Geometry without Topology

Yuri A. Rylov

Institute for Problems in Mechanics, Russian Academy of Sciences,
101-1, Vernadskii Ave., Moscow, 117526, Russia.
email: rylov@ipmnet.ru
Web site: $http://rsfq1.physics.sunysb.edu/\tilde{}rylov/yrylov.htm.$
or mirror Web site: $http://194.190.131.172/\tilde{}rylov/yrylov.htm.$

Abstract

The proper Euclidean geometry is considered to be metric space and described in terms of only metric and finite metric subspaces (σ-immanent description). Constructing the geometry, one does not use topology and topological properties. For instance, the straight, passing through points A and B, is defined as a set of such points R that the area $S(A,B,R)$ of triangle ABR vanishes. The triangle area is expressed via metric by means of the Hero's formula, and the straight appears to be defined only via metric, i.e. without a reference to (topological) concept of curve. (Usually, the straight is defined as the shortest curve, connecting two points A and B). Such a construction of geometry is free from such restrictions as continuity and dimensionality of the space which are generated by a use of topology but not by the geometry in itself. At such a description all information on the geometry properties (such as uniformity, isotropy, continuity and degeneracy) is contained in metric. The Riemannian geometry is constructed by two different ways: (1) by conventional way on the basis of metric tensor, (2) as a result of modification of metric in the σ-immanent description of the proper Euclidean geometry. The two obtained geometries are compared. The convexity problem in geometry and the problem of collinearity of vectors at distant points are considered. The nonmetric definition of curve is shown to be a concept of the proper Euclidean geometry which is inadequate to any non-Euclidean geometry

1 Introduction

There are several methods of the proper Euclidean geometry[1] description. The most old way of description is the axiomatic conception of the Euclidean geometry. The

[1]We use the term "Euclidean geometry" as a collective concept with respect to terms "proper Euclidean geometry" and "pseudo-Euclidean geometry". In the first case the eigenvalues of the metric tensor matrix have similar signs, in the second case they have different signs.

proper Euclidean geometry is described in terms of points, straights and planes, which are determined by their properties in terms of axioms. Some axioms describe properties of natural geometric objects (points, straights and planes), other axioms describe such properties of proper Euclidean geometry as uniformity, isotropy, continuity and degeneracy[2].

Real geometry of the space-time is uniform only approximately, and one needs to consider the geometries which should not be uniform, isotropic, continuous and degenerate. In other words, one needs to generalize and modify the proper Euclidean geometry. But it is quite impossible one to modify axioms of the proper Euclidean geometry, and one needs to describe the proper Euclidean geometry in the form, containing numerical characteristics which may be modified rather easily. Such numerical characteristic of the proper Euclidean geometry is the metric $\rho(P,Q)$, describing distance between any two points P and Q of the space. After modification of the Euclidean metric a new geometry appears, which may have other properties than uniformity, isotropy, continuity and degeneracy.

Usually for construction of (Riemannian) geometry one uses the following logical scheme

$$\begin{array}{ccccccc} & & & & & & \text{geometry} \\ \begin{array}{c}\text{coordinate}\\ \text{system}\end{array} & \longrightarrow & \begin{array}{c}\text{infinitesimal}\\ \text{distance}\end{array} & \longrightarrow & \begin{array}{c}\text{set of}\\ \text{geodesics}\end{array} & \begin{array}{c}\nearrow\\ \searrow\end{array} & \\ & & & & & & \begin{array}{c}\text{finite}\\ \text{distance}\end{array} \end{array} \tag{1.1}$$

As it follows from this scheme for construction of geometry one needs a coordinate system and a system of geodesics. The coordinate system is necessary for introduction of infinitesimal distance. The geodesic is defined as a shortest curve (line), connecting two points. Thus, the considered construction of geometry refers to the concept of a curve. The curve is a topological object, defined as a continuous mapping of the real axis onto the geometrical space of points. As a result the topology is considered usually to be a necessary element of geometry. According to (1.1) one cannot construct geometry without a use of topology (in the form of a curve). Actually the topology is only a mathematical tool, using for construction of geometry. To prove this, it is sufficient to construct geometry without a reference to the topological concept of a curve. We shall make this in the present paper.

The geometry is constructed in accord with the following logical scheme

$$\begin{array}{ccc} & \nearrow & \text{geometry} \\ \begin{array}{c}\text{finite}\\ \text{distance}\end{array} & & \\ & \searrow & \begin{array}{c}\text{set of}\\ \text{geodesics}\end{array} \end{array} \tag{1.2}$$

where geometry is constructed independently of a possibility of the geodesics construction. It is possible such a situation, when the geometry can be constructed,

[2]In general case the set of vectors, parallel to the given vector, forms a cone. Degeneracy of the proper Euclidean geometry means that this cone degenerates into a line.

whereas geodesics (the shortest curves) cannot. Such a situation is not exotic, because the real space-time geometry appears to be of such a kind. Timelike geodesics of the space-time are substituted by thin hallow tubes. Thickness of the tubes is microscopic. Describing macroscopic phenomena, one may neglect the tube thickness and substitute the tubes by lines. Then geometry may be considered as a degenerate one (the tubes degenerate into lines). Describing microscopic phenomena, one may not neglect the thickness of tubes, because the thickness of tubes (nondegeneracy of geometry) is a reason of quantum effects. Besides the geometry constructed in accord with the scheme (1.2) is free from such constraints as continuity and degeneracy, imposed by a use of the concept of a curve.

To carry out the idea of nondegenerate geometry, let us give some definitions which help us to formulate the problem of generalization and modification of the proper Euclidean geometry.

Definition 1.1 *The metric space $M = \{\rho, \Omega\}$ is a set Ω of points $P \in \Omega$ with the metric ρ given on $\Omega \times \Omega$*

$$\rho : \quad \Omega \times \Omega \to D_+ \subset \mathbb{R} \tag{1.3}$$

$$\rho(P, P) = 0, \qquad \rho(P, Q) = \rho(Q, P), \qquad \forall P, Q \in \Omega \tag{1.4}$$

$$D_+ = [0, \infty), \qquad \rho(P, Q) = 0, \quad \textit{if and only if} \quad P = Q, \qquad \forall P, Q \in \Omega \tag{1.5}$$

$$\rho(P, Q) + \rho(Q, R) \geq \rho(P, R), \qquad \forall P, Q, R \in \Omega \tag{1.6}$$

Definition 1.2 *Any subset $\Omega' \subset \Omega$ of points of the metric space $M = \{\rho, \Omega\}$, equipped with the metric ρ' which is a contraction $\rho|_{\Omega' \times \Omega'}$ of the mapping (1.3). on the set $\Omega' \times \Omega'$ is called the metric subspace $M' = \{\rho', \Omega'\}$ of the metric space $M = \{\rho, \Omega\}$.*

It is easy to see that the metric subspace $M' = \{\rho', \Omega'\}$ is a metric space.

Definition 1.3 *The metric space $M = \{\rho, \Omega\}$ is called finite, if the set Ω contains a finite number of points. The finite metric subspace $M(\mathcal{P}^n) = \{\rho, \mathcal{P}^n\}$ of $M = \{\rho, \Omega\}$, consisting of $n+1$ points $\mathcal{P}^n \equiv \{P_0, P_1, \ldots P_n\} \subset \Omega$, $\quad n = 0, 1, \ldots$ is called the nth order metric subspace.*

The proper Euclidean space may be considered to be a kind of metric space $E = \{\rho_E, \Omega\}$. Being a metric space, the proper Euclidean space and geometry on this space can be described in terms of only metric ρ and of finite metric subspaces. The finite metric subspaces $M(\mathcal{P}^n)$ are the simplest constituents of the metric space. Some properties of finite metric subspaces $M(\mathcal{P}^n)$ were investigated by Blumenthal [1], but he did not consider them to be primitive fundamental objects of metric space as we do. Metric space $M(\mathcal{P}^n)$, consisting of $n+1$ points and having nonvanishing length (concept of the length will be defined further), generates in the proper Euclidean space n-dimensional plane $\mathcal{L}_n(\mathcal{P}^n)$, which appears to be an attribute of $M(\mathcal{P}^n)$ and can be defined in terms of $M(\mathcal{P}^n)$.

Definition 1.4 *Elementary geometrical object is a set of points having some metric property.*

Definition 1.5 *Geometrical object is a set of points derived as joins and intersections of elementary geometrical objects.*

In other words, a geometrical object is a metric subspace $M_G = \{\rho, G\}$, $G \subset \Omega$ of metric space $M = \{\rho, \Omega\}$.

Definition 1.6 *Geometry is a totality of all propositions (definitions, axioms and theorems) on properties of geometrical objects.*

In other words, the geometry is a totality of all propositions on properties of all metric subspaces of the metric space $M = \{\rho, \Omega\}$.

Let us consider some examples of elementary geometrical objects.

Definition 1.7 *The sphere $\mathcal{S}(O;P)$, having its center at the point O and passing through the point P, is the set of points $R \in \Omega$ of the metric space $M = \{\rho, \Omega\}$, defined by the relation*

$$\mathcal{S}(O;P) = \{R | \rho(O,R) = \rho(O,P)\}, \qquad O,P,R \in \Omega$$

The basic points O and P, determining the sphere $\mathcal{S}(O;R)$, are not equivalent, because $\mathcal{S}(O;R)$ and $\mathcal{S}(R;O)$ are different elementary geometrical objects (different spheres). In particular, $P \in \mathcal{S}(O;R)$, but $O \notin \mathcal{S}(O;R)$. The sphere $\mathcal{S}(O;P)$ is an attribute of zeroth order metric subspaces $M(O)$ and $M(P)$ (or two points O, P).

