Interplay between Dynamic Systems Described by the Klein-Gordon and Dirac Equations

Yuri A.Rylov

Institute for Problems in Mechanics, Russian Academy of Sciences,
101, Vernadskii Ave., Moscow, 117526, Russia.

Abstract

It is shown that in the two-dimensional space-time the dynamic system, described by the free Klein-Gordon equation, turns to the dynamic system, described by the free Dirac equation, provided the current and the energy-momentum tensor are redefined in a proper way.

The purpose of the present paper is to compare dynamic properties of two dynamic systems $\mathcal{S}_{\rm KG}$ and $\mathcal{S}_{\rm D}$ in the two-dimensional space-time, where $\mathcal{S}_{\rm KG}$ is a dynamic system, described by the free Klein-Gordon equation and $\mathcal{S}_{\rm D}$ is the dynamic system, described by the free Dirac equation. Such a comparison of only dynamic properties is of interest in the light of the fact [1] that quantum effects can be explained in terms of dynamics only (i.e. without a reference to quantum axiomatics).

Conventionally the term dynamic system (dynamics) means a mathematical object, whose state is described by some dynamical variables, which may be real- or complex-valued quantities. Any physical quantity is a function of the state and can be expressed through the dynamic variables. Dynamic variables evolve according to dynamic equations which determine single-valuedly evolution of the state. Sometimes mathematical objects whose dynamical variables are operator-valued, or matrix-valued quantities are also considered as dynamic systems (so called quantum dynamics), although in this case the physical quantities cannot be expressed only through dynamic variables. For calculation of physical quantities one needs to introduce additional quantity (the state vector). As a result any physical quantity is expressed via dynamic variables and the state vector.

We shall use a more narrow definition. *Dynamic system is a set of dynamic variables, fully describing the state of that system, dynamic equations for them and expressions for the current j^l and the energy-momentum tensor T^{kl}*. Both dynamic equations and expressions for j^l and T^{kl} can be obtained from the action functional. In other words, any dynamic system is determined by the action functional which contains dynamic variables as its arguments.

Let us discuss in more detail the constraint that dynamic variables must fully describe the state of the system. The meaning of this constraint can be illustrated using an example of a free quantum particle $\mathcal{S}_{\rm S}$, described by the Schrödinger equation

$$i\hbar\frac{\partial\psi_{\rm S}}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\psi_{\rm S}=0. \tag{1}$$

Here $\psi_{\rm S}=\psi_{\rm S}(t,\mathbf{x})$ is a continuous complex dynamical variable, which fully discribes the state of the quantum particle $\mathcal{S}_{\rm S}$. Eq.(1) is a dynamic equation for the dynamic variable $\psi_{\rm S}$. If $\psi_{\rm S}$ is known, the state of $\mathcal{S}_{\rm S}$ and all physical quantities, relating to $\mathcal{S}_{\rm S}$ are determined.

The same quantum system $\mathcal{S}_{\rm S}$ can be described in the Heisenberg representation in terms of the position operator $\hat{\mathbf{q}}$ and the momentum operator $\hat{\mathbf{p}}$. The operators $\hat{\mathbf{q}}$ and $\hat{\mathbf{p}}$ are considered as quantum dynamic variables. These variables depend on time t and satisfy the dynamic equations

$$\frac{{\rm d}\hat{\mathbf{q}}}{{\rm d}t}=\frac{\hat{\mathbf{p}}}{m},\qquad \frac{{\rm d}\hat{\mathbf{p}}}{{\rm d}t}=0 \tag{2}$$

and the commutation relations

$$[q^\alpha,p_\beta]_-=i\hbar\delta^\alpha_\beta,\qquad \alpha,\beta=1,2,3 \tag{3}$$

As far as dynamic equations (2) remind formally Hamilton equation for a classical particle, the operators $\mathbf{q}$ and $\mathbf{p}$ are considered usually as quantum dynamic variables.

Quantum dynamic variables $\hat{\mathbf{q}}(t), \hat{\mathbf{p}}(t)$ are quite different from dynamic variables $\psi_{\mathrm{S}}(t,\mathbf{x})$ in the sense that $\hat{\mathbf{q}}$, $\hat{\mathbf{p}}$ cannot fully describe the state of $\mathcal{S}_{\mathrm{S}}$. In the Heisenberg representation the state of $\mathcal{S}_{\mathrm{S}}$ is determined by the state vector ψ_{H}. All physical quantities relating to $\mathcal{S}_{\mathrm{S}}$ are determined by quantum dynamic variables $\hat{\mathbf{q}}, \hat{\mathbf{p}}$ taken together with the state vector ψ_{H} (but not by dynamic variables only as it was in the Schrödinger representation (1)).

As we have shown, our definition of the dynamic system (and dynamic properties) disqualifies Heisenberg representation of a quantum dynamic system on the basis of its inability to describe the system state using only dynamic variables and the need for additional information in the state vector ψ_{H}. The underlying reason of this deficiency is that the Heisenberg description (2), (3) make use of quantum axiomatics (quantum principles) which in fact have a little to do with the dynamics of a system.

The Schrödinger description, including dynamic equation of the type (1) and expressions for j^l and T^{kl}, is sufficient for explanation and calculation of all real quantum effects [1], but it does not contain a concept of a single particle. To describe and to interpret the Schrödinger picture in terms of a single particle, one uses the statistical propositions which permit to calculate average values $\langle R\rangle_\psi$ of any physical quantity R at the state described by the wave function (state vector) ψ

$$\langle R\rangle_\psi = A^{-1}\int \psi^* \hat{R}\psi d\mathbf{x}, \qquad A = \int \psi^*\psi d\mathbf{x}, \tag{4}$$

where $\hat{R}$ is some linear operator associated with the physical quantity R. Statistical propositions (4) is some kind of a probabilistic construction that permits to speak about quantum phenomena in terms of a single particle. It is a kind of interpretation of quantum phenomena in terms of a single particle. As any interpretation this probabilistic construction contains some arbitrary elements which are not essential for calculation of quantum effects. Such an interpretation is not unique [1].

