Metric Space: Classification of Finite Subspaces instead of Constraints on Metric

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Abstract

A new method of metric space investigation, based on classification of its finite subspaces, is suggested. It admits one to derive information on metric space properties which is encoded in metric and to describe geometry in terms of only metric. The method admits one to remove constraints imposed usually on metric (the triangle axiom and nonnegativity of the squared metric). Elimination of the triangle axiom leads to "tubular generalization" of metric geometry (T-geometry), when the shortests are replaced by hallow tubes. Elimination of the second constraint admits one to use the metric space for description of the space-time and of other geometries with indefinite metric.

1 Introduction

The shortest (line), connecting two arbitrary points $P, Q \in \Omega$ of the metric space $\{\rho, \Omega\}$, is the basic geometrical object which is constructed usually in the metric space [1]. One can construct an angle, triangle, different polygons from segments of the shortest. Construction of two-dimensional and three-dimensional planes in the metric space is rather problematic. At any rate it is unclear how one could construct these planes, using the shortest as the main geometrical object. A possibility of the metric space description in terms of only the shortest is restricted. Although exhibiting ingenuity, such a description may be constructed. For instance, A.D. Alexandrov showed that internal geometry of two-dimensional boundaries of convex three-dimensional bodies may be represented in terms of metric [2]. Apparently, without introducing geometric objects which are analogs of two-dimensional plane, the solution of similar problem for three-dimensional boundaries of four-dimensional bodies is very difficult.

In reality in the scope of the metric geometry one can construct all geometric objects which can be constructed in the Euclidean geometry, including n-dimensional planes. It is necessary only to use more effective method of the metric space description, than that based on the use of the shortest. To formulate this method let us give necessary definitions.

Definition 1.1 *The metric space $M=\{\rho,\Omega\}$ is a set Ω of points $P\in\Omega$ with the metric ρ given on $\Omega\times\Omega$*

$$\rho: \quad \Omega\times\Omega\rightarrow D_{+}\subset\mathbb{R} \tag{1.1}$$

$$\rho(P,P)=0, \qquad \rho(P,Q)=\rho(Q,P), \qquad \forall P,Q\in\Omega \tag{1.2}$$

$$D_{+}=[0,\infty), \qquad \rho(P,Q)=0, \quad \text{if and only if} \quad P=Q, \qquad \forall P,Q\in\Omega \tag{1.3}$$

$$\rho(P,Q)+\rho(Q,R)\geq\rho(P,R), \qquad \forall P,Q,R\in\Omega \tag{1.4}$$

Definition 1.2 *Any subset $\Omega'\subset\Omega$ of points of the metric space $M=\{\rho,\Omega\}$, equipped with the metric ρ' which is a contraction $\rho|_{\Omega'\times\Omega'}$ of the mapping (1.1). on the set $\Omega'\times\Omega'$ is called the metric subspace $M'=\{\rho',\Omega'\}$ of the metric space $M=\{\rho,\Omega\}$.*

It is easy to see that the metric subspace $M'=\{\rho',\Omega'\}$ is a metric space.

Definition 1.3 *The metric space $M=\{\rho,\Omega\}$ is called a finite one, if the set Ω contains a finite number of points. The finite metric subspace $M(\mathcal{P}^n)=\{\rho,\mathcal{P}^n\}$ of $M=\{\rho,\Omega\}$, consisting of $n+1$ points $\mathcal{P}^n\equiv\{P_0,P_1,\ldots P_n\}\subset\Omega$ is called the nth order metric subspace.*

Definition 1.4 *A description is called σ-immanent, if it does not contain any references to objects or concepts other than subspaces of the metric space and its metric.*

The prefix σ associtės with the function σ, defined via metric ρ by the relation $\sigma=\frac{1}{2}\rho^2$. The function σ is called the world function. This name was suggested by Synge [3], who introduced it for the Riemannian space description and used it for description of the space-time geometry. In many cases a use of the world function σ instead of the metric ρ appears to be more convenient. σ-immanency of a description means that it is immanent to the metric space, and it is carried out in terms of its metric and subspaces.

The main idea of the suggested method is a description of the metric space in terms of its finite subspaces. They are rather simple geometric objects, consisting of several points. The finite metric subspaces $M(\mathcal{P}^n)$ are the simplest constituents of the metric space. Their properties were studied [4] from different points of view, but they were not considered as primitive geometrical objects.

Now they are considered as primitive objects of metric space. At first, one obtains σ-immanent description of any geometrical object of Euclidean space considered to be a metric space. Thereafter one replaces Euclidean metric by the metric of the metric space in question and obtains description the geometrical object which is an anolog of the Euclidean geometrical object. Any geometrical object of Euclidean geometry can be described in terms of points, straights and planes. In turn any n-dimensional Euclidean plane can be considered to be a nth order natural geometric object (nth order NGO), generated by $n+1$ points $\mathcal{P}^n$ which do not lie on one $(n-1)$-dimensional plane. Thus, the nth order NGO is considered to be an attribute of the nth order metric subspace $M(\mathcal{P}^n)$.

To construct the NGOs on the base of finite metric subspaces, it is sufficient to attribute some length to any finite metric subspace. One defines the length of the nth order metric subspace of Euclidean space by the relation $|M(\mathcal{P}^n)| = n!S_n(\mathcal{P}^n)$, where $S_n(\mathcal{P}^n)$ is the Euclidean volume of $(n+1)$-edr with vertices at the points $\mathcal{P}^n$.

By definition the length $|M(\mathcal{P}^n)| = \sqrt{F_n(\mathcal{P}^n)}$ of the nth order metric subspace $M(\mathcal{P}^n)$ is determined by the function F_n. The function F_n is a function of metric. The form of F_n is determined from the equation $\sqrt{F_n(\mathcal{P}^n)} = n!S_n(\mathcal{P}^n)$, written for the Euclidean space. Then the function F_n has the form

$$F_n: \quad \Omega^{n+1} \rightarrow \mathbb{R}, \qquad \Omega^{n+1} = \bigotimes_{k=1}^{n+1} \Omega, \qquad n = 1, 2, \ldots \tag{1.5}$$

$$F_n(\mathcal{P}^n) = \det ||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)||, \qquad P_0, P_i, P_k \in \Omega, \qquad i, k = 1, 2, ...n \tag{1.6}$$

$$\begin{aligned}(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k) &\equiv \Gamma(P_0, P_i, P_k) \equiv \sigma(P_0, P_i) + \sigma(P_0, P_k) - \sigma(P_i, P_k), \\ i, k &= 1, 2, ...n.\end{aligned} \tag{1.7}$$

where the function σ is defined via metric ρ by the relation

$$\sigma(P, Q) \equiv \frac{1}{2}\rho^2(P, Q), \qquad \forall P, Q \in \Omega. \tag{1.8}$$

and $\mathcal{P}^n$ denotes $n+1$ points $P_0, P_1, \ldots, P_n$ of Ω

$$\mathcal{P}^n = \{P_0, P_1, \ldots, P_n\} \subset \Omega \tag{1.9}$$

The suggested method has four characteristic properties which differ it from the conventional method of the metric space description.

1. Enabling to construct NGOs which are analogs of Euclidean planes, the new method admits one to derive information on manifold (dimensionality and coordinate system) from metric space. The manifold, considered usually as an independent construction, appears to be a derivative construction which can be built on the basis of metric.

2. The method based on mappings (1.5) does not use the triangle axiom (1.4). If this axiom is removed, the "tubular generalization" of the metric space appears. This generalization generates T-geometry (tubular geometry), where the shortests are replaced by hallow tubes. In this case the metric space $M = \{\rho, \Omega\}$ turns to the σ-space $V = \{\sigma, \Omega\}$, $\sigma = \frac{1}{2}\rho^2$. The geometry generated by the σ-space is called T-geometry.

3. The constraint (1.3) is also not used in the method based on mappings (1.5). Removing the constraint (1.3), one obtains the metric space with indefinite metric, which is indefinite in the sense that the world function σ is defined by the mapping

$$\sigma: \quad \Omega \times \Omega \to \mathbb{R}, \tag{1.10}$$

$$\sigma(P,P) = 0, \qquad \sigma(P,Q) = \sigma(Q,P), \qquad \forall P,Q \in \Omega. \tag{1.11}$$

and the metric $\rho = \sqrt{2\sigma}$ may be imaginary. In this case the σ-space $V = \{\sigma, \Omega\}$ can be used for a description of pseudo-Euclidean and pseudo-Riemannian geometry. In particular, it may be used for description of the space-time.

Any change of the world function of the proper Euclidean space may be considered as a deformation of the space. Under such a deformation of the Euclidean space the Euclidean straights turn to hallow tubes. (In general, it is possible such a case, when the straights turn to curves, remaining to be lines, but it is a very special case of deformation). The hallow tubes appear in the general case, because one equation, determining the tube, describes generally a surface. This fact explains the name of the geometry: tubular geometry, or T-geometry. From viewpoint of the more general T-geometry the conventional metric geometry is a degenerate geometry, where the tubes degenerate into lines (the shortests). The T-geometry is a natural geometry (which is nondegenerate, in general). It is the most general geometry. A strong argument in favour of T-geometry is the circumstance that on the basis of T-geometry one can construct such a space-time model, where quantum effects are explained as simple T-geometric effects, and the quantum constant is an attribute of the space-time [5].

