Geometry without Topology as a New Conception of Geometry

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Abstract

A geometric conception is a method of a geometry construction. The Riemannian geometric conception and a new T-geometric one are considered. T-geometry is built only on the basis of information included in the metric (distance between two points). Such geometric concepts as dimension, manifold, metric tensor, curve are fundamental in the Riemannian conception of geometry, and they are derivative in the T-geometric one. T-geometry is the simplest geometric conception (essentially only finite point sets are investigated) and simultaneously it is the most general one. It is insensitive to the space continuity and has a new property – nondegeneracy. Fitting the T-geometry metric with the metric tensor of Riemannian geometry, one can compare geometries, constructed on the basis of different conceptions. The comparison shows that along with similarity (the same system of geodesics, the same metric) there is a difference. There is an absolute parallelism in T-geometry, but it is absent in the Riemannian geometry. In T-geometry any space region is isometrically embeddable in the space, whereas in Riemannian geometry only convex region is isometrically embeddable. T-geometric conception appears to be more consistent logically, than the Riemannian one.

1 Introduction

Conception of geometry (geometric conception) is a method (a set of principles), which is used for construction of geometry. The proper Euclidean[1] geometry can be constructed on the basis of different geometric conceptions. For instance, one can use the Euclidean axiomatic conception (Euclidean axioms), or the Riemannian conception of geometry (dimension, manifold, metric tensor, curve). One can use metric conception of geometry (topological space, metric, curve). In any case one obtains the same proper Euclidean geometry. From point of view of this geometry it is of no importance which of possible geometric conceptions is used for the geometry construction. It means that the category of geometry conception is metageometric.

However, if we are going to generalize (to modify) the Euclidean geometry, it appears to be very important which of many possible geometric conceptions is used for the generalization. The point is that the generalization is some modification of original (fundamental) statements of geometry in the scope of the same geometric conception. As far as fundamental statements are different in different geometric conceptions, one is forced to modify different statements, that leads naturally to different results.

If one uses the Euclidean geometric conception, which contains only axioms and no numerical characteristics, the only possible modification consists in changing some axioms by other ones. In this case some new geometries appear which hardly may be considered to be a generalization of the Euclidean geometry. They are rather its different modifications.

Some fundamental statements of the Riemannian geometric conception contain numerical characteristics, as far as one sets the dimension n and metric tensor $g_{ik},\quad i,k=1,2,\dots n$, consisting of several functions of one point, i.e. of one argument $x=\{x^i\},\quad i=1,2,\dots n$. Varying n and g_{ik}, one obtains a class of Riemannian geometries, where each geometry is labelled by several functions of one point.

Recently a new geometric conception of the Euclidean geometry construction was suggested [1, 2]. The Euclidean geometry appears to be formulated in terms only of metric ρ, setting distance between all pairs of points of the space. Such a geometric conception is the most general in the sense, that all information on geometry is concentrated in one function of two points. It is evident that one function of two points contains more information, than several functions of one point (it is supposed that the set of points is continual). At some choice of the point set Ω, where the metric and geometry are set, the n-dimensional Euclidean geometry appears. At another choice of the metric another generalized geometry appears on the same set Ω. This geometry will be referred to as tubular geometry, or briefly T-geometry. All things being equal, the set of all T-geometries appears to be more powerful, than the

[1]We use the term "Euclidean geometry" as a collective concept with respect to terms "proper Euclidean geometry" and "pseudoeuclidean geometry". In the first case the eigenvalues of the metric tensor matrix have similar signs, in the second case they have different signs. The same interrelation take place between terms "Riemannian geometry", "proper Riemannian geometry" and "pseudo-Riemannian geometry".

set of all Riemannian geometries. This conception will be referred to as T-geometric conception, although the term "metric conception of geometry" fits more.

The point is that this term has been occupied. By the metric (or generalized Riemannian) geometry [4, 5, 6] is meant usually a geometry, constructed on the basis of the metric geometric conception, whose fundamental statements are topology and metric, i.e. the metric is set not on an arbitrary set of points, but on the topological space, where, in particular, concepts of continuity and of a curve are defined.

What actually is happen is that the metric geometric conception contains excess of fundamental statements. This excess appears as follows. Let us imagine that some conception A of Euclidean geometry contains some set of independent fundamental statements a. Let b be some set of corollaries of the fundamental statements a. Let us consider now the set $a \cup b$ as a set of fundamental statements of a geometric conception. It is another conception A' of Euclidean geometry. Its fundamental statements $a \cup b$ are not independent. Now one can obtain the conception A, or some other geometric conception, depending on how the fundamental statements $a \cup b$ are used. Now obtaining generalized geometries, one may not vary the fundamental statements independently. To avoid contradiction, one is to take into account mutual dependence of fundamental statements.

If we know nothing on mutual dependence of fundamental statements $a \cup b$, the geometric conception may appear to be eclectic. We risk to obtain contradictions, or artificial constraints on the generalized geometries obtained. In the case of the metric conception of the proper Euclidean geometry the statements on properties of the topological space and those of the curve are corollaries of metrical statements. They may be removed completely from the set of fundamental statements of the conception.

However, there are problems, connected with the fact that we have some preconceptions on what is the geometry, in general. In particular, it is a common practice to consider that the concept of the curve is an attribute of any geometry, which is used for description of the real space (or space-time). It is incorrect, and manifests itself, in particular, in imposition of some unjustified constraints (triangle inequality) on metric, which make the difficult situation. These preconceptions have a metalogic character. They are connected with association properties of human thinking. Overcoming of these preconceptions needs a serious analysis.

A cause for writing this paper is a situation, arising after appearance and discussion of papers on T-geometry [1, 2], which mean essentially a construction of a new geometric conception. Such a situation took place in the second half of XIXth century, when the non-Euclidean geometries appeared, and the most part of mathematical community considered sceptically applications of the Riemannian (and non-Euclidean) geometry to the real space geometry. Appearance of Riemannian geometries meant appearance of a new geometric conception. The reason of sceptical relation of the mathematical community to Riemannian geometry has not been analyzed up till now, although it was described in literature [3].

Appearance of a more general conception of geometry is important for applications of geometry. Geometry is a ground of the space-time model, and appearance of a new more general geometric conception poses the question as to whether the microcosm space-time geometry has been chosen optimally. If the existing space-time geometry is not optimal, it must be revised. The space-time geometry revision is to be accompanied by a revision of basic statements of physics as a science founded on the space-time model. For instance, appearance of Riemannian geometries and realization of the fact, that a new conception of geometry appears together with their appearance, has lead finally to a revision of the space-time conception and to creation of the general relativity theory.

Until appearance of T-geometries there was only one uniform isotropic geometry suitable for the space-time description. This is Minkowski geometry. An alternative to the Minkowski geometry to be anywhere reasonable did not exist. After realization of the fact that non-degenerate geometries (T-geometries) are as good as degenerate (Riemannian) geometries, a class of geometries suitable for description of uniform isotropic space-time appears. This class includes the Minkowski geometry. The uniform isotropic geometries of this class are labelled by a function of one argument. Geometries of the class differ in a value and character of nondegeneracy. All geometries of this class except for Minkowski geometry appear to be nondegenerate. In the nondegenerate geometry a motion of free particles appears to initially stochastic, whereas in the degenerate geometry it initially deterministic. It is well known, that motion of microparticles (electrons, protons, etc.) is stochastic. It seems incorrect to choose such a space-time model, where the microparticle motion is deterministic, and thereafter to introduce additional hypotheses (principles of quantum mechanics), providing stochasticity of microparticle motion. It is more reasonable to choose at once such a space-time geometry which provides the microparticle motion stochasticity. It is desirable to choose from the class of uniform nondegenerate geometries precisely that geometry, which agrees optimally with experimental data. If the complete agreement with experiment appears to be impossible, one can add supplementary hypotheses, as it is made in quantum mechanics. In any case the space-time geometry is to be chosen optimally. The choice of the Minkowski geometry as a space-time model for microcosm is not optimal certainly. A use of the Minkowski geometry as a space-time model for microcosm is explained by absence of alternative (i.e. essentially by a use of the Riemannian conception of geometry).

Thus, after appearance of a new conception of geometry and appearance of an alternative to the Minkowski geometry a revision of the space-time model is a logical necessity. This revision must be carried out independently of that whether the new version of the space-time model explains all quantum effects, or only part of them. In the last case one should add some hypotheses, explaining that part of experimental data, which are not explained by the revised space-time model. In any case one should use the most suitable space-time geometry among all possible ones.

Let us note that this conclusion does not agree with viewpoint of most of physicists, dealing with relativistic quantum theory. Many of them suppose that any revision of the existing space-time model is justified only in the case, if it explains at least one of experiments which cannot be explained by the existing theory. We agree with such a position, provided the existing theory modification does not concern principles of quantum theory and space-time model. At appearance of a more general conception of geometry one is forced to choose an optimal geometry independently of whether the new model solves all problems, or only a part of them. Another viewpoint, when one suggests either to solve all problems by means of a revision of the space-time geometry, or, if it appears to be impossible, to abandon from revision at all and to use certainly nonoptimal geometry, seems to be too extremistic.

Now results of application of nondegenerate geometry for the space-time description seem to be rather optimistic, because one succeeded to choose such a nondegenerate geometry, containing the quantum constant $\hbar$ as a parameter, that statistical description of stochastic particle motion in this space-time coincides with the quantum description in terms of Schrödinger equation in the conventional space-time model [7, 8]. Further development of the conception will show whether explanation of relativistic quantum effects is possible.

In the present paper a new geometric conception, based on the concept of distance and *only distance* is considered. In general, the idea of the geometry construction on the basis of the distance is natural and not new. The geometric conception, where the distance (metric) is a basic concept, is natural to be referred to as metric conception of geometry. Usually the term "metric geometry" is used for a geometry, constructed on the base of the metric space.

Definition 1.1 *The metric space $M = \{\rho, \Omega\}$ is the set Ω of points $P \in \Omega$, equipped by the metric ρ, setting on $\Omega \times \Omega$*

$$\rho: \quad \Omega \times \Omega \rightarrow D_+ \subset \mathbb{R}, \qquad D_+ = [0, \infty), \tag{1.1}$$

$$\rho(P,P) = 0, \qquad \rho(P,Q) = \rho(Q,P), \qquad \forall P, Q \in \Omega \tag{1.2}$$

$$\rho(P,Q) = 0, \quad \textit{if and only if} \quad P = Q, \qquad \forall P, Q \in \Omega \tag{1.3}$$

$$\rho(P,Q) + \rho(Q,R) \geq \rho(P,R), \qquad \forall P, Q, R \in \Omega \tag{1.4}$$

There is a generalization of metric geometry, known as distance geometry [9], which differs from the metric geometry in absence of constraint (1.4). The main problem of metric geometric conception is a construction of geometric objects, i.e. different sets of points in the metric space. For instance, to construct such a geometric object as the shortest, one is forced to introduce the concept of a curve as a continuous mapping of a segment of real axis on the space.

$$L: \ I \rightarrow \Omega, \qquad I = [0, 1] \subset \mathbb{R}, \tag{1.5}$$

The shortest, passing through points P and Q, is defined as a curve segment of the shortest length. On one hand, introduction of the concept of a curve means a rejection from the pure metric conception of geometry, as far as one is forced to introduce concepts, which do not defined via metric. On the other hand, if the concept of a curve is not introduced, it is not clear how to build such geometric objects which are analogs of Euclidean straight and plane. Without introduction of these objects the metric geometry looks as a very poor (slightly informative) geometry. Such a geometry cannot be used as a model of the real space-time.