Definition 1.8 *The circle cylinder $\mathcal{C}(P_1,P_2;P)$, passing through the point P, with axis, determined by the basic points P_1, P_2, is the set of points $R \in \Omega$ of the metric space $M = \{\rho, \Omega\}$, defined by the relation*

$$\mathcal{C}(P_1,P_2;P) = \{R | S_2(P_1,P_2,R) = S_2(P_1,P_2,P)\}, \qquad P_1,P_2,P,R \in \Omega$$

where $S_2(P_1,P_2,R)$ is the area of the triangle with vertices at the points P_1, P_2, R. If the areas of triangles $\triangle P_1P_2R$ and $\triangle P_1P_2P$ are equal, the heights (radii) dropped from the vertices R and P of these triangles onto their common base P_1P_2 (axis of the cylinder) are also equal. The triangle area $S_2(P_1,P_2,R)$ can be expressed via metric by the Hero's formula

$$S_2(A,B,C) = \sqrt{p(p-a)(p-b)(p-c)},$$

$$a = \rho(B,C), \qquad b = \rho(A,C), \qquad c = \rho(A,B), \qquad p = (a+b+c)/2$$

The circle cylinder $\mathcal{C}(P_1,P_2;P)$ is an attribute of two finite metric subspaces $M(P_1,P_2)$ and $M(P)$.

Definition 1.9 *The ellipsoid $\mathcal{E}(P_1, P_2; P)$, having its focuses at the basic points P_1, P_2 and passing through the point P is the set of points $R \in \Omega$ of the metric space $M = \{\rho, \Omega\}$, defined by the relation*

$$\mathcal{E}(P_1, P_2; P) = \{R | \rho(P_1, R) + \rho(P_2, R) = \rho(P_1, P) + \rho(P_2, P)\}, \qquad P_1, P_2, P, R \in \Omega$$

The ellipsoid $\mathcal{E}(P_1, P_2; P)$ is an attribute of two finite metric subspaces $M(P_1, P_2)$ and $M(P)$. If $P_1 \neq P$ and $P_2 \neq P$, the points $P_1, P_2 \notin \mathcal{E}(P_1, P_2; P)$, but the point $P \in \mathcal{E}(P_1, P_2; P)$. If the point $P = P_1$, the ellipsoid $\mathcal{E}(P_1, P_2; P)$ degenerates into segment $\mathcal{T}_{[P_1 P_2]}$ between the points P_1 and P_2 of the straight line $\mathcal{T}_{P_1 P_2}$, passing through the points P_1 and P_2. The segment $\mathcal{T}_{[P_1 P_2]}$ is defined as follows.

Definition 1.10 *The segment $\mathcal{T}_{[P_1 P_2]}$ of the straight between the basic points P_1, P_2 is the set of points $R \in \Omega$ of the metric space $M = \{\rho, \Omega\}$, defined by the relation*

$$\mathcal{T}_{[P_1 P_2]} = \{R | \rho(P_1, R) + \rho(P_2, R) - \rho(P_1, P_2) = 0\}, \qquad P_1, P_2, R \in \Omega \tag{1.7}$$

The segment $\mathcal{T}_{[P_1 P_2]}$ is an elementary geometrical object which does not depend on the order of points P_1, P_2. Besides both basic points $P_1, P_2 \in \mathcal{T}_{[P_1 P_2]}$. The segment $\mathcal{T}_{[P_1 P_2]}$ is an attribute of the first order metric subspace $M(P_1, P_2)$ in the sense that $\mathcal{T}_{[P_1 P_2]}$ is determined by the metric subspace $M(P_1, P_2)$ in itself. For instance, the sphere $\mathcal{S}(O; P)$ is determined by the points O, P of the metric subspace $M(O, P)$, but not by the metric subspace $M(O, P)$ in itself, and the sphere $\mathcal{S}(O; P)$ is not an attribute of the metric subspace $M(O, P)$, but it is an attribute of two zeroth order metric subspaces $M(O)$ and $M(P)$ (or two points O, P).

Definition 1.11 *The elemetary geometrical object which is an attribute of the nth order metric subspace $M(\mathcal{P}^n)$ is the nth order natural geometric object (the nth order NGO).*

Such geometrical objects as a point, an Euclidean straight, and an Euclidean plane are NGOs of the proper Euclidean geometry. The point P_0 is the zeroth order NGO $\mathcal{T}_{P_0}$ of the proper Euclidean geometry which is determined by the zeroth order metric subspace $M(P_0) = P_0$. The straight $\mathcal{T}_{P_0 P_1}$ of the proper Euclidean geometry is the first order NGO which is determined by the first order metric subspace $M(P_0, P_1)$. It means, in particular, that $\mathcal{T}_{P_0 P_1} = \mathcal{T}_{P_1 P_0}$.

The two-dimensional plane $\mathcal{T}_{P_0 P_1 P_2}$ of the proper Euclidean geometry is the second order NGO, determined by the second order metric subspace $M(P_0, P_1, P_2)$. It means that the NGO $\mathcal{T}_{P_0 P_1 P_2}$ does not depend on the order of basic points P_0, P_1, P_2, which determine $\mathcal{T}_{P_0 P_1 P_2}$. It does not always happen that the second order metric subspace $M(P_0, P_1, P_2)$ determines $\mathcal{T}_{P_0 P_1 P_2}$. Only $M(P_0, P_1, P_2) \not\subset \mathcal{T}_{P_0 P_1}$ enables to determine $\mathcal{T}_{P_0 P_1 P_2}$.

For explicit determination of the nth order NGO one needs to attribute a length $|M(\mathcal{P}^n)|$ to any nth order metric subspace $M(\mathcal{P}^n)$

Definition 1.12 *The squared length* $|M(\mathcal{P}^n)|^2$*of the nth order metric subspace* $M(\mathcal{P}^n) \subset \Omega$ *of the proper Euclidean space* $E = \{\rho_E, \Omega\}$ *is the real number.*

$$|M(\mathcal{P}^n)|^2 = (n!S_n(\mathcal{P}^n))^2 = F_n(\mathcal{P}^n)$$

where $S_n(\mathcal{P}^n)$ *is the Euclidean volume of the* $(n+1)$*-edr with vertices at points* $\mathcal{P}^n \equiv \{P_0, P_1, \ldots P_n\} \subset \Omega$.

In the proper Euclidean geometry the volume $S_n(\mathcal{P}^n)$ of the $(n+1)$-edr and the value $F_n(\mathcal{P}^n)$ of the function F_n, connected with it, can be expressed in terms of metric ρ by means of relations

$$F_n: \quad \Omega^{n+1} \to \mathbb{R}, \qquad \Omega^{n+1} = \bigotimes_{k=1}^{n+1} \Omega, \qquad n = 1, 2, \ldots \tag{1.8}$$

$$F_n(\mathcal{P}^n) = \det ||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)||, \qquad P_0, P_i, P_k \in \Omega, \qquad i, k = 1, 2, ...n \tag{1.9}$$

$$\begin{aligned} (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k) &\equiv \Gamma(P_0, P_i, P_k) \equiv \sigma(P_0, P_i) + \sigma(P_0, P_k) - \sigma(P_i, P_k), \\ i, k &= 1, 2, ...n. \end{aligned} \tag{1.10}$$

where the function σ is defined via metric ρ by the relation

$$\sigma(P, Q) \equiv \frac{1}{2}\rho^2(P, Q), \qquad \forall P, Q \in \Omega. \tag{1.11}$$

and $\mathcal{P}^n$ denotes $n+1$ points $P_0, P_1, \ldots, P_n$ of Ω

$$\mathcal{P}^n = \{P_0, P_1, \ldots, P_n\} \subset \Omega \tag{1.12}$$

The function σ, called world function [2], is very important quantity which may be used instead of metric ρ. In many cases a use of the function σ appears to be more convenient than a usage of metric ρ. The squared length $|M(\mathcal{P}^n)|^2 = F_n(\mathcal{P}^n)$ is calculated for the proper Euclidean space, but the expression (1.9) - (1.11) may be used for any finite subspaces of any metric space, because it contains only world function σ (metric ρ) and may be calculated for any metric space.

Definition 1.13 *A description is called* σ*-immanent, if it does not contain any references to objects or concepts other than finite subspaces of the metric space and its metric.*

Prefix σ in the term "σ-immanent" associates with the world function σ. Concept of σ-immanent description is very important for modification of the proper Euclidean geometry. Considering the proper Euclidean geometry to be a standard geometry and defining a geometrical object there in a σ-immanent way, one can use this definition in any metric space.

Note that definition of geometrical objects is a principal problem of the metric geometry, i.e. the geometry, generated by the metric space. The shortest (line), connecting two arbitrary points $P, Q \in \Omega$ of the metric space $\{\rho, \Omega\}$, is the basic geometrical object which is constructed usually in the metric space [3]. One can construct an angle, triangle, different polygons from segments of the shortest. Construction of two-dimensional and three-dimensional planes in the metric space is rather problematic. At any rate it is unclear how one could construct these planes, using the shortest as the main geometrical object. A possibility of the metric space description in terms of only the shortest is restricted. Although exhibiting ingenuity, such a description may be constructed. For instance, A.D. Alexandrov showed that internal geometry of two-dimensional boundaries of convex three-dimensional bodies may be represented in terms of metric [4]. Apparently, without introducing geometric objects which are analogs of two-dimensional plane, the solution of similar problem for three-dimensional boundaries of four-dimensional bodies is very difficult.

