Metrical approach to geometry and discrete space-time geometry

Yuri A.Rylov

Institute for Problems in Mechanics, Russian Academy of Sciences,
101-1, Vernadskii Ave., Moscow, 119526, Russia.
e-mail: rylov@ipmnet.ru
Web site: *http://rsfq1.physics.sunysb.edu/~rylov/yrylov.htm*
or mirror Web site:
http://gasdyn-ipm.ipmnet.ru/~rylov/yrylov.htm

Abstract

Metric approach to geometry admits one to construct a physical geometry, i.e. geometry which is described completely in terms and only in terms of the distance function ρ, or in terms of the world function $\sigma = 0.5\rho^2$. A discrete geometry is a special case of physical geometry, when there exist an elementary length λ_0 and all distances in the discrete geometry are larger, than the elementary length. A discrete geometry is obtained as a generalization of the proper Euclidean geometry. To produce such a generalization, a logical reloading in the description of the Euclidean geometry is to be produced. It means that the Euclidean geometry begins to be described in terms of the world function and only in terms of the world function. Such a description contains general geometric relations, which are valid in all physical geometries, and special relations, describing properties of the Euclidean world function. Replacing the Euclidean world function σ_{E} by the world function σ_{d} of the discrete geometry in all general geometric relations and ignoring special relations, one obtains general relations for the discrete geometry. As far as the form of σ_{d} is supposed to be known, the special relations for the discrete geometry are not needed.

Key words: elementary length; discrete space-time geometry; world function; logical reloading; geometrization of particle parameters; skeleton conception of particle dynamics;

1 Introduction

The proper Euclidean geometry as well as the geometry of Minkowski are continuous geometries. They are described by methods of differential geometry. However, there may exist discrete geometries, where the distance between any two points of the space-time is larger, than some elementary length λ_0. If characteristic scale of the problem is much larger, than the elementary length λ_0, one may set $\lambda_0 = 0$ and consider the space-time geometry as a continuous geometry. However, in microcosm, where characteristic scale is of the order of λ_0, one should consider a discrete space-time geometry, because the real space-time geometry may be discrete, and such a possibility is to be investigated.

At the conventional construction of the Euclidean geometry one uses such concepts as manifold, dimension, coordinate system, linear vector space, which might be used only in continuous (differential) geometries. A discrete geometry is considered as a generalization of the proper Euclidean geometry, because it is the only geometry, whose consistency has been proved. Constructing a discrete geometry as a generalization of the proper Euclidean geometry, one may not use above-mentioned concepts. The only concept, which may be used in the continuous geometry and in the discrete one, is the distance ρ. But the distance ρ is to be introduced as a fundamental quantity. In the Riemannian geometry the distance ρ is introduced as an integral along the geodesic from the infinitesimal distance

$$ds = \sqrt{g_{ik} dx^i dx^k}$$

Such a method of introduction of the distance ρ is inadequate in the discrete geometry, because it uses infinitesimal distance, which does not exist in the discrete geometry. Besides, in the case, when there are several geodesics, connecting two points, one obtains many-valued expressions for the distance or for the world function. Many-valued world function is inadmissible in a geometry.

To construct a discrete geometry, one needs to represent the proper Euclidean geometry in terms of the distance ρ (or in terms of the world function $\sigma = \frac{1}{2}\rho^2$) and to use this representation for generalization of the proper Euclidean geometry $\mathcal{G}_{\rm E}$ on the case of a discrete geometry $\mathcal{G}_{\rm d}$. Such a replacement of basic concepts of the Euclidean geometry means a logical reloading of the Euclidean geometry conception. Representation of a geometry in terms of a world function will be referred to as σ-immanent representation. The σ-immanent representation of the proper Euclidean geometry $\mathcal{G}_{\rm E}$ is always possible.

The distance function $\rho_{\rm d}$ of a discrete geometry $\mathcal{G}_{\rm d}$ satisfies the condition

$$|\rho_{\rm d}(P, Q)| \notin (0, \lambda_0), \qquad \forall P, Q \in \Omega \tag{1.1}$$

where Ω is the point set, where the geometry $\mathcal{G}_{\rm d}$ is given. In means that in the geometry $\mathcal{G}_{\rm d}$ there are no distances, which are shorter, than the elementary length λ_0. The distance $\rho_{\rm d}(P, Q) = 0$ is admissible. This condition takes place, if $P = Q$.

Note, that the condition (1.1) is a restriction on the values of the distance function, but not on values of its argument (points of Ω), although one considers usually a discrete geometry as a geometry on a lattice. It is true, that the geometries on a lattice are discrete geometries (they satisfy the relation (1.1)), but they form a very special case of the discrete geometries. Such a geometry is essentially a conventional differential geometry, given on a countable set of points, where the distances are the same as in the differential geometry, given on a continual set of points. Besides, such a discrete geometry cannot be uniform and isotropic. A general case of a discrete geometry takes place, when restrictions are imposed on the admissible values of the world function (distance function).

The simplest case of a discrete space-time geometry $\mathcal{G}_{\rm d}$ is obtained, if $\mathcal{G}_{\rm d} = \{\sigma_{\rm d}, \Omega_{\rm M}\}$ is given on the manifold $\Omega_{\rm M}$, where the geometry of Minkowski $\mathcal{G}_{\rm M} = \{\sigma_{\rm M}, \Omega_{\rm M}\}$ is given. The world function $\sigma_{\rm d}$ is chosen in the form

$$\sigma_{\rm d}(P,Q) = \sigma_{\rm M}(P,Q) + \frac{1}{2}\lambda_0^2 \operatorname{sgn}(\sigma_{\rm M}(P,Q)), \qquad \forall P,Q \in \Omega_{\rm M} \tag{1.2}$$

where $\sigma_{\rm M}$ is the world function of the geometry of Minkowski. It is easy to verify, that $\rho_{\rm d} = \sqrt{2\sigma_{\rm d}}$, defined by (1.2) satisfies the constraint (1.1). Such a discrete geometry is uniform and isotropic as well as the geometry of Minkowski.

Besides, in the discrete space-time geometry (1.2) a pointlike particle cannot be described by a world line, because any world line is a limit of the broken line, when lengths of its links tend to zero. But in the discrete geometry $\mathcal{G}_{\rm d}$ there are no infinitesimal lengths, and a pointlike particle is described by a world chain (broken line) instead of smooth world line. Description of a pointlike particle state by means of the particle position and its momentum becomes inadequate, because in the continuous (differential) space-time geometry the particle 4-momentum p_k is described by the relation

$$p_k = g_{kl}\frac{dx^l}{d\tau} = g_{kl}\lim_{d\tau\to 0}\frac{x^l(\tau + d\tau) - x^l(\tau)}{d\tau} \tag{1.3}$$

where $x^l = x^l(\tau)$, $l = 0,1,2,3$ is an equation of the world line. The limit in the formula (1.3) does not exist in $\mathcal{G}_{\rm d}$, and the 4-momentum p_k is not defined (at any rate in such a form). In general, the mathematical formalism of a differential geometry, based on the infinitesimal calculus (differential dynamic equations), is inadequate in the discrete space-time geometry, where infinitesimal distances are absent.

2 Metric approach to geometry

There is another circumstance, which prevents from constructing a discrete geometry. The proper Euclidean geometry is an axiomatizable geometry. It means, that all statements of the proper Euclidean geometry can be deduced from a system of several axioms (basic statements of the geometry). Usually one considers the axiomatizability of a geometry as an inherent property of any geometry. One believes that there are no nonaxiomatizable geometries. The reason of such a belief is rather simple. During two thousand years we knew the only geometry - the proper Euclidean geometry, which is axiomatizable. All differential geometries, constructed as a generalization of the proper Euclidean geometry, are also axiomatizable. One knows no other method of a geometry construction other, than the Euclidean method of the geometry deduction from some system of axioms. All differential geometries are constructed by means of this method. Mathematicians believe that any geometry is a logical construction. Such a discipline as the symplectic geometry is used in dynamics, but not for description of the geometric objects properties. Nevertheless it is called a geometry, because its structure reminds the structure of the Euclidean geometry.

In reality any geometry investigates a shape and a mutual disposition of geometrical objects in the space, or in the space-time. This property is an original property of a geometry. However, one used the only Euclidean method of the geometry construction during two thousand years, and as a result the axiomatizability of a geometry is considered now as an inherent property of any geometry, whereas a description of geometrical objects is considered as a secondary property of discipline, called geometry.

