Metrical conception of the space-time geometry

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Abstract

Primordially a geometry was a science on properties of geometrical objects and their mutual disposition. A use of the proper Euclidean geometry generated the axiomatic conception of geometry, where the geometry is considered as a logical construction. There is the metrical conception of a geometry, where the geometry is considered as a science on properties of geometric objects. In the framework of metrical conception the space-time geometries form a more powerful set of geometries, than those do in the framework of the axiomatic conception. It is important at the construction of the general relativity.

Key words: metric conception; axiomatic conception; tachyon gas; dark matter;

1 Introduction

Conception of the space-time geometry describes an interaction of concepts in the space-time geometry. Hierarchy of concepts is essential in the conception. There are two conceptions in the space-time geometry: (1) axiomatic conception and (2) metric conception.

Geometry has been arisen many years ago as a science on a shape of geometrical objects and on their mutual disposition in space. It was the proper Euclidean geometry $\mathcal{G}_{\rm E}$. Any geometrical object in $\mathcal{G}_{\rm E}$ can be constructed of blocks. Blocks are segments of straight line. Any geometrical object $\mathcal{O}$ can be filled by a set $\mathcal{S}$ of straight line segments L in such a way, that any point $\forall P \in \mathcal{O}$ belongs to one and only one segment $L \in \mathcal{S}$. Segments L have no common points. This property of $\mathcal{G}_{\rm E}$ can be used for construction of any geometrical object $\mathcal{O}$ of the Euclidean geometry $\mathcal{G}_{\rm E}$. Properties of the straight line segment can be formulated as some statements St_1. The rules of displacement of the straight line segments can be also formulated as some statements St_2. Using these statements St_1 and St_2, one can formulate the rules for construction of any geometrical object in $\mathcal{G}_{\mathrm{E}}$. Considering $St = St_1 \wedge St_2$ as basic statements (axioms) of $\mathcal{G}_{\mathrm{E}}$, one can obtain the rules of any geometrical object construction as a logical corollary of St and of definition of the geometric object. These rules can be formulated as some statements. The set of these statements forms the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$. Such a form of the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ presentation can be qualified as the axiomatic conception of $\mathcal{G}_{\mathrm{E}}$.

The Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ is considered formally as a logical construction founded on the set St of Euclidean axioms. Usually one does not consider the reasons, why the logical construction describes the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$. One believes, that any logical construction, containing axioms about properties of the simplest geometrical objects such as the straight line, describes some geometry $\mathcal{G}$ which may differ from $\mathcal{G}_{\mathrm{E}}$. The symplectic geometry has no relation to properties of geometrical objects. Nevertheless, it is treated as some kind of a geometry, because it is a logical construction, which is close to the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$.

For construction of a generalized geometry $\mathcal{G}$ one uses another set $\mathcal{S}_{\mathrm{g}}$ of axioms $\mathcal{A}_{\mathrm{g}}$. If the axioms $\mathcal{S}_{\mathrm{g}}$ are consistent, one obtains a generalized geometry $\mathcal{G}$, which is a logical construction in the framework of the axiomatic conception of a geometry. In such a way one constructed the geometry $\mathcal{G}_{\mathrm{M}}$ of the space-time, known as the geometry of Minkowski. Such a logical construction is possible, because axioms describing properties of straight lines are practically the same in $\mathcal{G}_{\mathrm{E}}$ and in $\mathcal{G}_{\mathrm{M}}$. However, if the properties of the straight line segments in the generalized geometry $\mathcal{G}$ differ from those in $\mathcal{G}_{\mathrm{E}}$, the logical construction of $\mathcal{G}$ becomes to be problematic. Let us imagine that the straight line segment L_{g} in $\mathcal{G}$ has the shape of a hallow tube. In this case a usage of the segments L_{g} as constructing blocks for construction of geometric objects becomes to be impossible. In the microcosm the real space-time geometry is discrete, and the straight line segments have the shape of hallow tubes. In this case axiomatic conception of the space-time geometry cannot describe the real space-time geometry. Then a connection between the geometry as a logical construction and geometry as a science on the shape of geometrical objects fails. It means that capacities of the axiomatic conception for construction of generalized geometries are restricted.