4. The method based on mappings (1.5) is insensitive to continuity of the σ-space. It can be used for continuous and discrete geometries.

In the second section definitions of main concepts and objects of σ-space are introduced. The third section is devoted to the formulation and proof of the theorem, stating that the Euclidean geometry can be described in terms of only metric. In the fourth section the role of the triangle axiom is discussed.

2 σ-space and its properties.

Definition 2.1 *σ-space $V = \{\sigma, \Omega\}$ is nonempty set Ω of points P with given on $\Omega \times \Omega$ real function σ*

$$\sigma: \quad \Omega \times \Omega \to \mathbb{R}, \qquad \sigma(P,P) = 0, \qquad \sigma(P,Q) = \sigma(Q,P) \qquad \forall P,Q \in \Omega. \tag{2.1}$$

The function σ is called world function, or σ-function.

Definition 2.2 . *Nonempty subset $\Omega' \subset \Omega$ of points of the σ-space $V = \{\sigma, \Omega\}$ with the world function $\sigma' = \sigma|_{\Omega' \times \Omega'}$, which is a contraction of σ on $\Omega' \times \Omega'$, is called σ-subspace $V' = \{\sigma', \Omega'\}$ of σ-space $V = \{\sigma, \Omega\}$.*

Further the world function $\sigma' = \sigma|_{\Omega'\times\Omega'}$, which is a contraction of σ will be designed by means of σ. Any σ-subspace of σ-space is a σ-space.

Definition 2.3 . *σ-space $V = \{\sigma, \Omega\}$ is called isometrically embedded in σ-space $V' = \{\sigma', \Omega'\}$, if there exists such a monomorphism $f : \Omega \to \Omega'$, that $\sigma(P, Q) = \sigma'(f(P), f(Q))$, $\forall P, \forall Q \in \Omega$, $f(P), f(Q) \in \Omega'$,*

Any σ-subspace V' of σ-space $V = \{\sigma, \Omega\}$ is isometrically embedded in it.

Definition 2.4 . *Two σ-spaces $V = \{\sigma, \Omega\}$ and $V' = \{\sigma', \Omega'\}$ are called to be isometric (equivalent), if V is isometrically embedded in V', and V' is isometrically embedded in V.*

Definition 2.5 *The σ-space $M = \{\rho, \Omega\}$ is called a finite σ-space, if the set Ω contains a finite number of points.*

Definition 2.6 . *The σ-subspace $M_n(\mathcal{P}^n) = \{\sigma, \mathcal{P}^n\}$of the σ-space $V = \{\sigma, \Omega\}$, consisting of $n + 1$ points $\mathcal{P}^n = \{P_0, P_1, ..., P_n\}$ is called the nth order σ-subspace .*

Definition 2.7 . *The σ-subspace $\overrightarrow{M_n}(\mathcal{P}^n) = \{\varepsilon, \sigma, \mathcal{P}^n\}$ of the σ-space $V = \{\sigma, \Omega\}$, consisting of $n+1$ points $\mathcal{P}^n = \{P_0, P_1, ..., P_n\}$ and equipped with orientation $\varepsilon = \pm 1$ is called the nth order oriented σ-subspace .*

Definition 2.8 . *The number $\sqrt{F_n(\mathcal{P}^n)}$, where $F_n(\mathcal{P}^n)$ is defined by relations (1.6)-(1.8), is called the length (volume) of the nth order σ-space $M_n(\mathcal{P}^n)$.*

Definition 2.9 . *The nth order multivector m_n is any mapping*

$$m_n : \qquad I_n \to \Omega, \qquad I_n \equiv \{0, 1, ..., n\} \tag{2.2}$$

The set I_n has a natural ordering, which generates the ordering of images $m_n(k) \in \Omega$ of points $k \in I_n$,

The ordered list of images of points in I_n has one-to-one connection with the multivector and may be used as the multivector identificator. Different versions of the point list will be used for writing the nth order multivector identificator:

$$\overrightarrow{P_0 P_1 ... P_n} \equiv \mathbf{P}_0 \mathbf{P}_1 ... \mathbf{P}_n \equiv \overrightarrow{\mathcal{P}^n}$$

Originals of points P_k in I_n are determined by the order of the point P_k in the list of identificator. Index of the point P_k has nothing to do with the original of P_k. Further we shall use identificator $\overrightarrow{P_0 P_1 ... P_n}$ of the multivector instead of the multivector. In this sense the nth order multivector $\overrightarrow{P_0 P_1 ... P_n}$ in the σ-space $V = \{\sigma, \Omega\}$ may be defined as the ordered set $\{P_l\}$, $l = 0, 1, \ldots n$ of $n + 1$ points $P_0, P_1, ..., P_n$, belonging to the σ-space V. The point P_0 is the origin of the multivector $\overrightarrow{P_0 P_1 ... P_n}$. Image $m_n(I_n)$ of the set I_n contains k points ($k \leq n + 1$). The set of all nth order multivectors m_n constitutes the set $\Omega^{n+1} = \bigotimes_{k=1}^{n+1} \Omega$, and any multivector $\overrightarrow{\mathcal{P}^n} \in \Omega^{n+1}$.

Definition 2.10 . *The scalar σ-product $(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})$ of two nth order multivectors $\overrightarrow{\mathcal{P}^n}$ and $\overrightarrow{\mathcal{Q}^n}$ is the real number*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = \det \|(\mathbf{P}_0\mathbf{P}_i.\mathbf{Q}_0\mathbf{Q}_k)\|, \qquad i,k = 1,2,...n, \qquad \overrightarrow{\mathcal{P}^n}, \overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1} \tag{2.3}$$

$$(\mathbf{P}_0\mathbf{P}_i.\mathbf{Q}_0\mathbf{Q}_k) \equiv \sigma(P_0,Q_i) + \sigma(Q_0,P_k) - \sigma(P_0,Q_0) - \sigma(P_i,Q_k), \qquad P_0,P_i,Q_0,Q_k \in \Omega \tag{2.4}$$

Operation of permutation of the multivector points can be effectively defined in the σ-space. Let us consider two nth order multivectors $\overrightarrow{\mathcal{P}^n} = \overrightarrow{P_0P_1P_2...P_n}$ and $\overrightarrow{\mathcal{P}^n_{(k\leftrightarrow l)}} = \overrightarrow{P_0P_1...P_{k-1}P_lP_{k+1}...P_{l-1}P_kP_{l+1}...P_n}$, $(n \geq 1)$, which is a result of permutation of points P_k, P_l, $(k < l)$. The scalar σ-product $(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})$ is defined by the relation (2.3). If $k = 0$, then the permutation $P_0 \leftrightarrow P_l$ changes the sign of the lth row of the determinant (2.3), and

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = -(\overrightarrow{\mathcal{P}^n_{(0\leftrightarrow l)}}.\overrightarrow{\mathcal{Q}^n}) \qquad l = 1,2,...n, \qquad \forall\overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}, \tag{2.5}$$

If $k \neq 0$, and $k < l$ then the permutation $P_k \leftrightarrow P_l$ transposes the lth row and the kth row of the determinant (2.3). The determinant changes its sign and, taking into account (2.5), one obtains

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = -(\overrightarrow{\mathcal{P}^n_{(k\leftrightarrow l)}}.\overrightarrow{\mathcal{Q}^n}) \qquad k \neq l, \qquad l,k = 0,1,2,...n, \qquad \forall\overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}, \tag{2.6}$$

As far as the relation (2.6) is valid for permutation of any two points of the multivector $\overrightarrow{\mathcal{P}^n}$ and for any multivector $\forall\overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}$, one may write

$$\overrightarrow{\mathcal{P}^n_{(i\leftrightarrow k)}} = -\overrightarrow{\mathcal{P}^n}, \qquad i,k = 0,1,...n, \qquad i \neq k, \qquad n \geq 1. \tag{2.7}$$

Thus, a change of the nth order multivector sign $(n \geq 1)$ (multiplication by the number $a = -1$) may be always defined as an odd permutation of points.

Let us consider the relation

$$\overrightarrow{\mathcal{P}^n}T\overrightarrow{\mathcal{Q}^n}: \qquad (\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = (\overrightarrow{\mathcal{R}^n}.\overrightarrow{\mathcal{Q}^n}), \qquad \forall\overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}, \tag{2.8}$$

between two nth order multivectors $\overrightarrow{\mathcal{P}^n} \in \Omega^{n+1}$ and $\overrightarrow{\mathcal{R}^n} \in \Omega^{n+1}$. The relation (2.8) is reflexive, symmetric and transitive, and it may be considered as an equivalence relation.

Definition 2.11 . *Two nth order multivectors $\overrightarrow{\mathcal{P}^n} \in \Omega^{n+1}$ and $\overrightarrow{\mathcal{R}^n} \in \Omega^{n+1}$ are equivalent $\overrightarrow{\mathcal{P}^n} = \overrightarrow{\mathcal{R}^n}$, if the relations (2.8) takes place.*

Definition 2.12 . *If the nth order multivector $\overrightarrow{\mathcal{N}^n}$ satisfies the relations*

$$(\overrightarrow{\mathcal{N}^n}.\overrightarrow{\mathcal{Q}^n}) = 0, \qquad \forall\overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}, \tag{2.9}$$

$\overrightarrow{\mathcal{N}^n}$ is the null nth order multivector.

