Pauli's Electron as a Dynamic System

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Abstract

A dynamic system $\mathcal{S}_P$ described by the Pauli equation for non-relativistic electron is investigated merely as a distributed dynamic system. No quantum principles are used. This system is shown to be a statistical ensemble of non-relativistic stochastic pointlike particles. The electron spin is shown to have a classical analog which is a collective (statistical) property of the ensemble (not a property of a single electron). Magnetic moment of the electron is a quantum property which has no classical analog. The magnetic moment is parallel to the spin only in the stationary state. In the arbitrary state the magnetic moment is not connected with the spin direction.

1 Introduction

In the present paper we consider different distributed dynamic systems: the system $\mathcal{S}_P$ described by the Pauli equation, the system $\mathcal{S}_S$ described by the Schrödinger equation and the statistical ensemble $\mathcal{E}_c$ of classical pointlike particles. The dynamic systems are investigated and compared between themselves. Such a comparison is possible, because the statistical ensemble is considered merely as a disributed dynamic system consisting of many independent non-interacting particles, but not as a construction which is described by a distribution function defined in the phase space of coordinates and momenta.

No quantum principles are used in the course of this investigation. Any dynamic system is studied merely as a dynamic system. Under some conditions the quantum properties of the systems $\mathcal{S}_P$ and $\mathcal{S}_S$ disappear and they turn to the statistical ensemble $\mathcal{E}_c$ of classical particles which can be investigated and interpreted in details. In addition the dynamic systems $\mathcal{S}_P$ and $\mathcal{S}_S$ are believed to describe a quantum electron and a spinless particle respectively and to be distinguished only by the spin. Comparing them, one can determine what is the spin from the mathematical viewpoint.

Such an investigation is of interest from the viewpoint of determination of scope limits of the quantum mechanics principles. Indeed, to determine where the QM principles stop to work, it is necessary to look at them from the more general point of view which is not use these principles.

In the investigation a special statistical ensemble technique (SET) is used. SET differs from the conventional quantum dynamics technique (QDT) connected closely with quantum principles (description in terms of wave functions ψ, linearity of dynamic equations in terms of ψ, use of linear operators). SET relates to QDT approximately in such a way, as the statistical physics relates to the axiomatic thermodynamics. For instance, it is well known that the temperature is a statistical (collective) property which cannot be applied to a system with several degrees of freedom. At the same time the temperature as a collective property cannot be understood from the viewpoint of thermodynamics, because the thermodynamics is an axiomatic construction which does not contain any statistical concepts.

In a like way QDT is an axiomatic construction which does not contain any statistical concepts (they appear in interpretation of quantum measurements, but not in the quantum dynamics). The conclusion that the electron spin is a collective property of the statistical ensemble (not of a single electron) cannot be understood from the viewpoint of quantum dynamics, and this conclusion is hard to be perceived, because now QDT is a dominating technique of investigating quantum systems. A rather curious psychlogical situation arises, when the old stereotype of thinking prevents from development of a new physical conception. Such a situation described by Jammer [1] took place, when the axiomatic thermodynamics was converting into the statistical physics.

Mathematically an application of SET to such quantum dynamic systems as $\mathcal{S}_P$ and $\mathcal{S}_S$ is based on some transformations of the action for the statistical ensemble of classical systems (see Appendix A). The transformations reduce the action of the classical statistical ensemble $\mathcal{E}_c$ to such a form, where a comparison with description in terms of wave function becomes possible. SET uses hydrodynamic variables and correlates with so called hydrodynamic interpretation of quantum mechanics [2-6]. There is a distinction. The hydrodynamic interpretation deals with dynamic equations which have the form of hydrodynamic equations. These equations contain spin variables only in the presence of an electromagnetic field. A transition to a description in terms of wave function needs an integration of these equations. (The two-component wave function can be considered merely as a way of description of an inviscid continuous medium [7], but not necessarily as a way of quantum description). Spin variables arise as a result of this integration (integration constants).

The papers [8-12] are associated with the present paper in some way. See also reviews [13,14] and reference therein.

SET deals with dynamic equations which have the form of partially integrated hydrodynamic equations, and a transition from these equations to a description in terms of a wave function does not need an integration. Dynamic equations used in SET contain spin variables (integration constants) even in an absence of the electromagnetic field.

It is well known that the Pauli equation describes quantum non-relativistic electrons. Conventionally a description of quantum systems arises as a result of quantization of classical ones. Principles of quantum mechanics applied to a classical system convert the system into a quantum system. For instance, let a classical system $\mathcal{S}_c$ consist of a charged non-relativistic particle moving in a given electromagnetic field $A_i = \{A_0, \mathbf{A}\} = \{A_i\}$, $i = 0,1,2,3$. $\mathcal{S}_c$ is described by the action

$$\mathcal{A}_c[\mathbf{x}] = \int \{\frac{m}{2}(\frac{d\mathbf{x}}{dt})^2 - \frac{e}{c}A_0 - \frac{e}{c}\mathbf{A}\frac{d\mathbf{x}}{dt}\}dt, \tag{1.1}$$

where $\mathbf{x} = \mathbf{x}(t)$, or by the Jacobi-Hamiltonian equation

$$-p_0 = \frac{1}{2m}(\mathbf{p} + \frac{e}{c}\mathbf{A})^2 + \frac{e}{c}A_0, \qquad p_0 = \frac{\partial S}{\partial t}, \qquad \mathbf{p} = \nabla \mathrm{S}. \tag{1.2}$$

Applying QM principles (quantizing the system and substituting momentum p_i by operators $-i\hbar\partial_i$) one obtains a dynamic equation for a wave function ψ describing a state of the quantum particle. The dynamic equation (Schrödinger equation) has the form

$$i\hbar\partial_t\psi = \{\frac{1}{2m}(-i\hbar\nabla + \frac{e}{c}\mathbf{A})^2 + \frac{e}{c}A_0\}\psi \tag{1.3}$$

where ψ is a complex function of coordinates and time. The dynamic equation (1.3) is known as an equation for a spinless non-relativistic particle. It can be also described by the action

$$\mathcal{A}_S[\psi^*, \psi] = \int [\frac{i\hbar}{2}(\psi^*\frac{\partial\psi}{\partial t} - \frac{\partial\psi^*}{\partial t}\psi) - \frac{e}{c}A_0\psi^*\psi - \frac{1}{2m}(i\hbar\partial_\alpha + \frac{e}{c}A_\alpha)\psi^* \times$$
$$(-i\hbar\partial_\alpha + \frac{e}{c}A_\alpha)\psi]d^4x, \tag{1.4}$$

where ψ^* means the complex conjugate to ψ.

A quantum non-relativistic electron is described by another dynamic equation (Pauli equation). The corresponding action has the form

$$\mathcal{A}_P[\psi^*, \psi] = \int [\frac{i\hbar}{2}(\psi^*\frac{\partial\psi}{\partial t} - \frac{\partial\psi^*}{\partial t}\psi) - \frac{e}{c}A_0\psi^*\psi - \frac{1}{2m}(i\hbar\partial_\alpha + \frac{e}{c}A_\alpha)\psi^* \times$$
$$(-i\hbar\partial_\alpha + \frac{e}{c}A_\alpha)\psi + \frac{e\hbar}{2mc}\varepsilon_{\alpha\beta\gamma}A_\beta\partial_\alpha(\psi^*\sigma_\gamma\psi)]d^4x, \tag{1.5}$$

where

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \qquad \psi^* = (\psi_1^*, \psi_2^*) \tag{1.6}$$

ψ_1, ψ_2 are complex functions, and ψ_1^*, ψ_2^* are complex conjugate functions. $\sigma = \{\sigma_0, \boldsymbol{\sigma}\} = \{\sigma_i\}$, $\quad i = 0,1,2,3$ are Pauli matrices

$$\sigma_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$
$$\sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{1.7}$$

and $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Chivita pseudo-tensor ($\varepsilon_{123} = 1$).

To obtain the Pauli equation as a result of quantization of some classical particle, one needs to suppose that the particle has a magnetic moment $\boldsymbol{\mu}$, interacting directly with the external magnetic field. The magnetic moment corresponds to an magnetic moment operator

$$\boldsymbol{\mu} = -\frac{i\hbar e}{2mc}\boldsymbol{\sigma} \tag{1.8}$$

and the wave function has two complex component. An additional term

$$\boldsymbol{\mu}\mathbf{H} = \boldsymbol{\mu}\nabla \times \mathbf{A} \tag{1.9}$$

which distinguishes the Pauli equation from the Schrödinger one arises in the action (1.5). Here the cross $\times$ means the vector product

Thus, dynamic systems $\mathcal{S}_S$ and $\mathcal{S}_P$ described respectively by actions (1.4), (1.5) can be considered as results of application of quantum principles to different classical dynamic systems $\mathcal{S}_{Scl}$ and $\mathcal{S}_{Pcl}$. These classical systems describe non-relativistic particles which differ not only in a magnetic moment, but also in a spin. The spin is not a classical quantity. In any case it is not known how to describe the electron spin in classical terms. In the action (1.5) the spin leads to an increase of the number of the wave function components (degrees of freedom). There are different point of view on the spin nature [15-22]. All these models consider the spin as a property of a single particle connected with some internal degrees of freedom.

Dynamic systems $\mathcal{S}_S$ and $\mathcal{S}_P$ [(1.4) and (1.5)] are distributed ones. They arise as a result of application of the QM principles to some concentrated classical dynamic systems. At the same time the systems $\mathcal{S}_S$ and $\mathcal{S}_P$ are dynamic systems which can be investigated independently of the way of their derivation, i.e. independently of the QM principles. Of course, one cannot be sure that all quantum properties of dynamic systems $\mathcal{S}_S$ and $\mathcal{S}_P$ are described by the actions (1.4) and (1.5). Maybe, some quantum properties display themselves in measurement procedures. Nevertheless an investigation of $\mathcal{S}_S$ and $\mathcal{S}_P$ simply as dynamic systems is of interest from the viewpoint how many quantum properties are taken into account by the actions of dynamic systems $\mathcal{S}_S$ and $\mathcal{S}_P$.

In our investigation of $\mathcal{S}_S$ and $\mathcal{S}_P$ one will use a standard dynamic system $\mathcal{E}_{cl}$ defined as an ensemble of independent non-relativistic classical pointlike particles moving in a given electromagnetic field. $\mathcal{E}_{cl}$ is described by the action

$$\mathcal{A}_{cl}[\mathbf{x}] = \int \{\frac{m}{2}(\frac{d\mathbf{x}}{dt})^2 - \frac{e}{c}A_0 - \frac{e}{c}\mathbf{A}\frac{d\mathbf{x}}{dt}\} dt d\boldsymbol{\xi}, \tag{1.10}$$

where $\mathbf{x} = \mathbf{x}(t, \boldsymbol{\xi})$ is a function of the time t, and Lagrangian coordinates $\boldsymbol{\xi} = \{\xi_\alpha\}$, $\alpha = 1, 2, 3$ labelling the particles of the ensemble. The action (1.10) is a sum (integral) of actions (1.1) of a single classical system. The dynamic system $\mathcal{E}_{cl}$ is a distributed system. It is a fluid constituted by quasi-parallelly moving non-interacting classical particles. On one hand the system $\mathcal{E}_{cl}$ can be interpreted easily in classical terms, on the other hand the dynamic systems $\mathcal{S}_S$ and $\mathcal{S}_P$ turn to $\mathcal{E}_{cl}$ under some conditions and can be interpreted on this base.

In the second section properties of the classical ensemble are investigated. The problem is formulated mathematically in the third section. The fourth section is devoted to investigation of properties of dynamic systems described by Pauli's and Schrödinger equations. In the fifth section some properties of the electron magnetic moment are investigated. Transformation properties of the Pauli equation written in the hydrodynamic variables are considered in the sixh section.