Note that constraints (1.5), (1.6), imposed on metric, are necessary only for constructing the shortest. The shortest, determined by two points P_1, P_2, may be replaced by the σ-immanent definition (1.7) of segment $\mathcal{T}_{[P_1P_2]}$, which coincides with the shortest in the metric space, described by the definition 1.1. This definition in itself does not need constraint (1.5), describing definiteness of the metric space, and constraint (1.6), describing one-dimensionality of the segment $\mathcal{T}_{[P_1P_2]}$. If the metric is not restricted by constraint (1.6), the segment $\mathcal{T}_{[P_1P_2]}$ takes the shape of a hallow tube, reminding ellipsoid, described by definition 1.9. If the constraint (1.6) is strengthened ($\leq$ is replaced by $<$), the segment $\mathcal{T}_{[P_1P_2]}$ degenerates into two points P_1, P_2. The case of the one-dimensional shortest is intermediate between the two cases.

In the case of the proper Euclidean space, considered to be a metric space, the first order NGO, defined by (1.7) is one-dimensional line. It is not clear whether one-dimensionality is a special property of the Euclidean geometry, or it is a property of any geometry in itself. We do not see, why one should insist on the one-dimensionality of the first order NGO $\mathcal{T}_{[P_1P_2]}$ in the case of an arbitrary modification of the proper Euclidean geometry. First, it is useful to consider the most general modification of the proper Euclidean geometry. Second, at the end of investigation, if it appears to be necessary, one can always reduce a degree of generalization, imposing additional constraints.

In the proper Euclidean space the n-dimensional plane (nth order NGO) $n = 1, 2, \ldots$ is defined as follows

Definition 1.14 *The nth order metric subspace $M(\mathcal{P}^n)$ of unvanishing length $|M(\mathcal{P}^n)|^2 = F_n(\mathcal{P}^n) \neq 0$ determines the nth order tube (the nth order NGO) $\mathcal{T}(\mathcal{P}^n)$ by means of the relation*

$$\mathcal{T}(\mathcal{P}^n) \equiv \mathcal{T}_{\mathcal{P}^n} = \left\{P_{n+1} | F_{n+1}\left(\mathcal{P}^{n+1}\right) = 0\right\}, \qquad P_i \in \Omega, \qquad i = 0, 1 \ldots n+1, \tag{1.13}$$

where the function F_n is defined by the relations (1.8), (1.10)

The nth order tube $\mathcal{T}_{\mathcal{P}^n}$ which is an analog of the n-dimensional Euclidean plane may be constructed in any metric space, as far as its definition 1.14 is σ-immanent.

It may be defined also in the metric space with omitted constraints (1.5), (1.6), imposed usually on the metric. We shall refer to such a generalized metric space as the σ-space. The geometry, generated by the σ-space, will be referred to as T-geometry (tubular geometry).

Definition 1.15 *σ-space $V = \{\sigma, \Omega\}$ is nonempty set Ω of points P with given on $\Omega \times \Omega$ real function σ*

$$\sigma : \quad \Omega \times \Omega \to \mathbb{R}, \qquad \sigma(P,P) = 0, \qquad \sigma(P,Q) = \sigma(Q,P) \qquad \forall P, Q \in \Omega. \quad (1.14)$$

The function σ is called world function, or σ-function. The metric ρ may be introduced in the σ-space by means of the relation (1.11). If σ is positive, metric ρ is also positive, but if σ is negative, the metric is imaginary.

Definition 1.16 . *Nonempty subset $\Omega' \subset \Omega$ of points of the σ-space $V = \{\sigma, \Omega\}$ with the world function $\sigma' = \sigma|_{\Omega' \times \Omega'}$, which is a contraction of σ on $\Omega' \times \Omega'$, is called σ-subspace $V' = \{\sigma', \Omega'\}$ of σ-space $V = \{\sigma, \Omega\}$.*

Further the world function $\sigma' = \sigma|_{\Omega' \times \Omega'}$, which is a contraction of σ will be designed by means of σ. Any σ-subspace of σ-space is a σ-space.

Definition 1.17 . *σ-space $V = \{\sigma, \Omega\}$ is called isometrically embeddable in σ-space $V' = \{\sigma', \Omega'\}$, if there exists such a monomorphism $f : \Omega \to \Omega'$, that $\sigma(P,Q) = \sigma'(f(P), f(Q)), \quad \forall P, \forall Q \in \Omega, \quad f(P), f(Q) \in \Omega'$,*

Any σ-subspace V' of σ-space $V = \{\sigma, \Omega\}$ is isometrically embeddable in it.

Definition 1.18 . *Two σ-spaces $V = \{\sigma, \Omega\}$ and $V' = \{\sigma', \Omega'\}$ are called to be isometric (equivalent), if V is isometrically embeddable in V', and V' is isometrically embeddable in V.*

Definition 1.19 *The σ-space $M = \{\rho, \Omega\}$ is called a finite σ-space, if the set Ω contains a finite number of points.*

Definition 1.20 . *The σ-subspace $M_n(\mathcal{P}^n) = \{\sigma, \mathcal{P}^n\}$of the σ-space $V = \{\sigma, \Omega\}$, consisting of $n+1$ points $\mathcal{P}^n = \{P_0, P_1, ..., P_n\}$ is called the nth order σ-subspace .*

All geometrical objects of T-geometry are obtained as follows. Geometrical objects of the proper Euclidean geometry are defined in the σ-immanent form. Then they may be considered to be definitions of corresponding geometrical objects in T-geometry. The world function σ of the proper Euclidean space satisfies some σ-immanent relations, describing special properties of the proper Euclidean geometry. Metric side of these relations had been formulated and proved by Menger [5]. Using our designations, we present this result in the form of theorem.

Theorem 1.1 *The σ-space $V = \{\sigma, \Omega\}$ is isomerically embeddable in n-dimensional Euclidean space E_n, if and only if any $(n+2)$th order σ-subspace $M(\mathcal{P}^{n+2}) \subset \Omega$ is isometrically embeddable in E_n.*

Unfortunately, the formulation of this theorem is not σ-immanent, as far as it contains a reference to n-dimensional Euclidean space E_n which is not defined σ-immanently. A more constructive version of the σ-space Euclideness conditions is formulated in the form of the following theorem.

Theorem 1.2 *The σ-space $V = \{\sigma, \Omega\}$ is the n-dimensional Euclidean space, if and only if the following three σ-immanent conditions are fulfilled.*

I.

$$\exists \mathcal{P}^n \subset \Omega, \qquad F_n(\mathcal{P}^n) \neq 0, \qquad F_{n+1}(\Omega^{n+2}) = 0, \tag{1.15}$$

II.

$$\sigma(P,Q) = \frac{1}{2} \sum_{i,k=1}^{n} g^{ik}(\mathcal{P}^n)[\Gamma(P_0,P_i,P) - \Gamma(P_0,P_i,Q)]$$
$$\times [\Gamma(P_0,P_k,P) - \Gamma(P_0,P_k,Q)], \qquad \forall P, Q \in \Omega \tag{1.16}$$

where $\Gamma(P_0, P_k, P)$ are defined by the relations (1.10). The quantities $g^{ik}(\mathcal{P}^n)$, $(i,k = 1,2,\dots n)$ are defined by the relations

$$\sum_{k=1}^{n} g_{ik}(\mathcal{P}^n) g^{kl}(\mathcal{P}^n) = \delta_i^l, \qquad i,l = 1,2,\dots n \tag{1.17}$$

where

$$g_{ik}(\mathcal{P}^n) = \Gamma(P_0,P_i,P_k), \qquad i,k = 1,2,\dots n \tag{1.18}$$

III. The relations

$$\Gamma(P_0,P_i,P) = x_i, \qquad x_i \in \mathbb{R}, \qquad i = 1,2,\dots n, \tag{1.19}$$

considered to be equations for determination of $P \in \Omega$, have always one and only one solution.

Remark 1.1 *For the Euclidean space to be the proper Euclidean the eigenvalues of the matrix $g_{ik}(\mathcal{P}^n) = \Gamma(P_0,P_i,P_k)$, $\quad i,k = 1,2,\dots n$ are to be of the same sign, otherwise the Euclidean space is pseudo-Euclidean.*

Remark 1.2 *The condition (1.15) is a corollary of condition (1.16). It is formulated as a separate condition in order to separate definition of dimension and that of the coordinate system.*

Let us note that all three conditions are written in σ-immanent form. Proof of this theorem can be found in [6]. Now we consider how results of this theorem can be used for construction of conventional description of the proper Euclidean space in some rectilinear coordinate system, starting from an abstract σ-space, satisfying conditions I - III of the theorem.

Let there be σ-space $V = \{\sigma, \Omega\}$, and it is known that conditions I - III of the theorem are fulfilled. Then the σ-space V is an Euclidean space, but the dimension n of the space is unknown. To determine the dimension n, let us take two different points $P_0, P_1 \in \Omega$, $F_1(\mathcal{P}^1) = 2\sigma(P_0, P_1) \neq 0$.

1. Let us construct the first order tube $\mathcal{T}(\mathcal{P}^1)$. If $\mathcal{T}(\mathcal{P}^1) = \Omega$, then dimension of the σ-space V $n = 1$. If $\Omega \backslash \mathcal{T}(\mathcal{P}^1) \neq \emptyset$, $\exists P_2 \in \Omega$, $P_2 \notin \mathcal{T}(\mathcal{P}^1)$, and hence, $F_2(\mathcal{P}^2) \neq 0$.

2. Let us construct the second order tube $\mathcal{T}(\mathcal{P}^2)$. If $\mathcal{T}(\mathcal{P}^2) = \Omega$, then $n = 2$, otherwise $\exists P_3 \in \Omega$, $P_3 \notin \mathcal{T}(\mathcal{P}^2)$, and hence, $F_3(\mathcal{P}^3) \neq 0$.

3. Let us construct the third order tube $\mathcal{T}(\mathcal{P}^3)$. If $\mathcal{T}(\mathcal{P}^3) = \Omega$, then $n = 3$, otherwise $\exists P_4 \in \Omega$, $P_4 \notin \mathcal{T}(\mathcal{P}^3)$, and hence, $F_4(\mathcal{P}^4) \neq 0$.