Any nth order multivector $\overrightarrow{\mathcal{P}^n}$, having two similar points $P_k = P_l, \quad (l \neq k)$, is the null nth order multivector $\overrightarrow{\mathcal{N}^n}$, because in this case the determinant in (2.3) has similar rows (kth and lth) and vanishes for $\forall \overrightarrow{\mathcal{Q}^n} \in \Omega^{n+1}$.

Any nth order multivector $\overrightarrow{\mathcal{P}^n}$ has either at least two similar points and belongs to the equivalence class $\xi\left(\mathcal{N}^n\right)$ of null multivector $\overrightarrow{\mathcal{N}^n}$, or has all different points and belongs to the equivalence class $\xi\left(\overrightarrow{M_n}(\mathcal{P}^n)\right)$, where $\overrightarrow{M_n}(\mathcal{P}^n) = \{\varepsilon, \sigma, \mathcal{P}^n\}$ is the nth order oriented σ-subspace of $V = \{\sigma, \Omega\}$, consisting of $n+1$ points $\mathcal{P}^n$, and ε is the orientation of this subset which takes values $\varepsilon = \pm 1$. Thus, the set Ω^{n+1} of all nth order multivectors may be presented in the form

$$\Omega^{n+1} = \xi\left(\mathcal{N}^n\right) \bigcup_{\overrightarrow{M_n}(\mathcal{P}^n)} \xi\left(\overrightarrow{M_n}(\mathcal{P}^n)\right) \tag{2.10}$$

Any equivalence class $\xi\left(\overrightarrow{M_n}(\mathcal{P}^n)\right)$ contains $\frac{1}{2}(n+1)!$ elements. The equivalence class $\xi\left(+1, \sigma, \mathcal{P}^n\right)$ can be obtained from the multivector $\overrightarrow{\mathcal{P}^n}$ by means of even permutation of points. Multivectors which are obtained from the multivector $\overrightarrow{\mathcal{P}^n}$ by means of the odd permutation belong to the equivalence class $\xi\left(-1, \sigma, \mathcal{P}^n\right)$. The factor set Ω^{n+1}/T is formed by the equivalence class $\xi\left(\mathcal{N}^n\right)$ of null multivectors $\overrightarrow{\mathcal{N}^n}$ and nth order oriented σ-subspaces $\overrightarrow{M_n}(\mathcal{P}^n)$. It is easy to see that the scalar σ-product $(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})$ of two non-null nth order multivectors $\overrightarrow{\mathcal{P}^n} \in \xi\left(\overrightarrow{M_n}(\mathcal{P}^n)\right)$ and $\overrightarrow{\mathcal{Q}^n} \in \xi\left(\overrightarrow{M_n}(\mathcal{Q}^n)\right)$ depends only on nth order oriented σ-subspaces $\overrightarrow{M_n}(\mathcal{P}^n)$, $\overrightarrow{M_n}(\mathcal{Q}^n)$. Thus, in reality the scalar σ-product $(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})$ describes mutual disposition of nth order oriented σ-subspaces $\overrightarrow{M_n}(\mathcal{P}^n)$ and $\overrightarrow{M_n}(\mathcal{Q}^n)$, which are constituents of the σ-space $V = \{\sigma, \Omega\}$. This result shows that as well σ-subspaces *the multivectors are natural costituents of the σ-space*, and description in terms of multivectors is equivalent to a description in terms of σ-subspaces.

This fact puts the question, if it is possible to express the scalar σ-product of $\overrightarrow{M_n}(\mathcal{P}^n)$ and $\overrightarrow{M_n}(\mathcal{Q}^n)$ directly in terms of nth order oriented σ-subspaces. In principle, this problem can be solved, but the solution will be very complicated and ineffective. The fact is that that multivectors associate with integer numbers, whereas the oriented σ-subspaces associate with natural numbers which do not contain zero. Descriptions in terms of only natural numbers seems ineffective. Even a record of a number in Roman numerals looks very complicated, because Roman numerals associate with nutural numbers and do not contain zero. Apparrently, the rule for multiplication of two numbers, written in Roman numerals, looks very complicated. Multivectors associate with Aribic numerals and contain null multivectors. Reduced to σ-subspaces, the multivectors generate the equivalence class $\xi\left(+1, M_n\right)$, which is an analog of positive number, the equivalence class $\xi\left(-1, M_n\right)$, which is an analog of negative numbers and the equivalence class $\xi\left(\mathcal{N}^n\right)$, which is an analog of zero.

Definition 2.13 . *The length* $|\overrightarrow{\mathcal{P}^n}|$ *of the multivector* $\overrightarrow{\mathcal{P}^n}$ *is the number*

$$|\overrightarrow{\mathcal{P}^n}| = \left\{ \begin{array}{ll} \mid \sqrt{(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n})} \mid = |\sqrt{F_n(\mathcal{P}^n)}|, & (\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n}) \geq 0 \\ i \mid \sqrt{(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n})} \mid = i|\sqrt{F_n(\mathcal{P}^n)}|, & (\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^n}) < 0 \end{array} \right. \qquad \overrightarrow{\mathcal{P}^n} \in \Omega^{n+1} \tag{2.11}$$

where the quantity $F_n(\mathcal{P}^n)$ *is defined by the relations (1.6), (1.7)*

Definition 2.14 . *Two nth order multivectors* $\overrightarrow{\mathcal{P}^n}$ $\overrightarrow{\mathcal{Q}^n}$ *are collinear* $\overrightarrow{\mathcal{P}^n} \parallel \overrightarrow{\mathcal{Q}^n}$, *if*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n})^2 = |\overrightarrow{\mathcal{P}^n}|^2 \cdot |\overrightarrow{\mathcal{Q}^n}|^2 \tag{2.12}$$

Definition 2.15 . *Two collinear nth order multivectors* $\overrightarrow{\mathcal{P}^n}$ *and* $\overrightarrow{\mathcal{Q}^n}$ *are similarly oriented* $\overrightarrow{\mathcal{P}^n} \uparrow\uparrow \overrightarrow{\mathcal{Q}^n}$ *(parallel), if*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = |\overrightarrow{\mathcal{P}^n}| \cdot |\overrightarrow{\mathcal{Q}^n}| \tag{2.13}$$

They have opposite orientation $\overrightarrow{\mathcal{P}^n} \uparrow\downarrow \overrightarrow{\mathcal{Q}^n}$ *(antiparallel), if*

$$(\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{Q}^n}) = -|\overrightarrow{\mathcal{P}^n}| \cdot |\overrightarrow{\mathcal{Q}^n}| \tag{2.14}$$

The special case of the first order multivector is especially interesting.

Definition 2.16 . *The vector* $\mathbf{PQ}$ *in the* σ*-space* V *is the first order multivector, or the ordered set* $\{P, Q\}$ *of two points* P, Q. *The point* P *is the origin, and* Q *is the end of the vector.*

Definition 2.17 . *According to (2.3) and (2.4) the scalar* σ*-product* $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ *of two vectors* $\mathbf{P}_0\mathbf{P}_1$ *and* $\mathbf{Q}_0\mathbf{Q}_1$ *is called the real number*

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) \equiv \sigma(P_0, Q_1) + \sigma(Q_0, P_1) - \sigma(P_0, Q_0) - \sigma(P_1, Q_1), \tag{2.15}$$

$$P_0, P_1, Q_0, Q_1 \in \Omega$$

In the case, when $Q_0 = P_0$, the relation (2.15) coincide with (1.7).

Definition 2.18 . *The length* $\mid \mathbf{PQ} \mid$ *of the vector* $\mathbf{PQ}$ *is the number*

$$\mid \mathbf{PQ} \mid = \sqrt{2\sigma(P, Q)} = \left\{ \begin{array}{ll} \mid \sqrt{(\mathbf{PQ}.\mathbf{PQ})} \mid, & (\mathbf{PQ}.\mathbf{PQ}) \geq 0 \\ i \mid \sqrt{(\mathbf{PQ}.\mathbf{PQ})} \mid, & (\mathbf{PQ}.\mathbf{PQ}) < 0 \end{array} \right. \qquad P, Q \in \Omega \tag{2.16}$$

Definition 2.19 . *Vectors* $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{Q}_0\mathbf{Q}_1$ *are parallel or antiparallel, if the following relations are fulfilled respectively*

$$\mathbf{P}_0\mathbf{P}_1 \uparrow\uparrow \mathbf{Q}_0\mathbf{Q}_1 : \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \mid \mathbf{P}_0\mathbf{P}_1 \mid \cdot \mid \mathbf{Q}_0\mathbf{Q}_1 \mid \tag{2.17}$$

$$\mathbf{P}_0\mathbf{P}_1 \uparrow\downarrow \mathbf{Q}_0\mathbf{Q}_1 : \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = - \mid \mathbf{P}_0\mathbf{P}_1 \mid \cdot \mid \mathbf{Q}_0\mathbf{Q}_1 \mid \tag{2.18}$$

