# Logical reloading as overcoming of crisis in geometry

Yuri A.Rylov

Institute for Problems in Mechanics, Russian Academy of Sciences,
101-1, Vernadskii Ave., Moscow, 119526, Russia.
e-mail: rylov@ipmnet.ru
Web site: $http : //rsfq1.physics.sunysb.edu/\tilde{}rylov/yrylov.htm$
or mirror Web site:
$http : //gasdyn-ipm.ipmnet.ru/\tilde{}rylov/yrylov.htm$

**Abstract**

Properties of the logical reloading in the Euclidean geometry are considered. The logical reloading is a logical operation which replaces one system of basic concepts of a conception by another system of basic concepts of the same conception. The logical reloading does not change propositions of the conception. However, generalizations of the conception are different for different systems of basic concepts. It is conditioned by the fact, that some systems of basic concepts contain not only propositions of the conception, but also some attributes of this conception description. Properties of the logical reloading are demonstrated in the example of the proper Euclidean geometry, whose generalization leads to different results for different system of basic concepts.

## 1 Introduction

Geometry studies the shape and mutual disposition of physical bodies, abstracting from other their properties. After such an abstraction the physical body turns in a geometrical object, i.e. in some subset of points of the space. Geometry is a science on a shape and on disposition of geometrical objects in a space or in a space-time. The space is a set of points. A geometrical object is a subset of points of the space.

The property of a geometry to be a science on mutual disposition of geometrical objects in the space or in the space-time will be called "geometricity". This special term is neccessary, because the contemporary geometry does not possess the property of "geometricity", in general. In other words, the contemporary geometry is not always is a science on mutual disposition of geometrical objects. Contemporary mathematics considers a geometry simply as a logical construction. For instance, the symplectic geometry is not a science on mutual dispositions of geometric objects. It is a logical construction, whose form reminds the form of the Euclidean geometry. In applications of geometry to physics and to mechanics only the geometricity of the space-time geometry is important. It is of no importance, whether or not the geometry is a logical construction. If the real space-time geometry is nonaxiomatizable, it means, that it is not a logical construction. However, such a geometry may not possess the property of geometricity.

Contemporary mathematicians do not recognize nonaxiomatizable geometries, which have the property of geometicity, but which are not a logical construction. This situation should be qualified as a crisis in geometry [1], which reminds the crisis, when mathematicians did not recognize non-Euclidean geometries of Lobachevski - Bolyai.

Aforetime the geometry studied disposition of geometrical objects in usual space. The time was considered as an additional characteristic of the physical bodies description. After creation of the relativity theory the space and the time are considered as a united event space (or space-time). It is a more general approach to description of the event space. Any point of the event space is an event, which occurs at some place and at some time.

The geometry is described completely, if the distance $\rho$ between any pair of points belonging to the space is given. The set $\Omega$ of points with a distance $\rho$, given on the set $\Omega$, is known as a metric space $\mathcal{M}$.

A use of the metric space in the physics and mechanics meets some problems. These problems lie in the definition of geometric objects in the metric space $\mathcal{M}$. The distance $\rho$ is supposed to satisfy the relations

$$\rho : \Omega \times \Omega \rightarrow [0, \infty), \qquad \rho(Q, P) = \rho(P, Q), \qquad \forall P, Q \in \Omega \tag{1.1}$$

$$\rho(P, Q) = 0, \qquad \text{iff} \quad P = Q \tag{1.2}$$

$$\rho(P, Q) + \rho(P, R) \geq \rho(R, Q), \qquad \forall P, Q, R \in \Omega \tag{1.3}$$

In the Euclidean geometry the distance has properties (1.1) - (1.3).

In the geometry of Minkowski the distance does not possess these properties. However, it would be very desirable to introduce a metric geometry (or some analog of metric geometry) for description of the space-time properties, because the metric geometry is free of such auxiliary concepts as coordinate system, dimension and such restriction as continuity. Metric geometry describes the geometric properties in terms of only distance, which is a true geometric concept.

After removal of the triangle axiom (1.3) the distance geometry arises [2]. Blumental failed to construct a straight line in terms of only distance. He was forced to introduce a straight as a continuous mapping of interval $(0, 1)$ onto the space (a point set). Such an introduction of nonmetric concept of mapping in geometry seems to be undesirable, because an auxiliary concept is introduced and the distance geometry ceases to be a pure metric geometry.

In general, a construction of geometrical objects is the main problem of the metric geometry. One can easily construct sphere and ellipsoid, because in the Euclidean geometry these geometrical objects are constructed directly in terms of distance. However, construction of other geometrical objects needs a use of some auxiliary means. For instance, a definition of a plane contains a reference to concept of linear independence of vectors. It is not quite clear, how to introduce this concept in terms of a distance.