4. Etc.

Continuing this process, one determines such $n+1$ points $\mathcal{P}^n$, that the condition $\mathcal{T}(\mathcal{P}^n) = \Omega$ and, hence, conditions (1.15) are fulfilled.

Then by means of relations

$$x_i(P) = \Gamma(P_0, P_i, P), \qquad i = 1, 2, \ldots n, \tag{1.20}$$

one attributes covariant coordinates $x(P) = \{x_i(P)\}$, $i = 1, 2, \ldots n$ to $\forall P \in \Omega$. Let $x = x(P) \in \mathbb{R}^n$ and $x' = x(P') \in \mathbb{R}^n$. Substituting $\Gamma(P_0, P_i, P) = x$ and $\Gamma(P_0, P_i, P') = x'_i$ in (1.16), one obtains the conventional expression for the world function of the Euclidean space in the rectilinear coordinate system

$$\sigma(P, P') = \sigma_E(x, x') = \frac{1}{2} \sum_{i,k=1}^{n} g^{ik}(\mathcal{P}^n)(x_i - x'_i)(x_k - x'_k) \tag{1.21}$$

where $g^{ik}(\mathcal{P}^n)$, defined by relations (1.18) and (1.17), is the contravariant metric tensor in this coordinate system.

Condition III of the theorem states that the mapping

$$x : \Omega \to \mathbb{R}^n$$

described by the relation (1.20) is a bijection, i.e. for $\forall y \in \mathbb{R}^n$ there exists such one and only one point $Q \in \Omega$, that $y = x(Q)$.

Thus, on the base of the world function, given on abstract set $\Omega \times \Omega$, one can determine the dimension n of the Euclidean space, construct rectilinear coordinate system with the metric tensor $g_{ik}(\mathcal{P}^n) = \Gamma(P_0, P_i, P_k)$, $\quad i, k = 1, 2, \ldots n$ and describe all geometrical objects which are determined in terms of coordinates. The Euclidean space and Euclidean geometry is described in terms and only in terms of world function (metric). Changing the world function, one obtains another σ-space and another (non-Euclidean) geometry. One should expect that another geometry is also described completely in terms of the world function. The properties of geometrical objects may appear other than the properties of these objects in the proper Euclidean geometry. For instance, in the Euclidean geometry $\mathcal{T}_{P_0P_1} \subset \mathcal{T}_{P_0P_1P_2}$, i.e. the straight, passing through the points P_0 and P_1, belongs to any two-dimensional plane, passing through these points. To prove these statement, one needs to use the relations (1.16). In the case of non-Euclidean geometry the relation $\mathcal{T}_{P_0P_1} \subset \mathcal{T}_{P_0P_1P_2}$ is invalid, in general.

Another example. Two circle cylinders $\mathcal{C}(P_0, P_1; P)$ and $\mathcal{C}(P_0, P_1'; P)$, $P_1' \in \mathcal{T}_{[P_0P_1]}$, $P_1' \neq P_1$, $P_1' \neq P_0$ coincide in the proper Euclidean geometry, but they are different geometrical objects in non-Euclidean geometry.

In the proper Euclidean geometry there exists geometrical object called line.

Definition 1.21 *The broken line $\mathcal{T}_{\rm br}$ is the set of connected straight segments $\mathcal{T}_{[P_iP_{i+1}]}$*

$$\mathcal{T}_{\rm br} = \bigcup_i \mathcal{T}_{[P_iP_{i+1}]} \tag{1.22}$$

The continuous line (or curve) is defined as a limit of the broken line $\mathcal{T}_{\rm br}$ at $P_i \to P_{i+1}$, $(i = 0, \pm 1, \pm 2, \ldots)$. The smooth line is defined as a limit of (1.22) at $P_i \to P_{i+1}$, $(i = 0, \pm 1, \pm 2, \ldots)$ under the constraint that $\cos \angle P_{i-1}P_iP_{i+1} \to -1$. Defining the segment $\mathcal{T}_{[P_iP_{i+1}]}$ by means of definition 1.10, one obtains metric definition of broken line (1.22). To obtain metric definition of the continuous line and that of smooth line, one needs to go to corresponding limits in (1.22). According to this definition the line is many-point geometrical object. This object is very complicated, because their points are given independently (i.e. there are many degrees of freedom).

On the other hand, in the proper Euclidean geometry there exists another (non-metric) definition of continuous line. The continuous line $\mathcal{L}$ is defined as a continuous mapping

$$L: \ I \to \Omega, \qquad I = [0,1] \subset \mathbb{R}. \tag{1.23}$$

Strictly, the geometrical object is a set $\mathcal{L} = L(I) \subset \Omega$ of points of the σ-space $V = \{\sigma, \Omega\}$, but not the mapping (1.23) in itself. However, as far as the number set I is fixed and the same in all cases, then with some stipulations one can consider the correspondence between the mapping L and the set of images $\mathcal{L} = L(I)$ to be one-to-one. Then one can label the geometrical objects (considered as σ-subspaces) by means of mappings (1.23) and identify the mapping (1.23) with the geometrical object $\mathcal{L}$, called curve (line).

In the proper Euclidean geometry the definition of line (1.23) agrees with the definition 1.22. But in non-Euclidean geometry definitions (1.23) and (1.22) do not agree, in general. Already in the Riemannian geometry an application of definition (1.23) as one of basic definitions poses problems.

In the Riemannian space the world function $\sigma_R(x, x')$ between the points x and

x' is determined by the relation [2]

$$\sigma_R(x,x') = \frac{1}{2}\left(\int\limits_{\mathcal{L}_{[xx']}} \sqrt{g_{ik}dx^i dx^k}\right)^2 \tag{1.24}$$

where $\mathcal{L}_{[xx']}$ denotes segment of geodesic connecting points x and x'. Let us use the world function (1.24) instead of the Euclidean world function (1.21) in the σ-immanent description of geometry. In other words, let us use for construction of geometry the logical scheme (1.2), but not (1.1). One obtains the σ-Riemannian geometry which is expected to be equivalent to the Riemannian geometry, because both the Riemannian geometry and the σ-Riemannian one are two generalizations of the Euclidean geometry, using the same world function which has to describe any geometry completely. In reality, using for geometry construction different logical schemes, the σ-Riemannian geometry and the Riemannian one coincide, but not at all points,.

The point is that the world function is a fundamental object of the σ-Riemannian geometry, whereas it is a derivative object in the Riemannian geometry, where the infinitesimal distance and the curve (line) are fundamental objects. The line $\mathcal{L}$, defined by nonmetric definition (1.23), is a complicated and fundamental structure of Riemannian geometry, which is absent in such a form in the σ-Riemannian geometry. The continuous line $\mathcal{L}$ in the σ-Riemannian geometry may be defined as a limit of the broken tube (1.22). But it is a derivative (not fundamental) geometrical object.

As a whole the situation looks as follows. The σ-Riemannian geometry is constructed σ-immanently, i.e. on the base of metric and does not need the nonmetric definition of line (1.23). The Riemannian geometry is constructed on the base of infinitesimal metric $dS = \sqrt{g_{ik}dx^i dx^k}$ (which coincide with the infinitesimal metric of the σ-Riemannian geometry) and uses the nonmetric definition of line (1.23) for definition of finite metric. As a result the finite metric of both geometries coincide, but only in the whole domain $D = \Omega$, where both geometries are defined. If one considers σ-Riemannian and Riemannian geometries in some subdomain $D' \subset D$, the finite metrics are defined in D' in different ways for these geometries. For σ-Riemannian geometry the finite metric in D' is defined as a cotraction of the finite metric in D, whereas for Riemannian geometry the finite metric is defined on the basis of system of geodesics inside D' which does not coincide, in general, with the system of geodesics in D. The geodesic segment $\mathcal{L}_{[xx']}$ which determines $\sigma_R(x,x')$ is a lengthy geometrical object, depending on the shape of the region D', where the Riemannian geometry is defined. As a result the finite metrics of both geometries may be different in $D' \subset D$, although they coincide in D.

Note that the nonmetric definition of line (1.23) needs additional constraints to be rather definite. Let us discuss these problems.

2 Riemannian space and convexity problem

The Riemannian space and the Riemannian geometry are introduced as follows. n-dimensional Riemannian space can be derived as a result of a generalization of the n-dimensional proper Euclidean space, written in a covariant form. Indeed, the n-dimensional Euclidean space $E_n = \{\mathbf{g}_E, K, \mathbb{R}^n\}$ is described by the infinitesimal distance written in the rectilinear coordinate system K

$$dS^2 = g_{ik}dx^i dx^k, \qquad g_{ik}= \text{diag}\,\{1, 1, ...1\} \tag{2.1}$$

$\mathbf{g}_E$ denotes the matrix $g_{ik}=$ diag $\{1, 1, ...1\}$ of the metric tensor. In the arbitrary curvilinear coordinate system $\tilde{K}$ the same distance have the form

$$dS^2 = \tilde{g}_{ik}\left(\tilde{x}\right) d\tilde{x}^i d\tilde{x}^k, \qquad \det \tilde{g}_{ik} \neq 0, \tag{2.2}$$

Here $\tilde{g}_{ik}\left(x\right)$ is constrained by the relation

$$\tilde{g}_{ik}\left(x\right) = \sum_{l=1}^{n} \frac{\partial f_l\left(x\right)}{\partial x^i} \frac{\partial f_l\left(x\right)}{\partial x^k}, \tag{2.3}$$

where $f_l : \mathbb{R}^n \to \mathbb{R}$, $l = 1, 2, \ldots n$ are n functions restricted by one condition $\det ||\partial f_i/\partial x^k|| \neq 0$, $i, k = 1, 2, ...n$. If $\tilde{g}_{ik}\left(x\right)$ does not satisfy the relation (2.3) the space stops to be Euclidean and becomes a Riemannian space $R_n = \left\{\tilde{\mathbf{g}}, \tilde{K}, \mathbb{R}^n\right\}$. Constraint (2.3) is a condition of the Euclideness of the space.