Besides, a construction of a geometrical object in $\mathcal{G}_{\mathrm{E}}$ and in the generalized geometry $\mathcal{G}$, obtained from $\mathcal{G}_{\mathrm{E}}$ in the framework of the axiomatic conception, needs a proof of numerous theorems. In other words, a construction of a geometrical object in $\mathcal{G}$ needs a repeating of the construction of this object in $\mathcal{G}_{\mathrm{E}}$. There is another way of a geometrical object $\mathcal{O}$ construction in $\mathcal{G}$. One constructs $\mathcal{O}_{\mathrm{E}}$ in $\mathcal{G}_{\mathrm{E}}$ and thereafter one deforms $\mathcal{G}_{\mathrm{E}}$ into $\mathcal{G}$. At this deformation the geometrical object $\mathcal{O}_{\mathrm{E}}$ in $\mathcal{G}_{\mathrm{E}}$ is deformed in the geometrical object $\mathcal{O}$ in $\mathcal{G}$. The deformation procedure of $\mathcal{O}_{\mathrm{E}}$ is much simpler, than the construction of $\mathcal{O}_{\mathrm{E}}$ and $\mathcal{O}$ from constructing blocks (it means a proof of theorems). Such a procedure of the geometric object construction is used in the metric conception of a geometry. This procedure (deformation) is very simple.

The fact is that, the Euclidean geometry can be presented completely in terms of its metric $\rho(P,Q)$. Here the quantity $\rho(P,Q)$ is the distance between $\forall P,Q\in\Omega$. Here Ω is the set of points, where the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ is given. Instead of ρ it is convenient to use the world function $\sigma(P,Q)=\frac{1}{2}\rho^2(P,Q)$. The world function is real even in the geometry of Minkowski, where the distance ρ is imaginary for spacelike intervals.

2 Metric conception of Euclidean geometry

Idea of metric conception is not new [1]. Unfortunately, the distance geometry of Blumenthal [1] is not a monistic conception, when a geometry is described completely in terms of distance and only in terms of distance. Blumenthal failed to construct a monistic conception of geometry, although the monism is very important for the metric conception construction.

In the metric conception the world function σ of a generalized geometry is defined as a single-valued function

$$\sigma: \quad \Omega\times\Omega\rightarrow\mathbb{R},\quad \sigma(P,Q)=\sigma(Q,P),\quad \sigma(P,P)=0,\quad \forall P,Q\in\Omega \tag{1}$$

Here Ω is the the point set, where the geometry is given. In the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ the world function σ_{E} has definite properties which can be described by additional restrictions.

The Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ is formulated in terms of the world function σ_{E} and only in terms of the world function as follows. Geometrical vector (g-vector) $\mathbf{PQ}=\{P,Q\}$ is defined as an ordered set of two points $P,Q\in\Omega$. The term "geometrical vector" is used in order to distingush it from the linvector u, which is defined as an element of the linear vector space $\mathcal{L}_n$, which is used in $\mathcal{G}_{\mathrm{E}}$, and $u\in\mathcal{L}_n$ In $\mathcal{G}_{\mathrm{E}}$ linvector and g-vector can coincide. However after deformation of $\mathcal{G}_{\mathrm{E}}$ they do not coincide, generally speaking.

Two g-vectors $\mathbf{PQ}$ and $\mathbf{RS}$ are equivalent ($\mathbf{PQ}\mathrm{eqv}\mathbf{RS}$), if their lengths are equal and they are in parallel ($\mathbf{PQ}\upuparrows\mathbf{RS}$)

$$(\mathbf{PQ}\upuparrows\mathbf{RS}):\quad (\mathbf{PQ}.\mathbf{RS})=|\mathbf{PQ}|\cdot|\mathbf{RS}| \tag{2}$$

Here $(\mathbf{PQ}.\mathbf{RS})$ is the scalar product of two g-vectors $\mathbf{PQ}$ and $\mathbf{RS}$, defined in terms of the world function in the form

$$(\mathbf{PQ}.\mathbf{RS})=\sigma(P,S)+\sigma(Q,R)-\sigma(P,R)-\sigma(Q,S) \tag{3}$$