Definition 2.20 . *Vectors* $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{Q}_0\mathbf{Q}_1$ *are collinear, if they are parallel, or antiparallel.*

$$\mathbf{P}_0\mathbf{P}_1 \parallel \mathbf{Q}_0\mathbf{Q}_1 : \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)^2 = \mid \mathbf{P}_0\mathbf{P}_1 \mid^2 \cdot \mid \mathbf{Q}_0\mathbf{Q}_1 \mid^2 \qquad (2.19)$$

Example 2.1. Let us consider D-dimensional point proper Euclidean space. It may be considered as a metric space $E_D = \{\rho, \mathbb{R}^D\}$ or as a σ-space $E_D = \{\sigma, \mathbb{R}^D\}$, the world function $\sigma = \frac{1}{2}\rho^2$ being given by the relations

$$\sigma(P,Q) = \sigma(x,y) = \frac{1}{2}\sum_{i,k=1}^{D} g_{ik}(x^i - y^i)(x^k - y^k), \qquad x, y \in \mathbb{R}^n, \qquad (2.20)$$

where $x = \{x^i\} \in \mathbb{R}^D$ and $y = \{y^i\} \in \mathbb{R}^D$, $(i = 1, 2, \ldots D)$ are contravariant coordinates of points P and Q respectively in some rectilinear coordinate system K. Here $g_{ik} = \text{const}$, $(i, k = 1, 2, \ldots D)$ is the metric tensor, $\det ||g_{ik}|| \neq 0$. Eigenvalues of the matrix g_{ik}, $i, k = 1, 2, \ldots D$ of the metric tensor are positive, and $\sum_{i,k=1}^{D} g_{ik}x^i x^k = 0$, if and only if $x = 0$. The above made definitions of the vector, its length, scalar product (σ-product) of two vectors and relations of collinearity agree with the use of these concepts for the Euclidean space. Indeed, the length of the vector $\mathbf{PQ}$ in the Euclidean space is

$$\mid \mathbf{PQ} \mid = \sqrt{g_{ik}(x^i - y^i)(x^k - y^k)} = \sqrt{2\sigma(P,Q)} \qquad (2.21)$$

that agrees with (2.16).

In the proper Euclidean space three vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0\mathbf{Q}_1$, and $\mathbf{P}_1\mathbf{Q}_1$ are coupled by the relation

$$\mid \mathbf{P}_1\mathbf{Q}_1 \mid^2 = \mid \mathbf{P}_0\mathbf{Q}_1 - \mathbf{P}_0\mathbf{P}_1 \mid^2 = \mid \mathbf{P}_0\mathbf{P}_1 \mid^2 + \mid \mathbf{P}_0\mathbf{Q}_1 \mid^2 - 2(\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{Q}_1) \qquad (2.22)$$

where $(\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{Q}_1)$ denotes the scalar product in the proper Euclidean space. It follows from (2.22)

$$(\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{Q}_1) = \frac{1}{2}\{\mid \mathbf{P}_0\mathbf{Q}_1 \mid^2 + \mid \mathbf{P}_0\mathbf{P}_1 \mid^2 - \mid \mathbf{P}_1\mathbf{Q}_1 \mid^2\} \qquad (2.23)$$

Substituting the point Q_1 by Q_0 in (2.23), one obtains

$$(\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{Q}_0) = \frac{1}{2}\{\mid \mathbf{P}_0\mathbf{Q}_0 \mid^2 + \mid \mathbf{P}_0\mathbf{P}_1 \mid^2 - \mid \mathbf{P}_1\mathbf{Q}_0 \mid^2\} \qquad (2.24)$$

Subtracting (2.24) from (2.23) and using the properties of the scalar product in the Euclidean space, one obtains

$$(\mathbf{P}_0\mathbf{P}_1, \mathbf{Q}_0\mathbf{Q}_1) = \frac{1}{2}\{\mid \mathbf{P}_0\mathbf{Q}_1 \mid^2 + \mid \mathbf{Q}_0\mathbf{P}_1 \mid^2 - \mid \mathbf{P}_0\mathbf{Q}_0 \mid^2 - \mid \mathbf{P}_1\mathbf{Q}_1 \mid^2\} \qquad (2.25)$$

Comparing (2.15) with (2.25) and using into account (2.16), one obtains that the scalar σ-product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ in the proper Euclidean space coincides with the scalar product $(\mathbf{P}_0\mathbf{P}_1, \mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{P}_0\mathbf{P}_1$, and $\mathbf{Q}_0\mathbf{Q}_1$.

In the proper Euclidean space the vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{P}_2$ are parallel or antiparallel, if cosine of the angle ϑ between them is equal respectively to 1 or -1. As far as

$$\cos\vartheta = (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_2) \mid \mathbf{P}_0\mathbf{P}_2 \mid^{-1} \cdot \mid \mathbf{P}_0\mathbf{P}_1 \mid^{-1}, \tag{2.26}$$

one obtains an accord with the definitions (2.17), (2.18).

In the proper Euclidean space the nth order simple m-vector (multivector) $\mathbf{m}$ is defined as an external (skew) product of vectors

$$\mathbf{m} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge ... \wedge \mathbf{e}_n \qquad \mathbf{e}_i = \mathbf{P}_0\mathbf{P}_i, \qquad i = 1, 2, ...n \tag{2.27}$$

The scalar product of two nth order simple m-vectors (multivectors) $\mathbf{m}$ and $\mathbf{q}$

$$\mathbf{q} = \bigwedge_{i=1}^{i=n} \mathbf{k}_i, \qquad \mathbf{k}_i = \mathbf{Q}_0\mathbf{Q}_i \qquad i = 1, 2, ...n \tag{2.28}$$

is defined by means of the relation

$$(\mathbf{m}, \mathbf{q}) = \det \|(\mathbf{e}_i, \mathbf{k}_l)\| = \det \|(\mathbf{P}_0\mathbf{P}_i, \mathbf{Q}_0\mathbf{Q}_l)\|, \qquad i, l = 1, 2, ...n \tag{2.29}$$

Comparing relations (2.29) and (2.3), one conclude that scalar product coincides with scalar σ-product, provided the simple m-vector coincides with the multivector.

In the proper Euclidean space, where the simple m-vector (multivector), defined in the form $\overrightarrow{\mathcal{P}^n} = \bigwedge_{i=1}^{i=n} \mathbf{P}_0\mathbf{P}_i$, is antisymmetric with respect to permutation of any two indices $i, k = 0, 1, ...n, \quad i \neq k$. It can be shown, using properties of the external product.

Definition 2.21 . *$n+1$ points $\mathcal{P}^n$, $P_i \in \Omega$ $(i = 0, 1, ...n)$ form the $(n+1)$-point tube σ-basis in σ-space, if the multivector $\overrightarrow{\mathcal{P}^n}$ has nonvanishing length*

$$|\overrightarrow{\mathcal{P}^n}|^2 \equiv F_n(\mathcal{P}^n) \neq 0. \tag{2.30}$$

Let us illustrate this definition of the tube σ-basis in the example of the D-dimensional proper Euclidean space. Let n vectors $\mathbf{e}_i = \mathbf{P}_0\mathbf{P}_i$, $i = 1, 2, \ldots n$ be given in D-dimensional proper Euclidean space ($n \leq D$). In this case the determinant (1.6) is the Gram's determinant

$$F_n(\mathcal{P}^n) = \det \parallel (\mathbf{e}_i.\mathbf{e}_k) \parallel = (n!S_n(\mathcal{P}^n))^2, \qquad i, k = 1, 2, \ldots n \tag{2.31}$$

and S_n is the volume of $(n+1)$-edr with vertices at points $\mathcal{P}^n$. Vanishing of this determinant is the necessary and sufficient condition of linear independence of vectors $\mathbf{e}_i, \quad (i = 1, 2, \ldots n)$ in the proper Euclidean space.

If the condition (2.30) is fulfilled, n vectors $\mathbf{e}_i, \quad (i = 1, 2, \ldots n)$ are linear independent and may serve as a basis in the n-dimensional plane $\mathcal{L}(\mathcal{P}^n)$, passing through $n+1$ points $\mathcal{P}^n$. In the proper Euclidean space the covariant components

$g_{ik}(\mathcal{P}^D)$ of metric tensor form the matrix of scalar products $(\mathbf{e}_i, \mathbf{e}_k)$ of basis vectors e_i, $(i = 1, 2, ...n)$. They may be calculated in the σ-immanent form

$$g_{ik}(\mathcal{P}^D) = (\mathbf{e}_i, \mathbf{e}_k) = (\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k) = \Gamma(P_0, P_i, P_k), \qquad i, k = 1, 2, \dots D, \tag{2.32}$$

using the expression (1.7) for calculation of the scalar σ-product of the vectors $\mathbf{e}_i = \mathbf{P}_0\mathbf{P}_i$, $\quad (i = 1, 2, \dots D)$.