Eliminating (2.3) one obtains a Riemannian space $R_n = \{\mathbf{g}, K, \mathbb{R}^n\}$, which is determined by the form of the metric tensor $g_{ik}(x)$. The world function is determined by the relation (1.24), where $\mathcal{L}_{[xx']} \subset \mathbb{R}^n$ is the geodesic segment of the geodesic $\mathcal{L}_{xx'} \subset \mathbb{R}^n$. This geodesic is an extremal of (1.24), considered as a functional of the curve $\mathcal{L} :\ x = x(\tau)$, written in the form

$$\sigma[x(\tau)] = \frac{1}{2}\left(\int_{\mathcal{L}} \sqrt{g_{ik}(x)\frac{dx^i}{d\tau}\frac{dx^k}{d\tau}}d\tau\right)^2 \tag{2.4}$$

The geodesic $\mathcal{L}_{xx'} :\ x = x(\tau)$ is described by the equations

$$\mathcal{L}_{xx'} : \qquad \frac{d^2x^i}{d\tau^2} + \gamma^i_{kl}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau} = 0, \qquad i = 1, 2, \ldots n \tag{2.5}$$

where

$$\gamma^i_{kl} = \gamma^i_{kl}(x) = \frac{1}{2}g^{ij}\left(g_{kj,l} + g_{lj,k} - g_{kl,j}\right) \tag{2.6}$$

is the Christoffel symbol, and comma before index l denotes differentiation with respect to x^l.

In particular, if g_{ik} =const, $i, k = 1, 2, \ldots n$, $g = \det ||g_{ik}|| \neq 0$, the world function is described by the relation (1.21), and the Riemannian space $R_n = \{\mathbf{g}, K, \mathbb{R}^n\}$

is the Euclidean space. Let us consider now the Riemannian space $R_n = \{\mathbf{g}_E, K, D\}$, where $D \subset \mathbb{R}^n$ is some region of the Euclidean space $E_n = \{\mathbf{g}_E, K, \mathbb{R}^n\}$. If the region D is convex, i.e. any segment $\mathcal{L}_{[xx']}$ of the straight $\mathcal{L}_{xx'}$, connecting the points $x, x' \in D$ belongs to D ($\mathcal{L}_{[xx']} \subset D$), then the world function of the Riemannian space $R_n = \{\mathbf{g}_E, K, D\}$ has the form (1.21) and the Riemannian space $R_n = \{\mathbf{g}_E, K, D\}$ can be embedded isometrically into the Euclidean space $E_n = \{\mathbf{g}_E, K, \mathbb{R}^n\}$.

If the region D is noncovex, then the system of geodesics of $R_n = \{\mathbf{g}_E, K, D\}$ is not a system of straight lines, and the world function (1.24) is not described by the relation (1.21).

Example. Let us consider two-dimensional proper Euclidean space, and rectilinear orthogonal coordinates on it. Let us consider the region $D:\ (x^1)^2 + (x^2)^2 \geq 1$. Geodesics of the Riemannian space $R_2' = \{\mathbf{g}_E, K, D\}$ looks as it is shown in Figure 1. After cutting a hole in the Euclidean plane the shape and length of geodesic segment between the points P and P' changes. World function $\sigma(P, P')$ between the points P and P' changes, and the part $R_2' = \{\mathbf{g}_E, K, D\}$ of the Euclidean plane $R_2 = \{\mathbf{g}_E, K, \mathbb{R}^2\}$ stops to be embeddable isometrically in $R_2 = \{\mathbf{g}_E, K, \mathbb{R}^2\}$. It seems to be rather strange, when part of the Euclidean plane cannot be embedded isometrically in the plane.

The problem of convexity is rather strong, and most of geometricians prefer to get around this problem, considering convex regions [4]. In the T-geometry the convexity problem is absent. Indeed, according to definition 1.16 any subset of a σ-space is always embeddable isometrically into the σ-space. From viewpoint of T-geometry, cutting a hole in the Euclidean plane $R_2 = \{\mathbf{g}_E, K, \mathbb{R}^2\}$, one does not change the system of geodesics (the first order NGOs), one cuts only holes in geodesics, making them discontinuous. Continuity is a property of coordinate systems, used in Riemannian geometry as the main tool of description. From viewpoint of T-geometry the convexity problem is a problem made artificially. Insisting on continuity of geodesics, one overestimates importance of the continuity for geometry and attributes the continuity of geodesics (the first order NGOs) to any Riemannian geometry, whereas the continuity of geodesics is a special property of the proper Euclidean geometry.

3 Riemannian geometry and one-dimensionality of the first order NGOs

Let us consider the n-dimensional pseudo-Euclidean space $E_n = \{\mathbf{g}_1, K, \mathbb{R}^n\}$ of the index 1, $\mathbf{g}_1$ =diag$\{1, -1, -1 \ldots -1\}$ to be a kind of n-dimensional Riemannian space[3]. The world function is defined by the relation (1.21)

$$\sigma_1(x,x') = \frac{1}{2}\sum_{i,k=1}^{n} g^{ik}\,(x_i - x'_i)\,(x_k - x'_k), \qquad g^{ik} = \text{diag}\,\{1,-1,-1\ldots-1\} \quad (3.1)$$

Geodesic $\mathcal{L}_{yy'}$ is a straight line, and it is considered in pseudo-Euclidean geometry to be the first order NGOs, determined by two points y and y'

$$\mathcal{L}_{yy'}: \quad x^i = \left(y^i - y'^i\right)\tau, \qquad i = 1,2,\ldots n, \qquad \tau \in \mathbb{R} \quad (3.2)$$

The geodesic $\mathcal{L}_{yy'}$ is called timelike, if $\sigma_1(y,y') > 0$, and it is called spacelike if $\sigma_1(y,y') < 0$. The geodesic $\mathcal{L}_{yy'}$ is called null, if $\sigma_1(y,y') = 0$.

The pseudo-Euclidean space $E_n = \{\mathbf{g}_1, K, \mathbb{R}^n\}$ generates the σ-space $V = \{\sigma_1, \mathbb{R}^n\}$, where the world function σ_1 is defined by the relation (3.1). The first order tube (NGO) $\mathcal{T}(x,x')$ in the σ-Riemannian space $V = \{\sigma_1, \mathbb{R}^n\}$ is defined by the relation (1.13)

$$\mathcal{T}(x,x') \equiv \mathcal{T}_{xx'} = \{r | F_2(x,x',r) = 0\}, \qquad \sigma_1(x,x') \neq 0, \qquad x,x',r \in \mathbb{R}^n, \quad (3.3)$$

$$F_2(x,x',r) = \left|\begin{array}{cc} (x'_i - x_i)(x'^i - x^i) & (x'_i - x_i)(r^i - x^i) \\ (r_i - x_i)(x'^i - x^i) & (r_i - x_i)(r^i - x^i) \end{array}\right| \quad (3.4)$$

Solution of equations (3.3), (3.4) gives the following result

$$\mathcal{T}_{xx'} = \left\{ r \left| \bigcup_{y\in\mathbb{R}^n}\bigcup_{\tau\in\mathbb{R}} r = (x'-x)\,\tau + y - x \wedge \Gamma(x,x',y) = 0 \wedge \Gamma(x,y,y) = 0 \right. \right\}, \quad (3.5)$$

$$x, x', y, r \in \mathbb{R}^n$$

where $\Gamma(x,x',y) = (x'_i - x_i)(y^i - x^i)$ is the scalar product of vectors $\overrightarrow{xy}$ and $\overrightarrow{xx'}$ defined by the relation (1.10). In the case of timelike vector $\overrightarrow{xx'}$, when $\sigma_1(x,x') > 0$, there is a unique null vector $\overrightarrow{xy} = \overrightarrow{xx} = \overrightarrow{0}$ which is orthogonal to the vector $\overrightarrow{xx'}$. In this case the $(n-1)$-dimensional surface $\mathcal{T}_{xx'}$ degenerates into the one-dimensional straight

$$\mathcal{T}_{xx'} = \left\{ r \left| \bigcup_{\tau\in\mathbb{R}} r = (x'-x)\,\tau \right. \right\}, \qquad \sigma_1(x,x') > 0, \qquad x,x',r \in \mathbb{R}^n, \quad (3.6)$$

Thus, for timelike vector $\overrightarrow{xx'}$ the first order tube $\mathcal{T}_{xx'}$ coincides with the geodesic $\mathcal{L}_{xx'}$. In the case of spacelike vector $\overrightarrow{xx'}$ the $(n-1)$-dimensional tube $\mathcal{T}_{xx'}$ contains the one-dimensional geodesic $\mathcal{L}_{xx'}$ of the pseudo-Euclidean space $E_n = \{\mathbf{g}_1, K, \mathbb{R}^n\}$.

[3]The term "Riemannian space" is considered to be a collective term with respect to concepts "proper Riemannian" and "pseudo-Riemannian". Matrix $\mathbf{g}$ of the metric tensor has eigenvalues of the same sign in the case of proper Riemannian space and of different signs in the case of pseudo-Riemannian one.

This difference poses the question what is the reason of this difference and what of the two generalization of the proper Euclidean geometry is more reasonable. Note that four-dimensional pseudo-Euclidean geometry is used for description of the real space-time. One can try to resolve this problem from experimental viewpoint. Free classical particles are described by means of timelike straight lines. At this point the pseudo-Euclidean geometry and the σ-pseudo-Euclidean geometry (T-geometry) lead to the same result. The spacelike straights are believed to describe the particles moving with superlight speed (so-called taxyons). Experimental attempts of taxyons discovery were failed. Of course, trying to discover taxyons, one considered them to be described by spacelike straights. On the other hand, the physicists believe that all what can exist does exist and may be discovered. From this viewpoint the failure of discovery of taxyons in the form of spacelike line justifies in favour of taxyons in the form of three-dimensional surfaces.

