Hamilton Variational Principle for Statistical ensemble of Deterministic Systems and its Application for Esemble of Stochastic Systems

Yuri A.Rylov

Institute for Problems in Mechanics, Russian Academy of Sciences,
101, bild.1 Vernadskii Ave., Moscow, 117526, Russia.
e-mail: rylov@ipmnet.ru

Abstract

Hamilton variational principle for a special type of statistical ensemble of deterministic dynamical systems is derived. This form of variational principle allows one to describe the statistical ensemble in terms of wave functions and provides a basis for a description of statistical ensemble of stochastic systems. It is shown that sometimes such a statistical description of stochastic particle motion appears to coincide with the quantum description in terms of Schrödinger equation.

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I Introduction

One derives the Hamilton variational principle, which appears to be an effective mathematical tool of a statistical description. It allows one to write dynamic equations for a statistical ensemble in terms of wave functions and to show that the quantum mechanical description in terms of the Schrödinger equation is a special case of statistical description.

Let a deterministic dynamic system $\mathcal{S}_{\mathrm{d}}$[1] be described by the Lagrangian $L(t,\mathbf{x},d\mathbf{x}/dt)$, where $\mathbf{x}=\{x^\alpha\} \quad \alpha=1,2,...n$, are generalized coordinates and $\mathbf{v}=d\mathbf{x}/dt=\{dx^\alpha/dt\} \quad \alpha=1,2,...n$ is generalized velocity. For brevity we shall speak about a particle with a position $\mathbf{x}$ and velocity $\mathbf{v}$. A statistical ensemble $\mathcal{E}_{\mathrm{gen}}\left[\mathcal{S}_{\mathrm{d}}\right]$ (i.e. a set of many independent identical systems $\mathcal{S}_{\mathrm{d}}$) is a deterministic dynamic system, whose state is described by the distribution function $F\left(t,\mathbf{x},\mathbf{p}\right)$, where $\mathbf{p}=\{\partial L/\partial(dx^\alpha/dt)\}, \quad \alpha=1,2,...n$ is the generalized momentum of $\mathcal{S}_{\mathrm{d}}$. The state $F\left(t,\mathbf{x},\mathbf{p}\right)$ of the general ensemble $\mathcal{E}_{\mathrm{gen}}\left[\mathcal{S}_{\mathrm{d}}\right]$ evolves according to the Liouville equation

$$\mathcal{E}_{\mathrm{gen}}\left[\mathcal{S}_{\mathrm{d}}\right]: \qquad \frac{\partial F}{\partial t}+\frac{\partial}{\partial x^\alpha}\left(\frac{\partial H}{\partial p_\alpha}F\right)-\frac{\partial}{\partial p_\alpha}\left(\frac{\partial H}{\partial x^\alpha}F\right)=0 \qquad (1.1)$$

where $H=H\left(t,\mathbf{x},\mathbf{p}\right)=\mathbf{vp}-L$ is the Hamilton function for the dynamic system $\mathcal{S}_{\mathrm{d}}$. Summation is produced over repeating Greek indices $(1-n)$. Dynamic systems $\mathcal{S}_{\mathrm{d}}$, constituting the statistical ensemble $\mathcal{E}_{\mathrm{gen}}\left[\mathcal{S}_{\mathrm{d}}\right]$, are called elements of this ensemble. Dynamic equation (1.1) for the statistical ensemble $\mathcal{E}_{\mathrm{gen}}\left[\mathcal{S}_{\mathrm{d}}\right]$ is obtained formally from the dynamic equations for elements $\mathcal{S}_{\mathrm{d}}$ of the statistical ensemble $\mathcal{E}_{\mathrm{gen}}\left[\mathcal{S}_{\mathrm{d}}\right]$.

We are interested in a special type $\mathcal{E}_{\mathrm{p}}\left[\mathcal{S}_{\mathrm{d}}\right]$ of the general ensemble $\mathcal{E}_{\mathrm{gen}}\left[\mathcal{S}_{\mathrm{d}}\right]$ which will be referred to as a pure statistical ensemble, because its state may be described by the wave function[2]. By definition a pure statistical ensemble $\mathcal{E}_{\mathrm{gen}}\left[\mathcal{S}_{\mathrm{d}}\right]$ of dynamical systems $\mathcal{S}_{\mathrm{d}}$ is such an ensemble, whose state $F_{\mathrm{p}}\left(t,\mathbf{x},\mathbf{p}\right)$ may be represented in the form

$$\mathcal{E}_{\mathrm{p}}\left[\mathcal{S}_{\mathrm{d}}\right]: \qquad F_{\mathrm{p}}\left(t,\mathbf{x},\mathbf{p}\right)=\rho\left(t,\mathbf{x}\right)\delta\left(\mathbf{p}-\mathbf{P}\left(t,\mathbf{x}\right)\right) \qquad (1.2)$$

where $\rho\left(t,\mathbf{x}\right)$ and $\mathbf{P}\left(t,\mathbf{x}\right)=\{P_\alpha\left(t,\mathbf{x}\right)\}$, $\alpha=1,2,...n$ are functions of only time t and generalized coordinates $\mathbf{x}$. In other words, the pure ensemble $\mathcal{E}_{\mathrm{p}}\left[\mathcal{S}_{\mathrm{d}}\right]$ is a dynamic system, considered in the configuration space of coordinates $\mathbf{x}$.

Dynamic equations for the pure ensemble $\mathcal{E}_{\mathrm{p}}\left[\mathcal{S}_{\mathrm{d}}\right]$ have the form

$$\frac{\partial}{\partial t}\rho+\frac{\partial}{\partial x^\alpha}\left(\rho\frac{\partial H}{\partial P_\alpha}\left(t,\mathbf{x},\mathbf{P}\right)\right)=0 \qquad (1.3)$$

$$\frac{\partial}{\partial t}\left(\rho P_\beta\right)+\frac{\partial}{\partial x^\alpha}\left(\rho P_\beta\frac{\partial H}{\partial P_\alpha}\left(t,\mathbf{x},\mathbf{P}\right)\right)+\rho\frac{\partial H}{\partial x^\beta}\left(t,\mathbf{x},\mathbf{P}\right)=0, \qquad \beta=1,2,...n \quad (1.4)$$

[1]The term "dynamic system" is used as a collective concept with respect to terms "nondeterministic (stochastic) dynamic system" and "deterministic dynamic system". Such an unusual teminology is used, because we need a collective concept for concepts "dynamic system' and "stochastic system".

[2]In quantum mechanics a pure state of a quantum system is called a such one which can be described by means of wave function (not by the dense matrix) [1].

They are obtained after substituting (1.2) into (1.1) and integration with respect to $\mathbf{p}$ with the multipliers $1, p_\beta, \quad \beta = 1, 2, ...n$ respectively. Interpreting ρ as a particle density and $\mathbf{v} = \partial H / \partial \mathbf{p}$ as a generalized velocity, the equation (1.3) is regarded as a continuity equation. Then (1.4) may be interpreted as generalized Euler equations for some ideal fluid without pressure.