Essentially the problem of constructing a pure metric conception of geometry is set as follows. Is it possible to construct on the basis of only metric such a geometry which were as informative as the Euclidean geometry? In other words, is it possible to construct the Euclidean geometry, setting in some way the metric on $\Omega\times\Omega$, where Ω is a properly chosen set of points? More concretely this problem is formulated as follows.

Let ρ_{E} be the metric of n-dimensional proper Euclidean space on $\Omega\times\Omega$. Is it possible on the base of information, contained in ρ_{E} to reconstruct the Euclidean geometry, i.e. to determine the dimension n, to introduce rectilinear coordinate system and metric tensor in it, to construct k-dimensional planes $k=1,2,\ldots n$ and to test whether the reconstructed geometry is proper Euclidean? If yes, and information, contained in metric is sufficient for construction of proper Euclidean geometry, the used prescriptions can be used for construction of a geometry with other metric. As a result each metric ρ corresponds to some metric geometry T_ρ, constructed *on the base of the metric and only metric*. Any such a geometry T_ρ is not less as informative as the proper Euclidean one in the sense, that any geometric object in proper Euclidean geometry corresponds to a geometric object in the metric geometry T_ρ, constructed according to the same prescriptions, as it is built in the proper Euclidean geometry. This geometric object may appear to bear little resemblance to its Euclidean analog. Besides, due to symmetry of the Euclidean space (presence of a motion group) different geometric objects in T_ρ may have the same Euclidean analog. For instance, in the Euclidean geometry any two different points P and Q, lying on the Euclidean straight L, determine this straight. In metric geometry T_ρ analogs of the Euclidean straight $\mathcal{T}_{PQ}$, $\mathcal{T}_{P_1Q_1}$, $P_1,Q_1\in\mathcal{T}_{PQ}$, determined by different pairs P,Q and P_1,Q_1, are different, in general, if the metric does not satisfy the condition (1.4).

There exists a positive solution of the stated problem, i.e. amount of information, contained in the metric, is sufficient for constructing the metric geometry which is not less informative, than the Euclidean one. Corresponding theorem has been proved [10].

Apparently K. Menger [11] succeeded to approach most closely to the positive solution of the mentioned problem, but he failed to solve it completely. The reason of his failure is some delusion, which may be qualified as "associative prejudice". An overcoming of this prejudice results a construction of new geometric conception, where all information on geometry is contained in metric. The new conception generates a class of T-geometries, which may be considered to be a generalization of conventional metric geometry on the base of metric space $M=\{\rho,\Omega\}$. Formally this generalization is approached at the expense of reduction of number of fundamental concepts, i.e. concepts necessary for the geometry construction and at the expense of elimination of constraints (1.3), (1.4), imposed on metric. Besides instead of metric

ρ one uses the quantity $\sigma = \frac{1}{2}\rho^2$, known as world function [12]. The world function is supposed to be real. It means that the metric ρ may be either nonnegative, or pure imaginary quantity. This extends capacities of geometry. Now one can consider the Minkowski geometry as a special case of T-geometry and use the T-geometry as a space-time geometry. The concept of the curve (1.5) is not used at the construction of geometry, i.e. it is not a fundamental concept, although as the geometry construction has been completed completed nothing prevents from introduction of the curve by means of the mapping (1.5).

But the curve L appears not to be an attribute of geometry. It is some additional object external with respect to geometry. A corollary of this is an appearance of a new geometry property, which is referred to as nondegeneracy. Euclidean and Riemannian geometries have no nondegeneracy. They are degenerate geometries. Associative prejudice is an delusion, appearing, when properties of one object are attributed by mistake to another object. Let us illustrate this in a simple example, which is perceived now as a grotesque. It is known that ancient Egyptians believed that all rivers flow towards the North. This delusion seems now to be nonsense. But many years ago it had weighty foundation. The ancient Egyptians lived on a vast flat plane and knew only one river the Nile, which flew exactly towards the North and had no tributaries on the Egyptian territory. The North direction was a preferred direction for ancient Egyptians who observed motion of heavenly bodies regularly. It was direction toward the fixed North star. They did not connect direction of the river flow with the plane slope, as we do now. They connected the direction of the river flow with the preferred spatial direction towards the North. We are interested now what kind of mistake made ancient Egyptians, believing that all rivers flow towards the North, and how could they to overcome their delusion.

Their delusion was not a logical mistake, because the logic has no relation to this mistake. The delusion was connected with associative property of human thinking, when the property A is attributed to the object B on the basis that in all known cases the property A accompanies the object B. Such an association may be correct or not. If it is erroneous, as in the given case, it is very difficult to discover the mistake. At any rate it is difficult to discover the mistake by means of logic, because such associations appear before the logical analysis, and the subsequent logical analysis is carried out on the basis of the existing associations. Let us imagine that in the course of a voyage an ancient Egyptian scientist arrived the Tigris, which is the nearest to Egypt river. He discovers a water stream which flows, first, not outright and, second, not towards the North. Does he discover his delusion? Most likely not. At any rate not at once. He starts to think that the water stream, flowing before him, is not a river. A ground for such a conclusion is his initial belief that "real" river is to flow, first, directly and, second, towards the North. Besides, the Nile was very important in the life of ancient Egyptians, and they were often apt to idolize the Nile. The delusion about direction of the river flow can be overcame only after that, when one has discovered sufficiently many different rivers, flowing towards different directions, and the proper analysis of this circumstance has been carried out.

Thus, to overcome the associating delusion, it is not sufficient to present another object B, which has not the property A, because one may doubt of whether the presented object is to be classified really as the object B. Another attendant circumstances are also possible.

If the established association between the object and its property is erroneous, one can say on associative delusion or on associative prejudice. The usual method of overcoming the associative prejudices is a consideration of wider set of phenomena, where the established association between the property A and the object B may appear to be violated, and the associative prejudice is discovered.

The associative prejudices are very stable. It is very difficult to overcome them, when they have been established, because they cannot be disproved logically. On the other hand, fixing incorrect correlations between objects of real world, the associative prejudices point out a wrong way for investigations.

Associative prejudices are known in history of science. For instance, the known statement of the Ptolemaic doctrine that the Earth is placed in the centre of universe, and heaven bodies rotate around it, is an example of the associative prejudice. In this case the property of being a centre of a planetary system is attributed to the Earth, whereas such a centre is the Sun. Overcoming of this prejudice was long and difficult, because in contrast to prejudice of ancient Egyptians it can be disproved neither logically, nor experimentally.

Another example of associative prejudice is the popular in XIXth century opinion that the Cartesian coordinate system is an attribute of geometry. This view point appeared, when the analytic geometry was discovered, and the Cartesian coordinate system became to serve as a tool at description of geometric objects of Euclidean geometry. Using analytic description of Euclidean geometry, many mathematicians of XIXth century applied Cartesian coordinates almost always and were inclined to believe that the Cartesian coordinates are an attribute of any geometry at all. On the other hand, non-uniform (Riemannian) geometry cannot be constructed in the Cartesian coordinate system. Any attempt of writing the Riemannian geometry metric tensor in a Cartesian coordinates turns non-uniform (Riemannian) geometry to uniform (i.e. Euclidean) geometry. In other words, the Cartesian coordinate system discriminates any non-uniform geometry. It is known [3] that mathematicians of XIXth century were biased against consideration of the Riemannian geometry as a really existing geometry. It seems that this scepsis in the relation of Riemannian geometry is connected with the associative prejudice, when the Cartesian coordinate system is considered to be an attribute of any geometry. As the coordinate system appears to be a way of the geometry description, but not its attribute, the scepsis disappears.

Now the viewpoint that the concept of the curve (1.5) is a fundamental concept (i.e. it is applied at construction of any geometry) holds much favor. This viewpoint is based on the circumstance that the curve is used at construction of all known (Riemannian and metric) geometries. Such a viewpoint is an associative prejudice (of the kind as the statement of ancient Egyptians that all rivers flow towards the North). To prove this, it is sufficient to construct a sufficiently informative geometry without using the concept of the curve (1.5). Such a geometry (T-geometry) has been constructed [1]. Constructing the new conception of geometry, its author did not think that he did not use the concept of the curve and overcame some prejudice. The point is that the metric $\rho(x,y)$, considered to be a function of two variable points x and y, contains much more information, than the metric tensor $g_{ik}(x)$, which is several functions of one variable point x. The author believed that information contained in metric is sufficient for constructing geometry, and he wants to construct a geometry on the base of only this information. It is possible, provided the concept of the curve is ignored. He did not suspect that he overcame the associative preconception on fundamental role of the curve and, hence, created a new conception of geometry. All this became clear well later at realization and discussion of the obtained results.

In the second section the T-geometric technique is described, and one shows that the Euclidean geometry can be formulated in terms of only metric. The method of the geometric objects, constructed in T-geometry, is described in the third section. The fourth section is devoted to the convexity problem. In the fifth and sixth sections one compares solutions of the parallelism problem in Riemannian and tubular geometries.

2 σ-space and T-geometry

T-geometry is constructed on σ-space $V = \{\sigma, \Omega\}$, which is obtained from the metric space after removal of constraints (1.3), (1.4) and introduction of the world function σ

$$\sigma(P,Q) \equiv \frac{1}{2}\rho^2(P,Q), \qquad \forall P,Q \in \Omega. \tag{2.1}$$

instead of the metric ρ:

Definition 2.1 *σ-space $V = \{\sigma, \Omega\}$ is nonempty set Ω of points P with given on $\Omega \times \Omega$ real function σ*

$$\sigma: \quad \Omega\times\Omega \rightarrow \mathbb{R}, \qquad \sigma(P,P) = 0, \qquad \sigma(P,Q) = \sigma(Q,P) \qquad \forall P,Q \in \Omega. \tag{2.2}$$

The function σ is known as the world function [12], or σ-function. The metric ρ may be introduced in σ-space by means of the relation (2.1). If σ is positive, the metric ρ is also positive, but if σ is negative, the metric is imaginary.