To interpret the structure of the set (3.5), describing the first order tube, let us take into account the zeroth order tube $\mathcal{T}_x$, determined by the point x in the σ-pseudo-Euclidean space is the light cone with the vertex at the point x (not the point x). Practically the first order tube consists of such sections of the light cones with their vertex $y \in \mathcal{L}_{xx'}$ that all vectors $\overrightarrow{yr}$ of these sections are orthogonal to the vector $\overrightarrow{xx'}$. In other words, the first order tube $\mathcal{T}_{xx'}$ consists of the zeroth order tubes $\mathcal{T}_y$ sections at y, orthogonal to $\overrightarrow{xx'}$, with $y \in \mathcal{L}_{xx'}$. For timelike $\overrightarrow{xx'}$ this section consists of one point, but for the spacelike $\overrightarrow{xx'}$ it is two-dimensional section of the light cone.

4 Collinearity in Riemannian and σ-Riemannian geometry

Let us return to the Riemannian space $R_n = \{\mathbf{g}, K, D\}, \quad D \subset \mathbb{R}^n$, which generates the world function $\sigma(x, x')$ defined by the relation (1.24). Then the σ-space $V = \{\sigma, D\}$ appears. it will be referred to as σ-Riemannian space. We are going to compare concept of collinearity (parallelism) of two vectors in the two spaces.

The world function $\sigma = \sigma(x, x')$ of both σ-Riemannian and Riemannian spaces satisfies the system of equations [7][4]

$$\begin{array}{llll} (1) & \sigma_l \sigma^{lj'} \sigma_{j'} = 2\sigma & (4) & \det \parallel \sigma_{i||k} \parallel \neq 0 \\ (2) & \sigma(x, x') = \sigma(x', x) & (5) & \det \parallel \sigma_{ik'} \parallel \neq 0 \\ (3) & \sigma(x, x) = 0 & (6) & \sigma_{i||k||l} = 0 \end{array} \tag{4.1}$$

where the following designations are used

$$\sigma_i \equiv \frac{\partial \sigma}{\partial x^i}, \qquad \sigma_{i'} \equiv \frac{\partial \sigma}{\partial x'^i}, \qquad \sigma_{ik'} \equiv \frac{\partial^2 \sigma}{\partial x^i \partial x'^k}, \qquad \sigma^{ik'} \sigma_{lk'} = \delta_l^i$$

[4]The paper [7] is hardly available for English speaking reader. Survey of main results of [7] in English may be found in [8]. See also [9]

Here the primed index corresponds to the point x, and unprimed index corresponds to the point x. Two parallel vertical strokes mean covariant derivative $\tilde{\nabla}_i^{x'}$ with respect to x^i with the Christoffel symbol

$$\Gamma_{kl}^{i} \equiv \Gamma_{kl}^{i}(x,x') \equiv \sigma^{is'}\sigma_{kls'}, \qquad \sigma_{kls'} \equiv \frac{\partial^3\sigma}{\partial x^k \partial x^l \partial x'^s}$$

For instance,

$$G_{ik} \equiv G_{ik}(x,x') \equiv \sigma_{i||k} \equiv \frac{\partial \sigma_i}{\partial x^k} - \Gamma_{ik}^{l}(x,x')\sigma_l \equiv \frac{\partial \sigma_i}{\partial x^k} - \sigma_{iks'}\sigma^{ls'}\sigma_l \qquad (4.2)$$

$$G_{ik||l} \equiv \frac{\partial G_{ik}}{\partial x^l} - \sigma_{ils'}\sigma^{js'}G_{jk} - \sigma_{kls'}\sigma^{js'}G_{ij}$$

Summation from 1 to n is produced over repeated indices. The covariant derivative $\tilde{\nabla}_i^{x'}$ with respect to x^i with the Christoffel symbol $\Gamma_{kl}^{i}(x,x')$ acts only on the point x and on unprimed indices. It is called the tangent derivative, because it is a covariant derivative in the Euclidean space $E_{x'}$ which is tangent to the Riemannian space R_n at the point x'. The covariant derivative $\tilde{\nabla}_{i'}^{x}$ with respect to x'^i with the Christoffel symbol $\Gamma_{k'l'}^{i'}(x,x')$ acts only on the point x' and on primed indices. It is a covariant derivative in the Euclidean space E_x which is tangent to the σ-Riemannian space V at the point x [7].

In general, the world function σ carries out the geodesic mapping $G_{x'} : R_n \to E_{x'}$ of the Riemannian space $R_n = \{\mathbf{g}, K, D\}$ on the Euclidean space $E_{x'} = \{\mathbf{g}, K_{x'}, D\}$, tangent to $R_n = \{\mathbf{g}, K, D\}$ at the point x' [7]. This mapping transforms the coordinate system K in R_n into the coordinate system $K_{x'}$ in $E_{x'}$. The mapping is geodesic in the sense that it conserves the lengths of segments of all geodesics, passing through the tangent point x' and angles between them at this point.

The tensor G_{ik}, defined by (4.2) is the metric tensor at the point x in the tangent Euclidean space $E_{x'}$. The covariant derivatives $\tilde{\nabla}_i^{x'}$ and $\tilde{\nabla}_k^{x'}$ commute identically, i.e. $(\tilde{\nabla}_i^{x'}\tilde{\nabla}_k^{x'} - \tilde{\nabla}_k^{x'}\tilde{\nabla}_i^{x'})A_{ls} \equiv 0$, for any tensor A_{ls} [7]. This shows that they are covariant derivatives in the flat space $E_{x'}$.

The system of equations (4.1) contains only world function σ and its derivatives, nevertheless the system of equations (4.1) is not σ-immanent, because it contains a reference to a coordinate system. It does not contain the metric tensor explicitly. Hence, it is valid for any Riemannian space $R_n = \{\mathbf{g}, K, D\}$. All relations written above are valid also for the σ-space $V = \{\sigma, D\}$, provided the world function σ is coupled with the metric tensor by relation (1.24).

σ-immanent expression for scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ in the proper Euclidean space has the form

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) \equiv \sigma(P_0,Q_1) + \sigma(Q_0,P_1) - \sigma(P_0,Q_0) - \sigma(P_1,Q_1) \qquad (4.3)$$

This relation can be easily proved as follows.

In the proper Euclidean space three vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0\mathbf{Q}_1$, and $\mathbf{P}_1\mathbf{Q}_1$ are coupled by the relation

$$|\,\mathbf{P}_1\mathbf{Q}_1\,|^2 = |\,\mathbf{P}_0\mathbf{Q}_1 - \mathbf{P}_0\mathbf{P}_1\,|^2 = |\,\mathbf{P}_0\mathbf{P}_1\,|^2 + |\,\mathbf{P}_0\mathbf{Q}_1\,|^2 - 2(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1) \qquad (4.4)$$

where $(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1)$ denotes the scalar product of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{Q}_1$ in the proper Euclidean space. It follows from (4.4)

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1) = \frac{1}{2}\{\mid \mathbf{P}_0\mathbf{Q}_1 \mid^2 + \mid \mathbf{P}_0\mathbf{P}_1 \mid^2 - \mid \mathbf{P}_1\mathbf{Q}_1 \mid^2\} \tag{4.5}$$

Substituting the point Q_1 by Q_0 in (4.5), one obtains

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_0) = \frac{1}{2}\{\mid \mathbf{P}_0\mathbf{Q}_0 \mid^2 + \mid \mathbf{P}_0\mathbf{P}_1 \mid^2 - \mid \mathbf{P}_1\mathbf{Q}_0 \mid^2\} \tag{4.6}$$

Subtracting (4.6) from (4.5) and using the properties of the scalar product in the Euclidean space, one obtains

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \frac{1}{2}\{\mid \mathbf{P}_0\mathbf{Q}_1 \mid^2 + \mid \mathbf{Q}_0\mathbf{P}_1 \mid^2 - \mid \mathbf{P}_0\mathbf{Q}_0 \mid^2 - \mid \mathbf{P}_1\mathbf{Q}_1 \mid^2\} \tag{4.7}$$

Taking into account that $\mid \mathbf{P}_0\mathbf{Q}_1 \mid^2 = 2\sigma(P_0, Q_1)$, one obtains the relation (4.3) from the relation (4.7).

Two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ are collinear $\mathbf{P}_0\mathbf{P}_1 || \mathbf{Q}_0\mathbf{Q}_1$ (parallel or antiparallel), provided $\cos^2\theta = 1$, where θ is the angle between the vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$. Taking into account that

$$\cos^2\theta = \frac{(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)^2}{(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1)(\mathbf{Q}_0\mathbf{Q}_1.\mathbf{Q}_0\mathbf{Q}_1)} = \frac{(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)^2}{|\mathbf{P}_0\mathbf{P}_1|^2 \cdot |\mathbf{Q}_0\mathbf{Q}_1|^2} \tag{4.8}$$

one obtains the following σ-immanent condition of the two vectors collinearity

$$\mathbf{P}_0\mathbf{P}_1 || \mathbf{Q}_0\mathbf{Q}_1 : \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)^2 = |\mathbf{P}_0\mathbf{P}_1|^2 \cdot |\mathbf{Q}_0\mathbf{Q}_1|^2 \tag{4.9}$$

The collinearity condition (4.9) is σ-immanent, because by means of (4.3) it can be written in terms of the σ-function only. Thus, this relation describes the vectors collinearity in the case of arbitrary σ-space.