Use of pure statistical ensembles is convenient in the case, when elements of the statistical ensemble are nondeterministic (stochastic) dynamic systems $\mathcal{S}_{\text{st}}$, for which there are no dynamic equations. Then the following procedure is used. The general statistical ensemble $\mathcal{E}_{\text{gen}}[\mathcal{S}_{\text{st}}]$ is considered to be consisting of elements $\mathcal{E}_{\text{p}}[\mathcal{S}_{\text{st}}]$, which consist in turn of elements $\mathcal{S}_{\text{st}}$, i.e.

$$\mathcal{E}_{\text{gen}}[\mathcal{S}_{\text{st}}] = \mathcal{E}_{\text{gen}}[\mathcal{E}_{\text{p}}[\mathcal{S}_{\text{st}}]] \tag{1.5}$$

There are no dynamic equations for dynamic system $\mathcal{S}_{\text{st}}$, but there are dynamic equations for a statistical ensemble $\mathcal{E}_{\text{p}}[\mathcal{S}_{\text{st}}]$. Derivation of dynamic equations for $\mathcal{E}_{\text{p}}[\mathcal{S}_{\text{st}}]$ is rather complicated informal procedure, which needs a special consideration for any kind of stochastic system $\mathcal{S}_{\text{st}}$. But if dynamic equations for dynamic systems $\mathcal{E}_{\text{p}}[\mathcal{S}_{\text{st}}]$ have been derived, dynamic equations for the statistical ensemble $\mathcal{E}_{\text{gen}}[\mathcal{E}_{\text{p}}[\mathcal{S}_{\text{st}}]]$ of deterministic dynamic systems $\mathcal{E}_{\text{p}}[\mathcal{S}_{\text{st}}]$ are obtained as a result of a formal procedure. Thus, complicated informal procedure of dynamic equations derivation for $\mathcal{E}_{\text{gen}}[\mathcal{S}_{\text{st}}]$ is divided into two parts: (1) derivation of dynamic equations for a more simple statistical ensemble $\mathcal{E}_{\text{p}}[\mathcal{S}_{\text{st}}]$ by means of informal procedure, (2) derivation of dynamic equations for $\mathcal{E}_{\text{gen}}[\mathcal{E}_{\text{p}}[\mathcal{S}_{\text{st}}]]$ by means of a formal procedure.

In this sense, a consideration of more simple pure statistical ensembles $\mathcal{E}_{\text{p}}$ simplifies investigation of stochastic systems.

Let us add to the Euler equations the following equations

$$\frac{dx^\beta}{dt} = \frac{\partial H}{\partial P_\beta}(t, \mathbf{x}, \mathbf{P}), \qquad \beta = 1, 2, ...n \tag{1.6}$$

describing motion of a particle in a given velocity field $\mathbf{v} = \partial H / \partial \mathbf{P}$. These equations can be rewritten in the form, known in hydrodynamics as Lin constraints [2]

$$\frac{\partial \xi_\beta}{\partial t} + \frac{\partial H}{\partial P_\alpha}(t, \mathbf{x}, \mathbf{P}) \partial_\alpha \xi_\beta = 0, \qquad \beta = 1, 2, ...n \tag{1.7}$$

where $\xi(t, \mathbf{x}) = \{\xi_\alpha(t, \mathbf{x})\}, \quad \alpha = 1, 2, ...n$ are n independent integrals of n equations (1.6).

The system of $2n + 1$ equations (1.3), (1.4), (1.7) forms a complete system of dynamic equations, describing evolution of dynamic systems $\mathcal{S}_{\text{d}}$, constituting the pure ensemble $\mathcal{E}_{\text{p}}[\mathcal{S}_{\text{d}}]$.

One can show that the system of $2n + 1$ equations (1.3), (1.4), (1.7) reduces to the form of a system of $n + 2$ equations

$$b_0[\partial_0 \varphi + g^\alpha(\boldsymbol{\xi}) \partial_0 \xi_\alpha] + H(\mathbf{x}, \mathbf{P}) = 0, \qquad \partial_k \equiv \frac{\partial}{\partial x^k}, \qquad k = 0, 1, ...n \tag{1.8}$$

$$\partial_0\rho+\partial_\alpha\left(\rho\frac{\partial H}{\partial P_\alpha}(t,\mathbf{x},\mathbf{P})\right)=0 \qquad (1.9)$$

$$\frac{\partial\xi_\beta}{\partial t}+\frac{\partial H}{\partial P_\alpha}(t,\mathbf{x},\mathbf{P})\,\partial_\alpha\xi_\beta=0,\qquad \beta=1,2,...n \qquad (1.10)$$

where φ is a new variable, and $\mathbf{P}$ is expressed now via n arbitrary functions $\mathbf{g}(\boldsymbol{\xi})=\{g^\alpha(\boldsymbol{\xi})\},\quad \alpha=1,2,...n$ of argument $\boldsymbol{\xi}$.

$$P_\beta=b_0\left(\partial_\beta\varphi+g^\alpha(\boldsymbol{\xi})\right)\partial_\beta\xi_\alpha,\qquad \beta=1,2,...n \qquad (1.11)$$

Practically it means that the system of $2n+1$ equations (1.9), (1.4), (1.10) is integrated partially. Here b_0 is an arbitrary constant, which can be incorporated in the variable φ and arbitrary functions $\mathbf{g}(\boldsymbol{\xi})$.

The system of $n+2$ equations (1.8), (1.9), (1.10) is remarkable in the relation that it can be described in terms of ψ function (wave function). But it is more convenient one to transform these equations, using the Hamilton variational principle.

We shall show that dynamic equations for the pure statistical ensemble $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]$ of deterministic dynamic systems $\mathcal{S}_\mathrm{d}$ are derived from the variational principle with the action

$$\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]:\qquad \mathcal{A}_E[\rho,\varphi,\boldsymbol{\xi}]=\int\rho\{-H(t,\mathbf{x},\mathbf{p})-b_0[\partial_0\varphi+g^\alpha(\boldsymbol{\xi})\partial_0\xi_\alpha]\}d^{n+1}x, \qquad (1.12)$$

$$p_\beta=b_0\left[\partial_\beta\varphi+g^\alpha(\boldsymbol{\xi})\partial_\beta\xi_\alpha\right],\qquad \beta=1,2,...,n,,\qquad \partial_i\equiv\frac{\partial}{\partial x^i} \qquad (1.13)$$

where $\rho,\varphi,\boldsymbol{\xi}$ are dependent variables, which are considered to be functions of $x=\{x^0,\mathbf{x}\}=\{t,\mathbf{x}\}$. $H(t,\mathbf{x},\mathbf{p})$ is the Hamiltonian function of $\mathcal{S}_\mathrm{d}$. b_0 is an arbitrary constant and $g^\alpha(\boldsymbol{\xi}),\quad \alpha=1,2,...,n$ are arbitrary functions of the argument $\boldsymbol{\xi}$. The dynamic variables $\varphi,\boldsymbol{\xi}$ are hydrodynamic (Clebsch) potentials. Clebsch [3, 4] had introduced them for a description of incompressible fluid. The variables $\varphi,\boldsymbol{\xi}$ are called potentials, because momentum $\mathbf{p}=\mathbf{P}(t,\mathbf{x})$ is expressed via derivatives of potentials $\varphi,\boldsymbol{\xi}$, as one can see from the relations (1.13). The Hamiltonian $H(t,\mathbf{x},\mathbf{p})$ is a function which determines the form of the action (1.12), and the variational principle based on the action (1.12) may be called the Hamilton variational principle.

The dynamic system $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]$ is an ideal fluidlike continuous medium without pressure. Variational principle for such a system in the Lagrangian coordinates $\boldsymbol{\xi}$, (where $\boldsymbol{\xi}$ are considered to be independent variables) is derived very simple. But a derivation of a variational principle for an ideal fluid in Euler coordinates $\mathbf{x}$ was being a problem for many years [5], because one tried to obtain a variational principle only for the Euler equations, i.e. for equations of the type (1.3), (1.4). The Euler equations form a closed subsystem of the complete system of dynamic equations. The variational principle generates only a complete system of dynamic equations (but not their closed subsystem). When the Lin constraints [2] were added to the Euler equations, the system of dynamic equations became complete, and the variational principle was written for different partial cases [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. The variational principle was written for an ideal fluid which can be considered to be a continuous set $\mathcal{S}[\mathcal{S}_\mathrm{d}]$ of nonrelativistic particles $\mathcal{S}_\mathrm{d}$, interacting between themselves via the pressure, which is a function of a collective variable (particle density) ρ and temperature T. Hamiltonian function of $\mathcal{S}_\mathrm{d}$ has the form

$$H(t,\mathbf{x},\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}),$$

The case, when the pressure is a function of ρ, $\nabla\rho$ and T, was considered in the paper [16]. Now the case of arbitrary Hamiltonian function of $\mathcal{S}_\mathrm{d}$ is considered. Parameters of the Hamiltonian (not necessarily the pressure) are considered to be functions of ρ and its space-time derivatives.