In general, there is a metrical approach to geometry, when a geometry is considered as a science, investigating a shape and a mutual disposition of geometrical objects. Such a geometry is known as a metric geometry (metric space), if it uses the triangle axiom. If the triangle axiom is not used, the geometry is called the distant geometry [1, 2]. It is supposed, that the distant geometry $\mathcal{G}_{\rm ds}=\{\sigma,\Omega\}$ is described completely by the world function $\sigma=\frac{1}{2}\rho^2$

$$\sigma: \quad \Omega\times\Omega\rightarrow\mathbb{R},\quad \sigma(P,Q)=\sigma(Q,P),\quad \forall P,Q\in\Omega \tag{2.1}$$

$$\rho(P,Q)\geq 0,\quad \forall P,Q\in\Omega,\quad \rho(P,Q)=\sqrt{2\sigma(P,Q)} \tag{2.2}$$

where Ω is the point set, where the geometry is given. The world function σ is used instead of the distance function ρ, because in the geometry of Minkowski the distance ρ may be either positive, or pure imaginary, whereas $\sigma=\frac{1}{2}\rho^2$ is always real.

At the metric approach to geometry, a geometry can be constructed on any point set (but not necessarily on a manifold) without a use coordinates. In the metric space the distance function ρ satisfies an additional constraint

$$\rho(P,Q)+\rho(P,R)\geq\rho(Q,R),\quad \forall P,Q,R\in\Omega \tag{2.3}$$

The condition (2.3) is known as the triangle axiom. This axiom admits one to introduce a straight line in the metric space as a shortest line between two points. In the distant geometry, where the constraint (2.3) is absent, one failed to introduce the straight line in terms of the distance function ρ. Blumental [2] introduced a curve as a continuous mapping $(0,1)\rightarrow\Omega$. The continuous mapping is an operation, which cannot be expressed only in terms of the distance function. As a result a purely metric approach to geometry, when geometry is described completely in terms of the distance function ρ, failed. The reason of this failure lies in the fact, that Blumental believed that the straight line has no thickness, whereas in reality in the distant geometry the straight line is a hollow tube. In reality the distant geometry is nonaxiomatizable geometry, which cannot be constructed by the Euclidean method.

What is on the bottom of the Euclidean method of the geometry construction? Let us get outside of this method. One cannot perceive the distance directly. One can perceive physical bodies. Geometrical object is an abstraction of space-time properties of a physical body. A physical body, evolving in the space-time, may pass from one space-time region with the space-time geometry $\{\sigma_1, \Omega_1\}$ to another space-time region with the space-time geometry $\{\sigma_2, \Omega_2\}$. We must have a possibility to recognize and to identify the same geometrical object in different space-time geometries. In order, that it should be possible, any geometrical object is to be described in terms of the distance function ρ and only in terms of ρ. Any geometrical object is described by its skeleton and its envelope. We consider a simple examples of geometrical objects. (The general definition of a geometrical object will be given later).

The simplest geometrical object is a sphere $\mathcal{SP}_{P_0P_1}$, determined by two points P_0, P_1 (skeleton). The point P_0 is a center of the sphere, P_1 is some point on the surface of the sphere. The points $\{P_0, P_1\}$ form the sphere skeleton. The surface of the sphere (its envelope) is a set of points

$$\mathcal{SP}_{P_0P_1} = \{R | \rho(P_0, R) = \rho(P_0, P_1)\}, \quad \rho = \sqrt{2\sigma} \tag{2.4}$$

The sphere is a hollow geometrical object in the sense, that there are internal points of the sphere, which do not belong to the sphere surface (envelope).

Another simple geometrical object is an ellipsoid $\mathcal{EL}_{F_1F_2P}$, determined by three points F_1, F_2, P. The points F_1, F_2 are focuses of the ellipsoid, and the point P is some point on the surface of the ellipsoid

$$\mathcal{EL}_{F_1F_2P} = \{R | \rho(F_1, R) + \rho(F_2, R) = \rho(F_1, P) + \rho(F_2, P)\}, \quad \rho = \sqrt{2\sigma} \tag{2.5}$$

If $F_1 \neq P \wedge F_2 \neq P$, the ellipsoid $\mathcal{EL}_{F_1F_2P}$ is a hollow geometrical object.

If $F_1 = P \vee F_2 = P$, the ellipsoid degenerates into a straight line segment $\mathcal{T}_{[P_0P_1]}$

$$\mathcal{T}_{[P_0P_1]} \equiv \mathcal{EL}_{P_0P_1P_1} = \mathcal{EL}_{P_0P_1P_0} = \{R | \rho(P_0, R) + \rho(P_1, R) = \rho(P_0, P_1)\} \tag{2.6}$$

The degenerate ellipsoid $\mathcal{EL}_{P_0P_1P_1}$ is a straight line segment $\mathcal{T}_{[P_0P_1]}$ by definition. This name is used, because in the proper Euclidean geometry a degenerate ellipsoid is a straight line segment. In other geometries the geometric object (2.6) may be a hollow geometrical object. It means, that it is not one-dimensional point set, as in the proper Euclidean geometry, but nevertheless we shall refer to it as a straight line segment.

The segment $\mathcal{T}_{[P_0P_1]}$ is determined by two points. All points of $\mathcal{T}_{[P_0P_1]}$ are points of the envelope, which consists of boundary points only. In the proper Euclidean geometry it is not a hollow geometrical object, because it has not internal points.

Is the straight line segment $\mathcal{T}_{[P_0P_1]}$ a hollow geometrical object in other distant geometries? It depends on the constraints (2.2),(2.3). If they are satisfied, the segment $\mathcal{T}_{[P_0P_1]}$ is entire (not hollow). If the distance function ρ does not satisfy the triangle axiom (2.3) the segment $\mathcal{T}_{[P_0P_1]}$ may be hollow. In other words, the segment $\mathcal{T}_{[P_0P_1]}$ may be a hollow tube.

Why is the segment entire, if the triangle axiom (2.3) is fulfilled? Let us consider a closed surface $\mathcal{S}$ defined by the relation

$$\mathcal{S}: \quad S_{P_0P_1}(R) = 0, \quad S_{P_0P_1}(R) = \rho(P_0, R) + \rho(P_1, R) - \rho(P_0, P_1) \tag{2.7}$$

Internal points R' (points inside the closed surface $\mathcal{S}$) satisfy the relation $S_{P_0P_1}(R') < 0$. External points R'' satisfy the relation $S_{P_0P_1}(R'') > 0$. If the triangle axiom is fulfilled, it may be written in the form

$$\rho(P_0, R) + \rho(P_1, R) \geq \rho(P_0, P_1), \quad \forall P_1, P_2, R \in \Omega \tag{2.8a}$$

It follows from (2.7) and (2.8a), that $S_{P_0P_1}(R') \geq 0, \quad \forall R' \in \Omega$. It means that the surface $\mathcal{S}$, which coincides with the segment $\mathcal{T}_{[P_0P_1]}$, cannot contain internal points.

Why it is important, whether or not the segment $\mathcal{T}_{[P_0P_1]}$ is hollow? Geometry is reduced to construction of geometrical objects and to investigation of their properties. In the proper Euclidean geometry all geometrical objects are constructed of blocks (point, straight segment). Blocks are to be simple entire (not hollow) geometrical objects. The segment $\mathcal{T}_{[P_0P_1]}$ is determined by two points, and it is entire in the proper Euclidean geometry. It may be used as a constructive block for construction of geometrical objects. For instance, in the proper Euclidean geometry a cube can be filled by straight segments placed in parallel with the cube edge in such a way, that any point of a cube belongs to one and only one segment. Such a situation is impossible, if the blocks are hollow geometrical objects. If the blocks are hollow tubes, one cannot fill the cube by these tubes in such a way, that any point of a cube belongs to one and only one tube. It means, that a cube cannot be constructed of hollow blocks. The same relates to any geometrical object.

The Euclidean method of the geometric object construction is based on the possibility of construction of any geometrical object from blocks. There is a finite number of rules, describing the blocks properties, and there is a finite number of rules for description of the blocks combinations at a construction of a geometrical object. Euclid formulated these rules in the form of axioms of a logical construction. Thus, the axiomatics of the proper Euclidean geometry describes the procedure of a construction of geometrical objects from blocks. If the segment $\mathcal{T}_{[P_0P_1]}$ is entire, the distant geometry is an axiomatizable geometry, because it can be realized as a geometry, where any geometric object can be constructed of blocks, i.e. by means of the Euclidean method.