Definition 2.22 . *The nth order tube $\mathcal{T}(\mathcal{P}^n)$, $(n = 0, 1, \dots)$, formed by $(n+1)$-point tube σ-basis $\mathcal{P}^n \subset \Omega$ is the set of points $R \in \Omega$*

$$\mathcal{T}(\mathcal{P}^n) \equiv \mathcal{T}_{\mathcal{P}^n} = \{R \mid F_{n+1}(\mathcal{P}^n, R) = 0\}, \qquad F_n(\mathcal{P}^n) \neq 0. \tag{2.33}$$

Let us note that always $\mathcal{P}^n \subset \mathcal{T}(\mathcal{P}^n)$.

The relation (2.33) may be written also in terms of multivector $\overrightarrow{\mathcal{P}^n}$

$$\mathcal{T}(\mathcal{P}^n) = \left\{ P_{n+1} \left| |\overrightarrow{\mathcal{P}^{n+1}}|^2 = 0 \right.\right\}, \qquad |\overrightarrow{\mathcal{P}^n}| \neq 0. \tag{2.34}$$

The tube $\mathcal{T}(\mathcal{P}^n)$ is the nth order natural geometrical object (NGO), i.e. the set of points, determined by geometry and parameters: $n+1$ points $\mathcal{P}^n$. The set of all possible NGOs is a set of σ-immanent geometric objects on the set Ω. Construction of the nth order tubes is connected with a classification of relatively simple $(n+2)$-point constructions. One considers $(n+2)$-point σ-subsets $M_{n+1}(\mathcal{P}^{n+1})$, having zero length and containing $n+1$ fixed points $\mathcal{P}^n$. Then the remaining point of $M_{n+1}(\mathcal{P}^{n+1})$ runs all points of the tube $\mathcal{T}(\mathcal{P}^n)$.

Definition 2.23 . *Section $\mathcal{S}_{n;P}$ of the tube $\mathcal{T}(\mathcal{P}^n)$ at the point $P \in \mathcal{T}(\mathcal{P}^n)$ is the set $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n))$ of points, belonging to the tube $\mathcal{T}(\mathcal{P}^n)$*

$$\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n)) = \{P' \mid \bigwedge_{l=0}^{l=n} \sigma(P_l, P') = \sigma(P_l, P)\}, \qquad P \in \mathcal{T}(\mathcal{P}^n) \qquad P' \in \Omega. \tag{2.35}$$

Note that $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n)) \subset \mathcal{T}(\mathcal{P}^n)$, because $P \in \mathcal{T}(\mathcal{P}^n)$. Indeed, belonging of the point P to $\mathcal{T}(\mathcal{P}^n)$ depends only on values of $n+1$ quantities $\sigma(P_l, P)$, $\ l = 0, 1, ...n$. According to(2.35) these quantities are the same for both points P and P'. Hence, any running point $P' \in \mathcal{T}(\mathcal{P}^n)$, if $P \in \mathcal{T}(\mathcal{P}^n)$.

In the proper Euclidean space the nth order tube is the n-dimensional plane, containing points $\mathcal{P}^n$, and its section $\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n))$ at the point P consists of one point P.

Lemma 2.1 *If $|\overrightarrow{\mathcal{P}^n}|^2 \neq 0$, the relation $|\overrightarrow{\mathcal{P}^n R}|^2 = F_{n+1}(\mathcal{P}^n, R) = 0$ is equivalent to the relation*

$$\overrightarrow{\mathcal{P}^n} || \overrightarrow{\mathcal{P}^{n-1}R} : \qquad (\overrightarrow{\mathcal{P}^n}.\overrightarrow{\mathcal{P}^{n-1}R})^2 = |\overrightarrow{\mathcal{P}^n}|^2 \cdot |\overrightarrow{\mathcal{P}^{n-1}R}|^2 \tag{2.36}$$

describing collinearity of two nth order multivectors $\overrightarrow{\mathcal{P}^n}$ and $\overrightarrow{\mathcal{P}^{n-1}R} \equiv \overrightarrow{P_0P_1...P_{n-1}R}$

This lemma states that if the $(n-1)$th order minor $M_{n\,n}$ of the element $a_{n\,n}$ of the nth order determinant Δ does not vanish, the relation $\Delta=0$ is equivalent to the relation $(M_{n\,n-1})^2=M_{n\,n}M_{n-1\,n-1}$ between the $(n-1)$th order minors of Δ. It is proved on the basis of the fact that the nth row of Δ is a linear combination of all other rows.

Using the lemma, the definition of the nth order tube $\mathcal{T}(\mathcal{P}^n)$ may be written in the form

$$\mathcal{T}(\mathcal{P}^n)\equiv\mathcal{T}_{\mathcal{P}^n}=\{R\mid\overrightarrow{\mathcal{P}^n}||\overrightarrow{\mathcal{P}^{n-1}R}\},\qquad F_n(\mathcal{P}^n)\neq 0. \tag{2.37}$$

Besides the lemma admits one to introduce a new geometric object $\mathcal{T}(\mathcal{P}^n;Q_0)$ which can be described as the nth order tube, passing through the point Q_0 collinearly to the multivector $\overrightarrow{\mathcal{P}^n}$. In the proper Euclidean space $\mathcal{T}(\mathcal{P}^n;Q_0)$ is n-dimensional plane, passing through the point Q_0, collinear to the n-dimensional plane $\mathcal{T}(\mathcal{P}^n)$.

Definition 2.24 *The nth order tube $\mathcal{T}(\mathcal{P}^n;Q_0)$, which passes through a given point Q_0 collinear to the nth order multivector $\overrightarrow{\mathcal{P}^n}$ of nonvanishing length $|\overrightarrow{\mathcal{P}^n}|^2=F_n(\mathcal{P}^n)\neq 0$, is the set of points $R\in\Omega$*

$$\mathcal{T}(\mathcal{P}^n;Q_0)\equiv\mathcal{T}_{Q_0}(\mathcal{P}^n)=\left\{R\left|F_{n+1}\left(\mathcal{P}^{n-1},R;Q_0\right)=0\right.\right\} \tag{2.38}$$

where $F_{n+1}\left(\mathcal{P}^{n-1},R;Q_0\right)$ is defined by the relation

$$F_{n+1}\left(\mathcal{P}^n,R;Q_0\right)=\det||a_{ik}||,\qquad i,k=1,2,...n+1 \tag{2.39}$$

$$\begin{aligned}
a_{i\,k}&=\left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k\right),\qquad i,k=1,2,...n\\
a_{i\,n+1}&=a_{n+1\,i}=\left(\mathbf{P}_0\mathbf{P}_i.\mathbf{Q}_0\mathbf{R}\right),\qquad i=1,2,...n\\
a_{n+1\,n+1}&=\left(\mathbf{Q}_0\mathbf{R}.\mathbf{Q}_0\mathbf{R}\right)
\end{aligned} \tag{2.40}$$

The determinant (2.39) is derived from the determinant $F_{n+1}\left(\mathcal{P}^n,R\right)$ of the form (1.6) by replacing the vector $\mathbf{P}_0\mathbf{R}$ by the vector $\mathbf{Q}_0\mathbf{R}$ in all scalar σ-products, containing the vector $\mathbf{P}_0\mathbf{R}$. This replacement transfers the origin of the running vector $\mathbf{P}_0\mathbf{R}$ to the point Q_0, whereas the running vector $\mathbf{P}_0\mathbf{R}=\mathbf{P}_0\mathbf{P}_{n+1}$ is defined by the condition $F_{n+1}(\mathcal{P}^{n+1})=0$, and it does not change. It is easy to see that $\mathcal{T}(\mathcal{P}^n;P_0)=\mathcal{T}(\mathcal{P}^n)$.

The zeroth and first order tubes are the most interesting and important. For F_1 one obtains from (1.6) and (1.7)

$$F_1(P_0,P_1)=2\sigma(P_0,P_1)$$

Then

$$\mathcal{T}(P_0)\equiv\mathcal{T}_{P_0}=\{P\mid\sigma(P_0,P)=0\}, \tag{2.41}$$

In the proper Euclidean space the zeroth order tube $\mathcal{T}_{P_0}=\{P_0\}$ consists of one point P_0, and its section $\mathcal{S}_{0;P_0}(\mathcal{T}_{p_0})=\{P_0\}$ consists of one point P_0 also. But in the pseudo-Euclidean space (for instance, in the space-time of the special relativity) $\mathcal{T}_{P_0}$ is the light cone with the vertex at the point P_0, and its section

$$\mathcal{S}_{0;P}(\mathcal{T}(P_0)) = \{P' \mid \sigma(P_0,P') = \sigma(P_0,P) = 0\} = \mathcal{T}_{P_0}$$

coincides with the light cone.

Describing the first order tubes, it is convenient to use the circumstance that the function $F_2(\mathcal{P}^2)$ can be presented in the form of a product

$$F_2(P_0,P_1,P_2) = S_+(P_0,P_1,P_2)S_2(P_0,P_1,P_2)S_2(P_1,P_2,P_0)S_2(P_2,P_0,P_1) \qquad (2.42)$$

where

$$S_+(P_0,P_1,P_2) \equiv S(P_0,P_1) + S(P_1,P_2) + S(P_0,P_2) \qquad (2.43)$$

$$S_2(P_0,P_1,P_2) \equiv S(P_0,P_1) + S(P_1,P_2) - S(P_0,P_2) \qquad (2.44)$$

Here $S = \sqrt{2\sigma}$ (one uses the designation S, but not ρ, because σ may be negative and S may be imaginary). S_+ vanishes, if and only if any term of the sum (2.43) vanishes. Then no two points form σ-basis, and the tube is not defined. The tube $\mathcal{T}(\mathcal{P}^2)$ may be presented as consisting of parts, and each multiplier in (2.42) (except for S_+) is responsible for one of these parts.