In the second section the Hamilton variational principle is derived. In the third section one shows how the Hamilton variational principle generates a description in terms of ψ-function (wave function). In the fourth section "conservative" stochastic systems are described on the basis of the Hamilton variational principle

II Variational principle

The action for the deterministic dynamic system $\mathcal{S}_\mathrm{d}$ has the form

$$\mathcal{S}_\mathrm{d}: \qquad \mathcal{A}_L[\mathbf{x}] = \int L(t,\mathbf{x},d\mathbf{x}/dt)dt, \tag{2.1}$$

A set of many independent systems $\mathcal{S}_\mathrm{d}$ forms a deterministic dynamic system (statistical ensemble) $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]$, whose action is a sum of actions (2.1). Let us label each system $\mathcal{S}_\mathrm{d}$ by parameters $\boldsymbol{\xi} = \{\xi_1,\xi_2,...\xi_n\}$, where n is the number of generalized coordinates. Usually the variables $\boldsymbol{\xi}$ are referred to as Lagrangian coordinates or generalized Lagrangian coordinates. One obtains for the action of the dynamic system $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]$

$$\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]: \qquad \mathcal{A}_L[\mathbf{x}] = \int \rho_0(\boldsymbol{\xi})\, L(t,\mathbf{x},d\mathbf{x}/dt)\mathrm{d}t\mathrm{d}\boldsymbol{\xi}, \qquad \mathrm{d}\boldsymbol{\xi} = \prod_{\alpha=1}^{\alpha=n}\mathrm{d}\xi_\alpha, \tag{2.2}$$

where $\rho_0(\boldsymbol{\xi})$ is some nonnegative weight function. For simplicity we set $\rho_0 = 1$, because in reality a use of the weight function is unessential. One can see this for a special case of the continuous dynamic system $\mathcal{S}[\mathcal{S}_\mathrm{d}]$, considered in paper [16], where the weight function was used. For further calculations it is useful one to introduce a fictitious Lagrangian time coordinate $\xi_0 = \xi_0(t,\boldsymbol{\xi})$, and rewrite the action (2.2) in the form

$$\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]: \qquad \mathcal{A}_L[\mathbf{x}] = \int L(x,\frac{\dot{\mathbf{x}}}{\dot{x}^0})\dot{x}^0\mathrm{d}^{n+1}\xi, \qquad \mathrm{d}^{n+1}\xi = \prod_{k=0}^{k=n}\mathrm{d}\xi_k, \tag{2.3}$$

where $x = \{x^0,\mathbf{x}\} = \{t,\mathbf{x}\} = \{x^k\}$, $k = 0,1,...n$, $\xi = \{\xi_0,\boldsymbol{\xi}\} = \{\xi_k\}$, $k = 0,1,...n$, and the point means differentiation with respect to ξ_0

$$\dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{d\xi_0}, \qquad \dot{x}^0 \equiv \frac{dx^0}{d\xi_0} \tag{2.4}$$

Now the action (2.3) is considered to be a functional of n dependent variables $\mathbf{x}$. The variable $t = x^0$ is some fixed function of ξ. The form of this function is unessential.

Let us consider the Lagrangian coordinates $\boldsymbol{\xi}$ to be dependent variables, which are functions of independent variables – coordinates $x = \{t, \mathbf{x}\} = \{x^0, \mathbf{x}\} = \{x^i\}, \quad i = 0, 1, ...n$ and transform the action to the form, where Eulerian coordinates x are independent variables. Let us introduce designations

$$J = \frac{\partial(\xi_0, \xi_1, ...\xi_n)}{\partial(x^0, x^1, ...x^n)} = \det||\xi_{i,k}||, \qquad \xi_{i,k} \equiv \frac{\partial \xi_i}{\partial x^k} \equiv \partial_k \xi_i \qquad i, k = 0, 1, ...n \tag{2.5}$$

$$j^i = \frac{\partial J}{\partial \xi_{0,i}} = \frac{\partial(x^i, \xi_1, \xi_2, ...\xi_n)}{\partial(x^0, x^1, ...x^n)} = J\dot{x}^i, \qquad i = 0, 1, ...n, \qquad \rho = j^0 = \frac{\partial J}{\partial \xi_{0,0}} \tag{2.6}$$

and relations

$$\mathrm{d}\xi_0 \mathrm{d}\boldsymbol{\xi} = J\mathrm{d}^{n+1}x \equiv J\prod_{i=0}^{i=n} \mathrm{d}x^i, \qquad \frac{dx^\alpha}{dt} = \frac{\partial(x^\alpha, \xi_1, ...\xi_n)}{\partial(t, \xi_1, ...\xi_n)} = \frac{j^\alpha}{j^0}, \qquad \alpha = 1, 2, ...n. \tag{2.7}$$

The action (2.3) transforms to the form

$$\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{d}}]: \qquad \mathcal{A}_E[\xi] = \int \mathcal{L}(x, j)\, \mathrm{d}^{n+1}x, \tag{2.8}$$

$$\mathcal{L}(x, j) = L(x^0, \mathbf{x}, \frac{\mathbf{j}}{j^0})j^0$$

where $j = \{j^0, \mathbf{j}\}$ is a function of $\xi_{i,k}$, defined by the relations (2.6). The action (2.8) is considered to be a functional of $n+1$ dependent variables ξ. In fact the variable ξ_0 is fictitious, and variation with respect to ξ_0 leads to an identity.

Let us introduce new variables $j = \{j^0, \mathbf{j}\} = \left\{j^k\right\}, \quad k = 0, 1, ...n$, defined by the relation (2.6) by means of the Lagrangian multipliers $p = \{p_k\}, \quad k = 0, 1, ...n$.

$$\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{d}}]: \qquad \mathcal{A}_E[\xi, p, j] = \int \left(\mathcal{L}(x, j) - p_i\left(j^i - \frac{\partial J}{\partial \xi_{0,i}}\right)\right) \mathrm{d}^{n+1}x, \tag{2.9}$$

Here and further a summation over repeated Latin super- and subindexes is made from 0 to n.

There are two ways for derivation of dynamic equations. The first way: the action (2.9) is varied with respect to variables $j, p, \boldsymbol{\xi}$, and after elimination of Lagrangian coordinates ξ the Eulerian equations (1.3), (1.4) appear. We are interested in another way, when dynamic equations $\delta\mathcal{A}_E/\delta\xi_i = 0$ are integrated in the form (1.13) and after elimination of variables p, j, one obtains dynamic equations of the Hamilton-Jacobi type for hydrodynamic potentials ξ.

Variation of the action (2.9) with respect to ξ_k leads to dynamic equations

$$\delta\xi_k: \qquad -\partial_l\left(p_i \frac{\partial^2 J}{\partial \xi_{0,i}\partial \xi_{k,l}}\right) = 0 \tag{2.10}$$

Being linear with respect to p, equations (2.10) can be solved in the form

$$p_k = b_0 \left(\partial_k \varphi + g^{\alpha}\left(\boldsymbol{\xi}\right)\right) \partial_k \xi_{\alpha}, \qquad k = 0, 1, ...n \tag{2.11}$$

where b_0 is an arbitrary constant, $g^{\alpha}\left(\boldsymbol{\xi}\right), \quad \alpha = 1, 2, ...n$ are arbitrary integration functions of the variables $\boldsymbol{\xi}$. After integration the fictitious variable ξ_0 stops to be fictitious, and φ is a new dependent variables which appears instead of ξ_0. Using Jacobian technique [16], one can verify by means of a direct substitution of (2.11) in (2.10) that (2.11) is a solution of (2.10). In particular, one needs the following identities

$$\partial_k \frac{\partial^2 J}{\partial \xi_{i,k} \partial \xi_{s,l}} \equiv 0, \qquad i, s, l = 0, 1, ...n. \tag{2.12}$$

$$\frac{\partial^2 J}{\partial \xi_{i,k} \partial \xi_{s,l}} \equiv \frac{1}{J} \left(\frac{\partial J}{\partial \xi_{i,k}} \frac{\partial J}{\partial \xi_{s,l}} - \frac{\partial J}{\partial \xi_{i,l}} \frac{\partial J}{\partial \xi_{s,k}} \right), \qquad i, k, l, s = 0, 1, ...n, \tag{2.13}$$

$$\xi_{l,k} \frac{\partial J}{\partial \xi_{s,k}} \equiv \delta_l^s J, \qquad \xi_{k,l} \frac{\partial J}{\partial \xi_{k,s}} \equiv \delta_l^s J, \qquad l, s = 0, 1, ...n, \tag{2.14}$$

$$\partial_k \frac{\partial J}{\partial \xi_{i,k}} \equiv \frac{\partial^2 J}{\partial \xi_{i,k} \partial \xi_{s,l}} \partial_k \partial_l \xi_s \equiv 0, \qquad i = 0, 1, ...n. \tag{2.15}$$