The length $|\mathbf{PQ}|$ of the g-vector $\mathbf{PQ}$ is defined by the relation

$$|\mathbf{PQ}|=\sqrt{2\sigma(P,Q)} \tag{4}$$

Here σ is the world function of $\mathcal{G}_{\mathrm{E}}$. Thus, the g-vectors $\mathbf{PQ}$ and $\mathbf{RS}$ are equivalent, if

$$(\mathbf{PQ}\mathrm{eqv}\mathbf{RS}):\quad (\mathbf{PQ}.\mathbf{RS})=|\mathbf{PQ}|\cdot|\mathbf{RS}|\wedge|\mathbf{PQ}|=|\mathbf{RS}| \tag{5}$$

The scalar product of two g-vectors is defined by (3). Equivalence of two g-vectors is defined by (5).

n g-vectors $\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2, ...\mathbf{P}_0\mathbf{P}_n$ are linear dependent, if and only if the Gram determinant

$$F_n(\mathcal{P}_n) = \det ||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)||, \quad i,k = 1,2,...n, \quad \mathcal{P}_n \equiv \{P_0, P_2, ...P_n\} \tag{6}$$

vanishes

$$F_n(\mathcal{P}_n) = 0 \tag{7}$$

The relations (4), (5), (7) are general geometric relations. They are valid in any generalized geometry $\mathcal{G}$ obtained as a result of deformation of $\mathcal{G}_{\rm E}$. Deformation of $\mathcal{G}_{\rm E}$ into $\mathcal{G}$ is obtained at the replacement $\sigma_{\rm E} \to \sigma$ in relations (4), (5), (7). Here $\sigma_{\rm E}$ is the world function of $\mathcal{G}_{\rm E}$, and σ is the world function of $\mathcal{G}$.

The special relations of the n-dimensional proper Euclidean geometry have the form [2]:

I. Definition of the metric dimension:

$$\exists \mathcal{P}_n \equiv \{P_0, P_1, ...P_n\} \subset \Omega, \qquad F_n(\mathcal{P}_n) \neq 0, \qquad F_k\left(\Omega^{k+1}\right) = 0, \qquad k > n \tag{8}$$

where $F_n(\mathcal{P}_n)$ is the n-th order Gram's determinant (6). g-vectors $\mathbf{P}_0\mathbf{P}_i$, $i = 1,2,...n$ are basic g-vectors of the rectilinear coordinate system K_n with the origin at the point P_0. The covariant coordinates $x_i(P)$ of the point P in the coordinate system K_n are defined by the relation

$$x_i(P) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}), \qquad i = 1,2,...n \tag{9}$$

The metric tensors $g_{ik}(\mathcal{P}_n)$ and $g^{ik}(\mathcal{P}_n)$, $i,k = 1,2,...n$ in K_n are defined by the relations

$$\sum_{k=1}^{k=n} g^{ik}(\mathcal{P}_n)\, g_{lk}(\mathcal{P}_n) = \delta_l^i, \qquad g_{il}(\mathcal{P}_n) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_l), \qquad i,l = 1,2,...n \tag{10}$$

II. Linear structure of the Euclidean space:

$$\sigma_{\rm E}(P,Q) = \frac{1}{2}\sum_{i,k=1}^{i,k=n} g^{ik}(\mathcal{P}_n)\,(x_i(P) - x_i(Q))\,(x_k(P) - x_k(Q)), \qquad \forall P,Q \in \Omega \tag{11}$$

where coordinates $x_i(P)$, $x_i(Q)$, $i = 1,2,...n$ of the points P and Q are covariant coordinates of the g-vectors $\mathbf{P}_0\mathbf{P}$, $\mathbf{P}_0\mathbf{Q}$ respectively in the coordinate system K_n.