Definition 2.2 . *Nonempty point set $\Omega' \subset \Omega$ of σ-space $V = \{\sigma, \Omega\}$ with the world function $\sigma' = \sigma|_{\Omega'\times\Omega'}$, which is a contraction σ on $\Omega' \times \Omega'$, is called σ-subspace $V' = \{\sigma', \Omega'\}$ of σ-space $V = \{\sigma, \Omega\}$.*

Further the world function $\sigma' = \sigma|_{\Omega'\times\Omega'}$, which is a contraction of σ will be denoted as σ. Any σ-subspace of σ-space is a σ-space.

Definition 2.3 . *σ-space $V = \{\sigma, \Omega\}$ is called isometrically embeddable in σ-space $V' = \{\sigma', \Omega'\}$, if there exists such a monomorphism $f : \Omega \to \Omega'$, that $\sigma(P,Q) = \sigma'(f(P), f(Q))$, $\forall P, \forall Q \in \Omega$, $f(P), f(Q) \in \Omega'$,*

Any σ-subspace V' of σ-space $V = \{\sigma, \Omega\}$ is isometrically embeddable in it.

Definition 2.4 . *Two σ-spaces $V = \{\sigma, \Omega\}$ and $V' = \{\sigma', \Omega'\}$ are called to be isometric (equivalent), if V is isometrically embeddable in V', and V' is isometrically embeddable in V.*

Definition 2.5 *The σ-space $M = \{\rho, \Omega\}$ is called a finite σ-space, if the set Ω contains a finite number of points.*

Definition 2.6 . *The σ-subspace $M_n(\mathcal{P}^n) = \{\sigma, \mathcal{P}^n\}$of the σ-space $V = \{\sigma, \Omega\}$, consisting of $n+1$ points $\mathcal{P}^n = \{P_0, P_1, ..., P_n\}$ is called the nth order σ-subspace .*

The T-geometry is a set of all propositions on properties of σ-subspaces of σ-space $V = \{\sigma, \Omega\}$. Presentation of T-geometry is produced on the language, containing only references to σ-function and constituents of σ-space, i.e. to its σ-subspaces.

Definition 2.7 *A description is called σ-immanent, if it does not contain any references to objects or concepts other, than finite subspaces of the metric space and its world function (metric).*

σ-immanence of description provides independence of the description on the method of description. In this sense the σ-immanence of a description in T-geometry reminds the concept of covariance in Riemannian geometry. Covariance of some relation in Riemannian geometry means that the considered relation is valid in all coordinate systems and, hence, describes only the properties of the Riemannian geometry in itself. Covariant description provides cutting-off from the coordinate system properties, considering the relation in all coordinate systems at once. The σ-immanence provides truncation from the methods of description by absence of a reference to objects, which do not relate to geometry itself (coordinate system, concept of curve, dimension).

The basic elements of T-geometry are finite σ-subspaces $M_n(\mathcal{P}^n)$, i.e. finite sets

$$\mathcal{P}^n = \{P_0, P_1, \ldots, P_n\} \subset \Omega \qquad (2.3)$$

The main characteristic of the finite σ-subspace $M_n(\mathcal{P}^n)$ is its length $|M(\mathcal{P}^n)|$

Definition 2.8 *The squared length $|M(\mathcal{P}^n)|^2$ of the nth order σ-subspace $M(\mathcal{P}^n) \subset \Omega$ of the σ-space $V = \{\sigma, \Omega\}$ is the real number.*

$$|M(\mathcal{P}^n)|^2 = (n! S_n(\mathcal{P}^n))^2 = F_n(\mathcal{P}^n)$$

where $S_n(\mathcal{P}^n)$ is the volume of the $(n+1)$-edr, whose vertices are placed at points $\mathcal{P}^n \equiv \{P_0, P_1, \ldots P_n\} \subset \Omega$, defined by means of relations

$$F_n: \quad \Omega^{n+1} \to \mathbb{R}, \qquad \Omega^{n+1} = \bigotimes_{k=1}^{n+1} \Omega, \qquad n = 1, 2, \ldots \tag{2.4}$$

$$F_n\left(\mathcal{P}^n\right) = \det || \left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k\right) ||, \qquad P_0, P_i, P_k \in \Omega, \qquad i, k = 1, 2, ...n \tag{2.5}$$

$$\begin{aligned}\left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k\right) &\equiv \Gamma\left(P_0, P_i, P_k\right) \equiv \sigma\left(P_0, P_i\right) + \sigma\left(P_0, P_k\right) - \sigma\left(P_i, P_k\right), \\ i, k &= 1, 2, ...n,\end{aligned} \tag{2.6}$$

where the function σ is defined via metric ρ by the relation (2.1) and $\mathcal{P}^n$ denotes $n+1$ points (2.3).

The meaning of the written relations is as follows. In the special case, when the σ-space is Euclidean space and its σ-function coincides with σ-function of Euclidean space, any two points P_0, P_1 determine the vector $\mathbf{P}_0\mathbf{P}_1$, and the relation (2.6) is a σ-immanent expression for the scalar product $(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)$ of two vectors. Then the relation (2.5) is the Gram's determinant for n vectors $\mathbf{P}_0\mathbf{P}_i$, $\quad i = 1, 2, \ldots n$, and $S_n(\mathcal{P}^n)$ is the Euclidean volume of the $(n+1)$-edr with vertices at the points $\mathcal{P}^n$.

The idea of constructing the T-geometry is very simple. All relations of proper Euclidean geometry are written in the σ-immanent form and declared to be valid for any σ-function. This results that any relation of proper Euclidean geometry corresponds some relation of T-geometry. It is important that in the relations, declared to be relations of T-geometry, only the properties (2.1) were used. The special properties of the Euclidean σ-function are not to be taken into account. The metric part of these relations was formulated and proved by K. Menger [11]. Let us present this result in our designations in the form of the theorem

Theorem 1 *The σ-space $V = \{\sigma, \Omega\}$ is isometrically embeddable in n-dimensional proper Euclidean space E_n, if and only if any $(n+2)$th order σ-subspace $M(\mathcal{P}^{n+2}) \subset \Omega$ is isometrically embeddable in E_n.*

Unfortunately, the formulation of this theorem is not σ-immanent, as far as it contains a reference to n-dimensional Euclidean space E_n which is not defined σ-immanently. A more constructive version of the σ-space Euclideaness conditions is formulated in the form

I.

$$\exists \mathcal{P}^n \subset \Omega, \qquad F_n(\mathcal{P}^n) \neq 0, \qquad F_{n+1}(\Omega^{n+2}) = 0, \tag{2.7}$$

II.

$$\sigma(P, Q) = \frac{1}{2} \sum_{i,k=1}^{n} g^{ik}(\mathcal{P}^n)[x_i\left(P\right) - x_i\left(Q\right)][x_k\left(P\right) - x_k\left(Q\right)], \qquad \forall P, Q \in \Omega, \tag{2.8}$$

where the quantities $x_i\left(P\right)$, $x_i\left(Q\right)$ are defined by the relations

$$x_i\left(P\right) = \left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}\right), \qquad x_i\left(Q\right) = \left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{Q}\right), \qquad i = 1, 2, ...n \tag{2.9}$$

The contravariant components $g^{ik}(\mathcal{P}^n)$, $(i,k=1,2,\dots n)$ of metric tensor are defined by its covariant components $g_{ik}(\mathcal{P}^n)$, $(i,k=1,2,\dots n)$ by means of relations

$$\sum_{k=1}^{n} g_{ik}(\mathcal{P}^n)g^{kl}(\mathcal{P}^n)=\delta_i^l, \qquad i,l=1,2,\dots n \tag{2.10}$$

where

$$g_{ik}(\mathcal{P}^n)=\Gamma(P_0,P_i,P_k), \qquad i,k=1,2,\dots n \tag{2.11}$$

III. The relations

$$\Gamma(P_0,P_i,P)=x_i, \qquad x_i\in\mathbb{R}, \qquad i=1,2,\dots n, \tag{2.12}$$

considered to be equations for determination of $P\in\Omega$, have always one and only one solution.
IIIa. The relations (2.12), considered to be equations for determination of $P\in\Omega$, have always not more than one solution.

Remark 1 *The condition (2.7) is a corollary of the condition (2.8). It is formulated in the form of a special condition, in order that a determination of dimension were separated from determination of coordinate system.*

The condition I determines the space dimension. The condition II describes σ-immanently the scalar product properties of the proper Euclidean space. Setting $n+1$ points $\mathcal{P}^n$, satisfying the condition I, one determines n-dimensional basis of vectors in Euclidean space. Relations (2.11), (2.10) determine covariant and contravariant components of the metric tensor, and the relations (2.9) determine covariant coordinates of points P and Q at this basis. The relation (2.8) determines the expression for σ-function for two arbitrary points in terms of coordinates of these points. Finally, the condition III describes continuity of the set Ω and a possibility of the manifold construction on it. Necessity of conditions I – III for Euclideaness of σ-space is evident. One can prove their sufficiency [10]. The connection of conditions I – III with the Euclideaness of the σ-space can be formulated in the form of a theorem.

Theorem 2 *The σ-space $V=\{\sigma,\Omega\}$ is the n-dimensional Euclidean space, if and only if σ-immanent conditions I – III are fulfilled.*

Remark 2 *For the σ-space were proper Euclidean, the eigenvalues of the matrix $g_{ik}(\mathcal{P}^n)$, $\quad i,k=1,2,\dots n$ must have the same sign, otherwise it is pseudoeuclidean.*

The theorem states that it is sufficient to know metric (world function) to construct Euclidean geometry. The information, contained in concepts of topological space and curve, which are used in metric geometry, appears to be excess.

Proof of this theorem can be found in [10]. A similar theorem for another (but close) necessary and sufficient conditions has been proved in ref. [1]. Here we show only constructive character of conditions I – III for proper Euclidean space. It means that starting from an abstract σ-space, satisfying conditions I – III, one can determine dimension n and construct a rectilinear coordinate system with conventional description of the proper Euclidean space in it. One construct sequentially straight, two-dimensional plane, etc...up to n-dimensional plane coincide with the set Ω. To construct all these objects, one needs to develop technique of T-geometry.