Eliminating p_k from relations (2.11) and (2.9), one obtains the action

$$\mathcal{A}_E[j, \varphi, \boldsymbol{\xi}] = \int \{\mathcal{L}(x, j) - b_0 j^i [\partial_i \varphi + g^{\alpha}(\boldsymbol{\xi}) \partial_i \xi_{\alpha}]\} \mathrm{d}^{n+1} x, \tag{2.16}$$

Let us introduce new variables

$$\mathbf{v} = \mathbf{j}/j^0, \qquad \rho = j^0$$

Then the action (2.16) transforms to the form

$$\mathcal{A}_E[\rho, \mathbf{v}, \varphi, \boldsymbol{\xi}] = \int \rho \{L(x, \mathbf{v}) - b_0[\partial_0 \varphi + g^{\alpha}(\boldsymbol{\xi}) \partial_0 \xi_{\alpha}]$$

$$-b_0 v^{\beta} [\partial_{\beta} \varphi + g^{\alpha}(\boldsymbol{\xi}) \partial_{\beta} \xi_{\alpha}]\} \mathrm{d}^{n+1} x, \tag{2.17}$$

Let us now eliminate the variables $\mathbf{v}$. Varying the action (2.17) with respect to v^{β}, one obtains

$$\frac{\delta \mathcal{A}_E}{\delta v^{\beta}} = \rho \left(\frac{\partial L(x, \mathbf{v})}{\partial v^{\beta}} - b_0 [\partial_{\beta} \varphi + g^{\alpha}(\boldsymbol{\xi}) \partial_{\beta} \xi_{\alpha}] \right) = 0, \qquad \beta = 1, 2, ...n \tag{2.18}$$

Let us use designations (2.11). The dynamic equations (2.18) take the form

$$\rho \left(\frac{\partial L(x, \mathbf{v})}{\partial v^{\beta}} - p_{\beta} \right) = 0, \qquad \beta = 1, 2, ...n \tag{2.19}$$

If the dynamic system $\mathcal{S}_{\mathrm{d}}$ is Hamiltonian, n equations (2.19), can be resolved with respect to $\mathbf{v}$ in the form $v^{\beta} = \partial H / \partial p_{\beta}, \quad \beta = 1, 2, ...n$, where

$$H = H\left(t, \mathbf{x}, \mathbf{p}\right) = p_{\alpha} v^{\alpha} - L(x, \mathbf{v}) \tag{2.20}$$

is the Hamilton function for the dynamic system $\mathcal{S}_{\mathrm{d}}$.

Substituting $\mathbf{v}$ in the action (2.17), one obtains

$$\mathcal{A}_E[\rho,\varphi,\boldsymbol{\xi}]=\int\rho\{-H\left(t,\mathbf{x},\mathbf{p}\right)-b_0[\partial_0\varphi+g^\alpha(\boldsymbol{\xi})\partial_0\xi_\alpha]\}\mathrm{d}^{n+1}x, \tag{2.21}$$

where $\mathbf{p}$ is determined by the relation (2.11).

Dynamic equations have the form

$$\delta\rho: \qquad H\left(\mathbf{x},\mathbf{p}\right)+b_0[\partial_0\varphi+g^\alpha(\boldsymbol{\xi})\partial_0\xi_\alpha]=0 \tag{2.22}$$

$$\delta\varphi: \qquad b_0\left(\partial_0\rho+\partial_\beta\left(\rho\frac{\partial H}{\partial p_\beta}\right)\right)=0 \tag{2.23}$$

$$\delta\xi_\alpha: \qquad \Omega^{\beta,\alpha}\rho\left(\xi_{\beta,0}+\frac{\partial H}{\partial p_\gamma}\xi_{\beta,\gamma}\right)=0, \qquad \Omega^{\alpha,\beta}\equiv\frac{\partial g^\alpha\left(\boldsymbol{\xi}\right)}{\partial\xi_\beta}-\frac{\partial g^\beta\left(\boldsymbol{\xi}\right)}{\partial\xi_\alpha} \tag{2.24}$$

where $\mathbf{p}$ is determined by the relation (2.11).

Let us consider a special case, when

$$\Omega^{\alpha,\beta}\equiv b_0\left(\frac{\partial g^\alpha\left(\xi\right)}{\partial\xi_\beta}-\frac{\partial g^\beta\left(\boldsymbol{\xi}\right)}{\partial\xi_\alpha}\right)\equiv 0 \tag{2.25}$$

Then

$$g^\alpha\left(\boldsymbol{\xi}\right)=\frac{\partial\phi\left(\boldsymbol{\xi}\right)}{\partial\xi_\alpha}, \qquad \alpha=1,2,...n,$$

equations (2.24) are satisfied identically, and $\mathbf{p}$ is a gradient

$$p_\beta=b_0\partial_\beta\left(\varphi+\phi\left(\boldsymbol{\xi}\right)\right), \qquad \beta=1,2,...n. \tag{2.26}$$

Equation (2.22) turns to the Hamilton-Jacobi equation for the variable $\Phi=b_0\left(\varphi+\phi\right)$

$$\partial_0\Phi+H\left(\mathbf{x},\mathbf{p}\right)=0, \qquad \mathbf{p}=\nabla\Phi \tag{2.27}$$

In this case the dynamic system $\mathcal{E}_{\mathrm{p}}\left[\mathcal{S}_{\mathrm{d}}\right]$, considered to be a generalized fluid, has an irrotational vector field of momentum $\mathbf{p}$ (irrotational flow).

The case, when condition (2.25) is not satisfied, the momentum vector field $\mathbf{p}$ is rotational (rotational flow). In this case equations (2.24) may be considered to be some kind of generalization of the Hamilton-Jacobi equation. For a real particle whose Hamiltonian has the form

$$H\left(\mathbf{x},\mathbf{p}\right)=\frac{\mathbf{p}^2}{2m}+V\left(\mathbf{x}\right) \tag{2.28}$$

the velocity $\mathbf{v}=\mathbf{p}/m$, and vector fields of $\mathbf{p}$ and $\mathbf{v}$ are simultaneously both rotational or irrotational. For a more general form of Hamiltonian the vector field of $\mathbf{p}$ may be irrotational, whereas the vector field of $\mathbf{v}$ is rotational, or vice versa.