Quantum principles (quantum axiomatics) can be derived from the statistical propositions (4) provided they take place for any quantities [2]. The statistical propositions agree with the dynamic equation (1) in such a way that any result which can be obtained from expressions for j^l and T^{kl} can be obtained also from the statistical propositions (4).

Quantum dynamic equations (2) in the Heisenberg representation cannot be derived from only dynamic equation (1). They can be obtained only as a result of combination of dynamic equation (1) with the statistical propositions (4). Thus, looking formally as dynamic equations, equations (2) contain implicitly statistical propositions (4) and, hence, quantum axiomatics. In particular, the dynamic equations (2) contain essentially the concept of linear operator, that is characteristic for the statistical propositions (4), but not for the dynamic equation (1).

In this paper only dynamic properties of two dynamic systems $\mathcal{S}_{\mathrm{KG}}$ and $\mathcal{S}_{\mathrm{D}}$ are compared, because these properties seem to be most important for calculation of quantum effects. In other words, the properties connected with dynamics (1) (but not with the statistical propositions) are investigated. In this relation our investigation differs from the well-known paper by Foldy-Wouthuysen [3] which investigates dynamic equations for $\mathcal{S}_{\mathrm{D}}$ in the Heisenberg representation and, hence, takes into account both dynamics and quantum axiomatics, contained in the Heisenberg dynamic equations (concept of a linear operator, commutation relations, etc.)

Note that the paper [1] is necessary only for a motivation of the presented investigation which is self-sufficient and does not need a reference to the paper [1] for its substantiation. Necessity of the investigation of only dynamics can be explained as follows.

Results of experiments with a single quantum particle are irreproducible. By definition it means that the single quantum particle is stochastic. Although a result of an experiment with a single stochastic particle is irreproducible, distributions of results of similar experiments with many independent identical stochastic particles are reproducible. Projecting many independent identical stochastic particles $\mathcal{S}_{\mathrm{st}}$ in the same space-time region, one obtains a cloud $\mathcal{E}[N,\mathcal{S}_{\mathrm{st}}]$ of N independent identical particles moving randomly. With the number N of particles tending to ∞, this cloud $\mathcal{E}[\infty,\mathcal{S}_{\mathrm{st}}]$ may be considered as a continuous medium, or a fluid. Only dynamics of the fluid is important for calculation of physical phenomena connected with the stochastic particle $\mathcal{S}_{\mathrm{st}}$ This statement is valid for any stochastic particle independently of the nature of the stochasticity. For instance, it is valid both for a quantum particle associated with the quantum (Madelung) fluid and for a Brownian particle associated with the Brownian fluid. In any case this fluid is a deterministic dynamic system, because experiments with this fluid $\mathcal{E}[\infty,\mathcal{S}_{\mathrm{st}}]$ are reproducible. Besides any reproducible experiment with the stochastic particle can be described in terms of the fluid $\mathcal{E}[\infty,\mathcal{S}_{\mathrm{st}}]$ without a reference to any probabilistic construction. Such constructions (probability density, or probability amplitude) are needed only for interpretation of the fluid motion in terms of a single stochastic particle. For instance, instead of the density ρ of the Brownian fluid, it is a common practice to speak about a probability density ρ of the Brownian particle position. In the case of the quantum (Madelung) fluid the interpretation in terms of a single particle is not so simple.

Thus, a stochastic particle associates with some kind of a fluid, and studying dynamics of this fluid, one investigates mean properties of the stochastic particle. The Madelung fluid is nondissipative, whereas the Brownian fluid is dissipative, and this is a reason of different behaviour of qunatum particles and Brownian ones.

Conventionally $\mathcal{S}_{\mathrm{KG}}$ associates with a spinless particle and a scalar wave function ψ, whereas $\mathcal{S}_{\mathrm{D}}$ associates with a particle of spin 1/2 and a spinor wave function ψ_{D}. Here only the two-dimensional space-time is considered, where spin is unessential. According to the quantum axiomatics the transformation properties of wave functions are connected with some internal properties of the described particle. If one abstracts from the quantum axiomatics, and considers $\mathcal{S}_{\mathrm{KG}}$ and $\mathcal{S}_{\mathrm{D}}$ simply as dynamic systems, one discovers that by means of a change of variables the two-component spinor ψ_{D} can be transformed into the one-component scalar ψ and some combination of derivatives of ψ. Under this transformation the Dirac equation for ψ_{D} transforms to the Klein-Gordon equation for ψ, but j^l and T^{kl} for $\mathcal{S}_{\mathrm{KG}}$ and $\mathcal{S}_{\mathrm{D}}$ do not transform one into other.

It means that from the dynamic point of view the dynamic systems $\mathcal{S}_{\mathrm{KG}}$ and $\mathcal{S}_{\mathrm{D}}$

differ only by definitions of j^l and T^{kl}. Redefining j^l and T^{kl}, one can turn $\mathcal{S}_{\rm D}$ to $\mathcal{S}_{\rm KG}$ and vice versa.

All this looks rather unexpected, if, *basing on the quantum axiomatics*, the transformation properties of the wave function are considered as internal properties of the described particle, because it is difficult to believe that internal properties of the particle can depend on a choice of dynamic variables.