A sphere $Sp_{O,P}$ with the center at the point $O$ and the point $P$ on the surface of the sphere is defined as a set of points $R$

$$Sp_{O,P} = \{R|\rho(O,R) = \rho(O,P)\} \tag{1.4}$$

An ellipsoid $El_{F_1F_2P}$ with focuses at the points $F_1, F_2$ and a point $P$ on the surface of the ellipsoid is defined as a set of points $R$

$$El_{F_1F_2P} = \{R|\rho(F_1,R) + \rho(F_2,R) = \rho(F_1,P) + \rho(F_2,P)\} \tag{1.5}$$

If the point $P$ on the surface of ellipsoid coincides with the focus $F_2$, the ellipsoid $El_{F_1F_2P}$ degenerates into segment $\mathcal{T}_{[F_1F_2]}$ of a straight line.

$$\mathcal{T}_{[F_1F_2]} = El_{F_1F_2F_2} = \{R|\rho(F_1,R) + \rho(F_2,R) = \rho(F_1,F_2)\} \tag{1.6}$$

In the proper Euclidean geometry the segment $\mathcal{T}_{[F_1F_2]}$ has no thickness (it is one-dimensional). However, if the triangle axiom (1.3) is not satisfied, the set $\mathcal{T}_{[F_1F_2]}$ is a non-one-dimensional surface.

Criterion of one-dimensionality may be formulated in terms of distance. The section $\mathcal{S}\left(P,\mathcal{T}_{[F_1F_2]}\right)$ is defined as a set of points $R$

$$\mathcal{S}\left(P,\mathcal{T}_{[F_1F_2]}\right) = \{R|\rho(F_1,R) = \rho(F_1,P) \wedge \rho(F_2,R) = \rho(F_2,P)\}, \qquad P \in \mathcal{T}_{[F_1F_2]} \tag{1.7}$$

The point $P \in \mathcal{S}\left(P,\mathcal{T}_{[F_1F_2]}\right)$ in evident way. By definition the segment (1.6) is one-dimensional (has no thickness), if any section of $\mathcal{T}_{[F_1F_2]}$ consists of one point

$$\mathcal{S}\left(P,\mathcal{T}_{[F_1F_2]}\right) = \{P\}, \qquad \forall P \in \mathcal{T}_{[F_1F_2]} \tag{1.8}$$

Let $\mathcal{S}\left(P,\mathcal{T}_{[F_1F_2]}\right)$ be a section of the segment $\mathcal{T}_{[F_1F_2]}$ at the point $P \in \mathcal{T}_{[F_1F_2]}$. One can show that the relation (1.8) takes place, if the distance satisfies the triangle axiom (1.3). In this case the segment $\mathcal{T}_{[F_1F_2]}$ of the straight line can be defined as a line of the shortest length, and this definition appears to be equivalent to definition (1.6).

In other words, if the straight has no thickness, one may use both definitions of the straight segment $\mathcal{T}_{[F_1F_2]}$: (1) the straight line is a shortest line and (2) definition (1.6). The two definitions are equivalent. However, the definition (1.6) may be used in the case, when the triangle axiom does not take place, whereas the first definition becomes to be incorrect.

We have the following problem: "Do such geometries exist, where the triangle axiom is not satisfied?" Of course, this question is interesting only in application to the real space-time geometry. It is of no interest for mathematicians, which may investigate some special class of geometries (satisfying the triangle axiom), remaining investigation of a more general geometries for later. Application of metric geometry to the space–time geometry needs also a refuse from the condition (1.2), which may not be used in geometries with indefinite metric, for instance, in geometry of Minkowski.

In the Riemannian space-time geometry we have instead of (1.1) - (1.3)

$$\sigma : \Omega \times \Omega \rightarrow \mathbb{R}, \qquad \sigma(P,P) = 0 \qquad \sigma(Q,P) = \sigma(P,Q), \qquad \forall P,Q \in \Omega \tag{1.9}$$

$$\sqrt{2\sigma(P,Q)} + \sqrt{2\sigma(P,R)} \leq \sqrt{2\sigma(R,Q)}, \qquad \forall P,Q,R \in \Omega \wedge \sigma(R,Q) > 0 \tag{1.10}$$

where $\sigma(P,Q)$ is the world function, connected with the distance $\rho(P,Q)$ by means of relation

$$\sigma(P,Q) = \frac{1}{2}\rho^2(P,Q) \tag{1.11}$$

In the space-time geometry the world function is always real, and the distance is positive for timelike interval ($\sigma(P,Q) > 0$), and it is imaginary for spacelike interval ($\sigma(P,Q) < 0$). In the Riemannian space-time geometry the conditions (1.9), (1.10) are fulfilled. Condition (1.10) for spacelike distances ($\sigma(P,Q) < 0$) is not fulfilled. It is not important, because spacelike world lines are not used in contemporary physics.

As soon as the mathematical technique of working with the world function has been developed [3, 4, 5], the question arises: "If the condition (1.10) is not fulfilled, is the space-time geometry non-Riemannian, or is there no geometry at all?" It was a very important question. On one hand, the metric geometry was insensitive to continuity, or discreteness. It was insensitive also to dimension of the space-time and to a choice of a coordinate system. On the other hand, if such non-Riemannian geometries exist, the habitual axiom of Euclidean (and Riemannian) geometry (the straight has no thickness) is violated.