Note that the action (2.21) contains information on initial values of momenta $\mathbf{p}$ of dynamic system $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{d}}]$. For instance, let us use standard initial conditions for variables $\varphi, \boldsymbol{\xi}$

$$\boldsymbol{\xi}(0,\mathbf{x}) = \mathbf{x}, \qquad \varphi(0,\mathbf{x}) = 0, \tag{2.29}$$

i.e. elements $\mathcal{S}_{\mathrm{d}}$ of $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{d}}]$ are labeled by their coordinates $\mathbf{x}$ at the moment $t = 0$. Then the integration functions $g^{\alpha}(\boldsymbol{\xi})$ are determined from the relation

$$\mathbf{p}(0,\mathbf{x}) = b_0 g^{\alpha}(\boldsymbol{\xi}) \nabla \xi_{\alpha} = b_0 \mathbf{g}(\boldsymbol{\xi}), \qquad \mathbf{g}(\boldsymbol{\xi}) = \{g^{\alpha}(\boldsymbol{\xi})\}, \qquad \alpha = 1,2,...n \tag{2.30}$$

Thus, one can consider that the action (2.21) contains complete information on motion of element $\mathcal{S}_{\mathrm{d}}$ of $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{d}}]$. Vice versa, if one knows how any element $\mathcal{S}_{\mathrm{d}}$ moves, one knows all about motion of the ensemble elements $\mathcal{S}_{\mathrm{d}}$ except for density of elements $\mathcal{S}_{\mathrm{d}}$ of $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{d}}]$, described by the quantity ρ. The variable ρ is a collective variable, containing information on density of elements $\mathcal{S}_{\mathrm{d}}$ of the statistical ensemble $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{d}}]$. Information on initial value of ρ is to be given additionally. Dynamic equations (2.22), (2.24), describing motion of single elements $\mathcal{S}_{\mathrm{d}}$ of the statistical ensemble $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{d}}]$, do not depend on the quantity ρ. In general, evolution of the statistical ensemble is insensitive to the number N of its elements. This property of the statistical ensemble is described by the relation

$$\mathcal{A}_E[a\rho, \mathbf{v}, \varphi, \boldsymbol{\xi}] = a\mathcal{A}_E[\rho, \mathbf{v}, \varphi, \boldsymbol{\xi}], \qquad a = \mathrm{const} > 0 \tag{2.31}$$

The property (2.31) is valid for any statistical ensemble $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{d}}]$ and $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{st}}]$.

III Description in terms of ψ-function

Let us introduce k-component complex function $\psi = \{\psi_{\alpha}\}$, $\alpha = 1,2,\ldots k$, defining it by the relations

$$\psi_{\alpha} = \sqrt{\rho} e^{i\varphi} u_{\alpha}(\boldsymbol{\xi}), \qquad \psi_{\alpha}^{*} = \sqrt{\rho} e^{-i\varphi} u_{\alpha}^{*}(\boldsymbol{\xi}), \qquad \alpha = 1,2,\ldots k$$

$$\psi^{*}\psi \equiv \sum_{\alpha=1}^{k} \psi_{\alpha}^{*}\psi_{\alpha}$$

where (*) means the complex conjugate, $u_{\alpha}(\boldsymbol{\xi})$, $\alpha = 1,2,\ldots k$ are functions of only variables $\boldsymbol{\xi}$. They satisfy the relations

$$-\frac{i}{2}\sum_{\alpha=1}^{k}\left(u_{\alpha}^{*}\frac{\partial u_{\alpha}}{\partial \xi_{\beta}} - \frac{\partial u_{\alpha}^{*}}{\partial \xi_{\beta}}u_{\alpha}\right) = g^{\beta}(\boldsymbol{\xi}), \qquad \beta = 1,2,...n, \qquad \sum_{\alpha=1}^{k} u_{\alpha}^{*}u_{\alpha} = 1 \tag{3.1}$$

k is such a natural number that equations (3.1) admit a solution. In general k may depend on the form of the arbitrary integration functions $\mathbf{g} = \{g^{\beta}(\boldsymbol{\xi})\}$, $\beta = 1,2,...n$.

It is easy to verify that

$$\rho = \psi^{*}\psi, \qquad p_l(\varphi, \xi\boldsymbol{\xi}) = -\frac{ib_0}{2\psi^{*}\psi}(\psi^{*}\partial_l\psi - \partial_l\psi^{*}\cdot\psi), \qquad l = 0,1,...n \tag{3.2}$$

The variational problem with the action (2.21) appears to be equivalent to the variational problem with the action functional

$$A[\psi,\psi^*]=\int\{\frac{ib_0}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-H\left(x,-\frac{ib_0}{2\psi^*\psi}(\psi^*\nabla\psi-\nabla\psi^*\cdot\psi)\right)\psi^*\psi\}\mathrm{d}^{n+1}x \tag{3.3}$$

where $\nabla=\{\partial_\alpha\}$, $\alpha=1,2,...n$

Note that the function ψ considered to be a function of independent variables $\{t,\mathbf{x}\}$ is very indefinite in the sense that the same state $\{\rho(t,\mathbf{x}),\mathbf{P}(t,\mathbf{x})\}$ of the statistical ensemble $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]$ may be described by different ψ-functions. There are two reasons for such an indefiniteness. First, the functions $u_\alpha(\boldsymbol{\xi})$ are not determined uniquely by differential equations (3.1). Second, their arguments $\boldsymbol{\xi}$ as functions of x are determined only to within the relabeling transformation

$$\xi_\alpha\rightarrow\tilde{\xi}_\alpha=\tilde{\xi}_\alpha(\boldsymbol{\xi}),\qquad D=\det\parallel\partial\tilde{\xi}_\alpha/\partial\xi_\beta\parallel=1,\qquad\alpha,\beta=1,2,...n \tag{3.4}$$

Description of the statistical ensemble $\mathcal{E}_\mathrm{p}[\mathcal{S}_\mathrm{d}]$ in terms of the function ψ is more indefinite, than the description in terms of the hydrodynamic potentials $\boldsymbol{\xi}$. Information about initial and boundary conditions, contained in functions $\mathbf{g}(\boldsymbol{\xi})$, is lost at the description in terms of the ψ-function.

Dynamic equations have the form

$$\delta\psi_\beta^*:\qquad\left[ib_0\partial_0-H+\frac{\partial H}{\partial p_\alpha}p_\alpha+\frac{ib_0}{2}\left(\frac{\partial H}{\partial p_\alpha}\nabla+\nabla\frac{\partial H}{\partial p_\alpha}\right)\right]\psi_\beta=0,\qquad\beta=1,2,...k \tag{3.5}$$

$$\delta\psi_\beta:\qquad\left[-ib_0\partial_0-H+\frac{\partial H}{\partial p_\alpha}p_\alpha-\frac{ib_0}{2}\left(\frac{\partial H}{\partial p_\alpha}\nabla+\nabla\frac{\partial H}{\partial p_\alpha}\right)\right]\psi_\beta^*=0,\qquad\beta=1,2,...k \tag{3.6}$$

where $H=H(x,\mathbf{p})$ and $\frac{\partial H}{\partial p_\alpha}$ are considered to be operators of multiplication by these quantities, and one has to substitute the expression (3.2) instead of $\mathbf{p}$ before action of operator ∇. In general, dynamic equations (3.5) (3.6) are not linear with respect to ψ-function, although they may be linear in some cases. In these interesting cases the dynamic equations may be solved rather simply.

The number k of the ψ-function components in the actions (3.3) is arbitrary. A formal variation of the action with respect to ψ_α and ψ_α^*, $\alpha=1,2,\ldots k$ leads to $2k$ real dynamic equations, but not all of them are independent. There are such combinations of variations $\delta\psi_\alpha$, $\delta\psi_\alpha^*$, $\alpha=1,2,\ldots k$ which do not change expressions (3.2). Such combinations of variations $\delta\psi_\alpha$, $\delta\psi_\alpha^*$, $\alpha=1,2,\ldots k$ do not change the action (3.3), and corresponding combinations of dynamic equations $\delta\mathcal{A}/\delta\psi_\alpha=0$, $\delta\mathcal{A}/\delta\psi_\alpha^*=0$ are identities that associates with a correlation between dynamic equations. Thus, increasing the number k, one increases the number of dynamic equations, but the number of independent dynamic equations remains the same. The number k is restricted underside by the condition that equations (3.1) have a solution. In other words, the minimal number k_m of the ψ-function components depends on the form of functions $\mathbf{g}(\boldsymbol{\xi})$, i.e. on initial conditions. This number k_m associates with the kinematic spin (k-spin) $s=2k_m+1$ of the ensemble state [16].

The ψ-function and k-spin remind quantum mechanical wave function and spin of a particle respectively. The ψ-function coincides with the wave function, if the dynamic equations (3.5), (3.6) become linear. It appears to be possible for statistical ensemble $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{st}}]$ of stochastic systems $\mathcal{S}_{\mathrm{st}}$. In this case the k-spin associates with the spin of the described particle, but the k-spin remains to be a property of the statistical ensemble $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{st}}]$ (i.e. a collective property), whereas in quantum mechanics the spin is considered to be a property of an individual particle.