If blocks are hollow, they cannot be used for construction of geometrical objects. In this case the distant geometry is nonaxiomatizable, because in this case one cannot use the Euclidean method for construction of geometric objects. Formally the segment $\mathcal{T}_{[P_0P_1]}$ is hollow, if the equivalence relation is intransitive. If the equivalence relation is transitive, the segment $\mathcal{T}_{[P_0P_1]}$ may be entire.

The constructive block $\mathcal{T}_{[P_0P_1]}$ is a directed object, whose direction is described by the vector $\mathbf{P}_0\mathbf{P}_1 = \overrightarrow{P_0P_1} = \{P_0, P_1\}$, which is an ordered set of two points. The point P_0 is the origin of the vector, the point P_1 is the end of the vector. Any vector $\mathbf{P}_0\mathbf{P}_1$ is described by its module

$$|\mathbf{P}_0\mathbf{P}_1| = \rho(P_0, P_1) = \sqrt{2\sigma(P_0, P_1)} \tag{2.9}$$

Vectors are directed quantities, and interrelation of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is described by an angle φ between them. In the proper Euclidean geometry there is a lot of vectors $\mathbf{Q}_0\mathbf{Q}_1$, which form the angle $\varphi \neq 0$ with the vector $\mathbf{P}_0\mathbf{P}_1$. However, in the proper Euclidean geometry there is only one vector $\mathbf{Q}_0\mathbf{Q}_1$ at the point Q_0 with fixed length $|\mathbf{Q}_0\mathbf{Q}_1|$, which forms with the vector $\mathbf{P}_0\mathbf{P}_1$ the angle $\varphi = 0$. By definition such a vector $\mathbf{Q}_0\mathbf{Q}_1$ is called the vector, which is parallel ($\mathbf{Q}_0\mathbf{Q}_1 \upuparrows \mathbf{P}_0\mathbf{P}_1$) to the vector $\mathbf{P}_0\mathbf{P}_1$.

Instead of the angle φ the interrelation of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ may be described by the scalar product $(\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_1)$ of these vectors, defined by the relation

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = |\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1| \cos\varphi \qquad (2.10)$$

In the proper Euclidean geometry the definition of the scalar product may be expressed in terms of the world function

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \sigma(P_0, Q_1) + \sigma(P_1, Q_0) - \sigma(P_1, Q_1) - \sigma(P_0, Q_0) \qquad (2.11)$$

As far as the definition of the scalar product is produced in terms of the world function, this definition may be used for any distant geometry.

Then condition of the vectors parallelism is obtained from (2.10) at $\varphi = 0$. It is written in the form

$$(\mathbf{Q}_0\mathbf{Q}_1 \upuparrows \mathbf{P}_0\mathbf{P}_1): \quad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = |\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1| \qquad (2.12)$$

In the proper Euclidean geometry all vectors $\mathbf{P}_0\mathbf{P}_1$,$\mathbf{P}_0\mathbf{P}'_1$, $\mathbf{P}_0\mathbf{P}''_1$, which are parallel to vector $\mathbf{Q}_0\mathbf{Q}_1$, are parallel between themselves. Such a situation is rather special. It is connected with a degenerate character of the proper Euclidean geometry. In the distant geometry vectors $\mathbf{P}_0\mathbf{P}_1$,$\mathbf{P}_0\mathbf{P}'_1$, $\mathbf{P}_0\mathbf{P}''_1$, which are parallel to vector $\mathbf{Q}_0\mathbf{Q}_1$, are not parallel between themselves, in general. This circumstance generates hollowness of straight segments $\mathcal{T}_{[P_0P_1]}$. It depends on properties of the world function σ, which describes a distant geometry completely.

In the proper Euclidean geometry two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ are equivalent by definition, if they are parallel ($\mathbf{P}_0\mathbf{P}_1 \upuparrows \mathbf{Q}_0\mathbf{Q}_1$) and their lengths are equal $|\mathbf{P}_0\mathbf{P}_1| = |\mathbf{Q}_0\mathbf{Q}_1|$

$$(\mathbf{P}_0\mathbf{P}_1 \text{eqv} \mathbf{Q}_0\mathbf{Q}_1): \quad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = |\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1| \wedge |\mathbf{P}_0\mathbf{P}_1| = |\mathbf{Q}_0\mathbf{Q}_1| \qquad (2.13)$$

This definition of two vectors equivalency (equality) together with the definitions (2.9), (2.11) formulates the equivalence of two vectors in terms of the world function and only in these terms. It does not refer to a dimension, to a coordinate system and other means of description. This definition of two vectors equivalence should be used in any distant geometry.

There are such distant geometries, where the straight segments $\mathcal{T}_{[P_0P_1]}$ are hollow tubes, and the definition (2.13), (2.11) appears to be intransitive and the distant geometry appears to be nonaxiomatizable. Some mathematicians object, that the definition (2.13), (2.11) cannot be used as an equivalence relation, because the equivalence relation is transitive by definition. They insist, that one should use another term for the definition (2.13), (2.11), (for instance, general equivalency). The reason of such an objection lies in the fact, that the mathematicians dealt only with axiomatizable geometries, which are logical constructions. Indeed, if one uses a logical construction, one can deduce conclusions, only if the equivalence relation is transitive, and from $a \backsim b$ and $b \backsim c$ it follows, that $a \backsim c$. If the the equivalence relation has not this property, one cannot deduce corollaries of theorems. Thus, if one insists on the transitivity of the equivalence relation, one insists on impossibility of nonaxiomatizable geometries, in particular, on impossibility of discrete space-time geometries, where the straight segments $\mathcal{T}_{[P_0P_1]}$ are hollow tubes. As a result the discrete geometries appear to be nonaxiomatizable.

The transitivity of the equivalence relation has been obtained from our experience of work with axiomatizable geometries (Euclidean geometry and its modifications). We have no authority to generalize this property to all space-time geometries. Whether or not the real space-time geometry is discrete, is a question of experimental data, but not a question of mathematical scholasticism. Another problem lies in the fact, that we could construct only axiomatizable geometries, and we could not construct discrete geometries. As a result we constructed only geometries on a lattice, which are not rigorous discrete geometries. How to construct discrete (nonaxiomatizable) geometries, we consider a few later.

3 Description of geometric objects

If the distant geometry includes indefinite geometries (like the geometry of Minkowski), the condition (2.2) is to be omitted, and description of the geometry is produced in terms of the world function. The geometry described completely by the world function (2.1) will be referred to as a physical geometry.

A geometrical object is a geometrical image of a physical body. Any geometrical object is some subset of points in the space-time. However, geometrical object is not an arbitrary set of points. Geometrical object is to be defined in the physical geometry in such a way, that similar geometrical objects (which are images of similar physical bodies) could be recognized in different space-time geometries.

Definition 3.1: A geometrical object $g_{\mathcal{P}_n,\sigma}$ of the geometry $\mathcal{G} = \{\sigma, \Omega\}$ is a subset $g_{\mathcal{P}_n,\sigma} \subset \Omega$ of the point set Ω. This geometrical object $g_{\mathcal{P}_n,\sigma}$ is a set of roots $R \in \Omega$ of the function $F_{\mathcal{P}_n,\sigma}$

$$g_{\mathcal{P}_n,\sigma} = \{R | F_{\mathcal{P}_n,\sigma}(R) = 0\}, \quad F_{\mathcal{P}_n,\sigma} : \quad \Omega \to \mathbb{R} \tag{3.1}$$

where $F_{\mathcal{P}_n,\sigma}$ depends on the point R via world functions of arguments $\{\mathcal{P}_n, R\} = \{P_0, P_1, ...P_n, R\}$

$$F_{\mathcal{P}_n,\sigma} \quad : \quad F_{\mathcal{P}_n,\sigma}(R) = G_{\mathcal{P}_n,\sigma}(u_1, u_2, ...u_s), \qquad s = \frac{1}{2}(n+1)(n+2) \tag{3.2}$$

$$u_l = \sigma(w_i, w_k), \qquad i, k = 0, 1, ...n+1, \quad l = 1, 2, ...\frac{1}{2}(n+1)(n+2) \tag{3.3}$$

$$w_k = P_k \in \Omega, \qquad k = 0, 1, ...n, \qquad w_{n+1} = R \in \Omega \tag{3.4}$$