We shall not use the term metric geometry with respect to geometry (1.9), because the term "metric geometry" is associated with the triangle axiom imposed on the metric (distance). We shall use the term "physical geometry" with respect to geometry, which is completely described by the world function $\sigma$, satisfying the condition (1.9). Another (more earlier) name of this geometry is "tubular geometry" (T-geometry), which arose because in T-geometry some straights are substituted by tubes. There exist such space-time isotropic T-geometries, where timelike tubes degenerate into one-dimensional straights. For instance, in the space-time geometry of Minkowski with world function

$$\sigma_{\mathrm{M}}(x,x') = \frac{1}{2}g_{ik}\left(x^i - x'^i\right)\left(x^k - x'^k\right), \qquad g_{ik} = \mathrm{diag}\left\{c^2, -1, -1, -1\right\} \tag{1.12}$$

the timelike straights are one-dimensional (have no thickness), and motion of free particles is deterministic.

However, a small deformation (change of $\sigma_{\mathrm{M}}$) of the space-time of Minkowski transforms timelike straight lines into tubes, and motion of free particles becomes stochastic. If the space-time deformation depends on the quantum constant in a proper way, the statistical description of stochastically moving free particles is equivalent to the quantum description in terms of the Schrödinger equation [6]. I have obtained this result only twenty five years ago after the question on non-Riemannian geometry had appeared.

This fact is connected with the logical reloading, which is a logical operation. This logical operation realizes a transition from one system of basic statements of a conception to another system of basic statements of the same conception. As a logical operation the logical reloading is essential only at a generalization of the existing conception. Such a generalizations are rather rear. As far as the logical reloading is used rather rear, and researchers possess this logical operation rather slightly.

Euclid has created his geometry as a logical construction. The Euclidean geometry has been taught in such a form for two thousands years. As a result almost all researchers believe, that a geometry is a logical construction. But what is connection between space properties and logic? Is a geometry a logical construction with necessity? Is the formal logic a necessary attribute of geometry? Why do mathematicians not recognize nonaxiomatizable geometries, which do not use the formal logic [1]? This paper is written to answer these questions.

# 2 Construction of geometrical objects in Euclidean geometry

Euclid investigates properties of the space, constructing geometrical objects. Investigation of the geometrical objects properties meant an investigation of properties of the space, because one may study geometry, only via properties of geometrical objects, placed in this space. In other words, investigation of a set of points is an investigation of properties of subsets of this point set. Geometry as a logical construction is a formalization of the process of the geometrical objects construction.

Euclid constructed geometrical objects from blocks. He used three sorts of blocks: (1) point, (2) segment of straight, (3) angle. Combining these blocks, Euclid constructed geometrical objects and investigated their properties. Constructing geometrical objects, Euclid used some rules. Some part of rules described properties of blocks, another part described process of combining blocks at construction of geometrical objects. The number of rules is finite, because the number of block sorts is finite. The Euclidean space, which had been investigated by Euclid, is uniform and isotropic. Blocks are not deformed at displacements, and the number of rules, describing displacement of blocks, is also finite. The rules of working with blocks admit one to construct geometrical objects from blocks mentally. Besides, these rules generate the rules of the complicate geometric object construction from other simpler geometrical objects. The simple geometric objects, constructed from blocks, are described by the rules known as axioms. More complicated rules (theorems) of the geometric objects construction from other geometrical objects are deduced from axioms by rules of formal logic. As a whole such a construction of geometric objects is perceived as a logical construction, where rules of the formal logic reflects rules of construction of geometric objects.

Usually one abstracts from the fact, that the logical construction of a geometry is a formalization of real construction of geometrical objects from blocks. The Euclidean geometry is presented directly as a logical construction. The connection between the geometry and the logical construction is considered to be so strong, that sometimes a logical construction, which is not connected with the space properties, is considered as some kind of a geometry. For instance, the symplectic geometry is treated as a kind of a geometry, although it is not describe space properties. It has only the form of Euclidean geometry with antisymmetric matrix of metric tensor.

The proper Euclidean geometry is uniform and isotropic. The blocks can be easily displaced without their deformation, and they are used for construction of geometrical objects. In the uniform geometry one can use a finite number of the block sorts. The corresponding logical construction contains finite number of axioms. All propositions of a uniform geometry can be deduced from the finite number of axioms, and this geometry may be qualified as an axiomatizable geometry.

If a geometry $\mathcal{G}$ is not uniform, one cannot use a finite number of the block sorts, because the blocks are deformed at a displacement. As a result two similar geometrical objects, constructed in the same way in different places of the space, will have different properties. We are forced to use infinite number of block sorts. The number of axioms will be infinite. Such a geometry should be qualified as nonaxiomatizable geometry.

One should use another way of construction of inhomogeneous geometries. The inhomogeneous geometry $\mathcal{G}$ is considered as a result of a deformation of some standard geometry $\mathcal{G}_{\mathrm{st}}$. The geometry $\mathcal{G}_{\mathrm{st}}$ is axiomatizable, and geometrical objects in $\mathcal{G}_{\mathrm{st}}$ are constructed from blocks. The standard geometry $\mathcal{G}_{\mathrm{st}}$ is supposed to be described completely by the world function $\sigma_{\mathrm{st}}$. In means that all propositions of $\mathcal{G}_{\mathrm{st}}$ can be expressed in terms of $\sigma_{\mathrm{st}}$. In particular, all geometrical objects in $\mathcal{G}_{\mathrm{st}}$ can be described in terms of $\sigma_{\mathrm{st}}$.