IV Pure Statistical Ensemble of stochastic particles

Let us consider statistical ensemble $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{st}}]$ of stochastic particles $\mathcal{S}_{\mathrm{st}}$. There are no dynamic equations for $\mathcal{S}_{\mathrm{st}}$, and one cannot derive dynamic equations for $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{st}}]$ from dynamic equations for $\mathcal{S}_{\mathrm{st}}$. But we believe that dynamic equations for $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{st}}]$ exist, because experiments with statistical ensembles of stochastic particles $\mathcal{S}_{\mathrm{st}}$ are reproducible. Let us consider motion of $\mathcal{S}_{\mathrm{st}}$ as a result of interaction between some deterministic particle $\mathcal{S}_{\mathrm{d}}$ and some stochastic agent, which makes motion of $\mathcal{S}_{\mathrm{d}}$ to be stochastic. To derive dynamic equations for $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{st}}]$, some supposition on properties of this stochastic agent are to be made, because one cannot derive dynamic equations for $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{st}}]$ from nothing. If $\mathcal{S}_{\mathrm{st}}$ is a Brownian particle, moving in a gas, one supposes that the Brownian particle collides with gas molecules, and these collisions make the Brownian particle motion to be stochastic. One supposes that these collisions are accidental and independent. As a result a motion of the Brownian particle may be considered to be a Markovian process. Dynamic system $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{st}}]$ appears to be dissipative in this case, and there is no variational principle for it.

However, there is another kind of stochastic agent which remains the statistical ensemble $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{st}}]$ to be a conservative deterministic dynamic system. Dynamic equations of this system can be derived from the Hamilton variational principle formulated in a proper way. This stochastic agent is known as a quantum stochasticity, which is an origin of quantum effects, described by quantum mechanics. Stochastic systems $\mathcal{S}_{\mathrm{st}}$, associated with such a kind of stochasticity will be referred to as conservative stochastic systems.

Let us consider some microparticle, for instance, electron. An individual electron is a stochastic system $\mathcal{S}_{\mathrm{st}}$, because experiments with a single electron are irreproducible. For instance, in a diffraction experiment a single electron, flying through a narrow hole in a diaphragm, hits a new point of a screen any time. But distribution of many independent electrons over the screen surface is reproducible. It means, that the statistical ensemble $\mathcal{E}_{\mathrm{p}}[\mathcal{S}_{\mathrm{st}}]$ of many $\mathcal{S}_{\mathrm{st}}$ is a deterministic dynamic system $\mathcal{S}_{\mathrm{S}}$. This deterministic dynamic system $\mathcal{S}_{\mathrm{S}}$ is described by the action

$$\mathcal{A}_{\mathrm{S}}[\psi^*, \psi] = \int \{\frac{i\hbar}{2}(\psi^*\partial_0\psi - \partial_0\psi^* \cdot \psi) - \frac{\hbar^2}{2m}\nabla\psi^* \cdot \nabla\psi\}\mathrm{d}^4x \tag{4.1}$$

where ψ is so called wave function. Dynamic equation, generated by the action (4.1)

$$i\hbar\partial_0\psi + \frac{\hbar^2}{2m}\nabla^2\psi = 0 \tag{4.2}$$

is known as the Schrödinger equation.

Why an electron is described in terms of wave function? What is the wave function? Principles of quantum mechanics answer these questions. The wave function is a fundamental mathematical object (a vector in the Hilbert space) which evolves according to linear dynamic equation. All relations of quantum mechanics are obtained from corresponding relations of classical ones, replacing position $\mathbf{x} \rightarrow$ operator of multiplication by $\mathbf{x}$ and momentum $\mathbf{p} \rightarrow -\mathbf{i}\hbar\nabla$. Quantum mechanics is an axiomatic theory, and quantum principles work very well in the non-relativistic case. Attempts of extension of the quantum mechanic principles to the relativistic case met problems. Investigators were forced to introduce corrections and new principles. All this leads to a suspicion that quantum mechanical principles are conceptually non-relativistic, and principles of quantum mechanics cannot be combined with the relativity principles.

We shall not use quantum mechanical principles. Instead we suppose, that the quantum stochasticity has a geometrical origin. Namely, we suppose, that the real space-time is described by the geometry of Minkowski only approximately. This approximation is valid only for large enough values of world function for the Minkowski space-time

$$\sigma_M(x,x') = \sigma_M(t,\mathbf{x},t',\mathbf{x}') = \frac{1}{2}\left(c^2(t-t')^2 - (\mathbf{x}-\mathbf{x}')^2\right) \tag{4.3}$$

where $x = \{t,\mathbf{x}\}$ and $x' = \{t',\mathbf{x}'\}$ are coordinates of two points in the space-time, and c is the speed of light. The world function σ is a way of geometry description [17]. Recently it was shown that the world function describes geometry completely [18, 19], and any change of world function leads to a change of geometry and vice versa. Geometry of the real space-time is described by the world function

$$\sigma(x,x') = \sigma_M(x,x') + D(\sigma_M(x,x')) \tag{4.4}$$

Here σ_M is the Minkowski world function (4.3), D is a distortion function

$$D = \begin{cases} d & \text{if} \quad \sigma_M > \sigma_0 \\ 0 & \text{if} \quad \sigma_M \leq 0 \end{cases} \qquad d = \frac{\hbar}{2bc} = \text{const}, \qquad b \approx 10^{-17}\text{g/cm} \tag{4.5}$$

where $\hbar$ is the quantum constant, and b is a new universal constant. σ_0 is a constant of the order $d \approx 10^{-21}\text{cm}^2$. Correction to the Minkowski world function is small for large values of σ_M, because the characteristic length $\sqrt{d} \approx 10^{-11}$cm is essential only in the microcosm. Geometry, described by the world function (4.4), is non-Riemannian. (It is called tubular geometry, or briefly T-geometry). In T-geometry a motion of free particles is stochastic, although the world function (4.4) and T-geometry in itself is not stochastic[3]. The stochasticity in T-geometry, described by the world function (4.4), depends on the particle mass. The stochasticity is large for particles of small masses. It is negligible for particles of macroscopic mass.

Construction of a geometry on the basis of only world function is a new conception of geometry. (See for details Ref. [18, 20, 21, 19]). Now from this conception we need only existence of a geometrical stochasticity. We identify the geometrical stochasticity with the quantum stochasticity and try to construct a statistical description of microparticles on the basis of Hamiltonian variational principle. Such an approach has many points in its favour. First, neither quantum principles, nor new principles are used. All results are logical corollaries of supposition about the form of the world function (4.4) of the space-time. Second, such an approach is relativistic from the outset, and one does not need to combine relativity principles with quantum mechanical ones and to solve problems, connected with this integration. Third, the wave function and spin appear to be attributes of statistical description (not fundamental objects, whose meaning is obscure). Fourth, interpretation of quantum mechanics appears to be single-valued. Different versions of quantum mechanics interpretation, which take place in the axiomatic quantum mechanics, are impossible under such an approach. In particular, there is a clear distinction between the individual stochastic system $\mathcal{S}_{\rm st}$ and statistically averaged system $\langle\mathcal{S}_{\rm st}\rangle$. (The last is the statistical ensemble $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm st}\right]$ normalized to one system). In quantum mechanics the same term is used for both $\mathcal{S}_{\rm st}$ and $\langle\mathcal{S}_{\rm st}\rangle$. It leads to misunderstanding and paradoxes such as paradox of the Schrödinger cat, or EPR paradox. Fifth, a use of the model approach (instead of axiomatic one) admits one to determine the domain, where the quantum principles are valid, and quantum mechanics works correctly.