Here $\mathcal{P}_n = \{P_0, P_1, ..., P_n\} \subset \Omega$ are $n+1$ points which are parameters, determining the geometrical object $g_{\mathcal{P}_n,\sigma}$

$$g_{\mathcal{P}_n,\sigma} = \{R | F_{\mathcal{P}_n,\sigma}(R) = 0\}, \qquad R \in \Omega, \qquad \mathcal{P}_n \in \Omega^{n+1} \tag{3.5}$$

$F_{\mathcal{P}_n,\sigma}(R) = G_{\mathcal{P}_n,\sigma}(u_1, u_2, ...u_s)$ is an arbitrary function of $\frac{1}{2}(n+1)(n+2)$ arguments u_k and of $n+1$ parameters $\mathcal{P}_n$. The set $\mathcal{P}_n = \{P_0, P_1, ...P_n\} \in \Omega^n$ of the geometric object parameters will be referred to as the skeleton of the geometrical object. The subset $g_{\mathcal{P}_n,\sigma} \subset \Omega$ will be referred to as the envelope of the skeleton. The skeleton is an analog of a frame of reference, attached rigidly to a physical body. Tracing the skeleton motion, one can trace the motion of the physical body. When a particle is considered as a geometrical object, its motion in the space-time is described by the skeleton $\mathcal{P}_n$ motion. At such an approach (the rigid body approximation) the shape of the envelope is of no importance.

Remark: An arbitrary subset Ω' of the point set Ω is not a geometrical object, in general. It is supposed, that physical bodies may have only a shape of a geometrical object, because only in this case one can identify identical physical bodies (geometrical objects) in different space-time geometries.

Existence of the same geometrical objects in different space-time regions, having different geometries, brings up the question on equivalence of geometrical objects in different space-time geometries. Such a question did not arise before, because one does not consider such a situation, when a physical body moves from one space-time region to another space-time region, having another space-time geometry. In general, mathematical technique of the conventional space-time geometry (differential geometry) is not applicable for simultaneous consideration of several different geometries of different space-time regions.

We can perceive the space-time geometry only via motion of physical bodies in the space-time, or via construction of geometrical objects corresponding to these physical bodies. As it follows from the *definition 3.1* of the geometrical object, the function $G_{\mathcal{P}_n,\sigma}$ as a function of its arguments u_k, $k = 1, 2, ...n(n+1)/2$ (of world functions of different points) is the same in all physical geometries. It means, that a geometrical object $\mathcal{O}_1$ in the geometry $\mathcal{G}_1 = \{\sigma_1, \Omega_1\}$ is obtained from the same geometrical object $\mathcal{O}_2$ in the geometry $\mathcal{G}_2 = \{\sigma_2, \Omega_2\}$ by means of the replacement $\sigma_2 \to \sigma_1$ in the definition of this geometrical object.

Definition 3.2: Geometrical object $g_{P'_n,\sigma'}$ ($\mathcal{P}'_n = \{P'_0, P'_1, ..P'_n\}$) in the geometry $\mathcal{G}' = \{\sigma', \Omega'\}$ and the geometrical object $g_{P_n,\sigma}$ ($\mathcal{P}_n = \{P_0, P_1, ..P_n\}$) in the geometry $\mathcal{G} = \{\sigma, \Omega\}$ are similar geometrical objects, if

$$\sigma'(P'_i, P'_k) = \sigma(P_i, P_k), \qquad i, k = 0, 1, ..n \tag{3.6}$$

and the functions $G'_{\mathcal{P}'_n,\sigma'}$ for $g_{P'_n,\sigma'}$ and $G_{\mathcal{P}_n,\sigma}$ for $g_{P_n,\sigma}$ in the formula (3.2) are the same functions of arguments $u_1, u_2, ...u_s$

$$G'_{\mathcal{P}'_n,\sigma'}(u_1, u_2, ...u_s) = G_{\mathcal{P}_n,\sigma}(u_1, u_2, ...u_s) \tag{3.7}$$

In this case

$$u_l \equiv \sigma(P_i, P_k) = u'_l \equiv \sigma'(P'_i, P'_k), \qquad i,k = 0,1,...n, \qquad l = 1,2,..n(n+1)/2 \tag{3.8}$$

The functions $F'_{\mathcal{P}'_n,\sigma'}$ for $g_{P'_n,\sigma'}$ and $F_{\mathcal{P}_n,\sigma}$ for $g_{P_n,\sigma}$ in the formula (3.2) have the same roots, if the relation (3.7) is fulfilled. As a result one-to-one connection between the geometrical objects $g_{P'_n,\sigma'}$ and $g_{P_n,\sigma}$ arises.

As far as the physical geometry is determined by its geometrical objects construction, a physical geometry $\mathcal{G} = \{\sigma, \Omega\}$ can be obtained from some known standard geometry $\mathcal{G}_{\rm st} = \{\sigma_{\rm st}, \Omega\}$ by means of a deformation of the standard geometry $\mathcal{G}_{\rm st}$. Deformation of the standard geometry $\mathcal{G}_{\rm st}$ is realized by the replacement $\sigma_{\rm st} \to \sigma$ in all definitions of the geometrical objects in the standard geometry. The proper Euclidean geometry is an axiomatizable geometry. It has been constructed by means of the Euclidean method as a logical construction. Simultaneously the proper Euclidean geometry is a physical geometry. It may be used as a standard geometry $\mathcal{G}_{\rm st}$. Construction of a physical geometry as a deformation of the proper Euclidean geometry will be referred to as the deformation principle [3]. The most physical geometries are nonaxiomatizable geometries. They can be constructed only by means of the deformation principle.

4 General geometric relations

Describing a physical geometry in terms of the world function, one should distinguish between general geometric relations and specific geometric relations. The general geometric relations are the linear vector space relations, which are written in terms of the world function. The general geometric relations are valid for any physical geometry.

The first general geometric definition is the definition of the scalar product of two vectors (2.11). Definition of the two vector equivalence (2.13) is also a general geometric relation.

Linear dependence of n vectors $\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2, ...\mathbf{P}_0\mathbf{P}_n$ is defined by the relation,

$$F_n(\mathcal{P}_n) = 0, \quad F_n(\mathcal{P}_n) \equiv \det ||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)||, \qquad i,k = 1,2,...n \tag{4.1}$$

where $\mathcal{P}_n = \{P_0, P_1, ...P_n\}$ and $F_n(\mathcal{P}_n)$ is the Gram's determinant. Vanishing of the Gram's determinant is the necessary and sufficient condition of the linear dependence of n vectors. Condition of linear dependence relates to the properties of the linear vector space. It seems rather meaningless to use it, if the linear vector space cannot be introduced. Nevertheless, the relation (4.1) written as a general geometric relation describes some general geometric properties of vectors, which in the proper Euclidean geometry transforms to the property of linear dependence. In particular, the space dimension of the proper Euclidean geometry is defined in terms of the world function by means of the relations of the type (4.1) as a maximal number of linear independent vectors, which is possible in the Euclidean space. This circumstance seems to be rather unexpected, because in conventional presentation of the Euclidean geometry the geometry dimension is postulated in the beginning of the presentation.

The general geometric relations describe either properties of the linear vector space concepts, or definition of geometrical objects. As we have seen, a definition of geometrical objects in the form of general geometric relations (i.e. in terms of the world function) is necessary to recognize the same physical body (and corresponding geometrical object) in different space-time geometries.

The general geometric relations are parametrized by the form of the world function. Changing the form of the world function, one obtains the general geometric relations at a new value of the parameter (new form of the world function).