Let us describe this relation for the case of σ-Riemannian geometry. Let coordinates of the points P_0, P_1, Q_0, Q_1 be respectively x, $x + dx$, x' and $x' + dx'$. Then writing (4.3) and expanding it over dx and dx', one obtains

$$\begin{aligned}(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) \quad &\equiv \quad \sigma(x, x' + dx') + \sigma(x', x + dx) - \sigma(x, x') - \sigma(x + dx, x' + dx') = \\ &\sigma + \sigma_{l'} dx'^{l'} + \frac{1}{2}\sigma_{l',s'} dx'^{l'} dx'^{s'} + \sigma + \sigma_i dx^i + \frac{1}{2}\sigma_{i,k} dx^i dx^k - \sigma \\ &-\sigma - \sigma_i dx^i - \sigma_{l'} dx'^{l'} - \frac{1}{2}\sigma_{i,k} dx^i dx^k - \sigma_{i,l'} dx^i dx'^{l'} - \frac{1}{2}\sigma_{l',s'} dx'^{l'} dx'^{s'}\end{aligned}$$

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = -\sigma_{i,l'} dx^i dx'^{l'} = -\sigma_{il'} dx^i dx'^{l'} \tag{4.10}$$

Here comma means differentiation. For instance, $\sigma_{i,k} \equiv \partial\sigma_i/\partial x^k$. One obtains for $|\mathbf{P}_0\mathbf{P}_1|^2$ and $|\mathbf{Q}_0\mathbf{Q}_1|^2$

$$|\mathbf{P}_0\mathbf{P}_1|^2 = g_{ik} dx^i dx^k, \qquad |\mathbf{Q}_0\mathbf{Q}_1|^2 = g_{l's'} dx'^{l'} dx'^{s'} \tag{4.11}$$

where $g_{ik} = g_{ik}(x)$ and $g_{l's'} = g_{l's'}(x')$. Then the collinearity condition (4.9) is written in the form

$$\left(\sigma_{il'}\sigma_{ks'} - g_{ik}g_{l's'}\right) dx^i dx^k dx'^{l'} dx'^{s'} = 0 \tag{4.12}$$

Let us take into account that in the Riemannian space the metric tensor $g_{l's'}$ at the point x' can be expressed via the world function σ of points x, x' by means of the relation [7]

$$g_{l's'} = \sigma_{il'}G^{ik}\sigma_{ks'}, \qquad g^{l's'} = \sigma^{il'}G_{ik}\sigma^{ks'} \tag{4.13}$$

where the tensor G_{ik} is defined by the relation (4.2), and G^{ik} is defined by the relation

$$G^{il}G_{lk} = \delta^i_k \tag{4.14}$$

Substituting the first relation (4.2) in (4.12) and using designation

$$u_i = -\sigma_{il'}dx'^{l'}, \qquad u^i = G^{ik}u_k = -\sigma^{il'}g_{l's'}dx^{l's'} \tag{4.15}$$

one obtains

$$\left(\delta^l_i\delta^s_k - g_{ik}G^{ls}\right) u_l u_s dx^i dx^k = 0 \tag{4.16}$$

The vector u_i is the vector $dx'_{i'} = g_{i'k'}dx'^{k'}$ transported parallelly from the point x' to the point x in the Euclidean space $E_{x'}$ tangent to the Riemannian space R_n. Indeed,

$$u_i = -\sigma_{il'}g^{l's'}dx'_s, \qquad \tilde{\nabla}^{x'}_k\left(-\sigma_{il'}g^{l's'}\right) \equiv 0, \qquad i,k = 1,2,\ldots n \tag{4.17}$$

and tensor $-\sigma_{il'}g^{l's'}$ is the operator of the parallel transport in $E_{x'}$, because

$$\left[-\sigma_{il'}g^{l's'}\right]_{x=x'} = \delta^{s'}_{i'}$$

and the tangent derivative of this operator is equal to zero identically. For the same reason, i.e. because of

$$\left[\sigma^{il'}g_{l's'}\sigma^{ks'}\right]_{x=x'} = g^{i'k'}, \qquad \tilde{\nabla}^{x'}_s(\sigma^{il'}g_{l's'}\sigma^{ks'}) \equiv 0$$

$G^{ik} = \sigma^{il'}g_{l's'}\sigma^{ks'}$ is the contravariant metric tensor in $E_{x'}$, at the point x.

The relation (4.16) contains vectors at the point x only . At fixed $u_i = -\sigma_{il'}dx'^{l'}$ it describes a collinearity cone, i.e. a cone of infinitesimal vectors dx^i at the point x parallel to the vector $dx'^{i'}$ at the point x'. Under some condition the collinearity cone can degenerates into a line. In this case there is only one direction, parallel to the fixed vector u^i. Let us investigate, when this situation takes place.

At the point x two metric tensors g_{ik} and G_{ik} are connected by the relation [7]

$$G_{ik}(x,x') = g_{ik}(x) + \int_x^{x'} F_{ikj''s''}(x,x'')\sigma^{j''}(x,x'')dx''^{s''}, \tag{4.18}$$

where according to [7]

$$\sigma^{i'} = \sigma^{li'}\sigma_l = G^{l'i'}\sigma_{l'} = g^{l'i'}\sigma_{l'} \tag{4.19}$$

Integration does not depend on the path, because it is produced in the Euclidean space $E_{x'}$. The two-point tensor $F_{ilk'j'} = F_{ilk'j'}(x,x')$ is the two-point curvature tensor, defined by the relation

$$F_{ilk'j'} = \sigma_{ilj'||k'} = \sigma_{ilj',k'} - \sigma_{sj'k'}\sigma^{sm'}\sigma_{ilm'} = \sigma_{i|l||k'||j'} \tag{4.20}$$

where one vertical stroke denotes usual covariant derivative and two vertical strokes denote tangent derivative. The two-point curvature tensor $F_{ilk'j'}$ has the following symmetry properties

$$F_{ilk'j'} = F_{lik'j'} = F_{ilj'k'}, \qquad F_{ilk'j'}(x,x') = F_{k'j'il}(x',x) \tag{4.21}$$

It is connected with the one-point Riemann-Ghristoffel curvature tensor r_{iljk} by means of relations

$$r_{iljk} = [F_{ikj'l'} - F_{ijk'l'}]_{x'=x} = f_{ikjl} - f_{ijkl}, \qquad f_{iklj} = [F_{ikj'l'}]_{x'=x} \tag{4.22}$$

In the Euclidean space the two-point curvature tensor $F_{ilk'j'}$ vanishes as well as the Riemann-Ghristoffel curvature tensor r_{iljk}.

Let us introduce designation

$$\Delta_{ik} = \Delta_{ik}(x,x') = \int\limits_x^{x'} F_{ikj''s''}(x,x'')\sigma^{j''}(x,x'')dx''^{s''} \tag{4.23}$$

and choose the geodesic $\mathcal{L}_{xx'}$ as the path of integration. It is described by the relation

$$\sigma_i(x,x'') = \tau\sigma_i(x,x') \tag{4.24}$$

which determines x'' as a function of parameter τ. Differentiating with respect to τ, one obtains

$$\sigma_{ik''}(x,x'')dx''^{k''} = \sigma_i(x,x')d\tau \tag{4.25}$$

Resolving equations (4.25) with respect to dx'' and substituting in (4.23), one obtains

$$\Delta_{ik}(x,x') = \sigma_l(x,x')\sigma_p(x,x')\int\limits_0^1 F_{ikj''s''}(x,x'')\sigma^{lj''}(x,x'')\sigma^{ps''}(x,x'')\tau d\tau \tag{4.26}$$

where x'' is determined from (4.24) as a function of τ. Let us set

$$F_{ik}^{\cdot\cdot lp}(x,x') = F_{ikj's'}(x,x')\sigma^{lj'}(x,x')\sigma^{ps'}(x,x') \tag{4.27}$$

then

$$G_{ik}(x,x') = g_{ik}(x) + \Delta_{ik}(x,x') \tag{4.28}$$

$$\Delta_{ik}(x,x') = \sigma_l(x,x')\sigma_p(x,x')\int\limits_0^1 F_{ik}^{..lp}(x,x'')\tau d\tau \tag{4.29}$$

Substituting g_{ik} from (4.28) in (4.16), one obtains

$$\left(\delta_i^l\delta_k^s - G^{ls}\left(G_{ik} - \Delta_{ik}\right)\right)u_l u_s dx^i dx^k = 0 \tag{4.30}$$

Let us look for solutions of equation in the form of expansion

$$dx^i = \alpha u^i + v^i, \qquad G_{ik}u^i v^k = 0 \tag{4.31}$$

Substituting (4.31) in (4.30), one obtains equation for v^i

$$G_{ls}u^l u^s\left[G_{ik}v^i v^k - \Delta_{ik}\left(\alpha u^i + v^i\right)\left(\alpha u^k + v^k\right)\right] = 0 \tag{4.32}$$

If the σ-Riemannian space $V = \{\sigma, D\}$ is σ-Euclidean, then as it follows from (4.29) $\Delta_{ik} = 0$. If $V = \{\sigma, D\}$ is the proper σ-Euclidean space, $G_{ls}u^l u^s \neq 0$, and one obtains two equations for determination of v^i

$$G_{ik}v^i v^k = 0, \qquad G_{ik}u^i v^k = 0 \tag{4.33}$$

The only solution

$$v^i = 0, \qquad dx^i = \alpha u^i, \qquad i = 1,2,\dots n \tag{4.34}$$

of (4.32) is a solution of the equation (4.30), where α is an arbitrary constant. In the proper Euclidean geometry the collinearity cone always degenerates into a line.