Let now $\mathcal{S}_{\rm st}$ be an electron or some other microparticle. The system $\mathcal{S}_{\rm st}$ is stochastic, but statistical ensemble $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm st}\right]$ of electrons $\mathcal{S}_{\rm st}$ is a conservative deterministic dynamic system, and some variational principle takes place for $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm st}\right]$. We suppose that in the zero approximation $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm st}\right]$ is described as a statistical ensemble $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm d}\right]$, where $\mathcal{S}_{\rm d}$ is a deterministic relativistic particle, described by the Hamiltonian

$$H\left(x,\mathbf{p}\right)=\sqrt{m^{2}c^{4}+\mathbf{p}^{2}c^{2}} \tag{4.6}$$

Then variational principle (2.21) for dynamic system $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm d}\right]$ has the form

$$\mathcal{A}_{E}[\rho,\varphi,\boldsymbol{\xi}]=\int\rho\{-\sqrt{m^{2}c^{4}+\mathbf{p}^{2}c^{2}}-b_{0}[\partial_{0}\varphi+g^{\alpha}(\boldsymbol{\xi})\partial_{0}\xi_{\alpha}]\}\mathrm{d}^{4}x, \tag{4.7}$$

where $\mathbf{p}$ is determined by (2.11), $n=3$. To obtain variational principle for $\mathcal{E}_{\rm p}\left[\mathcal{S}_{\rm st}\right]$, one should take into account stochastic component of the particle momentum. The mean value $\mathbf{p}_{\rm st}$ of the stochastic component has the form

$$\mathbf{p}_{\rm st}=\hbar\nabla\ln\rho \tag{4.8}$$

[3]It seems rather evident that the free particle motion is stochstic in the space-time with stochastic geometry [22, 23, 24], but a stochastic motion of a free particle in the deterministic space-time looks rather unexpected.

where $\hbar$ is the Planck quantum constant. In other words, the mean value $\mathbf{p}_{\mathrm{st}}$ of the stochastic component depends on the collective variable ρ which describes the state of the statistical ensemble. As far as the stochastic components and regular ones are supposed to be independent, a sum of squares of these components is to be used in the modified Hamiltonian.

The action for the dynamic system $\mathcal{E}_{\mathrm{p}}\left[\mathcal{S}_{\mathrm{st}}\right]$ has the form

$$\mathcal{A}_E[\rho,\varphi,\boldsymbol{\xi}]=\int\rho\{-\sqrt{m^2c^4+\mathbf{p}^2c^2+\hbar^2c^2\left(\nabla\ln\rho\right)^2}-b_0[\partial_0\varphi+g^{\alpha}(\boldsymbol{\xi})\partial_0\xi_{\alpha}]\}\mathrm{d}^4x, \quad (4.9)$$

where $\mathbf{p}$ is determined by (2.11), $n=3$. This fact may be interpreted also in the sense that the mass of the electron is modified

$$m^2\rightarrow m_{\mathrm{q}}^2=m^2+\frac{\hbar^2}{c^2}\left(\nabla\ln\rho\right)^2 \quad (4.10)$$

In fact, the supposition (4.10) is an origin of supposition (4.8). One supposes that consideration of geometric stochasticity leads only to a change of parameters of the system $\mathcal{S}_{\mathrm{d}}$ in $\mathcal{E}_{\mathrm{p}}\left[\mathcal{S}_{\mathrm{d}}\right]$. These parameters start to depend on the state density ρ of the ensemble. The system $\mathcal{S}_{\mathrm{d}}$ (4.6) has only one parameter – a mass m. The dependence (4.10) is obtained as a result of averaging over world lines of stochastic particles [20]. It is to be invariant with respect to a change of number of the ensemble elements, i.e. with respect to transformation

$$\rho\rightarrow a\rho,\qquad a=\mathrm{const}>0. \quad (4.11)$$

Note that the action (4.9) cannot be considered to be an action for the statistical ensemble of any deterministic dynamic systems $\mathcal{S}_{\mathrm{d}}$, because the effective Hamiltonian

$$H_{\mathrm{eff}}=\sqrt{m^2c^4+\mathbf{p}^2c^2+\hbar^2c^2\left(\nabla\ln\rho\right)^2}, \quad (4.12)$$

which enters in the Hamilton variational principle, depends now on the collective variable ρ describing the state of the whole ensemble, whereas Hamiltonian for any statistical ensemble $\mathcal{E}_{\mathrm{p}}\left[\mathcal{S}_{\mathrm{d}}\right]$ of deterministic dynamic systems $\mathcal{S}_{\mathrm{d}}$ has to depend only on variables of the dynamic system $\mathcal{S}_{\mathrm{d}}$, as it follows from (2.21). The action (4.9) is an action for a set of identical deterministic dynamic systems $\mathcal{S}_{\mathrm{d}}$, interacting between themselves. Hence, it is not a statistical ensemble of $\mathcal{S}_{\mathrm{d}}$. In the same time the action (4.9) may be considered to be an action for a statistical ensemble, because it has the main property of a statistical ensemble: not to depend on the number of elements of the statistical ensemble. Mathematically it means that if ρ is substituted by $a\rho$, $a=$const, $\mathcal{A}_E[\rho,\varphi,\boldsymbol{\xi}]$ is substituted by $a\mathcal{A}_E[\rho,\varphi,\boldsymbol{\xi}]$, or

$$\mathcal{A}_E[a\rho,\varphi,\boldsymbol{\xi}]=a\mathcal{A}_E[\rho,\varphi,\boldsymbol{\xi}] \quad (4.13)$$

It is easy to see that the action (4.9) has the property (4.13). Hence, the action (4.9) is an action of a statistical ensemble, but this ensemble cannot be a statistical ensemble of deterministic dynamic systems, because its elements interact between themselves and are not independent. It means that the statistical ensemble (4.9) is a statistical ensemble of nondeterministic dynamic systems. In other words, a statistical ensemble of nondeterministic systems can be considered to be a set of interacting deterministic systems, i.e. a stochasticity of dynamic systems is substituted by interaction of deterministic dynamic systems. Form of this interaction depends on the form of stochasticity. Considering different forms of interaction, satisfying the constraint (4.13), one can label and investigate different forms of stochasticity.

In fact we have no other way of the stochasticity description except for a reduction of the statistical ensemble $\mathcal{E}_{\mathrm{p}}\left[\mathcal{S}_{\mathrm{st}}\right]$ to a set $\mathcal{S}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right]$ of interacting deterministic dynamic systems $\mathcal{S}_{\mathrm{d}}$.

$$\mathcal{E}_{\mathrm{p}}\left[\mathcal{S}_{\mathrm{st}}\right]=\mathcal{S}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right] \tag{4.14}$$

Character of stochasticity is described by the way of interaction between deterministic dynamic systems $\mathcal{S}_{\mathrm{d}}$ in the set $\mathcal{S}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right]$ which is a deterministic dynamic system, consisting of interacting deterministic systems $\mathcal{S}_{\mathrm{d}}$. We shall refer to $\mathcal{S}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right]$ as a reduced statistical ensemble, consisting of elements $\mathcal{S}_{\mathrm{d}}$, although the set $\mathcal{S}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right]$ of $\mathcal{S}_{\mathrm{d}}$ is not a statistical ensemble of $\mathcal{S}_{\mathrm{d}}$ at all.

Thus, we know the only way of effective mathematical description and investigation of stochastic dynamic systems. This is a substitution of stochasticity by an interaction, i.e. reduction of the statistical ensemble $\mathcal{E}_{\mathrm{p}}\left[\mathcal{S}_{\mathrm{st}}\right]$ to a set $\mathcal{S}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right]$. One can describe properties of stochastic system $\mathcal{S}_{\mathrm{st}}$, only referring to the properties of the reduced statistical ensemble $\mathcal{S}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right]$. Different kinds of stochasticity are described by a consideration of different types of interaction between $\mathcal{S}_{\mathrm{d}}$ in $\mathcal{S}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right]$. We shall show that the action (4.9) describes a dynamic system which in the non-relativistic approximation is described by the Schrödinger equation. But there are another reduced statistical ensembles $\mathcal{S}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right]$ which have the Schrödinger equation as a dynamic equation.