It should stress in this connection that our results contradict by no means to the conventional results concerning the Dirac spinor field $\psi_{\rm D}$ and the Klein-Gordon scalar field ψ, because they concern quite different mathematical objects. Our results concern properties of dynamic systems $\mathcal{S}_{\rm D}$ and $\mathcal{S}_{\rm KG}$ taken in themselves, whereas the Dirac spinor field $\psi_{\rm D}$ and the Klein-Gordon field ψ are defined as mathematical objects satisfying some dynamic equations and in addition the constraints of quantum axiomatics. In particular, these constraints concern transformation properties of the field $\psi_{\rm D}$ and ψ. The $\psi_{\rm D}$ transforms as a spinor, and any transformation of $\psi_{\rm D}$ into a scalar is forbidden. Such a transformation is considered as incompatible with the quantum axiomatics. In the pure dynamics there are no such constraints. Any change of dependent variables is possible.

In the two-dimensional space-time the dynamic system $\mathcal{S}_{\rm KG}$ is described by the action

$$\mathcal{A}_{\rm KG}[\psi,\psi^*]=\frac{1}{2}\int(-m^2c^2\psi^*\psi+\hbar^2\partial_l\psi^*\partial^l\psi)d^2x,\qquad \partial_l\equiv\frac{\partial}{\partial x^l},\qquad l=0,1 \tag{5}$$

where ψ is a one-component complex variable, m is a mass of a particle, c and $\hbar$ are respectively the speed of the light and the Planck constant, and g_{ik}=diag$\{c^2,-1\}$, g^{ik}=diag$\{c^{-2},-1\}$ is the metric tensor in the two-dimensional space-time. There is a summation over repeated Latin indices 0-1. The dynamic equation has the form

$$\lambda^2\partial_l\partial^l\psi+\psi=0,\qquad \lambda\equiv\hbar/mc \tag{6}$$

The current j^l and the canonical energy-momentum tensor T_k^i are defined by the relations

$$j^l=\frac{i\hbar}{2}(\psi^*\partial^l\psi-\psi\partial^l\psi^*),\qquad l=0,1 \tag{7}$$

$$T_k^l=\frac{\hbar^2}{2}(\psi^{*l}\psi_k+\psi_k^*\psi^l-\delta_k^l\psi_i^*\psi^i)+\frac{m^2c^2}{2}\psi^*\psi\delta_k^l,\qquad l,k=0,1 \tag{8}$$

$$\psi_k\equiv\partial_k\psi,\qquad \psi^k\equiv\partial^k\psi\equiv g^{kj}\partial_j\psi, \tag{9}$$

The current j^l and the energy-momentum tensor T_k^l are attributes of the dynamic system $\mathcal{S}_{\rm KG}$, because they are sources of the electromagnetic field and of the gravitational field respectively. It means that the expression (7) for the current determines interaction of $\mathcal{S}_{\rm KG}$ with the external electromagnetic field by means of additional term $-ec^{-1}A_kj^k$ in the Lagrangian of the action (5). The canonical energy-momentum tensor (8) is symmetric and coincides with that derived by means of a variation with respect to metric tensor g_{ik}.

The dynamic system $\mathcal{S}_{\rm D}$ is described by the action

$$\mathcal{A}_{\rm D}[\psi_{\rm D},\psi_{\rm D}^*]=\int(-mc\bar\psi_{\rm D}\psi_{\rm D}+\frac{i}{2}\hbar\bar\psi_{\rm D}\gamma^l\partial_l\psi_{\rm D}-\frac{i}{2}\hbar\partial_l\bar\psi_{\rm D}\gamma^l\psi_{\rm D})d^2x \tag{10}$$

The current j^k_{D} and the canonical energy-momentum tensor $T^l_{\mathrm{D}k}$ have the form

$$j^l_{\mathrm{D}} = \bar{\psi}_{\mathrm{D}}\gamma^l\psi_{\mathrm{D}}, \qquad l = 0, 1, \tag{11}$$

$$T^l_{\mathrm{D}k} = \frac{i\hbar}{2}(\bar{\psi}_{\mathrm{D}}\gamma^l\partial_k\psi_{\mathrm{D}} - \partial_k\bar{\psi}_{\mathrm{D}}\gamma^l\psi_{\mathrm{D}}), \qquad l, k = 0, 1 \tag{12}$$

where

$$\psi_{\mathrm{D}} = \binom{\psi_+}{\psi_-}, \qquad \bar{\psi}_{\mathrm{D}} = \psi^*_{\mathrm{D}}c\gamma^0, \qquad \psi^* = (\psi^*_+, \psi^*_-) \tag{13}$$

$$\gamma^0 = c^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_0 = g_{00}\gamma^0 = c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{14}$$

Representation (14) of γ-matrices is chosen in such a way that the pseudo-scalar matrix $c\gamma^0\gamma^1$ is diagonal, and the wave functions $\psi_{\mathrm{D}} = \binom{\psi_+}{0}$, $\psi_{\mathrm{D}} = \binom{0}{\psi_-}$ are its eigenfunctions for any choice of ψ_+, ψ_-. In this case in virtue of Eq.(13) the Dirac equation

$$i\hbar\gamma^l\partial_l\psi_{\mathrm{D}} - mc\psi_{\mathrm{D}} = 0 \tag{15}$$

takes the form

$$\psi_+ = i\lambda\partial_+\psi_-, \qquad \psi_- = i\lambda\partial_-\psi_+, \tag{16}$$

$$\lambda \equiv \hbar/mc, \qquad \partial_\pm \equiv c^{-1}\partial_0 \pm \partial_1 \tag{17}$$