2 Classical Ensemble

The action (1.10) for a fluid is written in the Lagrangian coordinates (the time t and Lagrangian coordinates $\boldsymbol{\xi}$ are independent variables). The same action written in the Eulerian coordinates ($t, \mathbf{x}$ are independent variables) has the form

$$\mathcal{A}_{cl}[j, \varphi, \boldsymbol{\xi}] = \int \{ \frac{m j^\alpha j^\alpha}{2\rho} - \frac{e}{c} j^i A_i - \hbar j^i [\partial_i \varphi + g^\alpha(\boldsymbol{\xi}) \partial_i \xi_\alpha] \} d^4 x \tag{2.1}$$

$$j^0 = \rho, \qquad \mathbf{j} = \{j^\alpha\}, \qquad \alpha = 1,2,3, \qquad \boldsymbol{\xi} = \{\xi_\alpha\}, \qquad \alpha = 1,2,3,$$

where a summation is made over repeating indices: over Latin ones (0-3) and over Greek ones (1-3). All variables $j = \{\rho, \mathbf{j}\}$, φ , $\boldsymbol{\xi}$ are considered as functions of the Eulerian coordinates $(t, \mathbf{x})$. $g^\alpha(\boldsymbol{\xi})$ are arbitrary functions of Lagrangian coordinates ξ_α, $\alpha = 1,2,3$. Appearance of arbitrary functions is a result of integration of some dynamic equations arising in the course of transformation from the action (1.10) to the action (2.1) (see Appendix A). In turn a possibility of such an intergration is connected with an invariance of dynamic equation with respect to arbitrary transformation of Lagrangian coordinates $\boldsymbol{\xi}$

$$\xi_\alpha \to \tilde{\xi}_\alpha = \tilde{\xi}_\alpha(\boldsymbol{\xi}), \qquad \det \parallel \partial \tilde{\xi}_\beta / \partial \xi_\alpha \parallel = 1, \qquad \alpha, \beta = 1,2,3 \tag{2.2}$$

The action (2.1) arises from the action (1.10) as a result of resolving the relations $\mathbf{x} = \mathbf{x}(t, \boldsymbol{\xi})$ with respect to $\boldsymbol{\xi}$ and subsequent introduction of new variables

$$\rho = \det \parallel \partial \xi_\beta / \partial x^\alpha \parallel, \qquad \alpha, \beta = 1,2,3, \qquad \mathbf{j} = \rho \frac{d\mathbf{x}}{dt} \tag{2.3}$$

The variable ρ describes a concentration of particles in the ensemble $\mathcal{E}_{cl}$, and $\mathbf{j}$ describes a flux of the particles, as it follows from Eq.(2.3). 4-flux $j = \{\rho, \mathbf{j}\}$ is invariant with respect to the renumeration transformation (2.2).

3 Statement of the Problem

Our investigation is based on the following phenomenological statements.

(1) The distributed dynamic system $\mathcal{S}_P$ described by the action (1.5), (1.6) is associated in some way with the electron of the mass m and the charge e, moving in a given electromagnetic field.

(2) The distributed dynamic system $\mathcal{S}_S$ described by the action (1.4), is associated in some way with the spinless particle of the mass m and the charge e, moving in a given electromagnetic field.

(3) The mean concentration $j^0 = \rho$ and the flux $\mathbf{j}$ of particles in the dynamic systems $\mathcal{S}_P$ and $\mathcal{S}_P$ are defined by the relatioins

$$j^0 = \rho = \psi^* \psi, \tag{3.1}$$

$$j^\alpha = -\frac{i\hbar}{2m}(\psi^* \partial_\alpha \psi - \partial_\alpha \psi^* \psi) + \frac{e}{mc} A_\alpha \psi^* \psi, \qquad \alpha = 1, 2, 3, \tag{3.2}$$

The way of association between the systems $\mathcal{S}_P$ and a single quantum electron as well as between the system $\mathcal{S}_S$ and a single spinless particle is not fixed. It is not known whether $\mathcal{S}_P$ describes a single electron or a statistical ensemble of them. It is to be determined as a result of the investigation.

The unit spin vector $\mathbf{s}$ is defined for $\mathcal{S}_P$ my means of the relation

$$\mathbf{s} = \psi^* \boldsymbol{\sigma} \psi / \rho \tag{3.3}$$

A meaning of the spin arises from comparison of $\mathcal{S}_P$ and $\mathcal{S}_S$.

Let us note that formally the unit spin vector (3.3) can be introduced also for the system $\mathcal{S}_S$. Substituting one-component wave function ψ by the two-component one

$$\psi \to \psi \chi, \qquad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \text{ const}, \qquad \chi^* \chi = 1 \tag{3.4}$$

one obtains by means of Eq.(3.3) $\mathbf{s}$ =const, $\mathbf{s}^2 = 1$.

All results are obtained from the above stated three points on the base of logical conclusions and comparison of $\mathcal{S}_P$ and $\mathcal{S}_S$ between themselves and with the classical ensemble $\mathcal{E}_{cl}$ described by Eq.(2.1). Other suppositions are not used. In particular, all arguments based on a usage of the QM principles are rejected.

The notion of the statistical ensemble is crucial in our consideration. The way of its introduction [23] disinguishes slightly from the conventional way, when the statistical ensemble is considered as an instrument for calculation of average values. For classical systems it is a construction which is described by a distribution function in the phase space. For quantum ones it is a statistical operator, or the density matrix.

Here one uses a very simple and general conceptions of the statistical ensemble which is common for classical and quantum systems. One considers the statistical ensemble as a dynamic system consisting of many independent similar stochastic or dynamic systems [23]. Such a concept of the statistical ensemble can be founded as follows.

Pricise experiments with any stochastic system $\mathcal{S}_s$ are irreproducible. But physics, as all natural sciences, studies only reproducible phenomena. To obtain a reproducibility, a statistical description is used, i.e. many independent similar stochastic systems $\mathcal{S}_s$ are considered. The set of many independent similar systems (stochastic, or not) is known as a statistical ensemble $\mathcal{E}$. Experiments with the statistical ensemble $\mathcal{E}$ are reproducible, and $\mathcal{E}$ is a deterministic dynamic system. The state of

$\mathcal{E}$ evolves according to some dynamic equation, although there is no dynamic equation for a single stochastic system $\mathcal{S}_s$. Thus, instead of investigating dynamics of a stochastic system $\mathcal{S}_s$ (what is impossible) one studies dynamics of a statistical ensemble $\mathcal{E}$ (i.e. a deterministic dynamic system associated with the stochastic system $\mathcal{S}_s$). Investigating the ensemble $\mathcal{E}$, one enables to give some conclusions about the stochastic systems associated with $\mathcal{E}$.

Of course, a statistical ensemble $\mathcal{E}$ consisted of deterministic dynamic systems $\mathcal{S}_d$ can be constructed also. In this case the dynamics of $\mathcal{E}$ is determined by $\mathcal{S}_d$. In particular, a Lagrangian of $\mathcal{E}$ is a sum of Lagrangians of all systems $\mathcal{S}_d$.

Example 1. Let us consider a simple experiment. An electron flies out with a fixed energy from an electron gun, flies through a narrow hole at the diafragm and hits the screen at the point x_1. Repeating the experiment, one obtains another points of impact x_2, $x_3, \ldots x_n$. These points do not coincide. The experiment with a single electron is irreproducible. Then by definition a single electron is a stochastic system. Let the number N of experiments is very large ($N \gg 1$). The points of impact x_1, $x_2, \ldots x_N$ form some distribution $d_1(x)$ of points over the screen.

Let us produce another series of N experiments. If N is large enough, the disrtibution $d_2(x)$ of points over the screen coincides with $d_1(x)$. It means that experiments with many single electrons are reproducible, although experiment with a single electron is irreproducible. In other words, many independent stochastic electrons form a deterministic dynamic system (statistical ensemble). It is difficult to say what is a reason of such a change, but this fact takes place.

This statistical ensemble of stochastic electrons may be imagined as follows. Let there be a set of N independent similar arrangements, where one produces simultaneously N similar experiments with electrons. Let us project mentually all arrangements into one of them. One obtains a set of independent and non-interacting electrons moving along stochastic trajectories. These stochastic electrons form something like a continuous medium (fluid). This fluid is *a deterministic dynamic system.* A time evolution of this fluid is described by some dynamic equations. Although electrons of the fluid are charged, they do not interact between themselves via electromagnetic field, because they are a result of projection of electrons from spatially separated independent arrangements. At this stage it is of no importance what is a nature of the stochasticity, quantum or classical.

Thus, although one cannot follow the course of an evolution of a single stochastic system, it is possible to track the evolution of many independent similar stochastic systems, and by definition it is a statistical description of a stochastic system. The statistical ensemble $\mathcal{E}$, as any dynamic system, has a Lagrangian, an energy-momentum tensor, an angular momentum tensor, a flux of the particle, and, maybe, some other additive quantities characteristic for distributed dynamic systems. Calculating these quantities for the ensemble, one can obtain corresponding mean values for a single stochastic system $\mathcal{S}_{st}$, dividing by the number of systems in the ensemble.

It should note that such a statistical description does not use the QM principles in explicit way. All above considerations are valid for the ensemble of any stochastic systems (including the quantum ones and the Brownian particle), because they are based on the properties of an ensemble considered as a dynamic system.

The statistical ensemble $\mathcal{E}$ has two basic properties:
(1) The statistical ensemble $\mathcal{E}$ is a distributed dynamic system.
(2) The statistical ensemble $\mathcal{E}$ is an instrument for calculation of average values concerning the stochastic system $\mathcal{S}_s$.

Conventionally only the second property is used effectively. The first one is ignored usually. The rules of calculating averages are different for quantum and classical systems, because the associated statistical ensembles are different dynamic systems in the two cases. Being axiomatized and considered as principles, these rules devide the physical phenomena into two conceptually different parts: quantum phenomena and classical ones.

Here only the first property of the statisitcal ensemble (statistical ensemble as a dynamic system) is investigated. The second property (in the form of Eqs.(3.1), (3.2)) is used only for definition of the particle concentration and the particle flux (3.1), (3.2). All other rules of calculation of averages are ignored, because some of them cannot be tested experimentally and seem to be questionable. For instance, according to the QM principles the mean value of the nth power of the electron momentum x-component *at the state described by the wave function* ψ is defined by the rule

$$< p_x^n >_\psi = N^{-1} \int \psi^* (-i\hbar \partial_x)^n \psi d\mathbf{x}, \qquad N = \int \psi^* \psi d\mathbf{x}, \qquad n = 1, 2, \ldots$$

This relation cannot be tested experimentally, because a result of the measurement must be ascribed to a definite state ψ. Although one can measure $< p_x^n >_\psi$, it is not known to which state the obtained result is to be ascribed. The state ψ cannot be determined, because according to the time-energy uncertainty relation any measurement of the momentum needs some time. The state ψ changes essentially for this time. (See for details [24]). As far as the first property of the statistical ensemble does not axiomatized, there is no conceptual distinction between the classical and quantum ensembles (ensembles consisted of classical or quantum systems). Both types of the ensembles can be considered from a common point of view.

This simple consideration leads to the following conclusion. A position of a particle is an attribute of a single particle as a concentrated system, whereas a position distribution function is an attribute of a statistical ensemble as a distributed system. On this base one can conclude that $\mathcal{S}_P$ and $\mathcal{S}_S$ are rather statistical ensembles, than single particles.

But many scientists (maybe, most of them) believe that quantum properties and quantum principles reflect the nature of physical phenomena. We are afraid that many of them are not ready to accept the stated above approach. For this reason one prefers a more formal presentation based only on the three stated above points which do not provoke any objections.

It will be shown that under some initial conditions the systems $\mathcal{S}_P$ and $\mathcal{S}_S$ turn to the classical statistical ensemble $\mathcal{E}_{cl}$ described by the action (2.1). On this base one concludes that $\mathcal{S}_P$ and $\mathcal{S}_S$ are statistical ensembles under all other initial conditions, i.e. in the general case. As far as in the general case $\mathcal{S}_P$ and $\mathcal{S}_S$ cannot be statistical ensembles of classical systems because of the form of their actions, one concludes that $\mathcal{S}_P$ and $\mathcal{S}_S$ are statistical ensembles of stochastic systems. There is no action for a single stochastic system, and the action for the ensemble of stochastic systems can have any form which is a first order homogeneous function of ρ and $\mathbf{j}$.

4 Dynamic Systems Described by Equation of Pauli and Schrödinger

Using hydrodynamic variables instead of the wave function, it is easier to compare the systems $\mathcal{S}_S$ and $\mathcal{S}_P$ with $\mathcal{E}_{cl}$.