5 Specific properties of the n-dimensional Euclidean space

Along of general geometric properties, describing mainly definitions of the linear vector space, there are special geometric relations, describing properties of the world function. For instance, there are relations, which are necessary and sufficient conditions of the fact, that the world function σ is the world function of n-dimensional Euclidean space. They have the form [4]:

I. Definition of the dimension:

$$\exists\mathcal{P}^n \equiv \{P_0, P_1, ...P_n\} \subset \Omega, \qquad F_n(\mathcal{P}^n) \neq 0, \qquad F_k\left(\Omega^{k+1}\right) = 0, \qquad k > n \tag{5.1}$$

where $F_n(\mathcal{P}^n)$ is the n-th order Gram's determinant (4.1) Vectors $\mathbf{P}_0\mathbf{P}_i$, $i = 1, 2, ...n$ are basic vectors of the rectilinear coordinate system K_n with the origin at the point P_0. The metric tensors $g_{ik}(\mathcal{P}^n)$, $g^{ik}(\mathcal{P}^n)$, $i, k = 1, 2, ...n$ in K_n are defined by the relations

$$\sum_{k=1}^{k=n} g^{ik}(\mathcal{P}^n)\, g_{lk}(\mathcal{P}^n) = \delta_l^i, \qquad g_{il}(\mathcal{P}^n) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_l), \qquad i, l = 1, 2, ...n \tag{5.2}$$

$$F_n(\mathcal{P}^n) = \det ||g_{ik}(\mathcal{P}^n)|| \neq 0, \qquad i, k = 1, 2, ...n \tag{5.3}$$

II. Linear structure of the Euclidean space:

$$\sigma(P, Q) = \frac{1}{2}\sum_{i,k=1}^{i,k=n} g^{ik}(\mathcal{P}^n)\,(x_i(P) - x_i(Q))\,(x_k(P) - x_k(Q)), \qquad \forall P, Q \in \Omega \tag{5.4}$$

where coordinates $x_i(P)$, $x_i(Q)$, $i = 1, 2, ...n$ of the points P and Q are covariant coordinates of the vectors $\mathbf{P}_0\mathbf{P}$, $\mathbf{P}_0\mathbf{Q}$ respectively in the coordinate system K. The covariant coordinates are defined by the relation

$$x_i(P) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}), \qquad i = 1, 2, ...n \tag{5.5}$$

III: The metric tensor matrix $g_{lk}\left(\mathcal{P}^n\right)$ has only positive eigenvalues g_k

$$g_k > 0, \qquad k = 1, 2, ..., n \tag{5.6}$$

IV. The continuity condition: the system of equations

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}) = y_i \in \mathbb{R}, \qquad i = 1, 2, ...n \tag{5.7}$$

considered to be equations for determination of the point P as a function of coordinates $y = \{y_i\}, \;\; i = 1, 2, ...n$ has always one and only one solution. Conditions I – IV contain a reference to the dimension n of the Euclidean space, which is defined by the relations (5.1).

All relations I – IV are written in terms of the world function. They are constraints on the form of the world function of the proper Euclidean geometry. Constraints (5.1), determining the dimension via the form of the world function, look rather unexpected. They contain a lot of constraints imposed on the world function of the proper Euclidean geometry, and they are necessary. At the conventional approach to geometry one uses a very simple supposition: "Let the dimension of the Euclidean space be n." instead of numerous constraints (5.1).

In the vector representation of the proper Euclidean geometry, which is based on a use of the linear vector space, the dimension is considered as a primordial property of the linear vector space and as a primordial property of the Euclidean geometry. Situation, when the geometry dimension is different at different points of the space Ω, or when it is indefinite, is not considered. In the vector representation of the Euclidean geometry one does not distinguish between the general geometric relations and the specific relations of the geometry.

Instead of constraints (5.1) – (5.7) one may use an explicit form of the world function

$$\sigma_{\mathrm{E}}\left(x, x'\right) = \sum_{k=1}^{k=n}\left(x^k - x'^k\right)^2 \tag{5.8}$$

where $x^k, x'^k \in \mathbb{R}$, $k = 1, 2, ...n$ are Cartesian coordinates of points P and P' respectively. The relation (5.8) satisfies all constraints (5.1) – (5.7). It uses concepts of dimension and of coordinates as primordial concepts of geometry. Using the world function only in such an explicit form, one cannot imagine a generalized geometry without such concepts as a dimension and a coordinate system, although these concepts are only means of a geometry description.

In general, after the logical reloading to σ-representation (description in terms of the world function) the proper Euclidean geometry looks rather unexpected. Some concepts look very simple in the vector representation. The same concepts look complicated in the σ-representation and vice versa. As a result the proper Euclidean geometry in the σ-representation is perceived hardly. In the vector representation one has several fundamental quantities: dimension, coordinate system, linear dependence, whereas in the σ-representation there is only one fundamental quantity: world function. The dimension, the coordinate system and the linear dependence are derivative quantities. Agreement between these quantities is achieved in any geometry, because they are defined as some attributes of the world function.

6 Equivalence of physical geometries

Generalization of general geometric expressions (2.9) – (2.13) on the case of the discrete geometry $\mathcal{G}_{\rm d}$ is obtained by means of the replacement $\sigma_{\rm E} \rightarrow \sigma_{\rm d}$, where $\sigma_{\rm d}$ is the world function of the discrete geometry $\mathcal{G}_{\rm d}$. We are to be ready, that properties of concepts and relations (2.6), (2.11) – (2.13), (4.1)) in $\mathcal{G}_{\rm d}$ differ strongly from their properties in $\mathcal{G}_{\rm E}$. However, we have no alternative to these relations for definition of these geometrical quantities in a discrete geometry $\mathcal{G}_{\rm d}$.

Definition 6.1: The physical geometry $\mathcal{G} = \{\sigma, \Omega\}$ is a point set Ω with the single-valued function σ on it

$$\sigma : \qquad \Omega \times \Omega \rightarrow \mathbb{R}, \qquad \sigma(P,P) = 0, \qquad \sigma(P,Q) = \sigma(Q,P), \qquad P,Q \in \Omega \quad (6.1)$$

Definition 6.2: Two physical geometries $\mathcal{G}_1 = \{\sigma_1, \Omega_1\}$ and $\mathcal{G}_2 = \{\sigma_2, \Omega_2\}$ are equivalent ($\mathcal{G}_1 \text{eqv} \mathcal{G}_2$), if the point set $\Omega_1 \subseteq \Omega_2 \wedge \sigma_1(P,Q) = \sigma_2(P,Q)$, $\forall P,Q \in \Omega_1$, or $\Omega_2 \subseteq \Omega_1 \wedge \sigma_2(P,Q) = \sigma_1(P,Q)$, $\forall P,Q \in \Omega_2$

Remark: Coincidence of point sets Ω_1 and Ω_2 is not necessary for equivalence of geometries $\mathcal{G}_1$ and $\mathcal{G}_2$. If one demands coincidence of Ω_1 and Ω_2 in the case of equivalence of $\mathcal{G}_1$ and $\mathcal{G}_2$, then an elimination of one point P from the point set Ω_1 turns the geometry $\mathcal{G}_1 = \{\sigma_1, \Omega_1\}$ into geometry $\mathcal{G}_2 = \{\sigma_1, \Omega_1 \backslash P\}$, which appears to be not equivalent to the geometry $\mathcal{G}_1$. Such a situation seems to be inadmissible, because a geometry on a part $\omega \subset \Omega_1$ of the point set Ω_1 appears to be not equivalent to the geometry on the whole point set Ω_1.

According to definition the geometries $\mathcal{G}_1 = \{\sigma, \omega_1\}$ and $\mathcal{G}_2 = \{\sigma, \omega_2\}$ on parts of Ω, $\omega_1 \subset \Omega$ and $\omega_2 \subset \Omega$ are equivalent $(\mathcal{G}_1 \text{eqv} \mathcal{G})$, $(\mathcal{G}_2 \text{eqv} \mathcal{G})$ to the geometry $\mathcal{G}$, whereas the geometries $\mathcal{G}_1 = \{\sigma, \omega_1\}$ and $\mathcal{G}_2 = \{\sigma, \omega_2\}$ are not equivalent, in general, if $\omega_1 \nsubseteq \omega_2$ and $\omega_2 \nsubseteq \omega_1$. Thus, the relation of equivalence is intransitive, in general. The space-time geometry may vary in different regions of the space-time. It means, that a physical body, described as a geometrical object, may evolve in such a way, that it appears in regions with different space-time geometry.

The space-time geometry of Minkowski as well as the Euclidean geometry are continuous geometries. It is true for usual scales of distances. However, one cannot be sure, that the space-time geometry is continuous in microcosm. The space-time geometry may appear to be discrete in microcosm. We consider a discrete space-time geometry and discuss the corollaries of the suggested discreteness.