III: The metric tensor matrix $g_{lk}(\mathcal{P}^n)$ has only positive eigenvalues g_k

$$g_k > 0, \qquad k = 1,2,...,n \tag{12}$$

IV. The continuity condition: the system of equations

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}) = y_i \in \mathbb{R}, \qquad i = 1,2,...n \tag{13}$$

considered to be equations for determination of the point P as a function of coordinates $y = \{y_i\}$, $i = 1, 2, ...n$ has always one and only one solution. Conditions I – IV contain a reference to the dimension n of the Euclidean space, which is defined by the relations (8). The conditions I – IV are necessary and sufficient conditions of the fact that the world function is the world function of n-dimensional Euclidean geometry.

In the framework of the metric conception any generalized geometry is obtained by repalcement of the Euclidean world function σ_{E} by the world function σ of the generalized geometry $\mathcal{G}$ in relations (4), (5), (7). Such a replacement in the special relations of $\mathcal{G}_{\mathrm{E}}$ is not produced, because these relations describe properties of the world function σ_{E} of $\mathcal{G}_{\mathrm{E}}$. There is not yet established name for the generalized geometry $\mathcal{G}$, obtained in the framework of the metric conception. One uses the names T-geometry (tubular geometry) and physical geometry. Here I shall use the name physical geometry.

3 Definition of geometrical objects

Geometrical object $\mathcal{O} \subset \Omega$ is a subset of points in the point set Ω, where the geometry is given. In the physical geometry the geometric object $\mathcal{O}$ is described by means of the skeleton-envelope method. It means that any geometric object $\mathcal{O}$ is considered to be a set of intersections and joins of elementary geometric objects (EGO).

The finite set $\mathcal{P}_n \equiv \{P_0, P_1, ..., P_n\} \subset \Omega$ of parameters of the envelope function $f_{\mathcal{P}_n}$ is the skeleton of elementary geometric object (EGO) $\mathcal{E} \subset \Omega$. The set $\mathcal{E} \subset \Omega$ of points forming a boundary of EGO is called the envelope of its skeleton $\mathcal{P}_n$. The envelope function $f_{\mathcal{P}_n}$

$$f_{\mathcal{P}_n}: \qquad \Omega \rightarrow \mathbb{R}, \tag{14}$$

determining the boundary of EGO is a function of the running point $R \in \Omega$ and of parameters $\mathcal{P}^n \subset \Omega$. The envelope function $f_{\mathcal{P}_n}$ is supposed to be an algebraic function of s arguments $w = \{w_1, w_2, ...w_s\}$, $s = (n+2)(n+1)/2$. Each of arguments $w_k = \sigma(Q_k, L_k)$ is the world function σ of two points $Q_k, L_k \in \{R, \mathcal{P}_n\}$, either belonging to skeleton $\mathcal{P}_n$, or coinciding with the running point R. Thus, the boundary $\mathcal{E}$ of any elementary geometric object is determined by its skeleton $\mathcal{P}_n$ and its envelope function $f_{\mathcal{P}_n}$. The boundary $\mathcal{E}$ of a geometric object $\mathcal{O}$ is the set of zeros of the envelope function

$$\mathcal{E} = \{R | f_{\mathcal{P}_n}(R) = 0\} \tag{15}$$

The envelope $\mathcal{E}$ is a boundary of the point set $\mathcal{O}$, forming the geometric object.

Characteristic points of the EGO are the skeleton points $\mathcal{P}_n \equiv \{P_0, P_1, ..., P_n\}$. The simplest example of EGO is the segment $\mathcal{T}_{[P_0 P_1]}$ of the straight line between the points P_0 and P_1, which is defined by the relation

$$\begin{aligned} \mathcal{T}_{[P_0 P_1]} &= \{R | f_{P_0 P_1}(R) = 0\}, & (16) \\ f_{P_0 P_1}(R) &= \sqrt{2\sigma(P_0, R)} + \sqrt{2\sigma(R, P_1)} - \sqrt{2\sigma(P_0, P_1)} & (17) \end{aligned}$$

The set of points $\mathcal{T}_{[P_0P_1]}$, defined by (16), (17) is a segment of a straight line in any physical geometry. In the space-time it is a three-dimensional surface, generally speaking. In the space-time geometry of Minkowski $\mathcal{T}_{[P_0P_1]}$ degenerates into one-dimensional line for a timelike g-vector $\mathbf{P}_0\mathbf{P}_1$, but it remains a three-dimensional surface for a spacelike g-vector $\mathbf{P}_0\mathbf{P}_1$.