Let us transform actions (1.4), (1.5) to hydrodynamic variables instead of wave function ψ. Introducing new variables $j = \{\rho, \mathbf{j}\}$, φ, $\boldsymbol{\xi} = \{\xi_\alpha\}$, $\alpha = 1,2,3$ in the action (1.5) instead of ψ:

$$\psi = \rho^{1/2} e^{i\varphi} \boldsymbol{\sigma\eta}\chi, \qquad \psi^* = \chi^* \boldsymbol{\sigma\eta} e^{-i\varphi}\rho^{1/2}, \qquad \boldsymbol{\eta} = \boldsymbol{\xi}/\mid \boldsymbol{\xi} \mid$$

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \text{ const}, \qquad \chi^* = (\chi_1^*, \chi_2^*) = \text{ const} \tag{4.1}$$

$$\chi^*\chi = 1, \qquad \chi^*\boldsymbol{\sigma}\chi = \mathbf{b}, \qquad \mathbf{b}^2 = 1, \qquad \boldsymbol{\eta}^2 = 1,$$

one obtains the action (1.5) in terms of hydrodynamic variables

$$\mathcal{A}_P[\rho, \mathbf{j}, \varphi, \boldsymbol{\xi}] = \int (\mathcal{L}_{Pc} + \mathcal{L}_{sH} + \mathcal{L}_{Pq}) d^4x, \tag{4.2}$$

where

$$\mathcal{L}_{Pc} = \frac{m}{2\rho}\mathbf{j}^2 - \frac{e}{c} j^i A_i - \hbar j^i \partial_i \varphi - \hbar j^i f_i, \qquad j^0 \equiv \rho \tag{4.3}$$

$$\mathcal{L}_{sH} = -\frac{e\hbar}{2mc}\rho \mathbf{sH}, \qquad \mathbf{H} = \nabla \times \mathbf{A} \tag{4.4}$$

$$\mathcal{L}_{Pq} = \frac{\hbar^2\rho}{2m}\mathbf{f}^2 - \frac{\hbar^2\rho}{2m}(\partial_\alpha \eta_\beta)(\partial_\alpha \eta_\beta) - \frac{\hbar^2}{8m}\frac{(\nabla\rho)^2}{\rho} \tag{4.5}$$

Here $f = \{f_0, \mathbf{f}\}$ and $\mathbf{s}$ are functions of $\boldsymbol{\eta}$ and $\partial_i \boldsymbol{\eta}$ determined by the relations

$$\mathbf{s} = \{s_\alpha\}, \qquad s_\alpha = \psi^* \sigma_\alpha \psi / \rho = 2\eta_\alpha (b_\beta \eta_\beta) - b_\alpha, \qquad \alpha = 1,2,3; \tag{4.6}$$

$$f_i = \varepsilon_{\alpha\beta\gamma}\eta_\alpha \partial_i \eta_\beta b_\gamma, \qquad i = 0,1,2,3; \qquad f = \{f_0, \mathbf{f}\}, \tag{4.7}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the Levi-Chivita pseudo-tensor ($\varepsilon_{123} = 1, \quad \varepsilon^{123} = -1$). The unit vector $\mathbf{s}$ determines the electron spin direction, as one can see from Eq.(4.6). Mathematical details of this transformation are presented in Appendix B.

The reciprocal transformation from hydrodynamic variables $j = \{\rho, \mathbf{j}\}, \varphi, \boldsymbol{\xi}$ to the wave function ψ is defined by the relations (3.1), (3.2), (4.6) and

$$\boldsymbol{\xi}/\mid \boldsymbol{\xi} \mid = \boldsymbol{\eta} = (\mathbf{b} + \mathbf{s})[2(1 + \mathbf{bs})]^{-1/2} \tag{4.8}$$

Let us note that the Lagrangian $\mathcal{L}_{Pc}$ coincides with the Lagrangian of the classical ensemble (2.1) provided the arbitrary functions $g^\alpha(\boldsymbol{\xi})$ are chosen in the form

$$g^\alpha(\boldsymbol{\xi}) = - \mid \boldsymbol{\xi} \mid^{-2} \varepsilon_{\alpha\beta\gamma}\xi_\beta b_\gamma, \qquad \alpha = 1,2,3. \tag{4.9}$$

For the quasi-unform state of the system $\mathcal{S}_P$, when all dynamic variables (excluding the action $\hbar\varphi$) change slightly

$$\frac{\hbar}{mv} \mid \nabla u \mid \ll \mid u \mid, \qquad u = \rho, \eta_1, \eta_2, \eta_3; \qquad v = \mid \mathbf{j} \mid /\rho, \tag{4.10}$$

the term $\mathcal{L}_{Pq}$ of the action (4.2) is small as compared with $\mathcal{L}_{Pc}$ and can be neglected. If moreover the magnetic field vanishes, then $\mathcal{L}_{sH} = 0$, the Lagrangian $\mathcal{L}_P$ of the dynamic system $\mathcal{S}_P$ coincides with $\mathcal{L}_{Pc}$, and $\mathcal{S}_P$ turns into statistical ensemble of classical particles.

It means that in the quasi-uniform approximation (4.10) the dynamic system $\mathcal{S}_P$ is a *statistical ensemble* $\mathcal{E}_{cl}$ of classical particles (not a single classical particle). Then in the general case the dynamic system $\mathcal{S}_P$ is a *statistical ensemble* of quantum (stochastic) particles, (not a single quantum system).

Indeed, supposing that $\mathcal{S}_P$ is a single quantum system, one cannot understand, why in the quasi-uniform approximation $\mathcal{S}_P$ turns to an *ensemble* of classical systems, but not to a single classical system.

The action (1.4) for the dynamic system $\mathcal{S}_S$ can be also written in the hydrodynamic variables $j = \{\rho, \mathbf{j}\}$, φ. In this case a change of variables have the form

$$\psi = \sqrt{\rho}\exp(i\varphi), \qquad \psi^* = \sqrt{\rho}\exp(-i\varphi) \tag{4.11}$$

After the transformation the action (1.4) takes the form

$$\mathcal{A}_S[\rho, \mathbf{j}, \varphi] = \int (\mathcal{L}_{Sc} + \mathcal{L}_{Sq}) d^4x, \tag{4.12}$$

where

$$\mathcal{L}_{Sc} = \frac{m}{2\rho}\mathbf{j}^2 - \frac{e}{c} j^i A_i - \hbar j^i \partial_i \varphi, \qquad j^0 \equiv \rho \tag{4.13}$$

$$\mathcal{L}_{Sq} = -\frac{\hbar^2}{8m}\frac{(\nabla\rho)^2}{\rho} \tag{4.14}$$

Comparison of Lagrangians (4.13) and (2.1) shows that they coincide, if $g^\alpha = 0$ in Eq.(2.1). It means that in the quasi-uniform approximation (4.10) the system $\mathcal{S}_S$ is also a statistical ensemble of classical systems, and in general case $\mathcal{S}_S$ is a statistical ensemble of quantum systems. As far as in the quasi-uniform approximation both ensembles $\mathcal{S}_S$ and $\mathcal{S}_P$ reduce to the ensemble (2.1) which is equivalent to the ensemble (1.10), it means that in this approximation both ensembles $\mathcal{S}_S$ and $\mathcal{S}_P$ are constituted from similar dynamic systems $\mathcal{S}_d$, and each the system $\mathcal{S}_d$ is a non-relativistic pointlike classical particle moving in a given electromagnetic field.

Thus, in the quasi-uniform approximation (4.10) $\mathcal{S}_S$ and $\mathcal{S}_P$ differ in the structure of ensembles, but not in the dynamic systems they are constructed of. The classical parts (4.13) and (4.3) of the systems $\mathcal{S}_S$ and $\mathcal{S}_P$ are distinguished by the term

$$-\hbar j^i f_i = -\hbar j^i \varepsilon_{\alpha\beta\gamma}\eta_\alpha \partial_i \eta_\beta b_\gamma = -\hbar j^i \frac{\varepsilon_{\alpha\beta\gamma} s_\alpha \partial_i s_\beta b_\gamma}{2(1 + \mathbf{sb})} \tag{4.15}$$

which contains total derivatives $j^i\rho^{-1}\partial_i$ of only spin variables (4.8). This fact can be treated in the only way. Classical analog of a spin exists, and it is a collective (statistical) property determined by the ensemble structure. The spin is not a property of a single electron.

Note that in the semiclassical approximatioon the term (4.15) vanishes. Indeed, the semiclassical approximation (known also in the one-dimensional case as WKB-approximation [25-28]) is obtained by representation of the wave function in the form

$$\psi = \sqrt{\rho}\exp(\mathrm{iS}/\hbar), \qquad \rho = B/\mid\nabla S\mid \tag{4.16}$$

where the momentum $\mathbf{p} = \nabla S$ and B are supposed to change slowly. Then expanding S and B over powers of the constant $\hbar$, one can write a solution of dynamic equations in the form of asymptotic expansion. Semiclassical approximation used mainly for solving dynamic equations of quantum mechanics, including the case of particles with a spin [29,30]. But here we are interested in this approximation as the conventional way of a transition to the classical description.

It is easy to see that relations (4.16) coincides with Eq.(4.1), provided $S = \hbar\varphi$, $B = (\boldsymbol{\sigma\eta})\chi$. A formal transition in Eq.(4.2) to classical approximation is carried out , if one sets $\hbar = 0$. In this case the Lagrangians (4.4), (4.5) and the last term in Eq.(4.3) vanish, and the Lagrangian takes the form

$$\mathcal{L}_{Pqcl} = \frac{m}{2\rho}\mathbf{j}^2 - \frac{e}{c}j^iA_i - j^i\partial_iS \tag{4.17}$$

which coincides with the Lagrangian (4.13) for classical part of the system $\mathcal{S}_S$. The term (4.15) vanishes, because it contains the Planck constant as a factor, which cannot be removed by means of a transformation of the form $\boldsymbol{\eta} \to \boldsymbol{\eta}' = \sqrt{\hbar}\boldsymbol{\eta}$, because the vector $\boldsymbol{\eta}$ is normalized by the condition (4.1). If $\hbar \to 0$, then $\boldsymbol{\eta}' \to 0$, and the term (4.15) vanishes. Thus the quasi-uniform approximation is not equivalent to the semiclassical approximation, because the quasi-uniform approximation contains the term proportional to the quantum constant $\hbar$.

Note that according to the relation (4.10) the last term of the relation (4.3) is small as compared with the first term. But it is a small quantity of the first order, whereas all terms of Eq.(4.5) are small quantities of the second order. Usually the semiclassical approximation is considered as a formal expansion over powers of the quantum constant $\hbar$. From this point of view the Lagrangian (4.3) is not completely classical, because it contains the first order quantum correction. Hence, from this viewpoint the spin associated with this term is a quantum property. Thus, basing on SET, one can more exactly separate out the classical part of the Lagrangian, than it is made at an usage of the semiclassical approximation.

Let the electromagnetic field be absent $A_i = 0$, $i = 0, 1, 2, 3$. In the quasi-uniform approximation the actions for $\mathcal{S}_S$ and $\mathcal{S}_P$ have the form

$$\mathcal{A}_{Sc}[\rho, \mathbf{j}, a] = \int(\frac{m}{2\rho}\mathbf{j}^2 - j^i\partial_ia)d^4x, \tag{4.18}$$

$$\mathcal{A}_{Pc}[\rho, \mathbf{j}, a, \boldsymbol{\xi}] = \int(\frac{m}{2\rho}\mathbf{j}^2 - j^i\partial_ia - \hbar\boldsymbol{\xi}^{-2}j^i\varepsilon_{\alpha\beta\gamma}\xi_\alpha\partial_i\xi_\beta b_\gamma)d^4x, \tag{4.19}$$

where $a = \hbar\varphi$ is the action variable. $\mathcal{A}_{Sc}$ does not depend on the variable $\boldsymbol{\xi}$ and on the constant $\hbar$, whereas $\mathcal{A}_{Pc}$ depends on $\boldsymbol{\xi}$ and contains the quantum constant $\hbar$. Nevertheless $\mathcal{A}_{Pc}$ describes a statistical ensemble of classical systems. Appearance of $\hbar$ is unessential, because it enters in the arbitrary function g^{α} of Eq.(2.1).
The action (4.19) generates the dynamic equations

$$\delta a: \qquad \partial_0\rho + \nabla(\rho\mathbf{v}) = 0, \tag{4.20}$$

$$\delta\boldsymbol{\xi}: \qquad \partial_0\boldsymbol{\eta} + (\mathbf{v}\nabla)\boldsymbol{\eta} = 0, \tag{4.21}$$

$$\delta\rho: \qquad \partial_0 a = -m\mathbf{v}^2/2 - \hbar\varepsilon_{\alpha\beta\gamma}\eta_\alpha\partial_0\eta_\beta b_\gamma \tag{4.22}$$

$$\delta\mathbf{j}: \qquad \mathbf{v} = \mathbf{j}/\rho = \frac{1}{m}\nabla a + \frac{\hbar}{m}\varepsilon_{\alpha\beta\gamma}\eta_\alpha\nabla\eta_\beta b_\gamma \tag{4.23}$$

The equations (4.20)-(4.22) describe a time evolution of variables $\rho, \boldsymbol{\eta}, a$, whereas Eq.(4.23) does not contain time derivatives and determines the particle velocity via "potentials" $a, \boldsymbol{\eta}$. The time evolution of the velocity $\mathbf{v}$ is determined by the time evolution of potentials $a, \boldsymbol{\eta}$. Usinq Eq.(4.8), the equation (4.23) can be written in the form

$$\mathbf{v} = \frac{1}{m}[\nabla a + \frac{\hbar}{2(1+\mathbf{bs})}\varepsilon_{\alpha\beta\gamma}s_\alpha\nabla s_\beta b_\gamma], \tag{4.24}$$

where the velocity is expressed through the action a and the mean spin $\mathbf{s}$.