The inhomogeneous geometry $\mathcal{G}$ and all geometrical objects in $\mathcal{G}$ are constructed as result of a deformation of the standard geometry $\mathcal{G}_{\mathrm{st}}$. It means that in all descriptions of geometrical objects in $\mathcal{G}_{\mathrm{st}}$ the world function $\sigma_{\mathrm{st}}$ is substituted by the world function $\sigma$ of the geometry $\mathcal{G}$. As a result one obtains descriptions of all geometrical objects in $\mathcal{G}$ and, hence, one obtains a description of the geometry $\mathcal{G}$ in terms of the world function $\sigma$.

The method of the geometry $\mathcal{G}$ construction by means of a deformation of the standard geometry $\mathcal{G}_{\mathrm{st}}$ is called the deformation principle [7]. The proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ may be used as a standard geometry, because $\mathcal{G}_{\mathrm{E}}$ is a axiomatizable physical geometry. The term "physical geometry" means by definition, that the geometry may be described completely in terms of the world function $\sigma_{\mathrm{E}}$.

Constructing generalized geometries, one adopts conventionally from Euclid his method of the geometry construction. However, this method can be used only for construction of axiomatizable geometries. It does not work at construction of physical geometries, which are nonaxiomatizable, in general. We shall use the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ as the standard geometry $\mathcal{G}_{\mathrm{st}}$, and it means, that we adopt from Euclid his geometry, but not his method of the geometry construction.

At construction of generalized geometries it is important to compare geometrical objects in different geometries. In particular, one should be able to recognize similar objects in different geometries. The geometrical objects in different physical geometries are considered to be similar, if they are defined similar, i.e. if they have the same form in terms of the world functions. For instance, the segment $\mathcal{T}_{[P_0P_1]}$ is defined by the relation (1.6) in terms of distance. In terms of the world function it has the form

$$\mathcal{T}_{[P_0P_1]} = \left\{ R | \sqrt{2\sigma\left(P_0, R\right)} + \sqrt{2\sigma\left(R, P_1\right)} = \sqrt{2\sigma\left(P_0, P_1\right)} \right\} \tag{2.1}$$

The same form of definition of the segment $\mathcal{T}_{[P_0P_1]}$ has in all physical geometries. However, it does not mean, that properties of the segment $\mathcal{T}_{[P_0P_1]}$ are the same in all physical geometries, because properties of the world function are different in different physical geometries. For instance, in the geometry of Minkowski the timelike segment $\mathcal{T}_{[P_0P_1]}$ $(\sigma_{\mathrm{M}}\left(P_0, P_1\right) > 0)$ is one-dimensional, i.e. any its section, defined by the relation (1.7) consists of one point

$$\mathcal{S}\left(P, \mathcal{T}_{[P_1P_2]}\right) = \left\{P\right\}, \qquad \forall P \in \mathcal{T}_{[P_1P_2]} \tag{2.2}$$

In the deformed geometry of Minkowski $\sigma_{\mathrm{d}}$, described by the world function

$$\sigma_{\mathrm{d}} = \sigma_{\mathrm{M}} + d \cdot \mathrm{sgn}\left(\sigma_{\mathrm{M}}\right), \qquad d = \frac{\hbar}{2bc} = \mathrm{const} \tag{2.3}$$

the same timelike segment $\mathcal{T}_{[P_1P_2]}$ is a three-dimensional surface (tube). In the relation (2.3) $\sigma_{\mathrm{M}}$ is the world function of the Minkowski space-time, $\hbar$ is the quantum constant, $c$ is the speed of the light, and $b$ is some universal constant. At proper choice of $b$, the world function (2.3) describes the properties of the space-time in microcosm more effective, than the world function $\sigma_{\mathrm{M}}$. In the space-time geometry $\mathcal{G}_{\mathrm{d}}$, described by the world function $\sigma_{\mathrm{d}}$, world lines of free microparticles appear to be stochastic. Statistical description of these stochastic world lines is equivalent to quantum description in terms of the Schrödinger equation [6]. In other words, a use of the space-time geometry (2.3) instead of the geometry of Minkowski admits one to remove quantum principles.

Thus, the proper choice of the space-time geometry admits one to explain quantum effects as geometrical effects. In this explanation one uses only principles of classical dynamics. Quantum principles are not introduced, or they are obtained as corollaries of the physical geometry of the space-time. At such an approach the number of physical principles reduces. When the number of basic principles reduces, the physical theory becomes more perfect.

At axiomatic approach to geometry the properties of geometrical objects are obtained in the form of theorems, deduced from axiomatics. In physical geometries the properties of geometrical objects are obtained only after taking into account properties of the world function.