The physical geometry is formulated without a use of coordinates. It is important in the case, when a physical body travels from the space-time region with the geometry $\mathcal{G}_1$ to the region with the space-time geometry $\mathcal{G}_2$. Geometrical object $\mathcal{O}$ is a geometrical image of a physical body. The geometric object $\mathcal{O}$ is described by its skeleton $\mathcal{P}_n = \{P_0, P_1, ...P_n\}$ and its envelope function $f_{\mathcal{P}_n}$. All points of the skeleton are connected rigidly in the sense, that distances between the skeleton points are the same in any physical geometry

$$\sigma_1 (P_i, P_k) = \sigma_2 (P_i, P_k), \quad i, k = 0, 1, ...n$$

where σ_1 is the world function in the geometry $\mathcal{G}_1$ and σ_2 is the world function in the geometry $\mathcal{G}_2$. The envelope function (14) as a function of arguments w_s is the same in geometries $\mathcal{G}_1$ and $\mathcal{G}_2$, although values of some arguments w_s are different in geometries $\mathcal{G}_1$ and $\mathcal{G}_2$

The Riemannian geometry which is used as the most general geometry in the axiomatic conception cannot be described without a reference to a coordinate system. It is a defect of the axiomatic conception with respect to the metric conception. One cannot identify the same geometric object in different space-time geometries described in the axiomatic conception. It is also a defect of the axiomatic conception The set of physical space-time geometries is more powerful, than the set of Riemannian space-time geometries. It is also a defect of the axiomatic conception at its use as a space-time geometry..

In the general relativity, where the space-time geometry is determined by the matter distribution, one should use the metric conception for a description of the space-time geometry. A use of the metric conception leads to impossibility of the dark holes formation [3]. The reason of such an impossibility is the induced anti-gravitation, arising in the case, when the matter is very dense [4].

In the Minkiwski space-time existence of taxions depends on the conception of the space-time geometry. In the axiomatic conceptions existence of tachyons is considered to be impossible. In the metric conception world line of a tachyon wobbles with infinite amplitude, and detection of a single tachyon is impossible. However, the tachyon gas forms the dark matter and its gravitational influence can be detected [5, 6]. The metric conception of the space-time geometry solves easily the problem of the dark matter, which fails to be solved in the framework of the axiomatic conception.

4 Preference of axiomatic conception. Social reasons of such a preference

The axiomatic conception of the space-time geometry exists many years. The metric conception of the space-time geometry exists about twenty years. The scientific community does not perceive the metric conception, although nobody can take objection against it. What is a reason of such behaviour? The scientific community perceives easily very exotic new ideas, *provided they do not need a revision of the existing theory*. However, if one speaks about a new conception which needs a revision of the existing conception, the scientific community does not consider the new conception. A transition to a new conception is especially difficult, if the new conception uses a new mathematical formalism. There are examples of such a transition to a new conception in the science history. For instance, the transition from the Aristotelian conception of mechanics to the Newtonian conception of mechanics lasted almost hundred years, because the scientific community did not accept the concept of inertia. Papers by Boltzmann on foundation of axiomatic thermodynamics were not accepted, because one needs to revise the thermodynamic laws, which were exact in thermodynamics and which are fulfilled on the average in the Boltzmann's papers.

Conception of the particle dynamics, where the statistical ensemble is considered as the basic object of dynamics, admits one to describe quantum particles as classical stochastic particles [7]. In this classical conception of the quantum particles dynamics the quantum principles are not used, and there is no necessity to consider the quantum principles as the first laws of nature. For instance, one does not need to quantize the gravitational field. Nevertheless the problem of the gravitational field quantization is considered as one of the main problems of theoretical physics.

In the space-time geometry also there are problems with transition from axiomatic conception to metric conception of the space-time geometry. I believe that disinclination of the scientific community to consider new conception is conditioned by disinclination of revising the existing conception.

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