In the case of a spinless particle the dynamic equation are obtained from the action (4.18) and have the form

$$\delta a: \qquad \partial_0\rho + \nabla(\rho\mathbf{v}) = 0, \tag{4.25}$$

$$\delta\rho: \qquad \partial_0 a = -m\mathbf{v}^2/2, \tag{4.26}$$

$$\delta\mathbf{j}: \qquad \mathbf{v} = \mathbf{j}/\rho = \frac{1}{m}\nabla \mathrm{a}. \tag{4.27}$$

It easy to verify that in both cases the hydrodynamic variables $\rho, \mathbf{j}$ satisfy a system of the hydrodynamic equations (without pressure) of the form

$$\partial_0\rho + \nabla(\rho\mathbf{v}) = 0, \qquad \partial_0\mathbf{v} + (\mathbf{v}\nabla)\mathbf{v} = 0. \tag{4.28}$$

Hydrodynamic variables $\rho, \mathbf{v}$ of both ensembles of electons (4.19) and of spinless particles (4.18) satisfy Eq.(4.28). From the viewpoint of hydrodynamic equations (4.28) both dynamic systems are distinguished only by the initial conditions: (4.23) for electrons and (4.27) for spinless particles. Taking curl of relations (4.23) and (4.27), one obtains respectively

$$\nabla\times\mathbf{p} = \hbar\varepsilon_{\alpha\beta\gamma}(\nabla\eta_\alpha\times\nabla\eta_\beta)b_\gamma, \qquad \mathbf{p} = m\mathbf{v} \tag{4.29}$$

$$\nabla\times\mathbf{p} = 0, \tag{4.30}$$

where $\mathbf{p} = m\mathbf{v}$ is a momentum of a particle. As it follows from Eqs.(4.29), (4.30), the vorticity of the initial momentum field is determined by the spin. The momentum field of spinless particles is potential. It means that the spin $\mathbf{s}$ is a property of the electron ensemble as a whole (not of a single electron). The spin is a classical property, because it exists in the ensemble of classical systems.

In other words, there is a classical analog $\mathbf{s}$ of the electron spin operator $\boldsymbol{\sigma}$, and this spin is a collective (statistical) property of the ensemble, but not a manifestation of some inner degrees of freedom of a single electron.

Interaction of the spin with the magnetic field is described by the term (4.4) in the action (4.2), i.e the electron magnetic moment has a quantum nature, because the Lagrangian $\mathcal{L}_{Pc}+\mathcal{L}_{sH}$ has not the form (A.2) and the term (4.4) is not classical.

Let us note that separation of the quantum part $\mathcal{L}_{Pq}$ in the action (4.2) can be formalized by means of introducing special "quantum" variables $\boldsymbol{\kappa}_i = \{\kappa_i^1, \kappa_i^2, \kappa_i^3\}$, $i = 0,1,2,3$ which are responsible for quantum effects. Let us rewrite the part $\mathcal{A}_{Pq}$ of the action (4.2) corresponding to the Lagrangian (4.5) in the form

$$\mathcal{A}_{Pq}[\rho,\mathbf{j},\varphi,\boldsymbol{\xi},\kappa] = -\int \frac{\hbar^2\rho}{2m}[(\varepsilon_{\alpha\beta\gamma}\eta_\alpha\boldsymbol{\kappa}_\beta b_\gamma)\varepsilon_{\mu\nu\sigma}\eta_\mu b_\sigma(\boldsymbol{\kappa}_\nu - 2\nabla\eta_\nu)-$$
$$-\boldsymbol{\kappa}_\beta(\boldsymbol{\kappa}_\beta - 2\nabla\eta_\beta) - \frac{1}{4}\boldsymbol{\kappa}_0(\boldsymbol{\kappa}_0 - 2\nabla\log\rho)]d^4x \qquad (4.31)$$

Variation of Eq.(4.31) with respect to $\boldsymbol{\kappa}_i$, $i = 0,1,2,3$ leads to the equations

$$\boldsymbol{\kappa}_0 = \nabla\log\rho, \qquad (a_\mu a_\alpha - \delta_{\mu\alpha})(\boldsymbol{\kappa}_\alpha - \nabla\eta_\alpha) = 0, \qquad \mu = 1,2,3 \qquad (4.32)$$
$$a_\mu \equiv \varepsilon_{\alpha\mu\gamma}\eta_\alpha b_\gamma$$

Resolving Eq.(4.32) with respect to κ and substituting the solutions into Eq.(4.31), one obtains the quantum part (4.5) of the action (4.2). Suppressing variables $\boldsymbol{\kappa}$ ($\boldsymbol{\kappa}_i \equiv 0$, $i = 0,1,2,3$), the action (4.31) vanishes. It means that the variables κ are responsible for quantum effects, because vanishing κ lead to vanishing quantum part of the action. The classical part (4.3) and the quasi-quantum part (4.4) of the action remain. Separation of the action (4.2), carried out by the quantum variables κ, is invariant with respect to any change of dependent variables which does not contain κ. For instance, let the electromagnetic field vanish $A_i \equiv 0$. Returning to variables ψ^*,ψ,κ_i in the action (4.2) with $\mathcal{L}_{Pc},\mathcal{L}_{sH}$ and $\mathcal{L}_{Pq}$ defined respectively by Eqs. (4.3), (4.4) (with $A_i \equiv 0$) and (4.31), one obtains instead of Eq.(1.5)

$$\mathcal{A}_P[\psi^*,\psi,\kappa] = \int\{\frac{i\hbar}{2}(\psi^*\frac{\partial\psi}{\partial t} - \frac{\partial\psi^*}{\partial t}\psi) - \frac{\hbar^2}{2m}\nabla\psi^*\nabla\psi\}d^4x-$$
$$-\frac{\hbar^2}{2m}\int\psi^*\psi\{[\varepsilon_{\alpha\beta\gamma}\eta_\alpha(\boldsymbol{\kappa}_\beta - \nabla\eta_\beta)b_\gamma]^2-$$
$$-(\boldsymbol{\kappa}_\alpha - \nabla\eta_\alpha)(\boldsymbol{\kappa}_\alpha - \nabla\eta_\alpha) - \frac{1}{4}[\boldsymbol{\kappa}_0 - \nabla\log(\psi^*\psi)]^2\}dx^4, \qquad (4.33)$$

where

$$\boldsymbol{\eta} = (\psi^*\boldsymbol{\sigma}\psi + \mathbf{b}\psi^*\psi)\{2[(\psi^*\psi)^2 + \mathbf{b}(\psi^*\boldsymbol{\sigma}\psi)]\}^{-1/2}$$

and $\mathbf{b}$ is a constant unit vector. Eliminating κ_i from the action (4.33), one obtains the action (1.5) with $A_i = 0$. Setting $\boldsymbol{\kappa}_i \equiv 0$, $i = 0,1,2,3$ in Eq.(4.33), one obtains an action for the statistical ensemble of classical electrons in terms of wave functions.

5 Magnetic Moment of the Electron Ensemble

Let us consider a quasi-uniform (quasi-classical) electron ensemble. In such an ensemble the condition (4.10) takes place, and the terms containing the second power of derivatives (proportional to $\hbar^2$) can be neglected. It is equivalent formally to a calculation to within $\hbar^2$. Let us use the action of the form (B.5) neglecting there the terms proportional to $\hbar^2$.

$$\mathcal{A}_{Pm}[\rho, a, \boldsymbol{\xi}] = \int \{-\rho(\partial_0 a + \hbar f_0) - \frac{e}{c}\rho A_0 - \frac{\rho}{2m}(\nabla a + \frac{e}{c}\mathbf{A})^2 -$$

$$-\frac{\hbar}{m}(\nabla a + \frac{e}{c}\mathbf{A})[\rho\mathbf{f} + \frac{1}{2}\nabla \times (\mathbf{s}\rho)]\}d^4x + O(\hbar^2), \qquad \boldsymbol{\eta} = \boldsymbol{\xi}/\mid \boldsymbol{\xi} \mid \tag{5.1}$$

Here a is defined by the relation $a = \hbar\varphi$. $\mathbf{s}$ and $\mathbf{f}$ are functions of $\boldsymbol{\eta}$ defined by Eqs.(4.6), (4.7). The action $\mathcal{A}_{Pm}$ is not an action of a classical ensemble, because it cannot be written in the form (A.2).

The dynamic equations have the form

$$\frac{\delta \mathcal{A}_{Pm}}{\delta a} = \partial_0 \rho + \nabla[\frac{\rho}{m}(\nabla a + \frac{e}{c}\mathbf{A}) + \frac{\hbar}{m}\rho\mathbf{f}] = O(\hbar^2) \tag{5.2}$$

$$\frac{\delta \mathcal{A}_{Pm}}{\delta \rho} = -\partial_0 a - \frac{e}{c}A_0 - \hbar f_0 - \frac{1}{2m}(\nabla a + \frac{e}{c}\mathbf{A})^2 -$$

$$-\frac{\hbar}{m}(\nabla a + \frac{e}{c}\mathbf{A})\mathbf{f} + \frac{e\hbar}{2mc}\mathbf{s}\nabla \times \mathbf{A} = O(\hbar^2) \tag{5.3}$$

$$\frac{\delta \mathcal{A}_{Pm}}{\delta \xi_\alpha} = (\delta_{\alpha\mu} - \eta_\alpha\eta_\mu)\{-2\hbar\rho\varepsilon_{\mu\beta\gamma}[\partial_0\eta_\beta + \frac{1}{m}(\nabla a + \frac{e}{c}\mathbf{A})\nabla\eta_\beta]b_\gamma -$$

$$-\frac{e\hbar\rho}{mc}\varepsilon_{\nu\beta\gamma}\partial_\beta A_\gamma[\delta_{\mu\nu}(\mathbf{b}\boldsymbol{\eta}) + \eta_\nu b_\mu]\} = O(\hbar^2) \tag{5.4}$$

Obtaining Eq.(5.4), one uses Eqs.(5.2) and (4.6).