# 3 Logical reloading in the proper Euclidean geometry

There are three different equivalent representations of the proper Euclidean geometry [8]: (1) Euclidean representation (E-representation), (2) vector representation (V-representation) and (3) representation in terms of world function ($\sigma$-representation). Transformation from one representation to another one is a logical reloading, when basic concepts of the representation are changed. The $E$-reprsentation uses three blocks (point, segment, angle) for construction of geometric objects.

The $V$-representation uses two blocks (point and directed segment, or vector). Instead of angle one uses additional structure (linear vector space), which accomplishes function of the angle, describing direction of vectors. The angle is constructed from to segments, having a common point. If one formulates the rules of constructing an angle from two segments, one may reduce the number of sorts of blocks, remaining only point and segment. The block "angle" is replaced by the rule of its construction. As a result one obtains two blocks (point and segment) and some additional structure, which admits one to construct the angles from two segments. This additional structure is known as the linear vector space. Reduction of basic elements (blocks) is a logical reloading, which can be introduced in the proper Euclidean geometry. This logical reloading from $E$-representation to $V$-representation was conditioned by application of Euclidean geometry to physics and mechanics, where conception of a vector and coordinate system were used. Concept of the angle was not so essential, because it directivity of a vector may be described by scalar products of a vector with basic vectors of the coordinate system. A vector (directed segment) and coordinate system are attributes of the linear vector space, which is an auxiliary structure in $V$-representation.

Vector representation is based on the concept of linear vector space, which contains such concepts as, continuity, dimension, coordinate system, linear independence. These concepts are necessary for application in physics and mechanics, where they are used for description of the particle motion and evolution of the force fields. These concepts are used in $V$-representation as basic concepts or as properties of the linear vector space. Conventionally the linear vector space is considered as an attribute of the Euclidean geometry (but not as an attribute of Euclidean geometry description).

The $\sigma$-representation contains only one block (point). As far as a segment can be constructed of points, it is possible to reduce the number of block sorts remaining only one sort (point). Elimination of a segment (vector) is accompanied by the rules of the segment construction from the points. This reduction of the block sorts leads to the logical reloading (transition from $V$-representation to $\sigma$-representation). This transition to $\sigma$-representation is accompanied by introduction of a new structure (world function $\sigma$), which contains the rules of the segment construction from points (2.1). The world function describes a connection of two points of the space. It admits one to construct all attributes of the linear vector space in terms of the world function. World function admits one to obtain all concepts of the $V$-representation (dimension, coordinate system, metric tensor, linear vector space). Of course, construction of all attributes of the $V$-representation is possible, provided the world function is the world function of the Euclidean space. This world function satisfies some conditions, which appear to be rather strong.

**Definition 1** *Vector* $\mathbf{P}_0\mathbf{P}_1 = \overrightarrow{P_0P_1}$ *is an ordered set of two points* $P_0, P_1$. *The point* $P_0$ *is the origin of the vector and the point* $P_1$ *is its end. The length of vector* $\mathbf{P}_0\mathbf{P}_1$ *is*

$$|\mathbf{P}_0\mathbf{P}_1| = \sqrt{2\sigma (P_0, P_1)} \tag{3.1}$$

The crucial point of the $\sigma$-representation is the definition of scalar product in terms of the world function.

**Definition 2** *The scalar product* $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ *of two vectors* $\mathbf{P}_0\mathbf{P}_1$ *and* $\mathbf{Q}_0\mathbf{Q}_1$ *has the form.*

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \sigma (P_0, Q_1) + \sigma (P_1, Q_0) - \sigma (P_0, Q_0) - \sigma (P_1, Q_1) \tag{3.2}$$

If origin of $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is the same $Q_0 = P_0$, the relation (3.2) takes the form

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{Q}_1) = \sigma (P_0, Q_1) + \sigma (P_1, P_0) - \sigma (P_1, Q_1) \tag{3.3}$$

Together with the relation (3.1) the relation (3.3) realizes a formulation of the cosine theorem. In relations (3.1) - (3.3) the world function $\sigma$ is the world function of the Euclidean geometry.

The necessary and sufficient condition of linear dependence of $n$ vectors ,$\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0\mathbf{P}_2$,...$\mathbf{P}_0\mathbf{P}_n$, defined by $n+1$ points $\mathcal{P}^n \equiv \{P_0, P_1, ..., P_n\}$ in the proper Euclidean space, is a vanishing of the Gram's determinant

$$F_n (\mathcal{P}^n) \equiv \det ||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)|| , \qquad i, k = 1, 2, ...n \tag{3.4}$$

Expressing the scalar products $(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)$ in (3.4) via world function $\sigma_{\mathrm{E}}$ by means of relation (3.3), we obtain definition of linear dependence of $n$ vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0\mathbf{P}_2$,...$\mathbf{P}_0\mathbf{P}_n$ in the proper Euclidean space in the form

$$F_n (\mathcal{P}^n) = 0 \tag{3.5}$$

$$F_n (\mathcal{P}^n) \equiv \det ||\sigma (P_0, P_i) + \sigma (P_0, P_k) - \sigma (P_i, P_k)|| , \qquad i, k = 1, 2, ...n \tag{3.6}$$

The necessary and sufficient conditions of the fact, that a physical geometry, described by the world function $\sigma$, is $n$-dimensional proper Euclidean geometry, have the form of four conditions.