7 Discreteness and its manifestations

The simplest discrete space-time geometry $\mathcal{G}_{\rm d}$ is described by the world function (1.2). Density of points in $\mathcal{G}_{\rm d}$ with respect to point density in $\mathcal{G}_{\rm M}$ is described by the relation

$$\frac{d\sigma_{\rm M}}{d\sigma_{\rm d}} = \left\{ \begin{array}{lll} 0 & \text{if} & |\sigma_{\rm d}| < \frac{1}{2}\lambda_0^2 \\ 1 & \text{if} & |\sigma_{\rm d}| > \frac{1}{2}\lambda_0^2 \end{array} \right. \tag{7.1}$$

If the world function has the form

$$\sigma_{\mathrm{g}}=\sigma_{\mathrm{M}}+\frac{\lambda_0^2}{2}\left\{\begin{array}{lll}\operatorname{sgn}(\sigma_{\mathrm{M}}) & \text{if} & |\sigma_{\mathrm{M}}|>\sigma_0 \\ \frac{\sigma_{\mathrm{M}}}{\sigma_0} & \text{if} & |\sigma_{\mathrm{M}}|\leq\sigma_0\end{array}\right. \tag{7.2}$$

where $\sigma_0=$ const, $\sigma_0\geq 0$, the relative density of points has the form

$$\frac{d\sigma_{\mathrm{M}}}{d\sigma_{\mathrm{g}}}=\left\{\begin{array}{lll}\frac{2\sigma_0}{2\sigma_0+\lambda_0^2} & \text{if} & |\sigma_{\mathrm{g}}|<\sigma_0+\frac{1}{2}\lambda_0^2 \\ 1 & \text{if} & |\sigma_{\mathrm{g}}|>\sigma_0+\frac{1}{2}\lambda_0^2\end{array}\right. \tag{7.3}$$

If the parameter $\sigma_0\rightarrow 0$, the world function $\sigma_{\mathrm{g}}\rightarrow\sigma_{\mathrm{d}}$ and the point density (7.3) tends to the point density (7.1). The space-time geometry $\mathcal{G}_{\mathrm{g}}$, described by the world function σ_{g} is a geometry, which is a partly discrete geometry, because it is intermediate between the discrete geometry $\mathcal{G}_{\mathrm{d}}$ and the continuous geometry $\mathcal{G}_{\mathrm{M}}$. We shall refer to the geometry $\mathcal{G}_{\mathrm{g}}$ as a granular geometry.

Deflection of the discrete space-time geometry from the continuous geometry of Minkowski generates special properties of the geometry, which are corollaries of impossibility of the linear vector space introduction.

Let $\mathbf{P}_0\mathbf{P}_1$ be a timelike vector in $\mathcal{G}_{\mathrm{d}}$ $(\sigma_{\mathrm{d}}(P_0,P_1)>0)$. We try to determine a vector $\mathbf{P}_1\mathbf{P}_2$ at the point P_1, which is equivalent to vector $\mathbf{P}_0\mathbf{P}_1$. Vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_1\mathbf{P}_2$ may be considered as two adjacent links of a broken world line, describing a pointlike particle

Let for simplicity coordinates have the form

$$P_0=\{0,0,0,0\}, \qquad P_1=\{\mu,0,0,0\}, \qquad P_2=\left\{x^0,\mathbf{x}\right\}=\left\{x^0,x^1,x^2,x^3\right\} \tag{7.4}$$

In this coordinate system the world function of geometry Minkowski has the form

$$\sigma_{\mathrm{M}}(x,x')=\frac{1}{2}\left(\left(x^0-x^{0\prime}\right)^2-(\mathbf{x}-\mathbf{x}')^2\right) \tag{7.5}$$

and σ_{d} is determined by the relation (1.2). We are to determine coordinates x of the point P_1 from two equations (2.13), which can be written in the form

$$\sigma_{\mathrm{d}}(P_0,P_1)=\sigma_{\mathrm{d}}(P_1,P_2), \quad \sigma_{\mathrm{d}}(P_0,P_2)=4\sigma_{\mathrm{d}}(P_0,P_1) \tag{7.6}$$

After substitution of world function (1.2) one obtains

$$\frac{1}{2}\left(\left(x^0-\mu\right)^2-\mathbf{x}^2+\lambda_0^2\right)=\frac{1}{2}\left(\mu^2+\lambda_0^2\right) \tag{7.7}$$

$$\frac{1}{2}\left(\left(x^0\right)^2-\mathbf{x}^2+\lambda_0^2\right)=2\left(\left(x^0-\mu\right)^2-\mathbf{x}^2+\lambda_0^2\right) \tag{7.8}$$

Solution of these equations has the form

$$x^0=2\mu+\frac{3}{2}\frac{\lambda_0^2}{\mu}, \qquad \mathbf{x}^2=3\lambda_0^2\left(1+\frac{3\lambda_0^2}{4\mu^2}\right) \tag{7.9}$$

As a result the point P_2 has coordinates

$$P_2 = \left\{ 2\mu + \frac{3}{2}\frac{\lambda_0^2}{\mu}, r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta \right\}, \quad r = \lambda_0\sqrt{3 + \frac{9}{4}\frac{\lambda_0^2}{\mu^2}} \tag{7.10}$$

where θ and φ are arbitrary quantities. Thus, spatial coordinates of the point P_2 are determined to within $\sqrt{3}\lambda_0$. In the limit $\lambda_0 \to 0$ the point P_2 is determined uniquely. Two solutions

$$P_2' = \left\{ 2\mu + \frac{3}{2}\frac{\lambda_0^2}{\mu}, 0, 0, r \right\}, \quad P_2'' = \left\{ 2\mu + \frac{3}{2}\frac{\lambda_0^2}{\mu}, 0, 0, -r \right\}$$

are divided by spatial distance $i\,|\mathbf{P}_2'\mathbf{P}_2''| = 2r \approx 2\sqrt{3}\lambda_0$ ($\lambda_0 \ll \mu$). It is a maximal distance between two solutions $\mathbf{P}_2'$ and $\mathbf{P}_2''$.

If $\lambda_0 = 0$, then the discrete geometry turns to the geometry of Minkowski, and $P_2 = \{2\mu, 0, 0, 0\}$

$$x^0 = 2\mu, \quad x^1 = 0, \quad x^2 = 0, \quad x^3 = 0 \tag{7.11}$$

follow from one equation $\mathbf{x}^2 = 0$. It means, that the geometry of Minkowski is a degenerate geometry, because different solutions of the discrete geometry merge into one solution of the geometry of Minkowski.

Let us consider the same problem for spacelike vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_1\mathbf{P}_2$, when

$$P_0 = \{0, 0, 0, 0\}, \quad P_1 = \{0, l, 0, 0\}, \quad P_2 = \{ct, x, y, z\} \tag{7.12}$$

We have the same equations (7.6), but now we have another solution

$$x = 2l + \frac{3\lambda_0^2}{2l}, \quad c^2t^2 - y^2 - z^2 = r^2 = 3\lambda_0^2 + \frac{9}{4}\frac{\lambda_0^4}{l^2} \tag{7.13}$$

The point P_2 has coordinates

$$P_2 = \left\{ \sqrt{a_2^2 + a_3^2 + r^2}, 2l + \frac{3\lambda_0^2}{2l}, a_2, a_3 \right\}, \quad r^2 = 3\lambda_0^2\left(1 + \frac{3\lambda_0^2}{4l^2}\right) \tag{7.14}$$

where a_2 and a_3 are arbitrary quantities. The difference between two solutions P_2' and P_2''

$$P_2' = \left\{ \sqrt{a_2^2 + a_3^2 + r^2}, 2l + \frac{3\lambda_0^2}{2l}, a_2, a_3 \right\}, \quad P_2'' = \left\{ \sqrt{b_2^2 + b_3^2 + r^2}, 2l + \frac{3\lambda_0^2}{2l}, b_2, b_3 \right\}$$

may be infinitely large

$$|\mathbf{P}_2'\mathbf{P}_2''| = \sqrt{2a_2b_2 + 2a_3b_3 - 2\sqrt{r^2 + a_2^2 + a_3^2}\sqrt{r^2 + b_2^2 + b_3^2} + 2r^2}$$

This difference remains very large, even if $\lambda_0 \to 0$.

Thus, both the discrete geometry and the geometry of Minkowski are multivariant with respect to spacelike vectors. However, this circumstance remains to be unnoticed in the conventional relativistic particle dynamics, because the spacelike vectors do not used there.

Multivariance of the discrete geometry leads to intransitivity of the equivalence relation of two vectors. Indeed, if $(\mathbf{Q}_0\mathbf{Q}_1\text{eqv}\mathbf{P}_0\mathbf{P}_1)$ and $(\mathbf{Q}_0\mathbf{Q}_1\text{eqv}\mathbf{P}_0\mathbf{P}_1')$, but vector $(\mathbf{P}_0\mathbf{P}_1\overline{\text{eqv}}\mathbf{P}_0\mathbf{P}_1')$. It means intransitivity of the equivalence relation. Besides, it means that the discrete geometry is nonaxiomatizable, because in any logical construction the equivalence relation is transitive.

