Riemannian geometry does not use such non-metrical concept as the concept of a curve. The triangle axiom is an external constraint in the σ-Riemannian geometry in the sense, that it is not used in construction of the geometry. Avoiding the triangle axiom, we obtain a more general geometry, whose concepts are constructed without a reference to the triangle axiom.

The σ-Riemannian geometry is a more general construction than, the Riemannian one in the sense, that imposing some constraints on the σ-Riemannian geometry, one may to obtain the Riemannian geometry. For instance, in the physical geometry there are two sorts of straights

$$\mathcal{T}_{P_0P_1;P_0} \equiv \mathcal{T}_{P_0P_1} = \{R|\mathbf{P}_0\mathbf{P}_1 \parallel \mathbf{P}_0\mathbf{R}\} \tag{4.20}$$

and

$$\mathcal{T}_{P_0P_1;Q_0} = \{R|\mathbf{P}_0\mathbf{P}_1 \parallel \mathbf{Q}_0\mathbf{R}\} \tag{4.21}$$

The straights of the type (4.20) are single-variant (one-dimensional) in the σ-Riemannian geometry, whereas the straight of the type (4.21) are multivariant, in general. The straight of the type (4.20) is a special (degenerate) case of the straight (4.21), when the point Q_0 coincides with the point P_0. In the proper Euclidean geometry the straights of both types are single-variant (one-dimensional).

To obtain the Riemannian geometry from the σ-Riemannian geometry, one needs to remove the multivariant straights of the type (4.21) and to use only degenerate one-dimensional straights of the type (4.20). Thereafter the degenerate straights (4.20) are used for introduction of the concept of a geodesic. The world function is constructed as a secondary concept on the basis of the infinitesimal line element and of the geodesic. The world function is not used at the construction of the Riemannian geometry. The Riemannian geometry is constructed as an internal geometry of a surface in the proper Euclidean space. Eliminating multivariant straights (4.21) from the Riemannian geometry and leaving only degenerate straights, one cannot be sure that the way of Riemannian geometry construction, based on a use of only degenerate straights (geodesics), is consistent. A use of geodesics in the construction of geometry leads to the fact, that two-dimensional Euclidean plane with a hole cannot be isometrically embedded to the Euclidean plane without a hole. Besides, in the Riemannian geometry the absolute parallelism is absent, although it takes place in the σ-Riemannian geometry.

Thus, it seems, that the Riemannian geometry is only a part of the full physical geometry. The remaining part of the full geometry is cut by the constraint, that the degenerate straights (4.20) form the complete set of straights. This constraint is an internal constraint, which is used at the construction of the Riemannian geometry. It cannot be removed without destruction of the Riemannian geometry.

In application to the space-time the σ-Riemannian geometry is more effective, than the Riemannian geometry. The σ-Riemannian geometry is a more general geometry. To obtain a more general space-time geometry from the σ-Riemannian geometry, it sufficient only to remove the triangle axiom, which is an external constraint. We obtain immediately a space-time geometry of a general form. In the Riemannian geometry the triangle axiom is an internal constraint, which is used at the construction of the Riemannian geometry. Removing the triangle axiom, we lose the method of the Riemannian geometry construction.

Using the Riemannian geometry as a space-time geometry, one cannot imagine, that the space-time geometry may be responsible for quantum effects [3]. One cannot imagine that the space-time geometry may be responsible for a limited divisibility of physical bodies [4]. The unlimited divisibility, used in the Riemannian space-time geometry, generates an independence of the particle dynamics on the space-time geometry. In reality, the true space-time geometry must determine the particle dynamics [5].

References

[1] D. Hilbert, *Grundlagen der Geometrie.* 7 Auflage, B.G.Teubner, , Leipzig, Berlin, 1930.

[2] Yu.A.Rylov, Geometry without topology as a new conception of geometry. *Int. Jour. Mat. & Mat. Sci.* **30**, iss. 12, 733-760, (2002), (See also http://arXiv.org/abs/math.MG/0103002).

[3] Yu.A.Rylov, Non-Riemannian model of the space-time responsible for quantum effects. *Journ. Math. Phys.* **32(8)**, 2092-2098, (1991).

[4] Yu. A.Rylov Tubular geometry construction as a reason for new revision of the space-time conception. (Printed in *What is Geometry?* polimetrica Publisher, Italy, pp.201-235 http://www.polimetrica.com/polimetrica/406/), see also http;//afXiv.org/abs/physics/0504031

[5] Yu. A.Rylov, Deformation principle and further geometrization of physics http://arXiv.org/abs/0704.3003