It follows from Eq.(16) that both wave functions $\psi_\pm$ satisfy the free Klein-Gordon equation

$$\lambda^2\partial_l\partial^l\psi_\pm + \psi_\pm = 0 \tag{18}$$

Let us introduce the two-component differential operator

$$\mathcal{L}(w, \partial, \lambda) = \begin{pmatrix} \sqrt{w_+} + i\lambda\sqrt{w_-}\partial_+ \\ \sqrt{w_-} + i\lambda\sqrt{w_+}\partial_- \end{pmatrix}. \tag{19}$$

where $w_l = (w_0, w_1)$ is a constant timelike or null vector and

$$w_+ = c^{-1}w_0 + w_1, \qquad w_- = c^{-1}w_0 - w_1$$

Under the continuous Lorentz transformation

$$x^0 \to \tilde{x}^0 = x^0\cosh\chi + c^{-1}x^1\sinh\chi$$

$$x^1 \to \tilde{x}^1 = x^1\cosh\chi + cx^0\sinh\chi \tag{20}$$

the components $w_\pm$ and $\partial_\pm$ transforms as follows

$$w_+ \to \tilde{w}_+ = e^\chi w_+, \qquad w_- \to \tilde{w}_- = e^{-\chi}w_-$$

$$\partial_+ \to \tilde{\partial}_+ = e^\chi\partial_+, \qquad \partial_- \to \tilde{\partial}_- = e^{-\chi}\partial_- \tag{21}$$

According to Eqs. (20), (21) the differential operator $\mathcal{L}$ transforms as follows

$$\mathcal{L}(w, \partial, \lambda) \to \mathcal{L}(\tilde{w}, \tilde{\partial}, \lambda) = \begin{pmatrix} \sqrt{\tilde{w}_+} + i\lambda\sqrt{\tilde{w}_-}\tilde{\partial}_+ \\ \sqrt{\tilde{w}_-} + i\lambda\sqrt{\tilde{w}_+}\tilde{\partial}_- \end{pmatrix} =$$

$$= \begin{pmatrix} e^{\chi/2}(\sqrt{w_+} + i\lambda\sqrt{w_-}\partial_+) \\ e^{-\chi/2}(\sqrt{w_-} + i\lambda\sqrt{w_+}\partial_-) \end{pmatrix} = e^{-c\gamma^0\gamma^1\chi/2}\mathcal{L}(w,\partial,\lambda) \tag{22}$$

Under space reflections

$$x^0 \to \tilde{x}^0 = x^0, \qquad x^1 \to \tilde{x}^1 = -x^1 \tag{23}$$

one has

$$w_+ \to \tilde{w}_+ = w_-, \qquad w_- \to \tilde{w}_- = w_+,$$
$$\partial_+ \to \tilde{\partial}_+ = \partial_-, \qquad \partial_- \to \tilde{\partial}_- = \partial_+, \tag{24}$$
$$\mathcal{L}(w,\partial,\lambda) \to \mathcal{L}(\tilde{w},\tilde{\partial},\lambda) = c\gamma^0\mathcal{L}(w,\partial,\lambda) \tag{25}$$

Under time reflections

$$x^0 \to \tilde{x}^0 = -x^0, \qquad x^1 \to \tilde{x}^1 = x^1 \tag{26}$$

one can write

$$w_+ \to \tilde{w}_+ = e^{i\pi}w_-, \qquad w_- \to \tilde{w}_- = e^{-i\pi}w_+,$$
$$\partial_+ \to \tilde{\partial}_+ = e^{i\pi}\partial_-, \qquad \partial_- \to \tilde{\partial}_- = e^{-i\pi}\partial_+, \tag{27}$$
$$\mathcal{L}(w,\partial,\lambda) \to \mathcal{L}(\tilde{w},\tilde{\partial},\lambda) = e^{i\pi/2}\gamma^1\mathcal{L}(w,\partial,\lambda) \tag{28}$$

It means that the differential operator $\mathcal{L}(w,\partial,\lambda)$ transforms as a spinor under all transformations of the Lorentz group.

Let us form the two-component quantity

$$\psi_{\mathrm{D}} = \binom{\psi_+}{\psi_-} = \mathcal{L}(w,\partial,\lambda)\psi \tag{29}$$

If ψ is a scalar, satisfying the KG equation (6), then ψ_{D} is a spinor, satisfying the Dirac equation (15) for any choice of the timelike constant vector $w_l = (w_0, w_1)$. Vice versa, if the spinor ψ_{D} satisfies the Dirac equation (15), then the scalar ψ defined by Eq.(29) satisfies the KG equation (6) for any choice of the timelike vector w.

The transformation reciprocal to the transformation (29) can be presented in the explicit form

$$\psi = \frac{1}{2}\hat{Q}(\sqrt{w_+}\psi_+ + \sqrt{w_-}\psi_-) \tag{30}$$

$$c^{-1}\partial_0\psi = -\frac{i(1-c^{-1}w_0\hat{Q})}{2\lambda\sqrt{w_-}}\psi_+ - \frac{i(1+c^{-1}w_0\hat{Q})}{2\lambda\sqrt{w_+}}\psi_- \tag{31}$$

where $\hat{Q} = (w_1 + i\lambda w\partial_1)^{-1}$ is the operator reciprocal to the operator $w_1 + i\lambda w\partial_1$

$$\hat{Q}\psi(t,x) = (2\pi\hbar)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[i\hbar^{-1}k(x-x')](w_1 - \frac{kw}{mc})^{-1}\psi(t,x')dkdx' \tag{32}$$

$$w \equiv \sqrt{w_+w_-} = \sqrt{w_kw^k}$$

The state of $\mathcal{S}_{\mathrm{KG}}$ is described by ψ and $\partial_0\psi$ given at some moment of time. The relations (30), (31) express these quantities via components ψ_+ and ψ_- of the spinor ψ_{D}.