Let us set

$$\mathcal{T}_{[P_0P_1]} = \mathcal{T}_{[P_1P_0]} = \{P \mid S_2(P_0,P,P_1) = 0\}, \qquad (2.45)$$

$$\mathcal{T}_{P_0[P_1} = \mathcal{T}_{P_1]P_0} = \{P \mid S_2(P_0,P_1,P) = 0\}, \qquad (2.46)$$

Let us refer to $\mathcal{T}_{[P_0P_1]}$ as the tube segment between the points P_0, P_1, and to $\mathcal{T}_{P_0[P_1}$ as the tube ray outgoing from P_1 towards the point P_0.

It is evident from (2.42), (2.45), (2.46) that

$$\mathcal{T}_{P_0P_1} = \mathcal{T}_{P_0]P_1} \cup \mathcal{T}_{[P_0P_1]} \cup \mathcal{T}_{P_0[P_1} \qquad (2.47)$$

Using lemma 2.1 for $n = 1$ and the relation (2.19), the first order tube $\mathcal{T}_{P_0P_1}$ may be defined also as a set of such points P that $\mathbf{P}_0\mathbf{P} \parallel \mathbf{P}_0\mathbf{P}_1$.

$$\mathcal{T}_{P_0P_1} = \{P \mid \mathbf{P}_0\mathbf{P}_1 \parallel \mathbf{P}_0\mathbf{P}\} \qquad (2.48)$$

Then for the tube rays one can use the definitions

$$\mathcal{T}_{P_0[P_1} = \{P \mid \mathbf{P}_1\mathbf{P} \uparrow\downarrow \mathbf{P}_1\mathbf{P}_0\} \qquad (2.49)$$

$$\mathcal{T}_{[P_0P_1} = \{P \mid \mathbf{P}_0\mathbf{P} \uparrow\uparrow \mathbf{P}_0\mathbf{P}_1\} \qquad (2.50)$$

Definition 2.25 . *The oriented segment $\overrightarrow{\mathcal{T}_{[P_0P_1]}}$ of the first order tube, formed by the vector $\overrightarrow{P_0P_1} \in \Omega^2$ of unvinishing length is a totality $\{\overrightarrow{P_0P_1}, \mathcal{T}_{[P_0P_1]}\}$ of the vector $\overrightarrow{P_0P_1}$ and segment $\mathcal{T}_{[P_0P_1]}$, formed by this vector. The length of the oriented segment $\overrightarrow{\mathcal{T}_{[P_0P_1]}}$ is the quantity*

$$|\overrightarrow{\mathcal{T}_{[P_0P_1]}}| = |\overrightarrow{P_0P_1}| = \sqrt{2\sigma(P_0,P_1)}. \qquad (2.51)$$

Describing in terms of differential geometry, the geodesic in D-dimensional Riemannian space is considered as *special kind of a curve, having the following properties.*

(i) *Extremality.* The distance $(2\sigma)^{1/2}$, measured along the geodesic between two points is the shortest (extremal) as compared with the distance measured along other curves.

(ii) *Definiteness.* Any two different points of the geodesic determine uniquely the geodesic, passing through these points.

(iii) *Minimality of the section* (one-dimensionality). Any section of the geodesic consists of one point.

At the conventional approach the property (ii) is a corollary of the property (i) (for rather small regions of the space), but the property (iii) is the property of any curve (but not only of geodesic).

In T-geometry the geodesic is considered as a special kind of the tube, degenerating into a line. Then the properties (ii) and (iii) are supposed to be fulfilled. The property (i) is not defined, because the concept of line is not defined.

Let us try to determine the geodesic as the tube, having the properties of definiteness and of the section minimality at the same time.

Definition 2.26 . *The tube $\mathcal{T}(\mathcal{P}^n)$ has the definiteness property, if the condition*

$$\mathcal{T}(\mathcal{Q}^n) = \mathcal{T}(\mathcal{P}^n) \tag{2.52}$$

is fulfilled for any $(n+1)$-point tube σ-basis $\mathcal{Q}^n \subset \mathcal{T}(\mathcal{P}^n)$

Definition 2.27 . *The tube $\mathcal{T}(\mathcal{P}^n)$ has the section minimality property, if for $\forall P \in \mathcal{T}(\mathcal{P}^n)$*

$$\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n)) = \{P\}, \qquad \forall P \in \mathcal{T}(\mathcal{P}^n) \tag{2.53}$$

Definition 2.28 . *σ-space is extremal on the tube $\mathcal{T}(\mathcal{P}^n)$, if for $\mathcal{T}(\mathcal{P}^n)$ the conditions of definiteness and section minimality are fulfilled.*

Definition 2.29 . *σ-space is extremal on the set $\mathcal{T}$ of tubes $\mathcal{T}(\mathcal{P}^n)$, if it is extremal on any tube of the set $\mathcal{T}$.*

Definition 2.30 . *σ-space is extremal in the nth order, if it is extremal on all nth order tubes $\mathcal{T}(\mathcal{P}^n)$.*

Definition 2.31 . *The tube $\mathcal{T}(\mathcal{P}^n)$ is the geodesic tube $\mathcal{L}(\mathcal{P}^n)$, if the σ-space is extremal on the tube $\mathcal{T}(\mathcal{P}^n)$.*

3 Euclidean space as a special case of σ-space.

Definition 3.1 . *n-dimensional Euclidean space E_n is a set $\mathbb{R}^n$ of all ordered sets $x = \{x_1, x_2, \dots x_n\}$ of n real numbers on which for $\forall x \in \mathbb{R}^n$, $\forall y \in \mathbb{R}^n$ is given the real function σ:*

$$\sigma(x, y) = \frac{1}{2}\sum_{i,k=1}^{n} g^{ik}(x_i - y_i)(x_k - y_k), \qquad g^{ik} = const, \qquad i, k = 1, 2, \dots n \quad (3.1)$$

$$\det \parallel g_{ik} \parallel = (\det \parallel g^{ik} \parallel)^{-1} \neq 0 \qquad (3.2)$$

The function σ is called the world function or simply σ-function. n-dimensional Euclidean space E_n is at the same time a σ-space $E_n = \{\sigma, \mathbb{R}^n\}$.

Remark. The given definition is equivalent to the definition of n-dimensional Euclidean space E_n as n-dimensional linear space $\mathbb{R}^n$ of vectors $x = \{x_1, x_2, \dots, x_n\} \in \mathbb{R}^n$ with the scalar product (x, y) of vectors $x, y \in \mathbb{R}^n$ given on it

$$(x, y) = \sum_{i,k=1}^{n} g^{ik} x_i y_k = \sigma(0, x) + \sigma(0, y) - \sigma(x, y), \qquad g^{ik} = \text{const}, \qquad i, k = 1, 2, \dots n \qquad (3.3)$$

where σ is given by the relation (3.1).

The Euclidean space , considered as σ-space $E_n = \{\sigma, \Omega\}$, $(\Omega = \mathbb{R}^n)$, have the following properties

$$\exists \mathcal{P}^n \subset \Omega, \qquad F_n(\mathcal{P}^n) \neq 0, \qquad F_{n+1}(\Omega^{n+2}) = 0, \qquad (3.4)$$

$$\sigma(P, Q) = \frac{1}{2}\sum_{i,k=1}^{n} g^{ik}(\mathcal{P}^n)[\Gamma(P_0, P_i, P) - \Gamma(P_0, P_i, Q)]$$
$$\times[\Gamma(P_0, P_k, P) - \Gamma(P_0, P_k, Q)], \qquad \forall P, Q \in \Omega \qquad (3.5)$$

$$\Gamma(P_0, P, Q) = \sum_{i,k=1}^{n} g^{ik}(\mathcal{P}^n)\Gamma(P_0, P_i, P)\Gamma(P_0, P_k, Q), \qquad \forall P, Q \in \Omega, \qquad (3.6)$$

where $\mathcal{P}^n$ is some $(n+1)$-point tube σ-basis in $\Omega = \mathbb{R}^n$ in the sense of the definition (2.30), i.e. $F_n(\mathcal{P}^n) \neq 0$, and the quantities $\Gamma(P_0, P_k, P)$ are defined by the relations (1.7). $(n+1)$-point tube σ-basis $\mathcal{P}^n$ corresponds to the basis of n vectors

$$\mathbf{e}_i = \mathbf{P}_0\mathbf{P}_i, \qquad P_i \in \mathcal{P}^n, \qquad i = 1, 2, \dots n \qquad (3.7)$$

and

$$x_i = x_i(P) = (\mathbf{P}_0\mathbf{P}.\mathbf{e}_i) = \Gamma(P_0, P, P_i), \qquad i = 1, 2, \dots n, \qquad \forall P \in \Omega \qquad (3.8)$$

are covariant coordinates of the vector $\mathbf{P}_0\mathbf{P}$ in this basis. The quantities

$$g_{ik} = g_{ik}(\mathcal{P}^n) = (\mathbf{e}_i.\mathbf{e}_k) = \Gamma(P_0, P_i, P_k), \qquad i, k = 1, 2, \dots n \qquad (3.9)$$

are covariant components of the metric tensor in this basis $\mathcal{P}^n$. As far as the basis vectors $\mathbf{e}_i$ are linearly independent and $\det ||g_{ik}(\mathcal{P}^n)|| \neq 0$, $\mathcal{P}^n$ is the tube σ-basis, because of (2.30) the following condition is fulfilled

$$F_n(\mathcal{P}^n) \equiv \det \parallel g_{ik}(\mathcal{P}^n) \parallel \neq 0, \qquad i,k = 1,2,\dots n \tag{3.10}$$

and the contravariant components $g^{ik} = g^{ik}(\mathcal{P}^n)$ of the metric tensor are determined by means of the relations

$$\sum_{k=1}^{n} g_{ik}(\mathcal{P}^n) g^{kl}(\mathcal{P}^n) = \delta_i^l, \qquad i,l = 1,2,\dots n \tag{3.11}$$