I. Definition of the dimension of the geometry:

$$\exists \mathcal{P}^n \equiv \{P_0, P_1, ...P_n\} \subset \Omega, \qquad F_n (\mathcal{P}^n) \neq 0, \qquad F_k \left(\Omega^{k+1}\right) = 0, \qquad k > n \tag{3.7}$$

where $F_n(\mathcal{P}^n)$ is the Gram's determinant (3.6). Vectors $\mathbf{P}_0\mathbf{P}_i$, $i=1,2,...n$ are basic vectors of the rectilinear coordinate system $K_n$ with the origin at the point $P_0$. The metric tensors $g_{ik}(\mathcal{P}^n)$, $g^{ik}(\mathcal{P}^n)$, $i,k=1,2,...n$ in $K_n$ are defined by the relations

$$\sum_{k=1}^{k=n} g^{ik}(\mathcal{P}^n) g_{lk}(\mathcal{P}^n) = \delta_l^i, \qquad g_{il}(\mathcal{P}^n) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_l), \qquad i,l=1,2,...n \tag{3.8}$$

$$F_n(\mathcal{P}^n) = \det||g_{ik}(\mathcal{P}^n)|| \neq 0, \qquad i,k=1,2,...n \tag{3.9}$$

II. Linear structure of the Euclidean space:

$$\sigma(P,Q) = \frac{1}{2}\sum_{i,k=1}^{i,k=n} g^{ik}(\mathcal{P}^n)(x_i(P)-x_i(Q))(x_k(P)-x_k(Q)), \qquad \forall P,Q\in\Omega \tag{3.10}$$

where coordinates $x_i(P)$, $x_i(Q)$, $i=1,2,...n$ of the points $P$ and $Q$ are covariant coordinates of the vectors $\mathbf{P}_0\mathbf{P}$, $\mathbf{P}_0\mathbf{Q}$ respectively, defined by the relation

$$x_i(P) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}), \qquad i=1,2,...n \tag{3.11}$$

III: The metric tensor matrix $g_{lk}(\mathcal{P}^n)$ has only positive eigenvalues

$$g_k > 0, \qquad k=1,2,...,n \tag{3.12}$$

IV. The continuity condition: the system of equations

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}) = y_i \in \mathbb{R}, \qquad i=1,2,...n \tag{3.13}$$

considered to be equations for determination of the point $P$ as a function of coordinates $y=\{y_i\}$, $i=1,2,...n$ has always one and only one solution. Conditions I – IV contain a reference to the dimension $n$ of the Euclidean space.

Generalization of the Euclidean geometry in $V$-representation admits one to consider geometries with indefinite metric tensor (geometry of Minkowski), or geometry with metric tensor, which is different at different points of the space. However, in the $V$-representation one can consider only geometries, which have some dimension, and this dimension is the same at all points of the space. Besides, in $V$-representation one cannot distinguish dimension as the number of coordinates, describing manifold from the dimension as the number of linear independent vectors, although these concepts are different, in general.

In the $\sigma$-representation one can consider geometries, having no dimension, or having dimensions, which are different at different points of the space. This difference between representations arises, because in $V$-representation the dimension of space is considered as a primary property of a geometry, whereas in $\sigma$-representation the dimension is only a secondary property of a geometry (something like an attribute of the geometry description in $V$-representation). It is a secondary concept, determined by the form of the world function.

Logical reloading, transforming $V$-representation of the Euclidean geometry into the $\sigma$-representation, is very important from the point of view of possible generalization of the Euclidean geometry. At a generalization of Euclidean geometry in the $V$-representation the most restrictive properties (3.7) and (3.10) of the Euclidean geometry are to be conserved, because they are properties of the linear vector space, which is the main structure of the $V$-representation.

At a generalization of Euclidean geometry in $\sigma$-representation the linear vector space is not used, in general. The equivalence relation becomes intransitive. Summation of vectors, as well as multiplication of a vector by a real number become multivariant. In general, a vector cannot be presented as a sum of its components along the coordinate axes, although projection $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)\,/\,|\mathbf{Q}_0\mathbf{Q}_1|$ of a vector $\mathbf{P}_0\mathbf{P}_1$ onto any non-zero vector $\mathbf{Q}_0\mathbf{Q}_1$ is determined uniquely. Although many properties of vectors in the $\sigma$-representations appear to be multivariant and unaccustomed, these properties are real properties of space-time geometry.

One should know these real properties of the space-time geometry, because in the general relativity the space-time geometry is determined by the matter distribution. One cannot know the space-time geometry previously, and one should consider all possible geometries. In the general relativity one supposes, that the space-time geometry can be only a Riemannian geometry. Thus, the supposition, that the space-time geometry is a Riemannian geometry, is a mistake from the physical viewpoint. Generalization of the general relativity on the case of arbitrary physical space-time geometry shows, that the space-time geometry appears to be non-Riemannian even in the case of slight gravitational field of a heavy sphere [9].