Substituting relation (29) into Eq.(11), one obtains

$$j_{\mathrm{D}}^0 = c^{-1}(\psi_+^*\psi_+ + \psi_-^*\psi_-) = \frac{4}{m^2c^2}(T_l^0 w^l + wmcj^0)\mathrm{sgn}(w_0), \qquad w \equiv \sqrt{w_l w^l} \quad (33)$$

$$j_{\mathrm{D}}^1 = -\psi_+^*\psi_+ + \psi_-^*\psi_- = \frac{4}{m^2c^2}(T_l^1 w^l + wmcj^1)\mathrm{sgn}(w_0),$$

It means that the current j_{D} associated with the Dirac dynamic system is constructed of the components of the current j^l and the energy-momentum tensor T_k^l, associated with the Klein-Gordon dynamic system.

Using Eq.(12), (14), one can present components of $T_{\mathrm{D}k}^l$ in the form

$$T_{\mathrm{D}l}^0 = c^{-1}(j_{(+)l} + j_{(-)l}), \qquad T_{\mathrm{D}l}^1 = j_{(-)l} - j_{(+)l}, \qquad l = 0, 1 \quad (34)$$

where $j_{(\pm)}$ are quantities of the type of the current (7)

$$j_{(+)l} = \frac{i\hbar}{2}(\psi_+^*\partial_l\psi_+ - \psi_+\partial_l\psi_+^*), \qquad j_{(-)l} = \frac{i\hbar}{2}(\psi_-^*\partial_l\psi_- - \psi_-\partial_l\psi_-^*) \quad (35)$$

Let us note that the canonical energy-momentum tensor $T_{\mathrm{D}k}^l$ is symmetric in virtue of dynamic equations, although it is not symmetric identically. Indeed, using relations (16) between the components ψ_+ and ψ_- and Eq.(18), one can show that

$$j_{(+)1} + j_{(-)1} = c^{-1}(j_{(+)0} - j_{(-)0}). \quad (36)$$

It follows from Eqs.(34) and (36) that

$$c^2 T_{\mathrm{D}1}^0 = -T_{\mathrm{D}0}^1, \qquad T_{\mathrm{D}}^{01} = T_{\mathrm{D}}^{10} \quad (37)$$

To present the energy-momentum tensor $T_{\mathrm{D}k}^l$ in terms of ψ in an explicit form, one introduces expressions

$$\mathcal{T}_k^l(\chi^*, \psi) = \frac{m^2c^2}{2}\chi^*\psi\delta_k^l + \frac{\hbar^2}{2}(\chi^{*l}\psi_k + \chi_k^*\psi^l - \delta_k^l\chi_i^*\psi^i), \qquad l, k = 0, 1 \quad (38)$$

$$\chi_l \equiv \partial_l\chi, \qquad \psi_l \equiv \partial_l\psi$$

$$\mathcal{J}^l(\chi^*, \psi) = \frac{i\hbar}{2}(\chi^*\partial^l\psi - \psi\partial^l\chi^*), \qquad l = 0, 1 \quad (39)$$

where χ^* and ψ are formal arguments of the functions $\mathcal{T}_l^k$ and $\mathcal{J}^l$. According to Eqs. (11), (12) the energy-momentum tensor $T_{\mathrm{D}k}^l$ is a result of the operator $i\hbar\partial_k$ action on the wave function ψ_{D} in the expression (11) for j_{D}^l. Then according to Eq.(8) and Eqs.(38), (39) one obtains

$$T_{\mathrm{D}k}^l = \frac{2i\lambda}{mc}\mathrm{sgn}(w_0)\left\{\mathcal{T}_s^l(\psi^*, \psi_k)w^s - \mathcal{T}_s^l(\psi, \psi_k^*)w^s + \right.$$

$$+wmc[\mathcal{J}^l(\psi^*, \psi_k) - \mathcal{J}^l(\psi, \psi_k^*)]\Big\}, \qquad \psi_k \equiv \partial_k \psi \tag{40}$$

These expressions are relativistically covariant with respect to scalars ψ, ψ^* and the vector w_l.

Thus, if ψ satisfies the KG equation (6), the ψ describes the dynamic system $\mathcal{S}_{\rm KG}$, provided the current j^k and the energy-momentum tensor T_k^l are defined by Eqs. (7) and (8) respectively.

If ψ satisfies the same KG equation (6), but the current j^k and the energy-momentum tensor T_k^l are determined by Eqs. (33), and (38)–(40) respectively, the ψ describes the dynamic system $\mathcal{S}_{\rm D}$.

In other words, dynamic systems $\mathcal{S}_{\rm KG}$ and $\mathcal{S}_{\rm D}$ differ by definition of the current and the energy-momentum tensor (but not by their dynamic equations). Interaction with the electromagnetic field is determined by the form of the current. It is different for $\mathcal{S}_{\rm KG}$ and $\mathcal{S}_{\rm D}$.

Description of the dynamic system $\mathcal{S}_{\rm D}$ in terms of the scalar ψ contains a timelike or null vector w_l. Description in terms of the spinor $\psi_{\rm D}$ does not contain this vector w_l, because it is "hidden inside γ^l". A similar situation arises in the case of the conventional 4-dimensional Dirac equation [5]. In this case a constant timelike 4-vector arises, if the Dirac system is described in terms of scalar-tensor variables (not in terms of spinors).