Conditions (3.5) and (3.6) are equivalent, as it follows from (1.7).

Definition 3.2 . *σ-space $V = \{\sigma, \Omega\}$ have the structure of n-dimensional Euclidean space on Ω, if there exists such a $(n+1)$-point tube σ-basis $\mathcal{P}^n \subset \Omega$, that for $\forall P$, $\forall Q \in \Omega$ the condition (3.6) is fulfilled.*

σ-space V, having the structure of n-dimensional Euclidean space may be not Euclidean, because one-to-one correspondence between the points $P \in \Omega$ and their coordinates $x \in \mathbb{R}^n$ may be absent. For instance, two different points P and P' may have similar coordinates and be mapped on one point x of the Euclidean space $E_n = \{\sigma, \mathbb{R}^n\}$.

Finally, the third property of the Euclidean space $E_n = \{\sigma, \mathbb{R}^n\}$ is formulated as follows. The relations

$$\Gamma(P_0, P_i, P) = x_i, \qquad x_i \in \mathbb{R}, \qquad i = 1,2,\dots n, \tag{3.12}$$

considered as equations for determination of $P \in \Omega$, have always one and only one solution.

Let us note that all three conditions are written in σ-immanent form. They are necessary properties of the Euclidean space. In this connection one can put the question whether these conditions are also sufficient conditions for the σ-space $\{\sigma, \Omega\}$ were the Euclidean space. The following theorem answers this question.

Theorem 3.1 *For the σ-space $\{\sigma, \Omega\}$ were n-dimensional Euclidean space, it is necessary and sufficient that the conditions (3.4), (3.5) and (3.12) be fulfilled.*

Proof. *Necessity* of conditions (3.4), (3.5) and (3.12) is tested by the direct substitution of world function σ for n-dimensional Euclidean space $E_n = \{\sigma, \mathbb{R}^n\}$.

Sufficiency. Let $\mathcal{P}^n$ be some $(n+1)$-point tube σ-basis in σ-space $\{\sigma, \Omega\}$. Such a σ-basis exists in virtue of (3.4). Let $P, Q \in \Omega$ be two arbitrary points. Let us introduce their covariant coordinates in $\mathcal{P}^n$ by means of the relations of type (3.8)

$$x_i = \Gamma(P_0, P_i, P), \qquad y_i = \Gamma(P_0, P_i, Q), \qquad i = 1,2,\dots n, \qquad \forall P, Q \in \Omega \tag{3.13}$$

Then the relation (3.5) is rewritten in the form (3.1)

$$\sigma(x,y)=\frac{1}{2}\sum_{i,k=1}^{n} g^{ik}(x_i-y_i)(x_k-y_k), \qquad g^{ik}(\mathcal{P}^n)=\text{const}, \qquad i,k=1,2,\ldots n \tag{3.14}$$

In virtue of the condition (3.12) any point $P\in\Omega$ corresponds to one and only one point $x\in\mathbb{R}^n$ and vice versa. In other words, the σ-space $\{\sigma,\Omega\}$ is isometric to n-dimensional Euclidean space $E_n=\{\sigma,\mathbb{R}^n\}$.

Corollary of the theorem. n-dimensional Euclidean space and all its properties can be described σ-immanently (i.e. in terms of the world function). In other words, the T-geometry is rich and pithy enough to contain Euclidean geometry as a special case, when the world function is restricted by σ-immanent relation (3.5), or by equivalent relation (3.6). The Riemannian geometry can be presented in the σ-immanent form [6] also. This may be interpreted in the sense that T-geometry contains the Riemannian geometry as a special case. T-geometry is informative enough to contain other geometries. Apparently, it is rather difficult to construct a geometry which would not be contained in T-geometry. The fact is that that practically any geometry may be considered as a result of a deformation (variation of distances between points) of σ-subspace of the Euclidean space of rather high dimensionality. At such a variation the pithiness of geometry (i.e. the number of geometric objects, suppositions and theorems) does not reduce, because the number of tubes does not reduce. This number may only increase under deformation, because any tube of the Euclidean space determined by many σ-bases splits, in general, into several different tubes.

Pithiness of geometry depends on development of the mathematical technique of the geometry (not only on axioms of the geometry). Conventionally the metric geometry is considered as a geometry which is less pithy as compared with the Euclidean geometry. One connects usually the pithiness of the metric geometry with constraints (1.3), (1.4), which are essential for construction of the shortest and geometric objects, connected with it. Essentially, the pithiness of the metric geometry is connected with its mathematical technique. Using more effective mathematical technique, connected with the mappings (1.5), the pithiness of metric geometry (the number of geometrical objects) increases even under removing constraints (1.3), (1.4) on metric.

The mathematical technique of T-geometry is constructed without introducing the concept of continuity, and T-geometry is insensitive to continuity of the set Ω, where the geometry is given.

Giving up of constraints (3.12) leads to a violation of the mapping $\Omega\to\mathbb{R}^n$ reversibility. In particular, it is possible such a case, when the σ-space $\{\sigma,\Omega\}$ appears to be a σ-subspace of the Euclidean space E_n.

Definition 3.3 . *The Euclidean σ-space $E'=\{\sigma,\Omega'\}$ is the σ-space which can be isometrically embedded in the Euclidean space. n-dimensional Euclidean σ-space $E'_n=\{\sigma,\Omega'\}$ is σ-space which can be isometrically embedded in n-dimensional*

Euclidean space $E_n = \{\sigma, \mathbb{R}^n\}$, but cannot be isometrically embedded in $(n-1)$-dimensional Euclidean space $E_{n-1} = \{\sigma, \mathbb{R}^{n-1}\}$.

n-dimensional Euclidean σ-space is a σ-subspace of n-dimensional Euclidean space $E_n = \{\sigma, \mathbb{R}^n\}$.

From the tube definition (2.33) and the condition (3.4) it follows that n-dimensional Euclidean σ-space is the nth order tube $\mathcal{T}(\mathcal{P}^n) = \Omega$, generated by any $(n+1)$-point tube σ-basis $\mathcal{P}^n \subset \Omega$, the condition of the tube section minimality (2.53) being fulfilled. Then the following theorem takes place.

Theorem 3.2 . *Let $\mathcal{P}^n$ be $(n+1)$-point tube σ-basis in the σ-space $V = \{\sigma, \Omega\}$. For the tube $\mathcal{T}(\mathcal{P}^n)$ be n-dimensional Euclidean σ-space it is necessary and sufficient, that*

(1) σ-space $\mathcal{T}(\mathcal{P}^n)$ have the structure of n-dimensional Euclidean space on $\mathcal{T}(\mathcal{P}^n)$,

(2) Section of $\mathcal{T}(\mathcal{P}^n)$ be minimal at any point:

$$\mathcal{S}_{n;P}(\mathcal{T}(\mathcal{P}^n)) = \{P\}, \qquad \forall P \in \mathcal{T}(\mathcal{P}^n).$$

4 Triangle axiom as a condition of the first order tube degeneration

Let us study constraints, imposed on σ-space by the triangle inequality (1.4). Let us consider segment $\mathcal{T}_{[P_0P_1]}$ of the tube $\mathcal{T}_{P_0P_1}$, contained between basic points P_0, P_1. It is described by equations (2.44), (2.45).

For continuous σ-space the tube segment $\mathcal{T}_{[P_0P_1]}$ is some surface, containing points P_0, P_1 and determined by relations (2.44) and (2.45). This surface $\mathcal{T}_{[P_0P_1]}$ and the region outside the surface are described by the equation

$$S_2(P_0, R, P_1) \equiv \rho(P_0, R) + \rho(R, P_1) - \rho(P_0, P_1) \geq 0, \tag{4.1}$$

where R is the running point. Thus, the triangle inequality is fulfilled on the surface $\mathcal{T}_{[P_0P_1]}$ and outside it. The region inside the surface $\mathcal{T}_{[P_0P_1]}$ associates with the inequality $S_2(P_0, R, P_1) < 0$, that corresponds to a violation of the triangle axiom. In other words, in the metric space the first order tube segment $\mathcal{T}_{[P_0P_1]}$ has no inner points. It means degeneration of the tube into a line, or into a surface which has no inner points. This statement is valid for the first order tube ray (2.46) and for the first order tube as a whole. In this sense the metric geometry (i.e. geometry generated by the metric space) is a degenerate geometry.