Let us introduce the following designations

$$\dot{\eta}_\beta = \partial_0\eta_\beta + \frac{1}{m}(\nabla a + \frac{e}{c}\mathbf{A})\nabla\eta_\beta + O(\hbar), \qquad \dot{\boldsymbol{\eta}} = \{\dot{\eta}_\beta\} \qquad \beta = 1, 2, 3; \tag{5.5}$$

$$\mathbf{H} = \nabla \times \mathbf{A} \tag{5.6}$$

where the dot means a substantional derivative, and $\mathbf{H}$ is a magnetic field. The equation (5.4) reduces to the form

$$\dot{\boldsymbol{\eta}} \times \mathbf{b} - \boldsymbol{\eta}[f_0 + \frac{1}{m}(\nabla a + \frac{e}{c}\mathbf{A})\mathbf{f}] + \frac{e}{2mc}[\mathbf{H}(\mathbf{b}\boldsymbol{\eta}) + \mathbf{b}(\boldsymbol{\eta}\mathbf{H}) - 2\boldsymbol{\eta}(\mathbf{H}\boldsymbol{\eta})(\boldsymbol{\eta}\mathbf{b})] = O(\hbar) \tag{5.7}$$

Taking the scalar product of Eq.(5.7) and $\mathbf{b}$, one obtains

$$f_0 + \frac{1}{m}(\nabla a + \frac{e}{c}\mathbf{A})\mathbf{f} = \frac{e}{2mc}[(\mathbf{H}\mathbf{b}) - 2(\mathbf{H}\boldsymbol{\eta})(\boldsymbol{\eta}\mathbf{b}) + (\boldsymbol{\eta}\mathbf{H})/(\boldsymbol{\eta}\mathbf{b})] + O(\hbar) \tag{5.8}$$

Substituting Eq.(5.8) into Eq.(5.3) and using the relations (4.6), one obtains

$$-\partial_0 a - \frac{e}{c}A_0 - \frac{1}{2m}(\nabla a + \frac{e}{c}\mathbf{A})^2 + \frac{e\hbar}{2mc}(\boldsymbol{\eta}\mathbf{H})/(\boldsymbol{\eta}\mathbf{b}) = O(\hbar^2) \tag{5.9}$$

It is the Jacobi-Hamilton equation for an electron moving in a given electromagnetic field. The last term describes an interaction of the magnetic moment $\boldsymbol{\mu}$ of the electron with the magnetic field, the magnetic moment being determined by the expression

$$\boldsymbol{\mu} = -\frac{e\hbar}{2mc}\boldsymbol{\eta}/(\boldsymbol{\eta}\mathbf{b}) + \mathbf{q} = -\frac{e\hbar}{mc}\frac{\mathbf{s}+\mathbf{b}}{(\mathbf{s}+\mathbf{b})^2} + \mathbf{q} \tag{5.10}$$

where $\mathbf{q}$ is a vector orthogonal to $\mathbf{H}$. Substituting Eq.(5.8) into Eq.(5.7), solving Eq.(5.7) with respect to $\dot{\boldsymbol{\eta}}$ and taking into account that $\boldsymbol{\eta}\dot{\boldsymbol{\eta}} = 0$, one obtains

$$\dot{\boldsymbol{\eta}} = -\frac{\Omega}{2}[\mathbf{n} + \mathbf{b}(\mathbf{n}\boldsymbol{\eta})/(\mathbf{b}\boldsymbol{\eta})] \times \boldsymbol{\eta} \tag{5.11}$$

where $\mathbf{n}$ is the unit vector directed along the magnetic field $\mathbf{H}$

$$\Omega = \frac{eH}{mc}, \qquad \mathbf{H} = H\mathbf{n}, \qquad \mathbf{n}^2 = 1, \qquad H = \sqrt{\mathbf{H}^2}. \tag{5.12}$$

If $\mathbf{H}$ is constant, the equation (5.11) can be solved. In this case $\mathbf{n}$=const, H=const.

Let $\mathbf{nb} = \cos\alpha \neq \pm 1$. Let us project Eq.(5.11) onto vectors $\mathbf{b}, \mathbf{n}$, and $\mathbf{b}\times\mathbf{n}$. Let us introduce the designations

$$u = (\mathbf{b}\boldsymbol{\eta}), \qquad v = (\mathbf{n}\boldsymbol{\eta}), \qquad w = (\mathbf{b}\times\mathbf{n})\boldsymbol{\eta} \tag{5.13}$$

Then Eq.(5.11) reduces to the system of equations

$$\dot{u} = -\frac{\Omega}{2}w, \qquad \dot{v} = \frac{\Omega}{2}vw/u, \qquad \dot{w} = -\frac{\Omega}{2}(v^2 - u^2)/\mathrm{u}. \tag{5.14}$$

The general solution of the system has the form

$$\begin{aligned} u &= \sqrt{d_1 - (d_1^2 - d_2^2)^{1/2}\sin(\Omega t + \psi)} \\ v &= d_2[d_1 - (d_1^2 - d_2^2)^{1/2}\sin(\Omega t + \psi_0)]^{-1/2} \\ w &= (d_1^2 - d_2^2)^{1/2}[d_1 - (d_1^2 - d_2^2)^{1/2}\sin(\Omega t + \psi_0)]^{-1/2}\cos(\Omega t + \psi_0) \end{aligned} \tag{5.15}$$

where $d_1 > 0$, d_2 and ψ_0 are constant along the lines tangent to the vector j^i. As far as $\boldsymbol{\eta}$ is an unit vector, the additional constraint on the constants d_1 and d_2 arises:

$$d_1 = d_2\cos\alpha + \frac{1}{2}\sin^2\alpha \tag{5.16}$$

As it follows from Eq.(5.15), if $d_1^2 \neq d_2^2$, the projections of the vector $\boldsymbol{\eta}$ onto directions $\mathbf{b}, \mathbf{n}, \mathbf{b}\times\mathbf{n}$ oscillate with the frequency $\Omega = eH/\mathrm{mc}$. The potential energy $\boldsymbol{\mu}\mathbf{H}$, connected with the magnetic moment is described by the expression

$$\boldsymbol{\mu}\mathbf{H} = -\frac{e\hbar H}{2mc}(\boldsymbol{\eta}\mathbf{n})/(\boldsymbol{\eta}\mathbf{b}) = -\frac{1}{2}\frac{\hbar\Omega d_2}{d_1 - (d_1^2 - d_2^2)^{1/2}\sin(\Omega t + \psi_0)} \tag{5.17}$$

$\boldsymbol{\mu}\mathbf{H}$ oscillates around the mean value

$$< \boldsymbol{\mu}\mathbf{H} > = -\frac{\Omega}{2\pi}\int\limits_{-\pi/2\Omega}^{\pi/2\Omega} \frac{\hbar\Omega d_2}{d_1-(d_1^2-d_2^2)^{1/2}\sin(\Omega t+\psi_0)}dt = -\hbar\frac{\Omega}{2}\,\mathrm{sgn}(d_2/d_1) \quad (5.18)$$

$$d_2 = uv = (\boldsymbol{\eta}\mathbf{n})(\boldsymbol{\eta}\mathbf{b})$$

Using expression (4.6) for the unit spin vector $\mathbf{s}$ and relations (5.13), (5.15), one can determine projections of $\mathbf{s}$ onto vectors $\mathbf{b}, \mathbf{n}$, and $[\mathbf{b}\times\mathbf{n}]$. One obtains

$$\begin{gathered}(\mathbf{sb}) = 2u^2-1 = 2d_1-1-2(d_1^2-d_2^2)^{1/2}\sin(\Omega t+\psi_0)\\ (\mathbf{sn}) = 2uv-(\mathbf{nb}) = 2d_2-\cos\alpha \\ \mathbf{s}(\mathbf{b}\times\mathbf{n}) = 2wu = 2(d_1^2-d_2^2)^{1/2}\cos(\Omega t+\psi_0)\end{gathered} \quad (5.19)$$

The solution (5.19) is stationary, if

$$d_1 = \varepsilon d_2, \qquad \varepsilon = \pm 1. \quad (5.20)$$

Substituting d_1 from Eq.(5.20) into Eq.(5.16), one obtains

$$d_2 = \frac{1}{2}(\varepsilon+\cos\alpha) \quad (5.21)$$

In the stationary case it follows from Eqs.(5.17), (5.18)

$$\boldsymbol{\mu}\mathbf{H} = < \boldsymbol{\mu}\mathbf{H} > = -\frac{1}{2}\hbar\Omega\varepsilon \quad (5.22)$$

Substituting Eq.(5.21) into Eq.(5.19), one obtains

$$(\mathbf{sb}) = \varepsilon\cos\alpha, \qquad (\mathbf{sn}) = \varepsilon, \qquad \mathbf{s}[\mathbf{b}\times\mathbf{n}] = 0 \quad (5.23)$$

It means that in the stationary case the spin vector $\mathbf{s}$ is parallel, or antiparallel to the magnetic field $\mathbf{H}$. In this case

$$\boldsymbol{\mu}\mathbf{H} = -\frac{e\hbar}{2mc}\mathbf{sH}, \qquad \boldsymbol{\mu} = -\frac{e\hbar}{2mc}\mathbf{s} \quad (5.24)$$

In the non-stationary case the relations (5.24) do not take place, in general, although projections of the spin $\mathbf{s}$ onto the magnetic field direction do not oscillate in any case.

Now let us consider the case, when $\mathbf{b} \parallel \mathbf{H}$ and

$$(\mathbf{nb}) = \cos\alpha = \varepsilon = \pm 1 \quad (5.25)$$

In this case the equation (5.11) has the form

$$\dot{\boldsymbol{\eta}} = -\Omega\mathbf{n}\times\boldsymbol{\eta} \quad (5.26)$$

Its solution can be written in the form

$$\boldsymbol{\eta} = a_1\mathbf{e}_1\cos(\Omega t+\psi_0) + a_2\mathbf{e}_1\times\mathbf{n}\sin(\Omega t+\psi_0) + \sqrt{1-a_1^2-a_2^2}\mathbf{n} \tag{5.27}$$

where $\mathbf{e}_1$ is an arbitrary unit vector orthogonal to the vector $\mathbf{n}$, a_1 and a_2 are real numbers satisfying the constraint $a_1^2+a_2^2 \leq 1$. In this case one obtains according to Eqs. (5.10), (4.6), (5.25) and (5.27)

$$\boldsymbol{\mu}\mathbf{H} = -\frac{1}{2}\hbar\Omega\varepsilon, \qquad (\mathbf{sn}) = \varepsilon(1-2a_1^2-2a_2^2) \tag{5.28}$$

The relation (5.24) does not take place, in general, if at least one of the quantities a_1, a_2 does not vanish, i.e. the flow is not steady-state. In the stationary case, when $a_1 = a_2 = 0$, the relations (5.28), (5.12) are equivalent to Eq.(5.24).

Thus, the connection (5.24) between the spin and the magnetic moment is valid only for stationary states. In the non-stationary case there is no connection between $<\boldsymbol{\mu}\mathbf{H}>$ and $(\mathbf{sH})$. There is no connection even between sgn$<\boldsymbol{\mu}\mathbf{H}>$ and sgn$(\mathbf{sH})$.

6 Transformation Properties

The action (4.2) is a functional of three arguments $\psi(x), \psi^*(x), A(x)$. Two first arguments $\psi(x), \psi^*(x)$ are dynamic variables the action is varied with respect to. Only these variables are shown as arguments of the action $\mathcal{A}_P$. The quantities $A_i(x)$, $i = 0,1,2,3$ are considered as a given external field, i.e as given functions of coordinates x. In general, these functions are unlike in different coordinate systems.