Transitivity of the equivalence relation in the case of the proper Euclidean geometry is a corollary of the special conditions (5.1) – (5.7). In the case of the arbitrary physical geometry they are not satisfied in general.

Transport of a vector $\mathbf{P}_0\mathbf{P}_1$ to some point Q_0 leads to some indeterminacy of the result of this transport, because at the point Q_0 there are many vectors $\mathbf{Q}_0\mathbf{Q}_1$, $\mathbf{Q}_0\mathbf{Q}_1'$,..., which are equivalent to the vector $\mathbf{P}_0\mathbf{P}_1$.

Sum $\mathbf{Q}_0\mathbf{S}$ of two vectors $\mathbf{Q}_0\mathbf{Q}_1$ and $\mathbf{Q}_1\mathbf{S}$, when the end of one vector is an origin of the other, is defined by points Q_0 and S

$$\mathbf{Q}_0\mathbf{S} = \mathbf{Q}_0\mathbf{Q}_1 + \mathbf{Q}_1\mathbf{S} \tag{7.15}$$

Sum $\mathbf{Q}_0\mathbf{S}$ of two vectors $\mathbf{Q}_0\mathbf{Q}_1$ and $\mathbf{P}_0\mathbf{P}_1$ at the point Q_0 is defined by the relations

$$\mathbf{Q}_0\mathbf{S} = \mathbf{Q}_0\mathbf{Q}_1 + \mathbf{Q}_1\mathbf{S}, \qquad (\mathbf{Q}_1\mathbf{S}\text{eqv}\mathbf{P}_0\mathbf{P}_1) \tag{7.16}$$

In the discrete geometry the sum of two vectors is not unique, in general, because of multivariance of the equivalence relation.

Result of multiplication of a vector $\mathbf{P}_0\mathbf{P}_1$ by a real number a is not unique also. The result $\mathbf{P}_0\mathbf{S}$ of such a multiplication by a number a is defined by relations

$$\mathbf{P}_0\mathbf{P}_1 \upuparrows \mathbf{P}_0\mathbf{S} \wedge |\mathbf{P}_0\mathbf{S}| = a\,|\mathbf{P}_0\mathbf{P}_1| \tag{7.17}$$

or in terms of algebraic relations

$$((\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{S}) = a\,|\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{P}_0\mathbf{P}_1|) \wedge |\mathbf{P}_0\mathbf{S}| = a\,|\mathbf{P}_0\mathbf{P}_1| \tag{7.18}$$

Thus, results of vectors summation and of a multiplication of a vector by a real number are not unique, in general, in the discrete geometry. It means, that one cannot introduce a linear vector space in the discrete geometry.

Let the discrete geometry is described by n coordinates. Let the skeleton $\mathcal{P}_n = \{P_0, P_1, ...P_n\}$ determine n vectors $\mathbf{P}_0\mathbf{P}_k$, $k = 1, 2, ...n$, which are linear independent in the sense

$$F_n(\mathcal{P}_n) = \det \|(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)\| \neq 0 \qquad i, k = 1, 2, ...n \tag{7.19}$$

One can determine uniquely projections of a vector $\mathbf{Q}_0\mathbf{Q}_1$ onto vectors $\mathbf{P}_0\mathbf{P}_k$, $k = 1, 2, ...n$ by means of relations

$$\Pr(\mathbf{Q}_0\mathbf{Q}_1)_{\mathbf{P}_0\mathbf{P}_k} = \frac{(\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_k)}{|\mathbf{P}_0\mathbf{P}_k|} \tag{7.20}$$

However, one cannot reestablish the vector $\mathbf{Q}_0\mathbf{Q}_1$, using its projections onto vectors $\mathbf{P}_0\mathbf{P}_k$, $k = 1, 2, ...n$, because a summation of vector components is multivariant. Thus, all operations of the linear vector space are not unique in the discrete geometry.

Mathematical technique of continuous geometry is not adequate for application in a discrete geometry, because it is too special and adapted for a continuous (differential) geometry. This circumstance is especially important in a description of the elementary particle dynamics. The state of a particle cannot be described by its position and its momentum, because the limit (1.3) does not exist in a discrete geometry. Besides, dynamic equations cannot be differential equations.

8 Skeleton conception of particle dynamics

An elementary particle is a physical body. In the discrete space-time geometry a position of a physical body is described by its skeleton $\mathcal{P}_n = \{P_0, P_1, ..P_n\}$. Of course, such a description of a physical body position may be used in any space-time geometry. The skeleton is an analog of the frame of reference attached rigidly to the particle (physical body). Tracing the skeleton motion, one traces the physical body motion. Direction of the skeleton displacement is described by the leading vector $\mathbf{P}_0\mathbf{P}_1$.

The skeleton motion is described by a world chain $\mathcal{C}$ of connected skeletons

$$\mathcal{C} = \bigcup_{s=-\infty}^{s=+\infty} \mathcal{P}_n^{(s)} \tag{8.1}$$

Skeletons $\mathcal{P}_n^{(s)}$ of the world chain are connected in the sense, that the point P_1 of a skeleton is a point P_0 of the adjacent skeleton. It means

$$P_1^{(s)} = P_0^{(s+1)}, \qquad s = ...0, 1, ... \tag{8.2}$$

The vector $\mathbf{P}_0^{(s)}\mathbf{P}_1^{(s)} = \mathbf{P}_0^{(s)}\mathbf{P}_0^{(s+1)}$ is the leading vector, which determines the direction of the world chain.

If the particle motion is free, the adjacent skeletons are equivalent

$$\mathcal{P}_n^{(s)}\text{eqv}\mathcal{P}_n^{(s+1)} : \qquad \mathbf{P}_i^{(s)}\mathbf{P}_k^{(s)}\text{eqv}\mathbf{P}_i^{(s+1)}\mathbf{P}_k^{(s+1)}, \qquad i, k = 0, 1, ...n, \qquad s = ..0, 1, .. \tag{8.3}$$

If the particle is described by the skeleton $\mathcal{P}_n^{(s)}$, the world chain (8.1) has $n(n+1)/2$ invariants

$$\mu_{ik} = \left|\mathbf{P}_i^{(s)}\mathbf{P}_k^{(s)}\right|^2 = 2\sigma\left(P_i^{(s)}, P_k^s\right), \qquad i, k = 0, 1, ...n, \qquad s = ...0, 1, ... \tag{8.4}$$

which are constant along the whole world chain.

Equations (8.3) form a system of $n(n+1)$ difference equations for determination of nD coordinates of n skeleton points $\{P_1, P_2, ..P_n\}$, where D is the dimension of the space-time. The number of dynamical variables, liable for determination distinguishes, in general, from the number of dynamic equations. It is the main difference between the skeleton conception of particle dynamics and the conventional conception of particle dynamics, where the number of dynamic variables coincides with the number of dynamic equations.

In the case of pointlike particle, when $n = 1$, $D = 4$, the number of equations $n_e = 2$, whereas the number of variables $n_v = 4$. The number of equations is less, than the number of dynamic variables. In the discrete space-time geometry (1.2) the position of the adjacent skeleton is not uniquely determined. As a result the world chain wobbles. In the nonrelativistic approximation a statistical description of the stochastic world chains leads to the Schrödinger equations [5], if the elementary length λ_0 has the form

$$\lambda_0^2 = \frac{\hbar}{bc} \tag{8.5}$$

where $\hbar$ is the quantum constant, c is the speed of the light and b is a universal constant, connecting the particle mass m with the length μ of the world chain link

$$m = b\mu$$

Dynamic equations (8.3) are difference equations. At the large scale, when one may go to the limit $\lambda_0 = 0$, the dynamic equations (8.3) turn to the differential dynamic equations. In the case of pointlike particle ($n = 1$) and of the Kaluza-Klein five-dimensional space-time geometry these equation describe the motion of a charged particle in the given electromagnetic field. One can see in this example, that the space-time geometry "assimilates" the electromagnetic field. It means that one may consider only a free particle motion, keeping in mind, that the space-time geometry can "assimilate" all force fields.