In the case, when all first order tubes $\mathcal{T}_{P_0P_1}$ degenerate into corresponding basic points P_0, P_1, the triangle inequality (1.4) takes the form of a strong inequality

$$\rho(P_0, R) + \rho(R, P_1) > \rho(P_0, P_1), \qquad P_0 \neq R \neq P_1 \neq P_0, \qquad \forall P_0, P_1, R \in \Omega. \tag{4.2}$$

In this case it seems to be reasonable to call the T-geometry ultradegenerate.

Example 4.1. Let us consider two different σ-spaces (and two T-geometries) on the unit sphere

$$\Omega = \left\{\mathbf{x}\left|\mathbf{x}^2 \leq 1\right.\right\} \subset \mathbb{R}^3, \qquad \mathbf{x} = \left\{x^1, x^2, x^3\right\} \in \mathbb{R}^3, \qquad \mathbf{x}^2 \equiv \sum_{i=1}^{3} \left(x^i\right)^2 \tag{4.3}$$

σ-space $V_E = \{\sigma_E, \Omega\}$ generates the proper Euclidean geometry

$$\sigma_E : \quad \Omega \times \Omega \to [0, \infty) \subset \mathbb{R}, \qquad \sigma_E(\mathbf{x}, \mathbf{x}') = \frac{1}{2}|\mathbf{x} - \mathbf{x}'|^2, \qquad \mathbf{x}, \mathbf{x}' \in \Omega, \tag{4.4}$$

σ-space $V = \{\sigma, \Omega\}$ generates T-geometry on the same set Ω by means of relations

$$\sigma : \quad \Omega \times \Omega \to [0, \infty) \subset \mathbb{R}, \qquad \sigma(\mathbf{x}, \mathbf{x}') = 2\left(\arcsin\sqrt{\frac{\sigma_E(\mathbf{x}, \mathbf{x}')}{2}}\right)^2, \qquad \mathbf{x}, \mathbf{x}' \in \Omega \tag{4.5}$$

Along with the two σ-spaces in the sphere Ω one considers their σ-subspaces $V_{Es} = \{\sigma_E, \Sigma\}$ and $V_s = \{\sigma, \Sigma\}$ on the sphere surface $\Sigma = \partial\Omega = \{\mathbf{x} \mid \mathbf{x}^2 = 1\} \subset \Omega$. As far as Σ is a subset of the set Ω, the T-geometries $V_{Es} = \{\sigma_E, \Sigma\}$ and $V_s = \{\sigma, \Sigma\}$ are generated by T-geometries V_E and V.

Let us design the tubes in σ-space V_E by means of the symbol $\mathcal{L}$, the tubes in the σ-space V are denoted by the symbol $\mathcal{T}$. The first order tubes $\mathcal{L}_{AB} \subset \Omega$, $(A, B \in \Sigma)$ are straight lines in Ω, and they are determined by two basic points A, B in Σ. In other words, T-geometry V_E is degenerate in the first order on Ω, and it is ultradegenerate in the first order on Σ, because $\Sigma \cap \mathcal{L}_{AB}$, $(A, B \in \Sigma)$ consists of two points A and B. The first order tubes $\Omega \cap \mathcal{T}_{AB}$, $(A, B \in \Sigma)$ are nondegenerate tubes on Ω. They are surfaces, formed by a rotation of the unit radius circle, passing through points $A, B \in \Sigma$, around the axis $\mathcal{L}_{AB}$. (see. Figure 4.1). These tubes tangent to the sphere surface Σ along the circles of maximal radius. The segment of the tube between the points $A, B \in \Sigma$ has the property $\mathcal{T}_{[AB]} \cap \Omega \neq \emptyset$, whereas the remaining part of the tube $\mathcal{T}_{AB}$: $\left(\mathcal{T}_{AB} \backslash \mathcal{T}_{[AB]}\right) \cap (\Omega \backslash \Sigma) = \emptyset$, $A, B \in \Sigma$ is found inside the sphere surface Σ. As a result the segment $\Sigma \cap \mathcal{T}_{[AB]}$, $(A, B \in \Sigma)$ is the shortest in the σ-space $V_s = \{\sigma, \Sigma\}$. This shortest on the sphere surface Σ connects points $A, B \in \Sigma$. The remaining part of the tube $\Sigma \cap \mathcal{T}_{AB}$, $(A, B \in \Sigma)$ is a continuation of the shortest $\Sigma \cap \mathcal{T}_{[AB]}$, $(A, B \in \Sigma)$ on Σ. In other words, σ-space $V_s = \{\sigma, \Sigma\}$ generates the degenerate in the first order T-geometry on Σ. Thus, T-geometry $V = \{\sigma, \Omega\}$ is nondegenerate on Ω and it is degenerate on Σ.

On the other hand, T-geometry in $V_s = \{\sigma, \Sigma\}$ on the sphere surface Σ can be constructed on the basis of Euclidean geometry in $V_E = \{\sigma_E, \Omega\}$. To construct $V_s = \{\sigma, \Sigma\}$ on the basis of $V_E = \{\sigma_E, \Omega\}$, one can use extremal properties of geodesics.

Let us consider the second order tube $\mathcal{L}_{ABC}$ on Ω, $(A, B, C \in \Sigma)$. This tube is a two-dimensional plane, passing through the points $A, B, C \in \Sigma$. In $V_{Es} = \{\sigma_E, \Sigma\}$ the second order tube $\Sigma \cap \mathcal{L}_{ABC}$ has the form of a circle, passing through points $A, B, C \in \Sigma$.

To construct internal geometry in $V_s = \{\sigma, \Sigma\}$, it is necessary to determine the metric $\rho(A, B) = \sqrt{2\sigma(A, B)}$, $A, B \in \Sigma$ on $\Sigma \times \Sigma$, using the following way.

$$\rho(A, B) = \inf_{C \in \Sigma, C \neq A, C \neq B} l_C(A, B), \qquad A, B \in \Sigma, \tag{4.6}$$

where $l_C(A, B) \subset [0, \infty)$ is the length of the curve $\Sigma \cap \mathcal{L}_{ABC}$ between the points $A, B \in \Sigma$. Let us order the points $R \in (\Sigma \cap \mathcal{L}_{ABC}) \subset \Sigma$ of the curve $\Sigma \cap \mathcal{L}_{ABC} \subset \Sigma$, solving the equation

$$\rho_E(A, R)\tau = \rho_E(A, B), \qquad R \in (\Sigma \cap \mathcal{L}_{ABC}), \qquad \tau \in \mathbb{R} \tag{4.7}$$

with respect to the point R. Solution of this equation determines $R = R_{AB}(\tau, C) \in (\Sigma \cap \mathcal{L}_{ABC}) \subset \Sigma$ as a function of the parameter $\tau \in [0, 1]$ and of the point $C \in \Sigma$. Therewith $A = R_{AB}(0, C)$, $B = R_{AB}(1, C)$. The function $R = R_{AB}(\tau, C)$ has two branches $R = R_1(\tau, C)$ and $R = R_2(\tau, C)$. It should choose the branch with the less values of $\rho_E(A, R_{AB}(\tau, C))$ and determine $l_C(A, B)$ by means of the relation

$$l_C(A, B) = \int_0^1 \left[\frac{d\rho_E}{d\tau'}\left(R_{AB}(\tau, C), R_{AB}(\tau', C)\right)\right]_{\tau'=\tau} d\tau \tag{4.8}$$

Substituting (4.8) in (4.6), one obtains

$$\rho(A, B) = 2 \arcsin \frac{S_E(A, B)}{2}, \qquad A, B \in \Sigma \tag{4.9}$$

that corresponds to (4.5).

This example shows that, using Euclidean geometry inside Ω, one can consrtuct internal metric (and T-geometry) on the surface Σ of the sphere Ω. The extremal properties of geodesics (shortests) and the second order tubes $\mathcal{L}_{ABC} \subset \Omega$ are used essentially for construction of geometry on Σ. The second order tubes $\mathcal{L}_{ABC} \subset \Omega$ generate a system of curves on the sphere surface Σ. One chooses those ones among them which have the minimal length between points A, B.

Apparently, with proper stipulations this procedure of constructing metric space on the sphere surface with Euclidean space inside sphere can be generalized on the case of arbitrary body and arbitrary metric space inside it.

Thus, classification of finite metric subspaces, using the series of mappings (1.5), appears to be a very effective method of studying the metric space. This method admits one to describe the Euclidean geometry and the Riemannian one in terms of only metric (world function). Geometries, constructed on the basis of this classification do not use concept of continuity. They are insensitive to discreteness or continuity of the space. Concept of continuity may be introduced on the basis of metric (world function) by means of a proper parametrization of extremal tubes [6].

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