The action (4.2) is invariant with respect to transfomations of the rotation group and with respect to dynamic variables ψ, ψ^* and the field A_i. It means that at the infinitesimal transformation of coordinates

$$\begin{gathered} x^0 \to \tilde{x}^0 = x^0; \qquad x^\alpha \to \tilde{x}^\alpha = x^\alpha - \omega_{\alpha\beta}x^\beta + O(\omega^2), \qquad \alpha = 1,2,3; \\ \mathbf{x} \to \tilde{\mathbf{x}} = \mathbf{x} + \boldsymbol{\omega}\times\mathbf{x} + O(\omega^2); \\ \boldsymbol{\omega} = \{\frac{1}{2}\varepsilon_{\alpha\beta\gamma}\omega_{\beta\gamma}\}, \qquad \alpha = 1,2,3; \qquad \omega_{\alpha\beta} = \omega_{\beta\alpha}; \qquad \varepsilon_{123} = 1 \end{gathered} \tag{6.1}$$

which is followed by the transformation of variables ψ, ψ^*, A

$$\psi(x) \to \tilde{\psi}(\tilde{x}) = (\sigma_0 - \frac{i}{2}\boldsymbol{\omega\sigma})\psi(x) + O(\omega^2) = (\sigma_0 - \frac{i}{4}\varepsilon_{\alpha\beta\gamma}\omega_{\beta\gamma}\sigma_\alpha)\psi(x) + O(\omega^2)$$

$$\psi^*(x) \to \tilde{\psi}^*(\tilde{x}) = \psi^*(x)(\sigma_0 + \frac{i}{2}\boldsymbol{\omega\sigma}) + O(\omega^2) = \psi^*(x)(\sigma_0 + \frac{i}{4}\varepsilon_{\alpha\beta\gamma}\omega_{\beta\gamma}\sigma_\alpha) + O(\omega^2) \tag{6.2}$$

$$\begin{gathered} A_0(x) \to \tilde{A}_0(\tilde{x}) = A_0(x); \\ A_\alpha(x) \to \tilde{A}_\alpha(\tilde{x}) = A_\alpha(x) - \omega_{\alpha\beta}A_\beta(x) + O(\omega^2), \qquad \alpha = 1,2,3, \end{gathered} \tag{6.3}$$

the action (4.2) considered as a functional of arguments ψ, ψ^*, A remains invariant and conserves its form, i.e.

$$\mathcal{A}_P[\psi^*,\psi] \equiv \mathcal{F}[\psi^*,\psi,A] = \mathcal{F}[\tilde{\psi}^*,\tilde{\psi},\tilde{A}] \equiv \tilde{\mathcal{A}}_P[\tilde{\psi}^*,\tilde{\psi}] \tag{6.4}$$

$$\mathcal{A}_P[\psi^*,\psi] = \int \mathcal{L}_P[\psi^*(x),\psi(x),A(x)]d^4x \tag{6.5}$$

$$\tilde{\mathcal{A}}_P[\tilde{\psi}^*,\tilde{\psi}] = \int \mathcal{L}_P[\tilde{\psi}^*(\tilde{x}),\tilde{\psi}(\tilde{x}),\tilde{A}(\tilde{x})\}]d^4\tilde{x} \tag{6.5}$$

Here

$$\mathcal{L}_P[\psi^*(x),\psi(x),A(x)] = \frac{i\hbar}{2}[\psi^*(x)\frac{\partial\psi}{\partial t}(x) - \frac{\partial\psi^*}{\partial t}(x)\psi(x)] - \frac{e}{c}A_0\psi^*(x)\psi(x) -$$

$$-\frac{1}{2m}[i\hbar\partial_\alpha + \frac{e}{c}A_\alpha(x)]\psi^*(x)[-i\hbar\partial_\alpha + \frac{e}{c}A_\alpha(x)]\psi(x) +$$

$$\frac{e\hbar}{2mc}\varepsilon_{\alpha\beta\gamma}A_\beta(x)\partial_\alpha[\psi^*(x)\sigma_\gamma\psi(x)] \tag{6.6}$$

Let us note that the reference to arguments of the action $\mathcal{A}_P$ is essential here, because the form of functionals $\mathcal{F}$ considered as functionals of arguments ψ, ψ, A and $\tilde{\psi}$, $\tilde{\psi}^*$, $\tilde{A}$ is the same before and after the transformation, whereas the form of functionals $\mathcal{A}_P[\psi^*,\psi]$, $\tilde{\mathcal{A}}_P[\tilde{\psi}^*,\tilde{\psi}]$, considered as functionals of ψ,ψ^* and $\tilde{\psi},\tilde{\psi}^*$ is different, in general, because $A(x)$ and $\tilde{A}(\tilde{x})$ are different, in general, functions of their arguments.

A transformation from variables ψ,ψ^* to variables $\rho,\mathbf{j},\boldsymbol{\eta},\varphi$ is described by the relations (3.1)-(4.1). The relations (3.1), (3.2) express explicitly the variables $\rho,\mathbf{j}$ through ψ, ψ^*, and Eq.(4.1) expresses implicitly the variables φ, $\boldsymbol{\eta}$ through ψ,ψ^*. According to Eqs.(4.6), (4.8) an explicit expression of variables φ, $\boldsymbol{\eta}$ through ψ,ψ^* has the form

$$\boldsymbol{\eta} = (\mathbf{s}+\mathbf{b})[2(1+\mathbf{bs})]^{-1/2} = \frac{\psi^*\boldsymbol{\sigma}\psi + (\chi^*\boldsymbol{\sigma}\chi)(\psi^*\psi)}{\sqrt{2[(\psi^*\psi) + (\psi^*\boldsymbol{\sigma}\psi)(\chi^*\boldsymbol{\sigma}\chi)](\psi^*\psi)}} \tag{6.7}$$

$$\sin\varphi = \frac{\chi^*\psi - \psi^*\chi}{i\sqrt{2[(\psi^*\psi) + (\psi^*\boldsymbol{\sigma}\psi)(\chi^*\boldsymbol{\sigma}\chi)]}} \tag{6.8}$$

and contains in the explicit form the parameters χ (or $\mathbf{b}$) which are not dynamic variables. The dynamic variables ρ and $\mathbf{j}$ are expressed only through ψ, ψ^*, A_i, and the law of transformation of ρ and $\mathbf{j}$ at rotations (6.1) is determined completely by the transformation law (6.2), (6.3) of variables ψ,ψ^* and A_i.

$$\rho(x) \to \tilde{\rho}(\tilde{x}) = \rho(x) \tag{6.9}$$

$$\mathbf{j}(x) \to \tilde{\mathbf{j}}(\tilde{x}) = \mathbf{j}(x) + \boldsymbol{\omega}\times\mathbf{j}(x) + O(\omega^2) \tag{6.10}$$

The transformation law of φ, $\boldsymbol{\eta}$ is determined not only by the transformation law of variables ψ,ψ^*, but also by the transformation law of parameters χ at the rotation (6.1). Strictly, one may transform χ according to any representation of the rotation group (6.1).

Using relations (6.7), (6.8), one obtains

$$\delta\boldsymbol{\eta}=\tilde{\boldsymbol{\eta}}(\tilde{x})-\boldsymbol{\eta}(x)=\boldsymbol{\omega}\times\boldsymbol{\eta}+\frac{1}{2(\mathbf{b}\boldsymbol{\eta})}\mathcal{P}_{\perp}(\delta\mathbf{b}-\boldsymbol{\omega}\times\mathbf{b}) \tag{6.11}$$

where $\mathcal{P}_{\perp}$ is the projection operator on the plane orthogonal to the unit vector $\boldsymbol{\eta}$

$$\mathcal{P}_{\perp}\mathbf{b}=\mathbf{b}-\boldsymbol{\eta}(\boldsymbol{\eta}\mathbf{b}) \tag{6.12}$$

$$\begin{aligned}\delta\varphi=\tilde{\varphi}(\tilde{x})-\varphi(x)=-\tan(\varphi)\boldsymbol{\eta}(\delta\mathbf{b}-\boldsymbol{\omega}\times\mathbf{b})+\\ \frac{(\delta\chi^{*}-\chi^{*}i\boldsymbol{\sigma}\boldsymbol{\eta}/2)\psi-\psi^{*}(\delta\chi+i\boldsymbol{\sigma}\boldsymbol{\eta}\chi/2)}{2i\rho^{1/2}\cos(\varphi)(\mathbf{b}\boldsymbol{\eta})}\end{aligned} \tag{6.13}$$

Let χ be a scalar. Then according to Eq. (4.1)

$$\delta\chi=0,\qquad\delta\chi^{*}=0,\qquad\delta\mathbf{b}=0. \tag{6.14}$$

In this case φ is not a scalar, and $\boldsymbol{\eta}$ is not a vector. The law of transformation of these variables is rather complicated and includes not only the variables $\boldsymbol{\eta}$, φ, but other variables and constants.

Let now χ be a spinor, i.e.

$$\delta\chi=-\frac{i}{2}\boldsymbol{\sigma}\boldsymbol{\eta}\chi,\qquad\delta\chi^{*}=\frac{i}{2}\chi^{*}\boldsymbol{\sigma}\boldsymbol{\eta}, \tag{6.15}$$

Then according to Eq.(4.1) $\mathbf{b}$ is a vector

$$\delta\mathbf{b}=\boldsymbol{\omega}\times\mathbf{b} \tag{6.16}$$

It follows from Eqs.(6.13) and (6.11) that φ is a scalar and $\boldsymbol{\eta}$ is a vector

$$\delta\varphi=0,\qquad\delta\boldsymbol{\eta}=\boldsymbol{\omega}\times\boldsymbol{\eta}. \tag{6.17}$$

In this case the transformation laws for variables φ, $\boldsymbol{\eta}$ are rather simple, and variables φ, $\boldsymbol{\eta}$ transform only through themselves. Let us study this case.

Thus, we have

$$\chi\rightarrow\tilde{\chi}=(\sigma_0-\frac{i}{2}\boldsymbol{\omega}\boldsymbol{\sigma})\chi+O(\omega^2)$$

$$\chi^{*}\rightarrow\tilde{\chi}^{*}=\chi^{*}(\sigma_0+\frac{i}{2}\boldsymbol{\omega}\boldsymbol{\sigma})+O(\omega^2) \tag{6.18}$$

$$\mathbf{b}\rightarrow\tilde{\mathbf{b}}=\mathbf{b}+\boldsymbol{\omega}\times\mathbf{b}+O(\omega^2). \tag{6.19}$$

$$\varphi(x)\rightarrow\tilde{\varphi}(\tilde{x})=\varphi(x),$$

$$\boldsymbol{\eta}(x)\rightarrow\tilde{\boldsymbol{\eta}}(\tilde{x})=\boldsymbol{\eta}(x)+\boldsymbol{\omega}\times\boldsymbol{\eta}(x)+O(\omega^2), \tag{6.20}$$

The action (4.2)-(4.5) is invariant with respect to rotation group (6.1) and retains its form with respect to variables $\rho,\mathbf{j},\boldsymbol{\eta},\varphi,A,\mathbf{b}$. It means that the action (4.2) satisfies the conditions

$$\mathcal{A}_P[\rho,\mathbf{j},\varphi,\boldsymbol{\eta}]\equiv\mathcal{F}[\rho,\mathbf{j},\varphi,\boldsymbol{\eta},A,\mathbf{b}]=\mathcal{F}[\tilde{\rho},\tilde{\mathbf{j}},\tilde{\varphi},\tilde{\boldsymbol{\eta}},\tilde{A},\tilde{\mathbf{b}}]\equiv\tilde{\mathcal{A}}_P[\tilde{\rho},\tilde{\mathbf{j}},\tilde{\varphi},\tilde{\boldsymbol{\eta}}] \tag{6.21}$$

The action (4.2) considered as a functional $\mathcal{F}$ of dynamic variables and of the quantities $A, \mathbf{b}$ does not change its form after rotation. At the same time the same action considered as a functional $\mathcal{A}_P = \mathcal{F}_1$ of only dynamic variables $\rho, j, \varphi, \boldsymbol{\eta}$ at fixed external fields $A, \mathbf{b}$ changes its form, in general, because in this case $A(x), \mathbf{b}(x)$ and $\tilde{A}(\tilde{x}), \tilde{\mathbf{b}}(\tilde{x})$ are unlike, in general, functions of their arguments. In both cases A and $\mathbf{b}$ are transformed according to Eqs.(6.3), (6.18).

The constant unit vector $\mathbf{b}$ is to be considered as a constant external field determining some direction in the space. This field represent a "spin" (4.1) of the constant base spinor χ. The vector $\mathbf{b}$ is absent in the original action (1.4). Both the base spinor χ and the vector $\mathbf{b}$ can be chosen in an arbitrary way. But being chosen in some coordinate system, they are determined single-valuedly in any other coordinate system obtained by means of a rotation.

Appearance of the vector field $\mathbf{b}$ is connected with the fact that both the action (4.2)-(4.5) and (2.1) take into account the solution (A.20) of hydrodynamic equations of the form (A.18), (A.19). In general, solutions of a rotation invariant equation are not rotationally invariant. For instance, if u is a scalar, the wave equation

$$c^{-2}\partial_0^2 u - \nabla^2 u = 0 \tag{6.22}$$

is rotationally invariant. But its solution of the form of a plane wave

$$u_k = C\exp\{i\omega_0 t - \mathrm{i}\mathbf{k}\mathbf{x}\}, \qquad \omega_0 = c^{-1}\sqrt{\mathbf{k}^2} \tag{6.23}$$

is not rotationally invariant. At the rotation the solution turns into another solution $u_{\tilde{k}}$ with other parameters $\tilde{\mathbf{k}}$. Only the whole set of solutions u_k is invariant with respect to rotations, but not each solution singly. If we want to make any single solution u_k to be invariant with respect to rotation, we should follow any coordinate transformation (6.1) with the transformation (6.19) of parameters $\mathbf{k}$

$$\mathbf{k} \rightarrow \tilde{\mathbf{k}} = \mathbf{k} + \boldsymbol{\omega} \times \mathbf{k} + O(\omega^2). \tag{6.24}$$

Thus, the solution u_k determined by Eq.(6.19) is rotationally invariant with respect to simultaneous rotations of vectors $\mathbf{x}$, $\mathbf{k}$, (but not with respect to a rotation only of the vector $\mathbf{x}$). Let us remember that $\mathbf{x}$ is an independent variable, and $\mathbf{k}$ is a parameter.