The transformation properties of a field are usually considered to be its important characteristic. In this connection the statement that the dynamic system $\mathcal{S}_{\rm D}$, described usually in terms of a spinor field $\psi_{\rm D}$, can be also described in terms of a scalar field ψ looks rather unexpectedly. To clear the situation, following Anderson [4], let us introduce the concept of an absolute object. By the definition the absolute object is a quantity (or quantities) which is the same for all solutions of dynamic equations. The absolute objects arise, if the dynamic equations and expressions for j^k an T^{kl} are written in the relativistically covariant form. In many cases the absolute objects may be substituted by numbers, or some definite functions. But in this case the form of dynamic equations and expressions for j^k an T^{kl} stops to be covariant. The total symmetry of the dynamic system is determined by the symmetry of the absolute objects. For instance, the description of $\mathcal{S}_{\rm KG}$, by the action (1) [by the dynamic equation (6), the current (7) and the energy-momentum tensor (8)] contains the metric tensor g_{lk}=diag$\{c^2, -1\}$ which is an absolute object, because it is the same for all solutions of the dynamic equation (6). Components of g^{kl} are invariant with respect to Lorentz trnsformations. Symmetry of $\mathcal{S}_{\rm KG}$ is determined by the Lorentz group, and $\mathcal{S}_{\rm KG}$ is a relativistic dynamic system.

A description of the dynamic system $\mathcal{S}_{\rm D}$, described by the action (10) [by the dynamic equation (15), the current (11) and the energy-momentum tensor (12)] contains the absolute objects γ^l, $\quad l = 0, 1$. Such a description of $\mathcal{S}_{\rm D}$ does not contain the metric tensor directly. The last arises as a derivative absolute object via the relations

$$\gamma^l \gamma^k + \gamma^k \gamma^l = 2g^{kl}, \qquad k, l = 0, 1. \tag{41}$$

The same dynamic system $\mathcal{S}_{\rm D}$, described by the dynamic equation (6), the current (33), (7), (8) and the energy-momentum tensor (40), contains two absolute objects:

the metric tensor g^{ik} and the constant timelike vector w_l. Apparently, this fact would be interpreted in the sense that the change of variables (29), (19) replaces the absolute objects $\gamma^l, \quad l = 0, 1$ by the absolute objects g^{ik}, w_l. It correlates with the paper [5], where the same result was obtained for the Dirac dynamic system in the $4D$ space-time. Although in the paper [5] another change of variables was used, but that change of variables removed γ-matrices and lead also to appearance of a new absolute object: a constant timelike vector f^k instead of γ-matrices.

The Lorentz group is a symmetry group of the metric tensor g^{kl}, but it is not a symmetry group of the constant vector w_l. It means that the dynamic system $\mathcal{S}_{\rm D}$ described in terms of the scalar ψ is non-relativistic. It is connected with the fact that the timelike vector w_l means a preferred direction in the space-time, or a preferred coordinate system, or a split of the space-time into the space and the time.

At the same time the $\mathcal{S}_{\rm D}$ described in terms of the spinor $\psi_{\rm D}$ is considered conventionally as a relativistic dynamic system, because the absolute objects γ^l are considered as invariants with respect to the Lorentz group. Does it mean that one can convert a relativisitic dynamic system to non-relativistic one by means of a change of variables? Such a possibility seems rather doubtful. As far as the $\mathcal{S}_{\rm D}$, described in terms of $\psi_{\rm D}$, contains only $\gamma^l, \quad l = 0, 1$ as original absolute objects, a decision on relativistic character of $\mathcal{S}_{\rm D}$ depends completely on the symmetry group of the γ-matrices γ^l, or on whether or not γ^l are scalars under the transformations of the Lorentz group. $\gamma^l, \quad l = 0, 1$ are not objects of the space-time, and transformation property of γ^l under the Lorentz group is a rather subtle question.

There are two approaches to the Dirac equation. In the first approach [6] the wave function $\psi_{\rm D}$ is considered as a scalar function defined on the field of Clifford numbers γ^l,

$$\psi_{\rm D} = \psi_{\rm D}(x, \gamma)\Gamma, \qquad \overline{\psi}_{\rm D} = \Gamma\overline{\psi}_{\rm D}(x, \gamma), \tag{42}$$

where Γ is a constant nilpotent factor which has the property $\Gamma f(\gamma)\Gamma = a\Gamma$. Here $f(\gamma)$ is arbitrary function of γ^l and a is a complex number depending on the form of the function f. Within such an approach under the Lorentz transformations $\bar{\psi}_{\rm D}$, $\psi_{\rm D}$ transform as scalars and γ^l transform as components of a vector. In this case the symmetry group of γ^l is a subgroup of the Lorentz group, and $\mathcal{S}_{\rm D}$ is non-relativistic dynamic system. Then the matrix vector γ^l describes some preferred direction in the space-time.