Definition 2.9 *The finite σ-space $M_n(\mathcal{P}^n) = \{\sigma, \mathcal{P}^n\}$ is called oriented $\overrightarrow{M_n(\mathcal{P}^n)}$, if the order of its points $\mathcal{P}^n = \{P_0, P_1, \dots P_n\}$ is fixed.*

Definition 2.10 . *The nth order multivector m_n is the mapping*

$$m_n : \qquad I_n \to \Omega, \qquad I_n \equiv \{0, 1, ..., n\} \tag{2.13}$$

The set I_n has a natural ordering, which generates an ordering of images $m_n(k) \in \Omega$ of points $k \in I_n$. The ordered list of images of points in I_n has one-to-one connection with the multivector and may be used as the multivector identificator. Different versions of the point list will be used for writing the nth order multivector identificator:

$$\overrightarrow{P_0P_1...P_n} \equiv \mathbf{P}_0\mathbf{P}_1...\mathbf{P}_n \equiv \overrightarrow{\mathcal{P}^n}$$

Originals of points P_k in I_n are determined by the order of the point P_k in the list of identificator. Index of the point P_k has nothing to do with the original of P_k. Further we shall use identificator. $\overrightarrow{P_0P_1...P_n}$ of the multivector instead of the multivector. In this sense the nth order multivector $\overrightarrow{P_0P_1...P_n}$ in the σ-space $V = \{\sigma, \Omega\}$ may be defined as the ordered set $\{P_l\}$, $l = 0, 1, \dots n$ of $n+1$ points $P_0, P_1, ..., P_n$, belonging to the σ-space V. The point P_0 is the origin of the multivector $\overrightarrow{P_0P_1...P_n}$. Image $m_n(I_n)$ of the set I_n contains k points ($k \leq n+1$). The set of all nth order multivectors m_n constitutes the set $\Omega^{n+1} = \bigotimes_{k=1}^{n+1} \Omega$, and any multivector $\overrightarrow{\mathcal{P}^n} \in \Omega^{n+1}$.

Definition 2.11 . *The scalar σ-product $(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})$ of two nth order multivectors $\overrightarrow{\mathcal{P}^n}$ and $\overrightarrow{\mathcal{Q}^n}$ is the real number*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = \det \|(\mathbf{P}_0\mathbf{P}_i.\mathbf{Q}_0\mathbf{Q}_k)\|, \qquad i, k = 1, 2, ...n, \qquad \overrightarrow{\mathcal{P}^n}, \overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1} \tag{2.14}$$

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{Q}_0\mathbf{Q}_k) \equiv \sigma(P_0, Q_i) + \sigma(Q_0, P_k) - \sigma(P_0, Q_0) - \sigma(P_i, Q_k), \qquad P_0, P_i, Q_0, Q_k \in \Omega \tag{2.15}$$

Definition 2.12 . *The length $|\overrightarrow{\mathcal{P}^n}|$ of the nth order multivector $\overrightarrow{\mathcal{P}^n}$ is the number*

$$|\overrightarrow{\mathcal{P}^n}| = \begin{cases} |\sqrt{(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n})}|, & (\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n}) \geq 0 \\ i\,|\sqrt{(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n})}|, & (\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n}) < 0 \end{cases} \qquad \overrightarrow{\mathcal{P}^n} \in \Omega^{n+1} \tag{2.16}$$

In the case, when multivector does not contain similar points, it coincides with the oriented finite σ-subspace, and it is a constituent of σ-space. In the case, when at least two points of multivector coincide, the multivector length vanishes, and the multivector is considered to be null multivector. The null multivector is not a finite σ-subspace, but a use of null multivectors assists in creation of a more simple technique. In the case of manipulation with numbers, written in Arabic numerals (where zero is present) is simpler, than the same manipulation with numbers, written in Roman numerals (where zero is absent). Something like that takes place in the case of multivectors. Essentially, the multivectors are basic objects of T-geometry. As to continual geometric objects, which are analogs of planes, sphere ellipsoid, etc., they are constructed by means of skeleton-envelope method (see next section) with multivectors, or finite σ-subspaces used as skeletons. As a consequence the T-geometry is presented σ-immanently, i.e. without reference to objects, external with respect to σ-space.

Definition 2.13 . *Two nth order multivectors $\overrightarrow{\mathcal{P}^n}$ $\overrightarrow{\mathcal{Q}^n}$ are collinear $\overrightarrow{\mathcal{P}^n} \parallel \overrightarrow{\mathcal{Q}^n}$, if*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})^2 = |\overrightarrow{\mathcal{P}^n}|^2 \cdot |\overrightarrow{\mathcal{Q}^n}|^2 \tag{2.17}$$

Definition 2.14 . *Two collinear nth order multivectors $\overrightarrow{\mathcal{P}^n}$ and $\overrightarrow{\mathcal{Q}^n}$ are similarly oriented $\overrightarrow{\mathcal{P}^n} \uparrow\uparrow \overrightarrow{\mathcal{Q}^n}$ (parallel), if*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = |\overrightarrow{\mathcal{P}^n}| \cdot |\overrightarrow{\mathcal{Q}^n}| \tag{2.18}$$

They have opposite orientation $\overrightarrow{\mathcal{P}^n} \uparrow\downarrow \overrightarrow{\mathcal{Q}^n}$ (antiparallel), if

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = -|\overrightarrow{\mathcal{P}^n}| \cdot |\overrightarrow{\mathcal{Q}^n}| \tag{2.19}$$

Vector $\mathbf{P}_0\mathbf{P}_1 = \overrightarrow{\mathcal{P}^1}$ is the first order multivector.

Definition 2.15 *nth order σ-subspace $M(\mathcal{P}^n)$ of nonzero length $|M(\mathcal{P}^n)|^2 = F_n(\mathcal{P}^n) \neq 0$ determines the set of points $\mathcal{T}(\mathcal{P}^n)$, called nth order tube by means of relation*

$$\mathcal{T}(\mathcal{P}^n) \equiv \mathcal{T}_{\mathcal{P}^n} = \left\{P_{n+1} | F_{n+1}\left(\mathcal{P}^{n+1}\right) = 0\right\}, \qquad P_i \in \Omega, \qquad i = 0, 1 \ldots n+1, \tag{2.20}$$

where the function F_n is defined by the relations (2.4) – (2.6)

In arbitrary T-geometry the nth order tube is an analog of n-dimensional properly Euclidean plane.

Definition 2.16 . *Section $\mathcal{S}_{n;P}$ of the tube $\mathcal{T}(\mathcal{P}^n)$ at the point $P \in \mathcal{T}(\mathcal{P}^n)$ is the set $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n))$ of points, belonging to the tube $\mathcal{T}(\mathcal{P}^n)$*

$$\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n)) = \{P' \mid \bigwedge_{l=0}^{l=n} \sigma(P_l, P') = \sigma(P_l, P)\}, \qquad P \in \mathcal{T}(\mathcal{P}^n) \qquad P' \in \Omega. \tag{2.21}$$

Let us note that $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n)) \subset \mathcal{T}(\mathcal{P}^n)$, because $P \in \mathcal{T}(\mathcal{P}^n)$. Indeed, whether the point P belongs to $\mathcal{T}(\mathcal{P}^n)$ depends only on values of $n+1$ quantities $\sigma(P_l, P)$, $\;l = 0, 1, ...n$. In accordance with (2.21) these quantities are the same for both points P and P'. Hence, any running point $P' \in \mathcal{T}(\mathcal{P}^n)$, if $P \in \mathcal{T}(\mathcal{P}^n)$.

In the proper Euclidean space the nth order tube is n-dimensional plane, containing points $\mathcal{P}^n$, and its section $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n))$ at the point P consists of one point P.

Now we can construct the proper Euclidean space and rectilinear coordinate system in it on the basis of only σ-function. Let it is known that the σ-space $V = \{\sigma, \Omega\}$ is the proper Euclidean space, but its dimension is not known. To determine the dimension n, let us take two different points $P_0, P_1 \in \Omega$, $\;F_1(\mathcal{P}^1) = 2\sigma(P_0, P_1) \neq 0$.

1. Let us construct the first order tube $\mathcal{T}(\mathcal{P}^1)$. If $\mathcal{T}(\mathcal{P}^1) = \Omega$, then dimension of the σ-space V $\;n = 1$. If $\Omega \backslash \mathcal{T}(\mathcal{P}^1) \neq \emptyset$, $\;\exists P_2 \in \Omega$, $\;P_2 \notin \mathcal{T}(\mathcal{P}^1)$, and hence, $F_2(\mathcal{P}^2) \neq 0$.

2. Let us construct the second order tube $\mathcal{T}(\mathcal{P}^2)$. If $\mathcal{T}(\mathcal{P}^2) = \Omega$, then $\;n = 2$, otherwise $\exists P_3 \in \Omega$, $\;P_3 \notin \mathcal{T}(\mathcal{P}^2)$, and hence, $F_3(\mathcal{P}^3) \neq 0$.

3. Let us construct the third order tube $\mathcal{T}(\mathcal{P}^3)$. If $\mathcal{T}(\mathcal{P}^3) = \Omega$, then $n = 3$, otherwise $\exists P_4 \in \Omega$, $\;P_4 \notin \mathcal{T}(\mathcal{P}^3)$, and hence, $F_4(\mathcal{P}^4) \neq 0$.

4. Etc.

Continuing this process, one determines such $n+1$ points $\mathcal{P}^n$, that the condition $\mathcal{T}(\mathcal{P}^n) = \Omega$ and, hence, conditions (2.7) are fulfilled.

Then by means of relations

$$x_i(P) = \Gamma(P_0, P_i, P), \qquad i = 1, 2, \ldots n, \tag{2.22}$$

one attributes covariant coordinates $x(P) = \{x_i(P)\}$, $\;i = 1, 2, \ldots n$ to $\forall P \in \Omega$. Let $x = x(P) \in \mathbb{R}^n$ and $x' = x(P') \in \mathbb{R}^n$. Substituting $\Gamma(P_0, P_i, P) = x$ and $\Gamma(P_0, P_i, P') = x'_i$ in (2.8), one obtains the conventional expression for the world function of the Euclidean space in the rectilinear coordinate system

$$\sigma(P, P') = \sigma_E(x, x') = \frac{1}{2} \sum_{i,k=1}^{n} g^{ik}(\mathcal{P}^n)\,(x_i - x'_i)\,(x_k - x'_k) \tag{2.23}$$

where $g^{ik}(\mathcal{P}^n)$, defined by relations (2.11) and (2.10), is the contravariant metric tensor in this coordinate system.

Condition III of the theorem states that the mapping

$$x: \; \Omega \rightarrow \mathbb{R}^n$$

described by the relation (2.22) is a bijection, i.e. $\forall y \in \mathbb{R}^n$ there exists such one and only one point $Q \in \Omega$, that $y = x(Q)$.

Thus, on the base of the world function, given on abstract set $\Omega \times \Omega$, one can determine the dimension n of the Euclidean space, construct rectilinear coordinate system with the metric tensor $g_{ik}(\mathcal{P}^n) = \Gamma(P_0, P_i, P_k)$, $\qquad i, k = 1, 2, \ldots n$ and describe all geometrical objects which are determined in terms of coordinates. The Euclidean space and Euclidean geometry is described in terms and only in terms of world function (metric).