For instance, the reduced statistical ensemble $\mathcal{S}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right]$, described by the action

$$\mathcal{A}_E[\rho,\varphi,\boldsymbol{\xi},\kappa]=\int\rho\{-\sqrt{m^2c^4+\hbar^2c^2\left(\partial_l\kappa^l+\kappa^l\kappa_l\right)}-b_0[\partial_0\varphi+g^\alpha(\boldsymbol{\xi})\partial_0\xi_\alpha]\}\mathrm{d}^{n+1}x, \tag{4.15}$$

$$\kappa=\{\kappa_0,\kappa_1,\kappa_2,\kappa_3\}, \qquad \kappa^l=g^{lj}\kappa_j, \qquad l=0,1,2,3 \tag{4.16}$$

where $\mathcal{S}_{\mathrm{d}}$ interact via some relativistic κ-field, is described (under some conditions) by the Klein-Gordon equation[25]. In the non-relativistic approximation this equation reduces to the Schrödinger equation. There are another kinds of a reduced statistical ensemble $\mathcal{S}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right]$ which has the Schrödinger equation as a dynamic equation under some conditions. It is not clear a priori, which of these reduced statistical ensembles $\mathcal{S}_{\mathrm{red}}\left[\mathcal{S}_{\mathrm{d}}\right]$ is true. This question needs further investigation.

Let us return to the action (4.9) and represent it in terms of ψ-function. In general, introduction of interaction between $\mathcal{S}_{\mathrm{d}}$ in the action (4.9) is not relativistically covariant. In the non-relativistic approximation the action (4.9) has the form

$$\mathcal{A}_E[\rho,\varphi,\boldsymbol{\xi}]=\int\rho\{-mc^2-\frac{\mathbf{p}^2}{2m}-\frac{\hbar^2}{2m}\left(\nabla\ln\rho\right)^2-b_0[\partial_0\varphi+g^\alpha(\boldsymbol{\xi})\partial_0\xi_\alpha]\}\mathrm{d}^4x, \tag{4.17}$$

where $\mathbf{p}$ is determined by the relation (2.11).

In terms of the ψ-function the action (4.17) is written in the form

$$A[\psi,\psi^*] = \int\{\frac{ib_0}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-mc^2\rho-\frac{\hbar^2(\nabla\rho)^2}{2m\rho}+\frac{b_0^2}{8\rho m}(\psi^*\nabla\psi-\nabla\psi^*\cdot\psi)^2\}\mathrm{d}^4x, \tag{4.18}$$

where $\rho\equiv\psi^*\psi$.

Let the function ψ have k components. Regrouping components of the function ψ in the action (4.18), one obtains the action in the form

$$\mathcal{A}_E[\psi,\psi^*] = \int\{\frac{ib_0}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-\frac{b_0^2}{2m}\nabla\psi^*\cdot\nabla\psi$$

$$+\frac{b_0^2}{4}\sum_{\alpha,\beta=1}^{k}Q^*_{\alpha\beta,\gamma}Q_{\alpha\beta,\gamma}\rho+\frac{b_0^2-\hbar^2}{8\rho m}(\nabla\rho)^2-mc^2\rho\}\mathrm{d}^4x,\qquad \rho\equiv\psi^*\psi \tag{4.19}$$

where

$$Q_{\alpha\beta,\gamma}=\frac{1}{\psi^*\psi}\begin{vmatrix}\psi_\alpha & \psi_\beta\\ \partial_\gamma\psi_\alpha & \partial_\gamma\psi_\beta\end{vmatrix},\qquad \alpha,\beta=1,2,\ldots k\qquad \gamma=1,2,3 \tag{4.20}$$

and $Q^*_{\alpha\beta,\gamma}$ is complex conjugate to $Q_{\alpha\beta,\gamma}$.

In the simplest case, when the ψ-function has only one component, all quantities $Q_{11,\gamma}=0,\quad \gamma=1,2,3$, and the motion of the ensemble particles is irrotational. Then the action (4.19) reduces to the form

$$\mathcal{A}_E[\psi,\psi^*] = \int\{\frac{ib_0}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-\frac{b_0^2}{2m}\nabla\psi^*\cdot\nabla\psi$$

$$-mc^2\rho+\frac{b_0^2-\hbar^2}{8\rho m}(\nabla\rho)^2\}\mathrm{d}^4x,\qquad \rho\equiv\psi^*\psi \tag{4.21}$$

Dynamic equation, generated by the action (4.21) is nonlinear due to the last term in the action (4.21). Equating the arbitrary integration constant b_0 to $\hbar$ $(b_0=\hbar)$, one obtains instead of (4.21)

$$\mathcal{A}_E[\psi,\psi^*] = \int\{\frac{i\hbar}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)-\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi-mc^2\psi^*\cdot\psi\}\mathrm{d}^4x \tag{4.22}$$

Note that the equivalence of the actions (4.21) and (4.22) can be verified directly, making in (4.21) a change of variables

$$\psi\to\tilde{\psi}=|\psi|\exp\left(\frac{\hbar}{b_0}\log\frac{\psi}{|\psi|}\right), \tag{4.23}$$

which transforms the action (4.21) into the action (4.22)

The action (4.22) generates linear dynamic equation for the ψ-function.

$$\frac{\delta \mathcal{A}}{\delta \psi_\alpha^*} = i\hbar\partial_0\psi + \frac{\hbar^2}{2m}\nabla^2\psi - mc^2\psi = 0 \tag{4.24}$$

After substitution $\psi = \Psi \exp\left(\frac{mc^2 t}{i\hbar}\right)$ the dynamic equation (4.24) transforms to the Schrödinger equation for a free particle

$$i\hbar\partial_0\Psi + \frac{\hbar^2}{2m}\nabla^2\Psi = 0 \tag{4.25}$$

It means that in some cases the ψ-function can coincide with the wave function.

Let us compare actions (4.21) and (4.22). They differ in a choice of the integration constant b_0 and describe the same dynamic system. The action (4.21) contains only one quantum term (i.e. the term, containing the quantum constant $\hbar$). This term $-\hbar^2(\nabla\rho)^2/8\rho m$ describes the energy density of stochastic component of motion. Other terms, containing integration constant b_0, are usual dynamical terms. In the action (4.22) practically all terms contain the quantum constant $\hbar$, and should be interpreted as quantum terms. Extension of quantum properties to all dynamical terms is the price which is paid for linearity of dynamic equation (4.24).

Separation of quantum and classical properties is very simple in the action (4.21). Setting $\hbar = 0$ in (4.21), one suppresses all quantum properties, and remaining terms describe a pure ensemble of deterministic classical particles. In the action (4.22) one cannot set $\hbar = 0$, because all dynamic terms contain $\hbar$. Separation of the classical part of the action in (4.22) is rather complicated procedure. From viewpoint of statistical description a linearity of dynamic equation (4.24) is an accidental circumstance, connected with a special form $-\hbar^2(\nabla\rho)^2/8\rho m$ of the stochastic component energy density and with irrotational character of the quantum fluid flow. If this flow is rotational, and ψ-function has more, than one component, dynamic equations are not linear. One cannot be sure, that completely relativistic statistical description can reduce to a linear dynamic equation, because in this case the stochastic component energy density has the form, which differs from $-\hbar^2(\nabla\rho)^2/8\rho m$. Nevertheless there exists such a relativistic statistical description, which generates a linear dynamic equation for one-component ψ-function (the action (4.15) generates the Klein-Gordon equation for one-component ψ-function [25]).

Note that the term $\rho U_{\mathrm{st}} = \hbar^2(\nabla\rho)^2/8\rho m$, describing stochastic component energy density, becomes to be large, if there is a local increase of density ρ. Let δx be the linear size of the region of the density increase, then $|\nabla\rho| \approx \rho/\delta x$, and the stochastic component energy $U_{\mathrm{st}} \approx (\hbar/\delta x)^2/8m$ becomes very large, provided δx is small enough. After expansion of the quantum fluid the local density increase disappears and the stochastic component energy U_{st} transforms to the kinetic energy $p^2/2m$. The particle momentum p becomes to be of the order $p \approx \sqrt{2mU_{\mathrm{st}}} \approx \hbar/2\delta x$. This relation is an origin of the indeterminacy relation. It means that the well known indeterminacy relations is a corollary of the way as the stochastic component energy U_{st} depends on the statistical ensemble density ρ.

Of course, linearity of dynamic equations is very convenient and useful, but it does not mean that linearity of dynamic equations may be considered to be a principle for construction of relativistic quantum theory.

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