In the second (conventional) approach the $\psi_{\rm D}$ is considered as a spinor, and the $\gamma^l, \quad l = 0, 1$ are scalars with respect to the transformations of the Lorentz group. Analyzing the two approaches, Sommerfeld [7] considered the first approach as more reasonable. In the second case the analysis is rather difficult due to of non-standard transformations of γ^l and $\psi_{\rm D}$ under linear coordinate transformations T. Indeed, the transformation T for the vector j^l

$$\tilde{\overline{\psi}}_{\rm D}\tilde{\gamma}^l\tilde{\psi}_{\rm D} = \frac{\partial \tilde{x}^l}{\partial x^s}\overline{\psi}_{\rm D}\gamma^s\psi_{\rm D}, \tag{43}$$

where quantities marked by tilde ˜ mean quantities in the transformed coordinate system, can be written by two ways

$$(1): \qquad \tilde{\psi}_{\mathrm{D}} = \psi_{\mathrm{D}}, \qquad \tilde{\overline{\psi}}_{\mathrm{D}} = \overline{\psi}_{\mathrm{D}}, \qquad \tilde{\gamma}^l = \frac{\partial \tilde{x}^l}{\partial x^s}\gamma^s \tag{44}$$

$$(2): \qquad \tilde{\gamma}^l = \gamma^l, \qquad \tilde{\psi}_{\mathrm{D}} = S(\gamma, T)\psi_{\mathrm{D}}, \qquad \tilde{\overline{\psi}}_{\mathrm{D}} = \overline{\psi}_{\mathrm{D}} S^{-1}(\gamma, T), \tag{45}$$
$$S^*(\gamma, T)\gamma^0 = \gamma^0 S^{-1}(\gamma, T)$$

The relation (44) corresponds to the first approach and the relation (45) to the second one. Both ways (44) and (45) lead to the same result, provided

$$S^{-1}(\gamma, T)\gamma^l S(\gamma, T) = \frac{\partial \tilde{x}^l}{\partial x^s}\gamma^s \tag{46}$$

The second way (45) has two defects. First, the transformation law of ψ_{D} depends on γ, i.e. under linear coordinate transformation T components of ψ_{D} transform through ψ_{D} and γ^l, but not only through ψ_{D}. Second, the relation (46) is compatible with Eq.(41) only under transformations T between orthogonal coordinate systems, when components $g^{lk} = \{c^{-2}, -1\}$ of the metric tensor are invariant. In other words, at the second approach the relation (41) is not covariant with respect to arbitrary linear transformations of coordinates. In this case the symmetry group of the dynamic system does not coincide in general with the symmetry group of absolute objects. Due to the two defects of the conventional (second) approach an investigation becomes rather difficult. Within the second approach it is rather difficult to discover, where "the vector w^l is hidden". Within the first approach, when γ^l transforms only through γ^l, and ψ_{D} transforms only through ψ_{D}, it is clear that the vector w_l arises from the vector matrices γ^l.

Thus, for describing $\mathcal{S}_{\mathrm{D}}$ one may use a constant vector w_l and a scalar field ψ instead of the spinor ψ_{D}. The fact that ψ_{D} defined by the relations (19), (29) is a spinor follows directly from the fact that ψ is a scalar field and w_l is a constant vector. So defined ψ_{D} is a spinor independently of whether or not ψ satisfies the KG equation (6). But the fact that ψ_{D} defined by Eqs.(19), (29) satisfies the Dirac equation (15) is a corollary of the fact that ψ satisfies Eq.(6).

Let us note that Eqs. (29), (19) can be written in the form which does not depend on reperesentation of γ-matrices, defined by Eq.(41). Eq.(29) can be written in the form

$$\psi_{\mathrm{D}} = [\Pi_-(\sqrt{w_+} + i\lambda\sqrt{w_-}\partial_+) + \Pi_+(\sqrt{w_-} + i\lambda\sqrt{w_+}\partial_-)]\begin{pmatrix} \psi \\ \psi \end{pmatrix} \tag{47}$$

$$\Pi_{\pm} = \frac{1}{2}(1 \pm c\gamma^0\gamma^1) \tag{48}$$

Thus, the spinor field in the two-dimensional space-time can be considered as a combination of a constant timelike (or null) vector and a scalar field that associates with the Kramers transformation [8].

Dynamic systems $\mathcal{S}_{\mathrm{D}}$ and $\mathcal{S}_{\mathrm{KG}}$ differ only by the definition of the current j^l and the energy-momentum tensor T_k^l which are sources of the electromagnetic field and the gravitational field respectively. Both dynamic systems can be described either by the spinor field $\psi_{\rm D}$, or by the scalar field ψ. If one describes $\mathcal{S}_{\rm D}$ in terms of the scalar field ψ, the choice of the constant vector w_l is arbitrary that leads to some arbitrariness of expressions for j^l and T_k^l in terms of ψ. Description of $\mathcal{S}_{\rm KG}$ in terms of the spinor field $\psi_{\rm D}$ is formally possible also. But it contains too many absolute objects: g_{kl}, γ^l, and w_l.

Let us consider the case, when $\psi = ae^{\kappa}$, a =const is a real wave function. Then $j^k \equiv 0, \quad k = 0,1$, and the current vanishes in $\mathcal{S}_{\rm KG}$. In this case according to Eqs.(19), (33)-(35) the dynamic system $\mathcal{S}_{\rm D}$ is described by the following quantities:

$$j_{\rm D}^l = 2ae^{2\kappa}[w^l + \lambda^2(2\kappa^l\kappa_s w^s - w^l\kappa_s\kappa^s)]{\rm sgn}(w_0) \tag{49}$$

$$T_{{\rm D}k}^l = 0, \qquad l,k = 0,1 \tag{50}$$

Thus, the real ψ describes a vanishing current in the dynamic system $\mathcal{S}_{\rm KG}$ and a vanishing energy-momentum tensor in the dynamic system $\mathcal{S}_{\rm D}$. It means essentially that one interchanges roles of the current and energy-momentum in dynamic systems $\mathcal{S}_{\rm KG}$ and $\mathcal{S}_{\rm D}$. Indeed, in $\mathcal{S}_{\rm KG}$ the energy density $T_0^0 \geq 0$, but the particle density j^0 can be both positive and negative. Vice versa, in $\mathcal{S}_{\rm D}$ the particle density $j_{\rm D}^0 \geq 0$, but the energy density can be both positive and negative. The situation, when the energy density is non-negative, but the particle density can be negative, seems more preferable from physical viewpoint, than the situation, when the particle density is non-negative, but the energy density is negative. The negative j^0 can be interpreted as a density of antiparticles, but it is very difficult to interpret the states with negative energy of free particles. Such states should be only removed. (A possibility of the second quantization is not considered, because it uses the quantum axiomatics essentially, whereas here only dynamic properties are considered).