Conditions I – III, formulated in the σ-immanent form admit one to construct the proper Euclidean space, using only information, contained in world function. σ-immanence of the formulation admits one to state that information, contained in the world function, is sufficient for construction of any T-geometry. Substitution of condition III by the condition IIIa leads to a reduction of constraints. At the fulfillment of conditions I,II,IIIa the σ-space appears to be isometrically embeddable in n-dimensional Euclidean space. It may be piecewise continuous, or even discrete. Such a σ-space can be obtained, removing arbitrary number of points from n-dimensional Euclidean space.

3 Skeleton-envelope method of geometric objects construction

Definition 3.1 *Geometric object $\mathcal{O}$ is some σ-subspace of σ-space.*

In T-geometry a geometric object $\mathcal{O}$ is described by means of skeleton-envelope method. It means that any geometric object $\mathcal{O}$ is considered to be a set of intersections and joins of elementary geometric objects (EGO).

Definition 3.2 *Elementary geometric object $\mathcal{E}\subset\Omega$ is a set of zeros of the envelope function*

$$f_{\mathcal{P}^n}: \qquad \Omega\rightarrow\mathbb{R}, \qquad \mathcal{P}^n\equiv\{P_0,P_1,...P_n\}\subset\Omega \tag{3.1}$$

i.e..

$$\mathcal{E}=\mathcal{E}_f\left(\mathcal{P}^n\right)=\{R|f_{\mathcal{P}^n}\left(R\right)=0\} \tag{3.2}$$

The finite set $\mathcal{P}^n\subset\Omega$ of parameters of the envelope function $f_{\mathcal{P}^n}$ is skeleton of elementary geometric object (EGO) $\mathcal{E}\subset\Omega$. The set $\mathcal{E}\subset\Omega$ of points forming EGO is called the envelope of its skeleton $\mathcal{P}^n$. For continuous T-geometry the envelope $\mathcal{E}$ is usually a continual set of points. The envelope function $f_{\mathcal{P}^n}$, determining EGO is a function of the running point $R\in\Omega$ and of parameters $\mathcal{P}^n\subset\Omega$. The envelope function $f_{\mathcal{P}^n}$ is supposed to be an algebraic function of s arguments $w=\{w_1,w_2,...w_s\}$, $s=(n+2)(n+1)/2$. Each of arguments $w_k=\sigma\left(Q_k,L_k\right)$ is a σ-function of two arguments $Q_k,L_k\in\{R,\mathcal{P}^n\}$, either belonging to skeleton $\mathcal{P}^n$, or coinciding with the running point R.

Let us consider examples of some simplest EGOs.

$$\mathcal{S}(P_0,P_1)=\{R|f_{P_0P_1}\left(R\right)=0\}, \qquad f_{P_0P_1}\left(R\right)=\sqrt{2\sigma\left(P_0,P_1\right)}-\sqrt{2\sigma\left(P_0,R\right)} \tag{3.3}$$

is a sphere, passing through the point P_1 and having its center at the point P_0. Ellipsoid $\mathcal{EL}$, passing through the point P_2 and having the focuses at points P_0, P_1 ($P_0 \neq P_1$) is described by the relation

$$\mathcal{EL}(P_0, P_1, P_2) = \{R | f_{P_0 P_1 P_2}(R) = 0\}, \tag{3.4}$$

where the envelope function $f_{P_0 P_1 P_2}(R)$ is defined by the equation.

$$f_{P_0 P_1 P_2}(R) = \sqrt{2\sigma(P_0, P_2)} + \sqrt{2\sigma(P_1, P_2)} - \sqrt{2\sigma(P_0, R)} - \sqrt{2\sigma(P_1, R)} \tag{3.5}$$

If focuses P_0, P_1 coincide ($P_0 = P_1$), the ellipsoid $\mathcal{EL}(P_0, P_1, P_2)$ degenerates into a sphere $\mathcal{S}(P_0, P_2)$. If the points P_1, P_2 coincide ($P_1 = P_2$), the ellipsoid $\mathcal{EL}(P_0, P_1, P_2)$ degenerates into a segment of a straight line $\mathcal{T}_{[P_0 P_1]}$ between the points P_0, P_1.

$$\mathcal{T}_{[P_0 P_1]} = \mathcal{EL}(P_0, P_1, P_1) = \{R | f_{P_0 P_1 P_1}(R) = 0\}, \tag{3.6}$$

$$f_{P_0 P_1 P_1}(R) = S_2(P_0, R, P_1) \equiv \sqrt{2\sigma(P_0, P_1)} - \sqrt{2\sigma(P_0, R)} - \sqrt{2\sigma(P_1, R)} \tag{3.7}$$

In the proper Euclidean geometry $\mathcal{T}_{[P_0 P_1]}$ is simply a segment of the straight between the points P_0, P_1.

The most important and interesting EGOs arise, when values of the envelope function $f_{\mathcal{P}^n}(R)$ coincide with values of the function $F_{n+1}(\mathcal{P}^n, R)$, determined by relation (2.5) and proportional to the squared length of the finite σ-subspace, consisting of $n+2$ points $\mathcal{P}^n, R$. This object is called the nth order natural geometric object (NGO). It is defined by the relation (2.20). In the case of proper Euclidean geometry it coincides with n-dimensional plane.

Another functions f generate another envelopes of elementary geometrical objects for the given skeleton $\mathcal{P}^n$. For instance, the set of two points $\{P_0, P_1\}$ forms a skeleton not only for the tube $\mathcal{T}_{P_0 P_1}$, but also for the segment $\mathcal{T}_{[P_0 P_1]}$ of the tube (straight) (3.6), and for the tube ray $\mathcal{T}_{[P_0 P_1}$, which is defined by the relation

$$\mathcal{T}_{[P_0 P_1} = \{R | S_2(P_0, P_1, R) = 0\} \tag{3.8}$$

where the function S_2 is defined by the relation (3.7).

4 Interrelation between T-geometric and Riemannian conceptions of geometry

Definition 4.1 *The geometric conception is a totality of principles of the geometry construction.*

Let us compare the Riemannian conception of geometry and that of T-geometry. n-dimensional Riemannian geometry $R_n = \{\mathbf{g}, K, \mathcal{M}_n\}$ is introduced on n-dimensional manifold $\mathcal{M}_n$ in some coordinate system K by setting the metric tensor $g_{ik}(x)$, $i, k = 1, 2, \ldots n$. Thereafter, using the definition (1.5) of the curve, which always can be introduced on the manifold $\mathcal{M}_n$, one introduces concept of geodesic $\mathcal{L}_{[xx']}$ as the shortest curve connecting points with coordinates x and x'. In the Riemannian space $R_n = \{\mathbf{g}, K, \mathcal{M}_n\}$ one introduces the world function $\sigma_R(x, x')$ between points x and x', defined by the relation

$$\sigma_R(x, x') = \frac{1}{2}\left(\int\limits_{\mathcal{L}_{[xx']}} \sqrt{g_{ik}dx^i dx^k}\right)^2, \qquad (4.1)$$

where $\mathcal{L}_{[xx']}$ denotes a segment of geodesic, connecting points x and x'.

T-geometry can be introduced on any set Ω, including the manifold $\mathcal{M}_n$. To set T-geometry on $\mathcal{M}_n$, it is insufficient of the metric tensor $g_{ik}(x)$, $i, k = 1, 2, \dots n$ introduction, because it determines only first derivatives of world function at coinciding points

$$g_{ik}(x) = -\sigma_{ik'}(x, x) \equiv -\left[\frac{\sigma(x, x')}{\partial x^i \partial x'^k}\right]_{x'=x} \qquad (4.2)$$

This is insufficient for determination of the world function. For setting T-geometry in a way consistent with the Riemannian geometry, one should set $\sigma(x, x') = \sigma_R(x, x')$, where $\sigma_R(x, x')$ is defined by the relation (4.1). Now one can construct geometric objects by the method described above. The T-geometry, introduced in such a way, will be referred to as σ-Riemannian geometry, for distinguishing different conceptions (i.e. rules of construction) of geometry.

Note that the world function, consistent with Riemannian geometry on the manifold, may be set as a solution of equations in partial derivatives. For instance, the world function can be defined as the solution of the differential equation [12]

$$\sigma_i g^{ik}(x)\sigma_k = 2\sigma, \qquad \sigma_i \equiv \frac{\partial \sigma}{\partial x^i} \qquad i = 1, 2, \dots n, \qquad (4.3)$$

satisfying the conditions (2.2).

The basic geometric objects of Riemannian geometry – geodesic segments $\mathcal{L}_{[xx']}$ coincide with the first order NGOs in T-geometry – the tube segments $\mathcal{T}_{[xx']}$, defined by the relations (3.6). Thus one can say on partial coincidence of two geometric conceptions: Riemannian and σ-Riemannian ones. But such a coincidence is not complete. There are some difference which appears sometimes essential.

Let us consider the case, when the manifold $\mathcal{M}_n$ coincides with $\mathbb{R}^n$ and metric tensor g_{ik} =const, $i, k = 1, 2, \dots n$, $g = \det ||g_{ik}|| \neq 0$ is the metric tensor of the proper Euclidean space. The world function is described by the relation (2.23), and the proper Riemannian space $E_n = \{\mathbf{g}_E, K, \mathbb{R}^n\}$ is the proper Euclidean space. Here $\mathbf{g}_E$ denotes the metric tensor of the proper Euclidean space.

Now let us consider the proper Riemannian space $R_n = \{\mathbf{g}_E, K, D\}$, where $D \subset \mathbb{R}^n$ is some region of the proper Euclidean space $E_n = \{\mathbf{g}_E, K, \mathbb{R}^n\}$. If this region D is convex, i.e. any segment $\mathcal{L}_{[xx']}$ of straight, passing through points $x, x' \in D$, belongs to D ($\mathcal{L}_{[xx']} \subset D$), the world function of the proper Riemannian space $R_n =$

$\{\mathbf{g}_E, K, D\}$ has the form (2.23), and the proper Riemannian space $R_n = \{\mathbf{g}_E, K, D\}$ can be embedded isometrically to the proper Euclidean space $E_n = \{\mathbf{g}_E, K, \mathbb{R}^n\}$.

If the region D is not convex, the system of geodesics in the region $R_n = \{\mathbf{g}_E, K, D\}$ is not a system of straights, and world function (4.1) is not described by the relation (2.23). In this case the region D cannot be embedded isometrically in $E_n = \{\mathbf{g}_E, K, \mathbb{R}^n\}$, in general. It seems to be paradoxical that one (nonconvex) part of the proper Euclidean space cannot be embedded isometrically to it, whereas another (convex) part can.