Description (1.5) of the electron ensemble in terms of the wave function (spinors) contains implicitly some arbitrariness connected with the choice of functions g^α. It is a reason why one cannot interpret the wave function itself. Only the combinations of the wave function component which does not contain the functions g^α can be interpreted. Only a set of solutions labelled by the vector $\mathbf{b}$ is rotationally invariant. As one passes from the spinor description (1.5) of the electron ensemble to the tensor description (4.2)-(4.5), the solutions of the form (A.20) with the functions g^α (4.9) are used. In order the term $j^i g^\alpha(\boldsymbol{\xi})\partial_i\xi_\alpha$ of the action (4.2) be rotationally invariant, it is necessary to transform the parameter $\mathbf{b}$ as a vector simultaneously with the dynamic variable $\boldsymbol{\xi}$. It is a reason, why the action (4.2)-(4.5) is rotationally invariant with respect to variables $\rho, \mathbf{j}, \boldsymbol{\eta}, \varphi, A, \mathbf{b}$, but it is not such with respect to variables $\rho, \mathbf{j}, \boldsymbol{\eta}, \varphi$,A.

7 Concluding Remarks

Thus, comparing two distributed quantum systems $\mathcal{S}_P, \mathcal{S}_S$ between themselves and with the statistical ensemble $\mathcal{E}_c$ of classical particles, one concludes.
(1) The Pauli and Schrödinger equations can be considered as describing a statistical ensemble of quantum stochastic particles.
(2) The electron spin can be interpreted as a classical collective property of a statistical ensemble which is connected with the vorticity of the ensemble particle flow.
(3) In this classical collective interpretation the electron magnetic moment can be related to the electron spin only in the stationary state.

Let us stress that these statements are a result of a mathematical investigation of quantum systems described by the Pauli and Schrödinger equations considered simply as dynamic systems. No additional suppositions (including quantum principles) are used. Any objections based on the QM principles justify against the QM principles themselves.

APPENDIX A
Classical Ensemble in Lagrangian and Eulerian Coordinates

Let us show equivalency of two forms (1.10) and (2.1) of the action for the statistical ensemble of classical systems. This equivalency can be formulated in a more general form. Let $L(t, \mathbf{x}, d\mathbf{x}/dt)$, $\mathbf{x} = \{x^\alpha\}$, $\alpha = 1, 2, ..n$ be a Lagrangian of a classical system $\mathcal{S}_d$. Then the action for the statistical ensemble of the systems $\mathcal{S}_d$ has the form

$$\mathcal{A}_L[\mathbf{x}] = \int L(t, \mathbf{x}, d\mathbf{x}/dt) dt d\boldsymbol{\xi}, \qquad d\boldsymbol{\xi} = \prod_{\alpha=1}^{\alpha=n} d\xi_\alpha \tag{A.1}$$

where $\boldsymbol{\xi} = \{\xi_\alpha\}$, $\alpha = 1, 2, \dots n$ are Lagrangian coordinates labelling the systems of the ensemble, and $\mathbf{x} = \mathbf{x}(t, \boldsymbol{\xi})$ is a function of t and $\boldsymbol{\xi}$.

The variational problem with functional (A.1) is equivalent to the variational problem with the functional

$$\mathcal{A}_E[j, \varphi, \boldsymbol{\xi}] = \int \{\mathcal{L}(x, j) - \hbar j^i [\partial_i \varphi + g^\alpha(\boldsymbol{\xi}) \partial_i \xi_\alpha]\} d^{n+1}x, \qquad d^{n+1}x = \prod_{i=0}^{n} dx^i, \tag{A.2}$$

where $j = \{j^0, \mathbf{j}\}$, $\mathbf{j} = \{j^\alpha\}$, $\alpha = 1, 2, \dots n$, φ, $\boldsymbol{\xi} = \{\xi_\alpha\}$, $\alpha = 1, 2, \dots n$ are functions of $x = \{x^0, \mathbf{x}\}$, $x^0 = t$, $\mathbf{x} = \{x^\alpha\}$, $\alpha = 1, 2, \dots n$, and a summation is made over repeating indices: over Latin ones $(0 - n)$ and over Greek ones $(1 - n)$. All variables $j = \{\rho, \mathbf{j}\}$, φ, $\boldsymbol{\xi}$ are functions of x. $g^\alpha(\boldsymbol{\xi})$, $\alpha = 1, 2, \dots n$ are arbitrary functions of $\boldsymbol{\xi}$. The function $\mathcal{L}(x, j)$ is defined by the relation

$$\mathcal{L}(x, j) = j^0 L(x^0, \mathbf{x}, \mathbf{j}/j^0) \tag{A.3}$$

where L is the Lagrangian of a single system $\mathcal{S}_d$. The variables $j = \{j^0, \mathbf{j}\}$ are connected with the variables $\mathbf{x}(t, \boldsymbol{\xi})$, $d\mathbf{x}(t, \boldsymbol{\xi})/dt$ by means of relations

$$\rho = \det \parallel \xi_{\alpha,\beta} \parallel, \qquad \alpha, \beta = 1, 2, \dots n, \qquad \xi_{\beta,\alpha} \equiv \partial_\alpha \xi_\beta, \qquad \mathbf{j} = \rho \frac{d\mathbf{x}}{dt} \tag{A.4}$$

and the functions $\boldsymbol{\xi} = \boldsymbol{\xi}(x)$ are determined as a solution of the system of equations

$$x^\alpha = x^\alpha(t, \boldsymbol{\xi}), \qquad \alpha = 1, 2, \ldots n \tag{A.5}$$

Let us introduce the time Lagrangian coordinate ξ_0, supposing that $t = t(\xi_0)$, $x = \{t, \mathbf{x}\} = x(\xi)$, $\xi = \{\xi_0, \boldsymbol{\xi}\}$. Then the action (A.1) can be rewritten in the form

$$\mathcal{A}_L[x] = \int \mathcal{L}(x, \dot{x}) d^{n+1}\xi, \qquad \dot{x} \equiv \partial x / \partial \xi_0, \qquad d^{n+1}\xi = d\xi_0 d\boldsymbol{\xi}, \tag{A.6}$$

$$\mathcal{L}(x, \dot{x}) = L(x^0, \mathbf{x}, \dot{\mathbf{x}}/\dot{x}^0)\dot{x}^0. \tag{A.7}$$

According to Eq.(A.7) the Lagrangian $\mathcal{L}(x, \dot{x})$ is a first order homogeneous function of $\dot{\mathbf{x}}$, i.e.

$$\mathcal{L}(x, a\dot{x}) = a\mathcal{L}(x, \dot{x}), \tag{A.8}$$

where a is an arbitrary parameter.

Let the variational problem with the functional (A.6) take place. Let us show that it is equivalent to the variational problem with the functional (A.2).

Considering $\xi = \{\xi_i\}$, $i = 0, 1, ..n$ as functions of $x = \{x^i\}$, $i = 0, 1, \ldots n$, the variational problem with the action (A.6) can be formulated as a variational problem with the action

$$\mathcal{A}[\xi] = \int \mathcal{L}(x, \frac{\partial J}{\partial \xi_{0,i}}) d^{n+1}x, \qquad \xi = \xi(x) \tag{A.9}$$

where J is a Jacobian

$$J \equiv \frac{\partial(\xi_0, \xi_1, \ldots \xi_n)}{\partial(x^0, x^1, \ldots x^n)} \equiv \det \parallel \xi_{i,k} \parallel,$$

$$\xi_{i,k} \equiv \partial \xi_i / \partial x^k, \qquad i, k = 0, 1, \ldots n \tag{A.10}$$

and

$$\frac{\partial J}{\partial \xi_{0,i}} \equiv \frac{\partial(x^i, \xi_1, \xi_2, \ldots \xi_n)}{\partial(x^0, x^1, \ldots x^n)} \equiv \frac{\partial x^i}{\partial \xi_0} J, \qquad i = 0, 1, \ldots n \tag{A.11}$$

Deriving Eq.(A.9) from Eq.(A.6), the relations (A.8), (A.11) were used.

In the variational problem with the action (A.9)) the Lagrangian coordinates $\xi = \{\xi_i\}$, $i = 0, 1, \ldots n$ are dynamic variables considered as functions of coordinates $x = \{x^i\}$, $i = 0, 1, \ldots n$.

Let us use designations

$$j^i = \frac{\partial J}{\partial \xi_{0,i}}, \qquad i = 0, 1, \ldots n \tag{A.12}$$

Adding constraints (A.12) to the action (A.9), one does not change the variational problem, because Eq.(A.9) does not contain variables j^i, and Eq. (A.12) is not a constraint in reality. Let us introduce Eq.(A.12) into the action (A.9) by means of Lagrange multpliers $p = \{p_i\}$, $i = 0, 1, \ldots n$. j^i and p_i, $i = 0, 1, \ldots n$ mean respectively a current density and a momentum density in the space $\{t, \mathbf{x}\}$.

Then one obtains

$$\mathcal{A}[j, p, \xi] = \int [\mathcal{L}(x, j) - p_i(j^i - \frac{\partial J}{\partial \xi_{0,i}})] dx^{n+1}. \qquad (A.13)$$

Dynamic equations generated by the action (A.13) have the form

$$\frac{\delta \mathcal{A}}{\delta \xi_\alpha} = -\partial_k (p_l \frac{\partial^2 J}{\partial \xi_{0,l} \partial \xi_{\alpha,k}}) = 0, \qquad \alpha = 1, 2, \ldots n; \qquad (A.14)$$

$$\frac{\delta \mathcal{A}}{\delta j^i} = \frac{\partial \mathcal{L}}{\partial j^i} - p_i = 0, \qquad i = 0, 1, \ldots n \qquad (A.15)$$

Variation with respect to p_i leads to Eq.(A.12).

Using identities

$$\partial_k (\frac{\partial^2 J}{\partial \xi_{0,l} \partial \xi_{i,k}}) \equiv 0, \qquad i, l = 0, 1, \ldots n, \qquad \partial_k j^k = \partial_k \frac{\delta J}{\delta \xi_{0,i}} \equiv 0, \qquad (A.16)$$

$$\frac{\partial^2 J}{\partial \xi_{0,i} \partial \xi_{j,k}} \xi_{j,l} \equiv \delta_l^k \frac{\partial J}{\partial \xi_{0,i}} - \delta_l^i \frac{\partial J}{\partial \xi_{0,k}}, \qquad i, k, l = 0, 1, \ldots n, \qquad (A.17)$$

and eliminating the variables ξ, one can reduce equatons (A.12), (A.14), (A.15) to the conventional hydrodynamic form

$$\partial_i j^i = 0, \qquad (A.18)$$

$$j^i [\partial_i \frac{\partial \mathcal{L}(x, j)}{\partial j^k} - \partial_k \frac{\partial \mathcal{L}(x, j)}{\partial j^i}] = 0, \qquad k = 0, 1, \ldots n. \qquad (A.19)$$

Instead of elimination of variables ξ Eqs. (A.14), (A.15) can be integrated in the form

$$p_k = \frac{\partial \mathcal{L}(x, j)}{\partial j^k} = \hbar \partial_k \varphi + \hbar g^\beta(\boldsymbol{\xi}) \partial_k \xi_\beta; \qquad k = 0, 1, \ldots n, \qquad (A.20)$$

where $g^\beta(\boldsymbol{\xi})$, $\beta = 1, 2, \ldots n$ are arbitrary functions of the argument $\boldsymbol{\xi} = \{\xi_\alpha\}$, $\alpha = 1, 2, \ldots$n. φ is some new variable. $\hbar$ is the Planck constant which is introduced, for variables φ, $\boldsymbol{\xi} = \{\xi_\alpha\}$, $\alpha = 1, 2, \ldots n$ and the functions g^β were dimensionless. In this context $\hbar$ has no quantum meaning.