Dynamic equations (8.3) realize the skeleton conception of particle dynamics in the microcosm. The skeleton conception of dynamics distinguishes from the conventional conception of particle dynamics in the relation, that the number of dynamic equations may differ from the number of dynamic variables, which are to be determined. In the conventional conception of particle dynamics the number of dynamic equations (first order) coincides always with the number of dynamic variables, which are to be determined. As a result the motion of a particle (or of an averaged particle) appears to be deterministic. In the case of quantum particles, whose motion is stochastic (indeterministic), the dynamic equations are written for a statistical ensemble of indeterministic particles (or for the statistically averaged particle).

In the conventional conception of dynamics one can obtain dynamic equation for the statistically averaged particle (i.e. statistical ensemble normalized to one particle), but there are no dynamic equations for a single stochastic particle. In the skeleton conception of the particle dynamics there are dynamic equations for a single particle. These equations are many-valued (multivariant), but they do exist. In the conventional conception of the particle dynamics one can derive dynamic equations for the statistically averaged particle, which are a kind of equations for a fluid (continuous medium). But one cannot obtain dynamic equations for a single indeterministic particle [6].

The skeleton conception of the particle dynamics realizes a more detailed description of elementary particle. One may hope to obtain some information on the elementary particle structure.

We have now only two examples of the skeleton conception application. Considering compactification in the 5-dimensional discrete space-time geometry of Kaluza-Klein, and imposing condition of uniqueness of the world function, one obtains that the value of the electric charge of a stable elementary particle is restricted by the elementary charge [7]. This result has been known from experiments, but it could not be explained theoretically, because in the continuous space-time geometry nobody considers the world function as a fundamental quantity, and one does not demand its uniqueness.

Another example concerns structure of Dirac particles (fermions). Consideration in framework of skeleton conception [8] shows, that a world chain of a fermion is a (spacelike or timelike) helix with timelike axis. The averaged world chain of a free fermion is a timelike straight line. The helical motion of a skeleton generates an angular moment (spin) and magnetic moment. Such a result looks rather reasonable. In the conventional conception of the particle dynamics the spin and magnetic moment of a fermion are postulated without a reference to its structure.

9 Concluding remarks

Thus, the supposition on the space-time geometry discreteness seems to be more natural and reasonable, than the supposition on quantum nature of physical phenomena in microcosm. Discreteness is simply a property of the space-time, whereas quantum principles assume introduction of new essences.

Formalism of the discrete geometry is very simple. It does not contain theorems with complicated proofs. Nevertheless the discrete geometry and its formalism is perceived hardly. The discrete geometry was not developed in the twentieth century, although the discrete space-time was necessary for description of physical phenomena in microcosm. It was rather probably, that the space-time is discrete in microcosm. What is a reason of the discrete geometry disregard? We try to answer this important question.

The discrete geometry was not developed, because it could be obtained only as a generalization of the proper Euclidean geometry. But almost all concepts and quantities of the proper Euclidean geometry use essentially concepts of the continuous geometry. They could not be used for construction of a discrete geometry. Only world function (or distance) does not use a reference to the geometry continuity. Only coordinateless expressions (2.9) –(2.13) of basic quantities of the Euclidean geometry in terms of world function admit one to construct a discrete geometry and other physical geometries.

Assurance, that any geometry is to be axiomatizable, was the second obstacle on the way of the discrete geometry construction. The fact, that the proper Euclidean geometry is a degenerate geometry, was the third obstacle. In particular, being a physical geometry, the proper Euclidean geometry is an axiomatizable geometry, and this circumstance is an evidence of its degeneracy. It is very difficult to obtain a general conception as a generalization of a degenerate conception, because some different quantities of the general conception coincide in the degenerate conception. It is rather difficult to disjoint them. For instance, a physical geometry is multivariant, in general. Single-variant physical geometry is a degenerate geometry. In the physical geometry the straight segment (2.6) is a surface (tube), in general. In the degenerate physical geometry (the proper Euclidean geometry) the straight segment is a one-dimensional set. How can one guess, that a straight segment is a surface, in general? Besides, multivariance of the equivalence relation leads to nonaxiomatizability of geometry. But we learn only axiomatizable geometries in the last two thousand years. How can we guess, that nonaxiomatizable geometries exist? The straight way from the Euclidean geometry to physical geometries was very difficult, and the physical geometry has been obtained on an oblique way.

J.L.Synge [9, 10] has introduced the world function for description of the Riemannian geometry. I was a student. I did not know the papers of Synge, and I introduced the world function for description of the Riemannian space-time in general relativity. My approach differed slightly from the approach of Synge. In particular, I had obtained an equation for the world function of Riemannian geometry [11], which contains only the world function and their derivatives,

$$\frac{\partial\sigma\left(x,x'\right)}{\partial x^{i}}G^{ik'}\left(x,x'\right)\frac{\partial\sigma\left(x,x'\right)}{\partial x'^{k}}=2\sigma\left(x,x'\right),\qquad G^{ik'}\left(x,x'\right)G_{lk'}\left(x,x'\right)=\delta_{l}^{i},\quad (9.1)$$

where

$$G_{lk'}\left(x,x'\right)\equiv\frac{\partial^{2}\sigma\left(x,x'\right)}{\partial x^{l}\partial x'^{k}},\quad l,k=0,1,2,3$$

This equation was obtained as a corollary of definition of the world function of the Riemannian geometry as an integral along the geodesic, connecting points x and x'. This equation contains only world function and its derivatives, but it does not contain a metric tensor.

This equation put the question. Let a world function do not satisfy the equation. Does this world function describe a non-Riemannian geometry or it does describe no geometry? It was very difficult to answer this question. On one hand, the formalism, based on the world function, is a more developed formalism, than formalism based on a use of metric tensor, because a geodesic is described in terms of the world function by algebraic equation (2.6), whereas the same geodesic is described by differential equations in terms the metric tensor.

On the other hand, the geodesic, described by (2.6) is one-dimensional only in the Riemannian geometry. In general, one equation (2.6) in n-dimensional space describes a $(n-1)$-dimensional surface. I did not know, whether the surface is a generalization of a geodesic in any geometry. I was not sure, because in the Euclidean geometry a straight segment is one-dimensional be definition. I left this question unsolved and returned to it almost thirty years later, in the beginning of ninetieth.

When the string theory of elementary particles appeared, it becomes clear for me, that the particle may be described by means of a world surface (tube) but not only by a world line. As the particle world line associates with a geodesic, I decided, that a world tube may describe a particle. It meant that there exist space-time geometries, where straights (geodesics) are described by world tubes. The question on possibility of the physical space-time geometry has been solved for me finally, when the quantum description appeared to be a corollary of the space-time multivariance [5].

References

[1] K. Menger, Untersuchen über allgemeine Metrik, *Mathematische Annalen,* **100**, 75-113, (1928).

[2] L.M. Blumenthal, *Theory and Applications of Distance Geometry*, Oxford, Clarendon Press, 1953

[3] Yu.A.Rylov, Non-Euclidean method of the generalized geometry construction and its application to space-time geometry in *Pure and Applied Differential geometry* pp.238-246. eds. Franki Dillen and Ignace Van de Woestyne. Shaker Verlag, Aachen, 2007. See also *e-print Math.GM/0702552*

[4] Yu.A.Rylov, Geometry without topology as a new conception of geometry. *Int. Jour. Mat. & Mat. Sci.* **30**, *iss. 12, 733-760, (2002).*

[5] Yu.A.Rylov, Non-Riemannian model of the space-time responsible for quantum effects. *Journ. Math. Phys.* **32(8)**, 2092-2098, (1991)

[6] Yu. A. Rylov, Uniform formalism for description of dynamic, quantum and stochastic systems *e-print /physics/0603237v6.*

[7] Yu. A. Rylov, Discriminating properties of compactification in discrete uniform isotropic space-time. *e-print 0809.2516v2*

[8] Yu. A. Rylov, Geometrical dynamics: spin as a result of rotation with superluminal speed. *e-print 0801.1913.*

[9] J.L. Synge, A characteristic function in Riemannian space and its applications to the soution of geodesic triangles. *Proc. London Math. Soc.* **32**, *241, (1931).*

[10] J.L.Synge, *Relativity: the General Theory.* Amsterdam, North-Holland Publishing Company, 1960.

[11] Yu.A.Rylov, On a possibility of the Riemannian space description in terms of a finite interval. *Izvestiya Vysshikh Uchebnych Zavedenii, Matematika. No.3(28), 131-142.* (1962). (in Russian).