The convexity problem appears to be rather complicated, and most of mathematicians prefer to go around this problem, dealing only with convex regions [13]. In T-geometry there is no convexity problem. Indeed, according to definition 2.2 subset of points of σ-space is always embeddable isometrically in σ-space. From viewpoint of T-geometry a removal of any region $R_n = \{\mathbf{g}_E, K, D\}$ from the proper Euclidean space $R_n = \{\mathbf{g}_E, K, \mathbb{R}^n\}$ cannot change shape of geodesics (first order NGOs). It leads only to holes in geodesics, making them discontinuous. The continuity is a property of the coordinate system, used in the proper Riemannian geometry as a main tool of description. Using continuous coordinate systems for description, we transfer constraints imposed on coordinate system to the geometry itself.

Insisting on continuity of geodesics, one overestimates importance of continuity for geometry and attributes continuous geodesics (the first order NGOs) to any proper Riemannian geometry, whereas the continuity is a special property of the proper Euclidean geometry. From viewpoint of T-geometry the convexity problem is an artificial problem. Existence of the convexity problem in the Riemannian conception of geometry and its absence in T-geometric conception means that the second conception of geometry is more perfect.

5 Riemannian geometry and one-dimensionality of the first order tubes

Let us consider the n-dimensional pseudoeuclidean space $E_n = \{\mathbf{g}_1, K, \mathbb{R}^n\}$ of the index 1, $\mathbf{g}_1 = \text{diag}\{1, -1, -1 \ldots -1\}$ to be a kind of n-dimensional Riemannian space. The world function is defined by the relation (2.23)

$$\sigma_1(x, x') = \frac{1}{2} \sum_{i,k=1}^{n} g^{ik} \left(x_i - x_i'\right)\left(x_k - x_k'\right), \qquad g^{ik} = \text{diag}\left\{1, -1, -1 \ldots -1\right\} \quad (5.1)$$

Geodesic $\mathcal{L}_{yy'}$ is a straight line, and it is considered in pseudoeuclidean geometry to be the first order NGOs, determined by two points y and y'

$$\mathcal{L}_{yy'}: \quad x^i = \left(y^i - y'^i\right)\tau, \qquad i = 1, 2, \ldots n, \qquad \tau \in \mathbb{R} \quad (5.2)$$

The geodesic $\mathcal{L}_{yy'}$ is called timelike, if $\sigma_1(y, y') > 0$, and it is called spacelike if $\sigma_1(y, y') < 0$. The geodesic $\mathcal{L}_{yy'}$ is called null, if $\sigma_1(y, y') = 0$.

The pseudoeuclidean space $E_n = \{\mathbf{g}_1, K, \mathbb{R}^n\}$ generates the σ-space $V = \{\sigma_1, \mathbb{R}^n\}$, where the world function σ_1 is defined by the relation (5.1). The first order tube (NGO) $\mathcal{T}(x, x')$ in the σ-Riemannian space $V = \{\sigma_1, \mathbb{R}^n\}$ is defined by the relation (2.20)

$$\mathcal{T}(x, x') \equiv \mathcal{T}_{xx'} = \{r | F_2(x, x', r) = 0\}, \qquad \sigma_1(x, x') \neq 0, \qquad x, x', r \in \mathbb{R}^n, \tag{5.3}$$

$$F_2(x, x', r) = \begin{vmatrix} (x'_i - x_i)(x'^i - x^i) & (x'_i - x_i)(r^i - x^i) \\ (r_i - x_i)(x'^i - x^i) & (r_i - x_i)(r^i - x^i) \end{vmatrix} \tag{5.4}$$

Solution of equations (5.3), (5.4) gives the following result

$$\mathcal{T}_{xx'} = \left\{ r \left| \bigcup_{y \in \mathbb{R}^n} \bigcup_{\tau \in \mathbb{R}} r = (x' - x)\tau + y - x \wedge \Gamma(x, x', y) = 0 \wedge \Gamma(x, y, y) = 0 \right. \right\}, \tag{5.5}$$
$$x, x', y, r \in \mathbb{R}^n$$

where $\Gamma(x, x', y) = (x'_i - x_i)(y^i - x^i)$ is the scalar product of vectors $\overrightarrow{xy}$ and $\overrightarrow{xx'}$ defined by the relation (2.6). In the case of timelike vector $\overrightarrow{xx'}$, when $\sigma_1(x, x') > 0$, there is a unique null vector $\overrightarrow{xy} = \overrightarrow{xx} = \overrightarrow{0}$ which is orthogonal to the vector $\overrightarrow{xx'}$. In this case the $(n-1)$-dimensional surface $\mathcal{T}_{xx'}$ degenerates into the one-dimensional straight

$$\mathcal{T}_{xx'} = \left\{ r \left| \bigcup_{\tau \in \mathbb{R}} r = (x' - x)\tau \right. \right\}, \qquad \sigma_1(x, x') > 0, \qquad x, x', r \in \mathbb{R}^n, \tag{5.6}$$

Thus, for timelike vector $\overrightarrow{xx'}$ the first order tube $\mathcal{T}_{xx'}$ coincides with the geodesic $\mathcal{L}_{xx'}$. In the case of spacelike vector $\overrightarrow{xx'}$ the $(n-1)$-dimensional tube $\mathcal{T}_{xx'}$ contains the one-dimensional geodesic $\mathcal{L}_{xx'}$ of the pseudoeuclidean space $E_n = \{\mathbf{g}_1, K, \mathbb{R}^n\}$.

This difference poses the question what is the reason of this difference and what of the two generalization of the proper Euclidean geometry is more reasonable. Note that four-dimensional pseudoeuclidean geometry is used for description of the real space-time. One can try to resolve this problem from experimental viewpoint. Free classical particles are described by means of timelike straight lines. At this point the pseudoeuclidean geometry and the σ-pseudoeuclidean geometry (T-geometry) lead to the same result. The spacelike straights are believed to describe the particles moving with superlight speed (so-called taxyons). Experimental attempts of taxyons discovery were failed. Of course, trying to discover taxyons, one considered them to be described by spacelike straights. On the other hand, the physicists believe that all what can exist does exist and may be discovered. From this viewpoint the failure of discovery of taxyons in the form of spacelike line justifies in favor of taxyons in the form of three-dimensional surfaces.

To interpret the structure of the set (5.5), describing the first order tube, let us take into account the zeroth order tube $\mathcal{T}_x$, determined by the point x in the σ-pseudoeuclidean space is the light cone with the vertex at the point x (not the point x). Practically the first order tube consists of such sections of the light cones with their vertex $y \in \mathcal{L}_{xx'}$ that all vectors $\overrightarrow{yr}$ of these sections are orthogonal to the vector $\overrightarrow{xx'}$. In other words, the first order tube $\mathcal{T}_{xx'}$ consists of the zeroth order tubes $\mathcal{T}_y$ sections at y, orthogonal to $\overrightarrow{xx'}$, with $y \in \mathcal{L}_{xx'}$. For timelike $\overrightarrow{xx'}$ this section consists of one point, but for the spacelike $\overrightarrow{xx'}$ it is two-dimensional section of the light cone.

6 Collinearity in Riemannian and σ-Riemannian geometry

Let us return to the Riemannian space $R_n = \{\mathbf{g}, K, D\}, \quad D \subset \mathbb{R}^n$, which generates the world function $\sigma(x, x')$ defined by the relation (4.1). Then the σ-space $V = \{\sigma, D\}$ appears. it will be referred to as σ-Riemannian space. We are going to compare concept of collinearity (parallelism) of two vectors in the two spaces.

The world function $\sigma = \sigma(x, x')$ of both σ-Riemannian and Riemannian spaces satisfies the system of equations [14][2]

$$\begin{array}{llll} (1) & \sigma_l \sigma^{lj'} \sigma_{j'} = 2\sigma & (4) & \det \parallel \sigma_{i||k} \parallel \neq 0 \\ (2) & \sigma(x, x') = \sigma(x', x) & (5) & \det \parallel \sigma_{ik'} \parallel \neq 0 \\ (3) & \sigma(x, x) = 0 & (6) & \sigma_{i||k||l} = 0 \end{array} \qquad (6.1)$$

where the following designations are used

$$\sigma_i \equiv \frac{\partial \sigma}{\partial x^i}, \qquad \sigma_{i'} \equiv \frac{\partial \sigma}{\partial x'^i}, \qquad \sigma_{ik'} \equiv \frac{\partial^2 \sigma}{\partial x^i \partial x'^k}, \qquad \sigma^{ik'} \sigma_{lk'} = \delta_l^i$$

Here the primed index corresponds to the point x', and unprimed index corresponds to the point x. Two parallel vertical strokes mean covariant derivative $\tilde{\nabla}_i^{x'}$ with respect to x^i with the Christoffel symbol

$$\Gamma_{kl}^i \equiv \Gamma_{kl}^i(x, x') \equiv \sigma^{is'} \sigma_{kls'}, \qquad \sigma_{kls'} \equiv \frac{\partial^3 \sigma}{\partial x^k \partial x^l \partial x'^s}$$

For instance,

$$G_{ik} \equiv G_{ik}(x, x') \equiv \sigma_{i||k} \equiv \frac{\partial \sigma_i}{\partial x^k} - \Gamma_{ik}^l(x, x') \sigma_l \equiv \frac{\partial \sigma_i}{\partial x^k} - \sigma_{iks'} \sigma^{ls'} \sigma_l \qquad (6.2)$$

$$G_{ik||l} \equiv \frac{\partial G_{ik}}{\partial x^l} - \sigma_{ils'} \sigma^{js'} G_{jk} - \sigma_{kls'} \sigma^{js'} G_{ij}$$

Summation from 1 to n is produced over repeated indices. The covariant derivative $\tilde{\nabla}_i^{x'}$ with respect to x^i with the Christoffel symbol $\Gamma_{kl}^i(x, x')$ acts only on the point x and on unprimed indices. It is called the tangent derivative, because it is a covariant

[2]The paper [14] is hardly available for English speaking reader. Survey of main results of [14] in English may be found in [15]. See also [2]

derivative in the Euclidean space $E_{x'}$ which is tangent to the Riemannian space R_n at the point x'. The covariant derivative $\tilde{\nabla}_{i'}^{x}$ with respect to x'^i with the Christoffel symbol $\Gamma_{k'l'}^{i'}(x,x')$ acts only on the point x' and on primed indices. It is a covariant derivative in the Euclidean space E_x which is tangent to the σ-Riemannian space R_n at the point x [14].

In general, the world function σ carries out the geodesic mapping $G_{x'}: R_n \to E_{x'}$ of the Riemannian space $R_n = \{\mathbf{g}, K, D\}$ on the Euclidean space $E_{x'} = \{\mathbf{g}, K_{x'}, D\}$, tangent to $R_n = \{\mathbf{g}, K, D\}$ at the point x' [14]. This mapping transforms the coordinate system K in R_n into the coordinate system $K_{x'}$ in $E_{x'}$. The mapping is geodesic in the sense that it conserves the lengths of segments of all geodesics, passing through the tangent point x' and angles between them at this point.