Mathematics does not consider problems of the geometry application to physics and to mechanics. Mathematicians may investigate only a part of possible geometries (for instance, only axiomatizable geometries), and this consideration of a part of all possible geometries is not a mistake from the mathematical viewpoint. However, if mathematician believes, that nonaxiomatizable geometries are impossible and tries to deduce a nonaxiomatizable geometry from some new axiomatics, it becomes to be a mistake. The obtained geometry appears to be inconsistent. By the way, already the Riemannian geometry appears to be inconsistent [1].

# 4 Corollaries of the logical reloading

In application to the Euclidean geometry the logical reloading means a transition from the conventional $V$-representation to $\sigma$-representation. As a result the main structure of the $V$-representation (linear vector space) is replaced by the structure of the $\sigma$-representation (world function). The geometry and all geometrical quantities are defined via the world function $\sigma$, and only via $\sigma$. In particular, vector, which is defined in $V$-representation as an element of the linear vector space, is defined in $\sigma$-representation by relations (3.1) - (3.6). Of course, if conditions of Euclideaness take place, all results, obtained in $V$-representation, coincide with results, obtained in $\sigma$-representation. If conditions (3.7) - (3.10) are not fulfilled, the geometry ceases to be Euclidean, and a linear vector space cannot be introduced, in general. However, results, obtained in $\sigma$-representation have a sense, and they contain many unexpected properties.

The most important property of the physical geometry, generated by a violation of conditions (3.9) and (3.10), is multivariance [10]. Multivariance of a geometry with respect to vector $\mathbf{P}_0\mathbf{P}_1$ and the point $Q_0$ means by definition, that at the point $Q_0$ there are many vectors $\mathbf{Q}_0\mathbf{Q}_1$, $\mathbf{Q}_0\mathbf{Q}_1'$,..., which are equivalent to vector $\mathbf{P}_0\mathbf{P}_1$.

Equivalency ($\mathbf{P}_0\mathbf{P}_1\text{eqv}\mathbf{Q}_0\mathbf{Q}_1$) of vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is defined as follows

$$(\mathbf{P}_0\mathbf{P}_1\text{eqv}\mathbf{Q}_0\mathbf{Q}_1), \quad \text{if} \quad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = |\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1| \wedge |\mathbf{P}_0\mathbf{P}_1| = |\mathbf{Q}_0\mathbf{Q}_1| \quad (4.1)$$

If vector $\mathbf{P}_0\mathbf{P}_1$ is given, and one looks for vector $\mathbf{Q}_0\mathbf{Q}_1$ at the point $Q_0$, which is equivalent to vector $\mathbf{P}_0\mathbf{P}_1$, one needs to solve two equations (4.1) with respect to position of the point $Q_1$. In the proper Euclidean geometry two equations (4.1) have always a unique solution independently of dimension of the Euclidean space. It means that at the point $Q_0$ there is one and only one vector $\mathbf{Q}_0\mathbf{Q}_1$, which is equivalent to vector $\mathbf{P}_0\mathbf{P}_1$. The proper Euclidean geometry is single-variant. In an arbitrary physical geometry $\mathcal{G}$ two equations (4.1) may have many solutions. Then at the point $Q_0$ there are many vectors $\mathbf{Q}_0\mathbf{Q}_1$, which are equivalent to vector $\mathbf{P}_0\mathbf{P}_1$. It means that the geometry $\mathcal{G}$ is multivariant. In this case the equivalence relation is intransitive, and geometry $\mathcal{G}$ is nonaxiomatizable. Thus, nonaxiomatizability of a physical geometry is a corollary of its multivariance.

The multivariance is a natural property of a physical (nonaxiomatizable) geometry. Multivariance is absent, when conditions (3.9) and (3.10) are fulfilled. In axiomatizable geometries the property of multivariance is absent. The axiomatizable geometries have been well studied and considered usually as "true geometries". However, the axiomatizability is not a feature of a "true geometry". The axiomatizability is a property of the most studied geometries. In general, a physical geometry is multivariant, and the multivariance is a natural property of the space-time geometry.

As we have seen in introduction, the timelike straight segments in the real space-time geometry (2.3) are not one-dimensional, and this property is a corollary of the geometry multivariance with respect to timelike vectors.

The multivariance leads to a splitting of geometrical objects. We consider this effect in the example of a circular cylinder. In the proper Euclidean geometry it is defined by its axis and a point $P$ on its surface. Let $F_1$, $F_2$ be two points on the axis of the circular cylinder. The cylinder $Cl_{PF_1F_2}$ is defined as a set of points $R$

$$Cl_{PF_1F_2} = \{R | S_{F_1F_2R} = S_{F_1F_2P}\} \tag{4.2}$$

where $S_{F_1F_2R}$ is the area of the triangle with vertices at the points $F_1, F_2, R$, which is calculated by means of the Heron's formula via side lengths of the triangle. Let the areas $S_{F_1F_2R}$ and $S_{F_1F_2P}$ are expressed via world functions of corresponding points. Let $\mathcal{T}_{[F_1F_2]}$ be the straight segment between points $F_1, F_2$ and the point $F_3 \in \mathcal{T}_{[F_1F_2]}$. Let $F_3 \neq F_1$, then in the proper Euclidean geometry the shape of the circular cylinder depends only on the axis, but not on a choice of points on this axis, and