Let now the space $V = \{\sigma, D\}$ be the σ-pseudo-Euclidean space of index 1, and the vector u^i be timelike, i.e. $G_{ik}u^i u^k > 0$. Then equations (4.33) also have the solution (4.34). If the vector u^i is spacelike, $G_{ik}u^i u^k < 0$, then two equations (4.33) have non-trivial solution, and the collinearity cone does not degenerate into a line. The collinearity cone is a section of the light cone $G_{ik}v^i v^k = 0$ by the plane $G_{ik}u^i v^k = 0$. If the vector u^i is null, $G_{ik}u^i u^k = 0$, then equation (4.32) reduces to the form

$$G_{ik}u^i u^k = 0, \qquad G_{ik}u^i v^k = 0 \tag{4.35}$$

In this case (4.34) is a solution, but besides there are spacelike vectors v^i which are orthogonal to null vector u^i and the collinearity cone does not degenerate into a line.

In the case of the proper σ-Riemannian space $G_{ik}u^i u^k > 0$, and equation (4.32) reduces to the form

$$G_{ik}v^i v^k - \Delta_{ik}\left(\alpha u^i + v^i\right)\left(\alpha u^k + v^k\right) = 0 \tag{4.36}$$

In this case $\Delta_{ik} \neq 0$ in general, and the collinearity cone does not degenerate. Δ_{ik} depends on the curvature an on the distance between the points x and x'. The more space curvature and the distance $\rho(x,x')$, the more the collinearity cone aperture.

In the curved proper σ-Riemannian space there is an interesting special case, when the collinearity cone degenerates . In any σ-Riemannian space the following equality takes place [7]

$$G_{ik}\sigma^k = g_{ik}\sigma^k, \qquad \sigma^k \equiv g^{kl}\sigma_l \tag{4.37}$$

Then it follows from (4.28) that

$$\Delta_{ik}\sigma^k = 0 \tag{4.38}$$

It means that in the case, when the vector u^i is directed along the geodesic, connecting points x and x', i.e. $u^i = \beta\sigma^i$, the equation (4.36) reduces to the form

$$(G_{ik} - \Delta_{ik})\, v^i v^k = 0, \qquad u^i = \beta\sigma^i \tag{4.39}$$

If Δ_{ik} is small enough as compared with G_{ik}, then eigenvalues of the matrix $G_{ik} - \Delta_{ik}$ have the same sign, as those of the matrix G_{ik}. In this case equation (4.39) has the only solution (4.34), and the collinearity cone degenerates.

5 Discussion

Thus, we see that in the σ-Riemannian geometry at the point x there are many vectors parallel to given vector at the point x'. This set of parallel vectors is described by the collinearity cone. Degeneration of the collinearity cone into a line, when there is only one direction, parallel to the given direction, is an exception rather than a rule, although in the proper Euclidean geometry this degeneration takes place always. Nonuniformity of space destroys the collinearity cone degeneration. In the proper Riemannian geometry, where the world function satisfies the system (4.1), one succeeded in conserving this degeneration for direction along the geodesic, connecting points x and x'. This circumstance is very important for degeneration of the first order NGOs into geodesic, because degeneration of NGOs is connected closely with the collinearity cone degeneration.

Indeed, definition of the first order tube (1.13), or (3.3) may be written also in the form

$$\mathcal{T}\left(\mathcal{P}^1\right) \equiv \mathcal{T}_{P_0P_1} = \left\{R \mid \mathbf{P}_0\mathbf{P}_1 || \mathbf{P}_0\mathbf{R}\right\}, \qquad P_0, P_1, R \in \Omega, \tag{5.1}$$

where collinearity $\mathbf{P}_0\mathbf{P}_1 || \mathbf{P}_0\mathbf{R}$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{R}$ is defined by the σ-immanent relation (4.9), which can be written in the form

$$\mathbf{P}_0\mathbf{P}_1 || \mathbf{P}_0\mathbf{R}: \quad F_2\left(P_0, P_1, R\right) = \begin{vmatrix} (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R}) \\ (\mathbf{P}_0\mathbf{R}.\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{R}.\mathbf{P}_0\mathbf{R}) \end{vmatrix} = 0 \tag{5.2}$$

The form (5.1) of the first order tube definition allows one to define the first order tube $\mathcal{T}(P_0, P_1; Q_0)$, passing through the point Q_0 collinear to the given vector $\mathbf{P}_0\mathbf{P}_1$. This definition has the σ-immanent form

$$\mathcal{T}(P_0, P_1; Q_0) = \left\{R \mid \mathbf{P}_0\mathbf{P}_1 || \mathbf{Q}_0\mathbf{R}\right\}, \qquad P_0, P_1, Q_0, R \in \Omega, \tag{5.3}$$

where collinearity $\mathbf{P}_0\mathbf{P}_1||\mathbf{Q}_0\mathbf{R}$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{R}$ is defined by the σ-immanent relations (4.9), (4.7). In the proper Euclidean space the tube (5.3) degenerates into the straight line, passing through the point Q_0 collinear to the given vector $\mathbf{P}_0\mathbf{P}_1$.

Let us define the set $\omega_{Q_0} = \{\mathbf{Q}_0\mathbf{Q})|Q \in \Omega\}$ of vectors $\mathbf{Q}_0\mathbf{Q}$. Then

$$\mathcal{C}(P_0, P_1; Q_0) = \{\mathbf{Q}_0\mathbf{Q}|Q \in \mathcal{T}(P_0, P_1; Q_0)\} \subset \omega_{Q_0} \tag{5.4}$$

is the collinearity cone of vectors $\mathbf{Q}_0\mathbf{Q}$ collinear to vector $\mathbf{P}_0\mathbf{P}_1$. Thus, the one-dimensionality of the first order tubes and the collinearity cone degeneration are connected phenomena.

In the Riemannian geometry the very special property of the proper Euclidean geometry (the collinearity cone degeneration) is considered to be a property of any geometry and extended to the case of Riemannian geometry. The line $\mathcal{L}$, defined as a continuous mapping (1.23) is considered to be the most important geometrical object. This object is considered to be more important, than the metric, and metric in the Riemannian geometry is defined in terms of the shortest lines. Use of line as a basic concept of geometry is inadequate for description of geometry and poses problems, which appears to be artificial. For instance, the convexity problem, when elimination of part of the point set Ω generates variation of properties of other regions is a result of the metric definition via concept of the line. Although choosing the world function in the proper way (satisfying equations (4.1)), one succeeded in conserving the collinearity cone degeneration for geodesic lines, but for distant points x and x' the collinearity cone does not degenerate, and the absolute parallelism is absent in the Riemannian geometry. Instead of the cone of collinear vectors one introduces concept of parallel transport of a vector, where the result depends on the path of the transport. Practically, it means that one vector of the vector cone is chosen and it is attributed to some curve connecting the points x and x'.

Being a special case of T-geometry, the σ-Riemannian geometry does not use the nonmetric concept of line at all. Here the nonmetric line is a special geometrical object characteristic for the proper Euclidean geometry which is a result of the collinearity cone degeneration. Instead of the continuous mapping (1.23) one uses the mapping

$$m_n: \quad I_n \to \Omega, \qquad I_n = \{0, 1, \ldots n\} \subset \mathbb{Z} \tag{5.5}$$

which determines geometrical object m_n, called the nth order multivector. [6]. The nth order multivector may be considered to be some generalization of the nth order σ-subspace $M(\mathcal{P}^n)$, and definition (5.5) of multivector appears to be σ-immanent. Application of mappings (5.5) is sufficient for description of any geometry, because all geometric objects are determined as subsets of the space Ω (not as mappings). Use of such complicated mappings as (1.23) is not necessary. For instance, to investigate the properties of the first order tube $\mathcal{T}_{P_0P_1} \subset \Omega$ (geodesic), one needs to investigate the set $\mathcal{T} = \{P_0\} \otimes \{P_1\} \otimes \mathcal{T}_{P_0P_1} \subset \Omega^3$, satisfying the condition $F_3(\mathcal{T}) = 0$. Here the mapping F_3 is known and fixed. Only zeros of the function F_3, having the form $\mathcal{T} = \{P_0\} \otimes \{P_1\} \otimes \mathcal{T}_{P_0P_1}$, are investigated. Power of the set $\mathcal{T}$ is much less than the power of the set of all mappings (1.23), and investigation of $\mathcal{T}$ is not so complicated as investigation of mappings (1.23).

One can reduce the power of the set of all mappings (1.23), imposing some additional restrictions on mapping (1.23), but nothing can change the fact that the mapping (1.23) is an attribute of the proper Euclidean geometry and is not an attribute of a geometry in itself. The convexity problem confirms this. The real space-time may appear not to have property of the collinearity cone degeneration [11]. Insisting on the mapping (1.23) as the main tool of geometry investigation, one closes the door for real investigations of geometry and shows a wrong way for them.

Besides purely logical arguments in favour of the T-geometry approach there are arguments of applied character. The fact is that application of T-geometry to the space-time model construction leads to new encouraging results [10, 11]. Consideration of uniform isotropic continuous model with zeroth curvature leads to a class of models, distinguishing by the shape of the tube. This class contains the well known Minkowski model, for which the timelike tubes degenerate into lines and which is not optimal, because it does not enable to describe quantum phenomena without using the quantum principles. Other (nondegenerate) models of this class have the following properties: (1) geometrization of mass of a particle described by the broken tube (1.22), (2) stochasticity of the world tube of a free particle which is conditioned by the collinearity cone non-degeneracy.

It turns out that it is possible one to choose optimal space-time model, for which the statistical description of stochastic free particle tubes coincides with the quantum description in terms of the Schrödinger equation. The quantum constant $\hbar$ appears to be a space-time property, introducing some "elementary length" (it is connected with the thickness of the particle world tube). As a result one does not need the quantum principles, and the quantum theory looks as a conception, created for compensation of our incorrect ideas on the space-time geometry at small distances.

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