The tensor G_{ik}, defined by (6.2) is the metric tensor at the point x in the tangent Euclidean space $E_{x'}$. The covariant derivatives $\tilde{\nabla}_i^{x'}$ and $\tilde{\nabla}_k^{x'}$ commute identically, i.e. $(\tilde{\nabla}_i^{x'}\tilde{\nabla}_k^{x'} - \tilde{\nabla}_k^{x'}\tilde{\nabla}_i^{x'})A_{ls} \equiv 0$, for any tensor A_{ls} [14]. This shows that they are covariant derivatives in the flat space $E_{x'}$.

The system of equations (6.1) contains only world function σ and its derivatives, nevertheless the system of equations (6.1) is not σ-immanent, because it contains a reference to a coordinate system. It does not contain the metric tensor explicitly. Hence, it is valid for any Riemannian space $R_n = \{\mathbf{g}, K, D\}$. All relations written above are valid also for the σ-space $V = \{\sigma, D\}$, provided the world function σ is coupled with the metric tensor by relation (4.1).

σ-immanent expression for scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ in the proper Euclidean space has the form

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) \equiv \sigma(P_0, Q_1) + \sigma(Q_0, P_1) - \sigma(P_0, Q_0) - \sigma(P_1, Q_1) \qquad (6.3)$$

This relation can be easily proved as follows.

In the proper Euclidean space three vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0\mathbf{Q}_1$, and $\mathbf{P}_1\mathbf{Q}_1$ are coupled by the relation

$$|\mathbf{P}_1\mathbf{Q}_1|^2 = |\mathbf{P}_0\mathbf{Q}_1 - \mathbf{P}_0\mathbf{P}_1|^2 = |\mathbf{P}_0\mathbf{P}_1|^2 + |\mathbf{P}_0\mathbf{Q}_1|^2 - 2(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1) \qquad (6.4)$$

where $(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1)$ denotes the scalar product of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{Q}_1$ in the proper Euclidean space. It follows from (6.4)

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1) = \frac{1}{2}\{|\mathbf{P}_0\mathbf{Q}_1|^2 + |\mathbf{P}_0\mathbf{P}_1|^2 - |\mathbf{P}_1\mathbf{Q}_1|^2\} \qquad (6.5)$$

Substituting the point Q_1 by Q_0 in (6.5), one obtains

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_0) = \frac{1}{2}\{|\mathbf{P}_0\mathbf{Q}_0|^2 + |\mathbf{P}_0\mathbf{P}_1|^2 - |\mathbf{P}_1\mathbf{Q}_0|^2\} \qquad (6.6)$$

Subtracting (6.6) from (6.5) and using the properties of the scalar product in the proper Euclidean space, one obtains

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \frac{1}{2}\{|\mathbf{P}_0\mathbf{Q}_1|^2 + |\mathbf{Q}_0\mathbf{P}_1|^2 - |\mathbf{P}_0\mathbf{Q}_0|^2 - |\mathbf{P}_1\mathbf{Q}_1|^2\} \qquad (6.7)$$

Taking into account that $\mid \mathbf{P}_0\mathbf{Q}_1 \mid^2 = 2\sigma(P_0, Q_1)$, one obtains the relation (6.3) from the relation (6.7).

Two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ are collinear $\mathbf{P}_0\mathbf{P}_1 || \mathbf{Q}_0\mathbf{Q}_1$ (parallel or antiparallel), provided $\cos^2\theta = 1$, where θ is the angle between the vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$. Taking into account that

$$\cos^2\theta = \frac{(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)^2}{(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1)(\mathbf{Q}_0\mathbf{Q}_1.\mathbf{Q}_0\mathbf{Q}_1)} = \frac{(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)^2}{|\mathbf{P}_0\mathbf{P}_1|^2 \cdot |\mathbf{Q}_0\mathbf{Q}_1|^2} \tag{6.8}$$

one obtains the following σ-immanent condition of the two vectors collinearity

$$\mathbf{P}_0\mathbf{P}_1 || \mathbf{Q}_0\mathbf{Q}_1 : \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)^2 = |\mathbf{P}_0\mathbf{P}_1|^2 \cdot |\mathbf{Q}_0\mathbf{Q}_1|^2 \tag{6.9}$$

The collinearity condition (6.9) is σ-immanent, because by means of (6.3) it can be written in terms of the σ-function only. Thus, this relation describes the vectors collinearity in the case of arbitrary σ-space.

Let us describe this relation for the case of σ-Riemannian geometry. Let coordinates of the points P_0, P_1, Q_0, Q_1 be respectively x, $x + dx$, x' and $x' + dx'$. Then writing (6.3) and expanding it over dx and dx', one obtains

$$\begin{aligned}
(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) &\equiv \sigma(x, x' + dx') + \sigma(x', x + dx) - \sigma(x, x') - \sigma(x + dx, x' + dx') \\
&= \sigma_{l'} dx'^{l'} + \frac{1}{2}\sigma_{l',s'} dx'^{l'} dx'^{s'} + \sigma_i dx^i + \frac{1}{2}\sigma_{i,k} dx^i dx^k \\
&\quad - \sigma_i dx^i - \sigma_{l'} dx'^{l'} - \frac{1}{2}\sigma_{i,k} dx^i dx^k - \sigma_{i,l'} dx^i dx'^{l'} - \frac{1}{2}\sigma_{l',s'} dx'^{l'} dx'^{s'}
\end{aligned}$$

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = -\sigma_{i,l'} dx^i dx'^{l'} = -\sigma_{il'} dx^i dx'^{l'} \tag{6.10}$$

Here comma means differentiation. For instance, $\sigma_{i,k} \equiv \partial\sigma_i/\partial x^k$. One obtains for $|\mathbf{P}_0\mathbf{P}_1|^2$ and $|\mathbf{Q}_0\mathbf{Q}_1|^2$

$$|\mathbf{P}_0\mathbf{P}_1|^2 = g_{ik} dx^i dx^k, \qquad |\mathbf{Q}_0\mathbf{Q}_1|^2 = g_{l's'} dx'^{l'} dx'^{s'} \tag{6.11}$$

where $g_{ik} = g_{ik}(x)$ and $g_{l's'} = g_{l's'}(x')$. Then the collinearity condition (6.9) is written in the form

$$(\sigma_{il'}\sigma_{ks'} - g_{ik} g_{l's'}) dx^i dx^k dx'^{l'} dx'^{s'} = 0 \tag{6.12}$$

Let us take into account that in the Riemannian space the metric tensor $g_{l's'}$ at the point x' can be expressed via the world function σ of points x, x' by means of the relation [14]

$$g_{l's'} = \sigma_{il'} G^{ik} \sigma_{ks'}, \qquad g^{l's'} = \sigma^{il'} G_{ik} \sigma^{ks'} \tag{6.13}$$

where the tensor G_{ik} is defined by the relation (6.2), and G^{ik} is defined by the relation

$$G^{il} G_{lk} = \delta^i_k \tag{6.14}$$

Substituting the first relation (6.2) in (6.12) and using designation

$$u_i = -\sigma_{il'} dx'^{l'}, \qquad u^i = G^{ik} u_k = -\sigma^{il'} g_{l's'} dx^{s'} \tag{6.15}$$

one obtains

$$\left(\delta_i^l\delta_k^s - g_{ik}G^{ls}\right) u_l u_s dx^i dx^k = 0 \qquad (6.16)$$

The vector u_i is the vector $dx'_{i'} = g_{i'k'}dx'^{k'}$ transported parallelly from the point x' to the point x in the Euclidean space $E_{x'}$ tangent to the Riemannian space R_n. Indeed,

$$u_i = -\sigma_{il'}g^{l's'}dx'_s, \qquad \tilde{\nabla}_k^{x'}\left(-\sigma_{il'}g^{l's'}\right) \equiv 0, \qquad i,k = 1,2,\ldots n \qquad (6.17)$$

and tensor $-\sigma_{il'}g^{l's'}$ is the operator of the parallel transport in $E_{x'}$, because

$$\left[-\sigma_{il'}g^{l's'}\right]_{x=x'} = \delta_{i'}^{s'}$$

and the tangent derivative of this operator is equal to zero identically. For the same reason, i.e. because of

$$\left[\sigma^{il'}g_{l's'}\sigma^{ks'}\right]_{x=x'} = g^{i'k'}, \qquad \tilde{\nabla}_s^{x'}(\sigma^{il'}g_{l's'}\sigma^{ks'}) \equiv 0$$

$G^{ik} = \sigma^{il'}g_{l's'}\sigma^{ks'}$ is the contravariant metric tensor in $E_{x'}$, at the point x.

The relation (6.16) contains vectors at the point x only . At fixed $u_i = -\sigma_{il'}dx'^{l'}$ it describes a collinearity cone, i.e. a cone of infinitesimal vectors dx^i at the point x parallel to the vector $dx'^{i'}$ at the point x'. Under some condition the collinearity cone can degenerates into a line. In this case there is only one direction, parallel to the fixed vector u^i. Let us investigate, when this situation takes place.

At the point x two metric tensors g_{ik} and G_{ik} are connected by the relation [14]

$$G_{ik}(x,x') = g_{ik}(x) + \int_x^{x'} F_{ikj''s''}(x,x'')\sigma^{j''}(x,x'')dx''^{s''}, \qquad (6.18)$$

where according to [14]

$$\sigma^{i'} = \sigma^{li'}\sigma_l = G^{l'i'}\sigma_{l'} = g^{l'i'}\sigma_{l'} \qquad (6.19)$$

Integration does not depend on the path, because it is produced in the Euclidean space $E_{x'}$. The two-point tensor $F_{ilk'j'} = F_{ilk'j'}(x,x')$ is the two-point curvature tensor, defined by the relation

$$F_{ilk'j'} = \sigma_{ilj'||k'} = \sigma_{ilj',k'} - \sigma_{sj'k'}\sigma^{sm'}\sigma_{ilm'} = \sigma_{i|l||k'||j'} \qquad (6.20)$$

where one vertical stroke denotes usual covariant derivative and two vertical strokes denote tangent derivative. The two-point curvature tensor $F_{ilk'j'}$ has the following symmetry properties

$$F_{ilk'j'} = F_{lik'j'} = F_{ilj'k'}, \qquad F_{ilk'j'}(x,x') = F_{k'j'il}(x',x) \qquad (6.21)$$

It is connected with the one-point Riemann-Ghristoffel curvature tensor r_{iljk} by means of relations

$$r_{iljk} = [F_{ikj'l'} - F_{ijk'l'}]_{x'=x} = f_{ikjl} - f_{ijkl}, \qquad f_{iklj} = [F_{ikj'l'}]_{x'=x} \tag{6.22}$$

In the Euclidean space the two-point curvature tensor $F_{ilk'j'}$ vanishes as well as the Riemann-Ghristoffel curvature tensor r_{iljk}.