$$Cl_{PF_1F_2} = Cl_{PF_1F_3}, \qquad F_3 \in \mathcal{T}_{[F_1F_2]} \tag{4.3}$$

However, in the multivariant geometry, in general, $Cl_{PF_1F_2} \neq Cl_{PF_1F_3}$ and in the multivariant geometry there are many cylinders, corresponding to one circular cylinder in the proper Euclidean geometry. From viewpoint of $V$-representation it is interpreted as a splitting of the Euclidean cylinder in a multivariant geometry. From viewpoint of $\sigma$-representation the fact, that shape of cylinders $Cl_{PF_1F_2}$ and $Cl_{PF_1F_3}$ are different, in general, is natural. From this viewpoint the equation (4.3) means a degeneration of cylinders in the Euclidean geometry. Interpretation of (4.3) as a degeneration is a more correct geometrical interpretation, because it does not use such an auxiliary structure as the linear vector space.

# 5 Concluding remarks

Conventional approach to the space-time geometry, when a geometry is considered to be a logical construction is poor. To obtain a true description of the space-time geometry, one needs to realize a logical reloading and transit to perception of a geometry as a science on shape and mutual disposition of geometrical objects. Using this approach and considering the proper Euclidean geometry, one obtains three different representations of the Euclidean geometry. These representations differ in the number of block sorts, using for construction of geometrical objects. Reduction of block sorts is compensated by introduction of an additional structure, which describes the rules of the eliminated block construction. In the Euclidean geometry a transition from one representation to another one may be obtained by means of the formal logic. This circumstance admits one to interpret the logical reloading as a logical operation.

In the inhomogeneous geometries, where blocks are deformed, a transformation from one representation to another one becomes to be impossible, in general. In this case one uses the deformation principle, which works only in $\sigma$-representation. After deformation of the standard (Euclidean) geometry the obtained geometry appears to be nonaxiomatizable, in general.

Logical reloading in the proper Euclidean geometry does not change this geometry. However, capacities of generalization of the proper Euclidean geometry are different in different representations of this geometry. Maximal capacity of generalization appears in the $\sigma$-representation, where the geometry is described completely in terms of world function $\sigma$, which is a function of two points of the space. Generalization of the proper Euclidean geometry in the $\sigma$-representation admits one to construct multivariant geometries, which are nonaxiomatizable. Appearance of multivariant (nonaxiomatizable) geometries shows, that the conventional approach, when a geometry is a logical construction may be used only in some special cases. Logical reloading to $\sigma$-representation restores the old conception of geometry, as a science on a shape and mutual disposition of geometrical objects. In other words, the logical reloading to the $\sigma$-representation is not a new idea. It is a return to old idea of metric geometry, which was not work without a use of the deformation principle. Being equipped by the deformation principle, the metric geometry turns into the physical geometry, which is an excellent tool for description of the space-time.

A use of the physical geometry admits one to describe space-time geometry of microcosm, where the geometry is multivariant and cannot be described in terms of conventional Riemannian geometry. The physical space-time geometry is effective in cosmology, where the space-time geometry is non-Riemannian and cannot be described correctly, if one supposes, that the space-time geometry is Riemannian.

It is interesting, that dynamics of particles is described by finite difference equations (but not differential). Even dynamic equations for gravitational field have the form of finite equations (but not differential). It is connected with the fact, that the physical space-time geometry may be discrete. Differential equations cannot be used effectively in the space-time geometry, which may be discrete in some regions.

# References


[1] Yu. A. Rylov, New crisis in geometry? *e-print /math.GM/0503261*

[2] L.M. Blumenthal, *Theory and Applications of Distance Geometry*, Oxford, Clarendon Press, 1953.

[3] J.L. Synge, *Relativity: The General Theory*, North-Holland, Amsterdam, 1960.

[4] Yu. A. Rylov, Description of Riemannian space by means of a finite interval. *Izvestiya Vysshich Uchebnych Zavedeniyi* Ser. Math. No.3, 132, 1962, (in Russian).

[5] Yu.A. Rylov, Relative gravitational field and conservation laws in general relativity. *Ann. Phys.* (Leipzig) **12**, 329, (1964).

[6] Yu.A. Rylov, Non-Riemannian model of space-time responsible for quantum effects. *J. Math. Phys.* **32**, 2092, (1991).

[7] Yu.A. Rylov, Non-Euclidean method of the generalized geometry construction and its application to space-time geometry. in *Pure and Applied Differential geometry* pp.238-246. eds. Franki Dillen and Ignace Van de Woestyne. Shaker Verlag, Aachen, 2007.

[8] Yu. A. Rylov, Different conceptions of Euclidean geometry. *e-print 0709.2755*

[9] Yu. A. Rylov, Relativistic nearness of events and deformation principle as tool of the relativity theory generalization on the arbitrary space-time geometry. *e-print 0910.3582v5.*

[10] Yu. A. Rylov, Multivariance as a crucial property of microcosm. *Concepts of Physics 6, iss.1, 89 -117, (2009).* See also *e-print* 0806.1716.