Thus, the dynamic system $\mathcal{S}_{\rm D}$ has two defects: (1) states with a negative energy, (2) incompatibility of the definition of the current and the energy-momentum tensor with the relativity principle. The KG dynamic system $\mathcal{S}_{\rm KG}$ seems to be preferable, than $\mathcal{S}_{\rm D}$, provided one does not appeal to the quantum axiomatics.

To investigate the dynamic systems $\mathcal{S}_{\rm KG}$ and $\mathcal{S}_{\rm D}$ in the non-relativistic approximation, let us make the transformation

$$\psi = m^{-1/2}\exp(-i\hbar^{-1}mc^2t)\Psi \tag{51}$$

Then the action (5) turns to the action

$$\mathcal{A}_{\rm KG}[\Psi,\Psi^*] = \int\{\frac{i\hbar}{2}(\Psi^*\partial_0\Psi - \partial_0\Psi^*\Psi) - \frac{\hbar^2}{2m}\partial_1\Psi^*\partial_1\Psi + \frac{\hbar^2}{2m}\partial_0\Psi^*\partial_0\Psi\}d^2x \tag{52}$$

If characteristic frequencies ω of the wave function Ψ are small with respect to $mc^2/\hbar$, the last term in the action (52) is small with respect to others. Formally it is of the order of c^{-2}. It can be neglected. Then one obtains the action for the non-relativistic Schrödinger equation.

Substituting Eq.(51) into Eqs.(8), (9), one obtains the following expressions for the current j^k and the energy-momentum T_l^k

$$j^0 = \Psi^*\Psi + \frac{i\hbar}{2mc^2}(\Psi^*\partial_0\Psi - \partial_0\Psi^*\Psi) \tag{53}$$

$$j^1 = -\frac{i\hbar}{2m}(\Psi^*\partial_1\Psi - \partial_1\Psi^*\Psi) \tag{54}$$

$$T_0^0 = mc^2\Psi^*\Psi + \frac{\hbar^2}{m}\Psi_1^*\Psi_1 + \frac{\hbar^2}{4m}\partial_k\partial^k(\Psi^*\Psi) \tag{55}$$

$$T_1^0 = mj_1 + m\frac{\lambda^2}{2}(\Psi_1^*\Psi_0 + \Psi_0^*\Psi_1) \tag{56}$$

$$T_0^1 = -mc^2 j_1 + mc^2\frac{\lambda^2}{2}(\Psi_1^*\Psi_0 + \Psi_0^*\Psi_1) \tag{57}$$

$$T_1^1 = -\frac{\hbar^2}{m}\Psi_1^*\Psi_1 - \frac{\hbar^2}{4m}\partial_k\partial^k(\Psi^*\Psi) \tag{58}$$

where one uses for brevity designations $\Psi_k \equiv \partial_k\Psi, \quad k = 0, 1$. All relations (52)-(58) are exact relativistic expressions. The last terms of Eqs.(52), (53), (55), (56), (58) have either the order $O(c^{-2})$, or have the form of a divergence. They can be neglected in the non-relativistic approximation. Eq.(56) can be written in the approximate form

$$T_0^0 = mc^2 j^0 + \frac{\hbar^2}{2m}\Psi_1^*\Psi_1 + O(c^{-2}), \tag{59}$$

where j^0 is determined by Eq.(53) to within c^{-2}.

Now let us calculate in terms of Ψ the current and the energy-momentum tensor of the system $\mathcal{S}_{\mathrm{D}}$. One obtains after calculations

$$j_{\mathrm{D}}^0 = j^0 + \frac{i\hbar}{2mc}(\Psi^*\partial_1\Psi - \partial_1\Psi^*\Psi) + O(c^{-2}) \tag{60}$$

$$j_{\mathrm{D}}^1 = j^1 - \frac{i\hbar}{2mc}(\Psi^*\partial_0\Psi - \partial_0\Psi^*\Psi) + O(c^{-2}) \tag{61}$$

where j^k is determined by relations (53), (54). The difference between j_{D} and j is of the order c^{-1} and vanishes in the non-relativistic approximation, when $c \to \infty$.

References

[1] Yu.A. Rylov, *Found. Phys.* **28**, 245, (1998).

[2] Neumann, J. *Mathematische Grundlagen der Quantenmechanik*, (Springer, Berlin, 1932) chp. 3.

[3] Foldy L.L. and Wouthuysen, S.A. *Phys. Rev.*, **78**, 29, (1950).

[4] Anderson, J.L. *Principles of Relativity Physics*, Academic Press. New York, 1967, p. 83.

[5] Rylov, Yu.A. *Advances Appl. Cliff. Algebras*, **5**, 1, 1995.

[6] Sauter, F. *Zs. Phys.*, **63**, 803, (1930); **64**, 295, (1930).

[7] Sommerfeld A., *Atombau and Spektrallinien*, v. 2, (Friedr. Vieweg und Sohn, Braunschweig, 1951) chp. 4, sec. 6.

[8] Kramers, H.A. *Quantum mechanics*, (Noth-Holland Publishing Company, Amsterdam, 1957), p.270.