Let us introduce designation

$$\Delta_{ik} = \Delta_{ik}(x,x') = \int_x^{x'} F_{ikj''s''}(x,x'')\sigma^{j''}(x,x'')dx''^{s''} \tag{6.23}$$

and choose the geodesic $\mathcal{L}_{xx'}$ as the path of integration. It is described by the relation

$$\sigma_i(x,x'') = \tau\sigma_i(x,x') \tag{6.24}$$

which determines x'' as a function of parameter τ. Differentiating with respect to τ, one obtains

$$\sigma_{ik''}(x,x'')dx''^{k''} = \sigma_i(x,x')d\tau \tag{6.25}$$

Resolving equations (6.25) with respect to dx'' and substituting in (6.23), one obtains

$$\Delta_{ik}(x,x') = \sigma_l(x,x')\sigma_p(x,x') \int_0^1 F_{ikj''s''}(x,x'')\sigma^{lj''}(x,x'')\sigma^{ps''}(x,x'')\tau d\tau \tag{6.26}$$

where x'' is determined from (6.24) as a function of τ. Let us set

$$F_{ik}^{\;\;..lp}(x,x') = F_{ikj's'}(x,x')\sigma^{lj'}(x,x')\sigma^{ps'}(x,x') \tag{6.27}$$

then

$$G_{ik}(x,x') = g_{ik}(x) + \Delta_{ik}(x,x') \tag{6.28}$$

$$\Delta_{ik}(x,x') = \sigma_l(x,x')\sigma_p(x,x') \int_0^1 F_{ik}^{\;\;..lp}(x,x'')\tau d\tau \tag{6.29}$$

Substituting g_{ik} from (6.28) in (6.16), one obtains

$$\left(\delta_i^l\delta_k^s - G^{ls}\left(G_{ik} - \Delta_{ik}\right)\right) u_l u_s dx^i dx^k = 0 \tag{6.30}$$

Let us look for solutions of equation in the form of expansion

$$dx^i = \alpha u^i + v^i, \qquad G_{ik}u^i v^k = 0 \tag{6.31}$$

Substituting (6.31) in (6.30), one obtains equation for v^i

$$G_{ls}u^l u^s \left[G_{ik}v^i v^k - \Delta_{ik}\left(\alpha u^i + v^i\right)\left(\alpha u^k + v^k\right)\right] = 0 \tag{6.32}$$

If the σ-Riemannian space $V = \{\sigma, D\}$ is σ-Euclidean, then as it follows from (6.29) $\Delta_{ik} = 0$. If $V = \{\sigma, D\}$ is the proper σ-Euclidean space, $G_{ls}u^l u^s \neq 0$, and one obtains two equations for determination of v^i

$$G_{ik}v^i v^k = 0, \qquad G_{ik}u^i v^k = 0 \tag{6.33}$$

The only solution

$$v^i = 0, \qquad dx^i = \alpha u^i, \qquad i = 1, 2, \dots n \tag{6.34}$$

of (6.32) is a solution of the equation (6.30), where α is an arbitrary constant. In the proper Euclidean geometry the collinearity cone always degenerates into a line.

Let now the space $V = \{\sigma, D\}$ be the σ-pseudoeuclidean space of index 1, and the vector u^i be timelike, i.e. $G_{ik}u^i u^k > 0$. Then equations (6.33) also have the solution (6.34). If the vector u^i is spacelike, $G_{ik}u^i u^k < 0$, then two equations (6.33) have non-trivial solution, and the collinearity cone does not degenerate into a line. The collinearity cone is a section of the light cone $G_{ik}v^i v^k = 0$ by the plane $G_{ik}u^i v^k = 0$. If the vector u^i is null, $G_{ik}u^i u^k = 0$, then equation (6.32) reduces to the form

$$G_{ik}u^i u^k = 0, \qquad G_{ik}u^i v^k = 0 \tag{6.35}$$

In this case (6.34) is a solution, but besides there are spacelike vectors v^i which are orthogonal to null vector u^i and the collinearity cone does not degenerate into a line.

In the case of the proper σ-Riemannian space $G_{ik}u^i u^k > 0$, and equation (6.32) reduces to the form

$$G_{ik}v^i v^k - \Delta_{ik}\left(\alpha u^i + v^i\right)\left(\alpha u^k + v^k\right) = 0 \tag{6.36}$$

In this case $\Delta_{ik} \neq 0$ in general, and the collinearity cone does not degenerate. Δ_{ik} depends on the curvature an on the distance between the points x and x'. The more space curvature and the distance $\rho(x, x')$, the more the collinearity cone aperture.

In the curved proper σ-Riemannian space there is an interesting special case, when the collinearity cone degenerates . In any σ-Riemannian space the following equality takes place [14]

$$G_{ik}\sigma^k = g_{ik}\sigma^k, \qquad \sigma^k \equiv g^{kl}\sigma_l \tag{6.37}$$

Then it follows from (6.28) that

$$\Delta_{ik}\sigma^k = 0 \tag{6.38}$$

It means that in the case, when the vector u^i is directed along the geodesic, connecting points x and x', i.e. $u^i = \beta\sigma^i$, the equation (6.36) reduces to the form

$$\left(G_{ik} - \Delta_{ik}\right) v^i v^k = 0, \qquad u^i = \beta\sigma^i \tag{6.39}$$

If Δ_{ik} is small enough as compared with G_{ik}, then eigenvalues of the matrix $G_{ik} - \Delta_{ik}$ have the same sign, as those of the matrix G_{ik}. In this case equation (6.39) has the only solution (6.34), and the collinearity cone degenerates.

7 Discussion

Thus, we see that in the σ-Riemannian geometry at the point x there are many vectors parallel to given vector at the point x'. This set of parallel vectors is described by the collinearity cone. Degeneration of the collinearity cone into a line, when there is only one direction, parallel to the given direction, is an exception rather than a rule, although in the proper Euclidean geometry this degeneration takes place always. Nonuniformity of space destroys the collinearity cone degeneration. In the proper Riemannian geometry, where the world function satisfies the system (6.1), one succeeded in conserving this degeneration for direction along the geodesic, connecting points x and x'. This circumstance is very important for degeneration of the first order NGOs into geodesic, because degeneration of NGOs is connected closely with the collinearity cone degeneration.

Indeed, definition of the first order tube (2.20), or (5.3) may be written also in the form

$$\mathcal{T}\left(\mathcal{P}^1\right) \equiv \mathcal{T}_{P_0P_1} = \left\{R \,|\, \mathbf{P}_0\mathbf{P}_1 || \mathbf{P}_0\mathbf{R}\right\}, \qquad P_0, P_1, R \in \Omega, \tag{7.1}$$

where collinearity $\mathbf{P}_0\mathbf{P}_1||\mathbf{P}_0\mathbf{R}$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{R}$ is defined by the σ-immanent relation (6.9), which can be written in the form

$$\mathbf{P}_0\mathbf{P}_1||\mathbf{P}_0\mathbf{R}: \quad F_2\left(P_0, P_1, R\right) = \left|\begin{array}{cc} (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{R}) \\ (\mathbf{P}_0\mathbf{R}.\mathbf{P}_0\mathbf{P}_1) & (\mathbf{P}_0\mathbf{R}.\mathbf{P}_0\mathbf{R}) \end{array}\right| = 0 \tag{7.2}$$

The form (7.1) of the first order tube definition allows one to define the first order tube $\mathcal{T}(P_0, P_1; Q_0)$, passing through the point Q_0 collinear to the given vector $\mathbf{P}_0\mathbf{P}_1$. This definition has the σ-immanent form

$$\mathcal{T}(P_0, P_1; Q_0) = \left\{R \,|\, \mathbf{P}_0\mathbf{P}_1 || \mathbf{Q}_0\mathbf{R}\right\}, \qquad P_0, P_1, Q_0, R \in \Omega, \tag{7.3}$$

where collinearity $\mathbf{P}_0\mathbf{P}_1||\mathbf{Q}_0\mathbf{R}$ of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{R}$ is defined by the σ-immanent relations (6.9), (6.7). In the proper Euclidean space the tube (7.3) degenerates into the straight line, passing through the point Q_0 collinear to the given vector $\mathbf{P}_0\mathbf{P}_1$.

Let us define the set $\omega_{Q_0} = \{\mathbf{Q}_0\mathbf{Q})|Q \in \Omega\}$ of vectors $\mathbf{Q}_0\mathbf{Q}$. Then

$$\mathcal{C}(P_0, P_1; Q_0) = \{\mathbf{Q}_0\mathbf{Q}|Q \in \mathcal{T}(P_0, P_1; Q_0)\} \subset \omega_{Q_0} \tag{7.4}$$

is the collinearity cone of vectors $\mathbf{Q}_0\mathbf{Q}$ collinear to vector $\mathbf{P}_0\mathbf{P}_1$. Thus, the one-dimensionality of the first order tubes and the collinearity cone degeneration are connected phenomena.

In the Riemannian geometry the very special property of the proper Euclidean geometry (the collinearity cone degeneration) is considered to be a property of any geometry and extended to the case of Riemannian geometry. The line $\mathcal{L}$, defined as a continuous mapping (1.5) is considered to be the most important geometric object. This object is considered to be more important, than the metric, and metric in the Riemannian geometry is defined in terms of the shortest lines. Use of line as a basic concept of geometry is inadequate for description of geometry and poses problems, which appears to be artificial.

First, extension of curves introduces nonlocal features in the geometry description. Nonlocalilty of description manifests itself: (1) in violation of isometrical embeddability of nonconvex regions in the space, from which they are cut, (2) in violation of absolute parallelism of vectors at different points of space. These unnatural properties of Riemannian geometry are corollaries of the metric definition via concept of a curve. In σ-Riemannian geometry such properties of Euclidean geometry as absolute parallelism and isometrical embeddability of nonconvex regions conserve completely. All this is a manifestation of negation of nondegeneracy. as a natural property of geometry. But one fails to remove nondegeneracy of non-uniform geometry. It exists for spacelike vectors even in the Minkowski geometry.

As far as one cannot remove nondegeneracy from Riemannian geometry, it seems reasonable to recognize that the nondegeneracy is a natural geometric property, and T-geometric conception is more perfect, than the Riemannian conception of geometry. A corollary of this conclusion is a reconstruction of local description and absolute parallelism (the last may be useful for formulation of integral conservation laws in a curved space-time). Besides, the T-geometric conception is essentially simpler, than the Riemannian one. It has simpler structure and uses simpler method of description. The fundamental mapping (2.13), introducing multivector in T-geometry is essentially simpler than fundamental mapping (1.5), introducing the curve in Riemannian geometry. The mapping (2.13) deals with finite objects. It does not contain any references to limiting processes, or limits whose existence should be provided. Finally, the T-geometry is not sensitive to that, whether the real space-time is continuous, or only fine-grained. This is important also, because it seems not to be tested experimentally.

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