Indeed, let us substitute relation (A.20) into Eq.(A.14) and use the first Eq.(A.16), antisymmetry of $\partial^2 J / \partial \xi_{0,l} \partial \xi_{\alpha,k}$ and symmetry of $\partial_k \partial_l \varphi$ with respect to indices l, k. Then one obtains

$$\partial_k [\frac{\partial^2 J}{\partial \xi_{0,l} \partial \xi_{\alpha,k}} g^\beta(\boldsymbol{\xi}) \xi_{\beta,l}] = 0, \qquad \alpha = 1, 2, \ldots n. \qquad (A.21)$$

Now using the identity

$$\frac{\partial^2 J}{\partial \xi_{0,l}\partial \xi_{\alpha,k}}\xi_{\beta,l} \equiv -\delta_\beta^\alpha \frac{\partial J}{\partial \xi_{0,k}}, \qquad \alpha,\beta = 1,2\ldots n, \qquad k = 0,1,\ldots n, \tag{A.22}$$

and the second relation (A.16), one can reduce Eq.(A.21) to the form

$$-\delta_\beta^\alpha \frac{\partial J}{\partial \xi_{0,k}}\xi_{\gamma,k}\frac{\partial g^\beta(\boldsymbol{\xi})}{\partial \xi_\gamma} = 0, \qquad \alpha = 1,2,\ldots n. \tag{A.23}$$

Due to the identity

$$\frac{\partial J}{\partial \xi_{0,k}}\xi_{l,k} \equiv \delta_l^0 J, \qquad l = 0,1,\ldots n \tag{A.24}$$

the equation (A.23) is satisfied for any functions $g^\beta(\boldsymbol{\xi})$, $\beta = 1,2,\ldots$n. Thus, the relation (A.20) is a general solution of the system of equations (A.14).

Let us substitute Eq.(A.20) into the action (A.13) and use that due to Eq.(A.24)

$$\frac{\partial J}{\partial \xi_{0,k}}p_k = \hbar \frac{\partial(\varphi,\xi_1,\xi_2,\ldots\xi_n)}{\partial(x^0,x^1,\ldots x^n)}. \tag{A.25}$$

Then the action (A.13) turns to Eq.(A.2). The term (A.25) is omitted, because it does not give any contribution into dynamic equations. Thus, Eq.(A.2) is a corollary of Eq.(A.1).

Now let the variational problem with the functional (A.2) take place. Corresponding dynamic equations have the form

$$\frac{\delta \mathcal{A}}{\delta \varphi} = \hbar \partial_i j^i = 0, \qquad i = 0,1,\ldots n; \tag{A.26}$$

$$\frac{\delta \mathcal{A}}{\delta j^i} = \frac{\partial \mathcal{L}}{\partial j^i} - \hbar\partial_i\varphi - \hbar g^\alpha(\boldsymbol{\xi})\xi_{\alpha,i} = 0, \qquad i = 0,1,\ldots n; \tag{A.27}$$

$$\frac{\delta \mathcal{A}}{\delta \xi_\alpha} = -\hbar[\frac{\partial g^\beta(\boldsymbol{\xi})}{\partial \xi_\alpha} - \frac{\partial g^\alpha(\boldsymbol{\xi})}{\partial \xi_\beta}]j^i\partial_i\xi_\beta = 0, \qquad \alpha = 1,2,\ldots n; \tag{A.28}$$

The relations (A.12) satisfy Eqs. (A.26), (A.28) identically, and usage of them as additional constraints does not change the variational problem. Substituting (A.12) into Eq.(A.2) and using identities (A.24), (A.25), one obtains Eq.(A.9) which is equivalent to Eq.(A.1).

APPENDIX *B*
Transformation of the Pauli Equation to the Hydrodynamic Form

Let us transform the action (1.5) for the Pauli equation to the hydrodynamic form (4.2)-(4.5). Let us substitute expression (4.1) for ψ into Eq.(1.5). Using designations (4.6), (4.7) and relations between the Pauli matrices

$$\sigma_0\sigma_\alpha = \sigma_\alpha\sigma_0 = \sigma_\alpha, \qquad \alpha = 1,2,3,$$

$$\sigma_\alpha \sigma_\beta = \delta_{\alpha\beta} + i\varepsilon_{\alpha\beta\gamma}\sigma_\gamma, \qquad \alpha,\beta = 1,2,3 \tag{B.1}$$

which permit to reduce any functions of σ to their linear combinations, one obtains after calculations

$$[i\hbar\partial_\alpha + \frac{e}{c}A_\alpha)\psi^*][-i\hbar\partial_\alpha + \frac{e}{c}A_\alpha)\psi] = \frac{m^2}{\rho}\{\mathbf{j} - \frac{\hbar}{m}[\mathbf{f}\rho + \frac{1}{2}\nabla \times (\mathbf{s}\rho)]\}^2 +$$

$$2\hbar\rho\mathbf{f}(\hbar\nabla\varphi + \frac{e}{c}\mathbf{A}) + \hbar^2\rho(\partial_\alpha\eta_\beta\partial_\alpha\eta_\beta) + \frac{\hbar^2}{4}\frac{(\nabla\rho)^2}{\rho}, \tag{B.2}$$

$$\frac{i\hbar}{2}(\psi^*\frac{\partial\psi}{\partial t} - \frac{\partial\psi^*}{\partial t}\psi) = -\hbar\rho\frac{\partial\varphi}{\partial t} - \hbar f_0\rho, \tag{B.3}$$

$$\mathbf{j} = \frac{\hbar}{m}[\rho\nabla\varphi + \rho\varepsilon_{\alpha\beta\gamma}\eta_\alpha\nabla\eta_\beta b_\gamma + \frac{1}{2}\nabla \times (\rho\mathbf{s})] + \frac{e}{mc}\mathbf{A}\rho. \tag{B.4}$$

Substituting the expressions (B.2), (B.3) into the action (1.5). one obtains the action as a functional of variables $\rho, \varphi, \boldsymbol{\eta}$

$$\mathcal{A}_P[\rho,\varphi,\boldsymbol{\eta}] = \int [-\frac{\rho}{2m}(\hbar\nabla\varphi + \frac{e}{c}\mathbf{A})^2 - \frac{\hbar}{m}(\hbar\nabla\varphi + \frac{e}{c}\mathbf{A})\rho\mathbf{f} - \frac{e\hbar}{2mc}\mathbf{A}\nabla\times(\mathbf{s}\rho) -$$

$$-\hbar\rho(\partial_0\varphi + f_0) - \frac{e}{c}\rho A_0 - \frac{\hbar^2\rho}{2m}(\partial_\alpha\eta_\beta\partial_\alpha\eta_\beta) - \frac{\hbar^2(\nabla\rho)^2}{8m\rho}]d^4x \tag{B.5}$$

where $\mathbf{f}$ and $\mathbf{s}$ are functions of $\boldsymbol{\eta}$ defined by the relations (4.6), (4.7).

Now let us use new variables $\rho, \mathbf{j}, \varphi, \boldsymbol{\eta}$. We shall consider the relation (B.4) as an additional costraint in the variational problem (B.5). In fact Eq. (B.4) is not a constraint on variables ρ, φ, $\boldsymbol{\eta}$, and does not change extremals of the functional. Addition of the relation (B.4) as an additional constraints means introduction of new variables $\mathbf{j}$. Adding the constraints (B.4) by means of Lagrange multiplyers, one obtains

$$\mathcal{A}_P[\rho,\mathbf{j},\varphi,\boldsymbol{\eta},\mathbf{p}] = \int \{-\hbar\rho(\partial_0\varphi + f_0) - \frac{e}{c}\rho A_0 - \frac{m}{2\rho}[\mathbf{j} - \frac{\hbar}{m}[\mathbf{f}\rho + \frac{1}{2}\nabla \times (\mathbf{s}\rho)]]^2 -$$

$$-\hbar\mathbf{f}(\hbar\nabla\varphi + \frac{e}{c}\mathbf{A})\rho - \frac{\hbar^2\rho}{2m}(\partial_\alpha\eta_\beta)(\partial_\alpha\eta_\beta) - \frac{\hbar^2}{8m}\frac{(\nabla\rho)^2}{\rho} - \frac{e\hbar}{2mc}\mathbf{A}\nabla\times(\mathbf{s}\rho) +$$

$$\mathbf{p}[\mathbf{j} - \frac{\hbar}{m}(\rho\nabla\varphi + \rho\varepsilon_{\alpha\beta\gamma}\eta_\alpha\nabla\eta_\beta b_\gamma - \frac{1}{2}\nabla \times (\rho\mathbf{s})) - \frac{e\rho}{mc}\mathbf{A}]\}d^4x \tag{B.6}$$

Varying the action (B.6) with respect to $\mathbf{j}$, one obtains

$$\frac{\delta\mathcal{A}}{\delta\mathbf{j}} = -\frac{m}{\rho}\{\mathbf{j} - \frac{\hbar}{m}[\mathbf{f}\rho + \frac{1}{2}\nabla \times (\mathbf{s}\rho)]\} + \mathbf{p} = 0 \tag{B.7}$$

Eliminating $\mathbf{p}$ from Eq.(B.7) and substituting into Eq.(B.6), one obtains after simplification the action of the form (4.2)-(4.5).

REFERENCES

[1] M.Jammer, *The conceptual development of quantum mechanics.* McGrow-Hill book Company, 1967. Sec.1.2
[2] E.Madelung, *Z. Phys.* **40**, 322 (1926).
[3] T.Takabayasi, *Progr. Theor. Phys.* **9**, 187 (1953).
[4] L.Janossy, M.Ziegler, *Acta Phys. Hung.* **16**, 37, (1963).
[5] L.Janossy, M.Ziegler, *Acta Phys. Hung.* **16**, 345 (1964).
[6] L.Janossy, M.Ziegler-Naray, *Acta Phys. Hung.* **20**, 233 (1966).
[7] Yu.A.Rylov, *J. Math. Phys.* **30**, 2516 (1989).
[8] J.E.Moyal, *Proc. Cambridge Philos. Soc.* **45** 99 (1949).
[9] D.Bohm, *Phys.Rev.* **85**, 166 (1952).
[10] D.Bohm, *Phys.Rev.* **85**, 180 (1952).
[11] L.de la Pena-Auerbach and L.Garcia-Colin Leopoldo, *J.Math. Phys.* **9**, 916 (1968).
[12] L.de la Pena-Auerbach and A.M.Cetto, *Found. Phys.* **5**, 355 (1975).
[13] S.Kaliski, *Proc. Vibr. Phys.* **3**, 1 (1970).
[14] F.J.Belinfante, *A Survey of Hidden-Variables Theories*, Pergamon, Oxford, 1973.
[15] R.H.Young, *Am. J. Phys.* **44**, 581 (1976).
[16] T.W.Marshall, *Proc. Roy. Soc.* **A276**, 475 (1963).
[17] A.O.Barut and A.J.Bracken, *Phys. Rev.D.* **23**, 2454 (1982).
[18] A.O.Barut and N.Zanghi, *Phys. Rev. Lett.* **52**, 2009 (1984).
[19] J.C.Aron, *Found Phys.* **11**, 863 (1981).
[20] A.Rueda, Cavalleri G. *Nuovo Cimento C.* **6**, 239 (1983).
[21] L.de la Peña and A.Jauregui, *Found. Phys.* **12**, 441 (1982).
[22] G.Spavieri, *Found. Phys.* **20**, 45 (1990).
[23] Yu.A.Rylov, *Phys. Essays.* **4**, 300 (1991).
[24] Yu.A.Rylov, in *Uncertainty Principle and Foundation of Quantum Mechanics.* W.C.Price and S.S.Chissick, eds. (Wiley, New York 1977), p.109.
[25] H.Jeffreys, *Proc London Math. Soc.* **23**, 428 (1923).
[26] G.Wentzel, *Zs. Phys.* **38**, 518 (1926).
[27] H.A.Kramers, *Zs. Phys.* **39**, 828 (1926).
[28] L.Brillouin, *Compt. Rend.* **183**, 24 (1926).
[29] V.P.Maslov and M.V.Fedoryuk, *Semiclassical approximation for quantum mechanics equations.* Nauka, Moscow, 1976. (in Russian)
[30] V.G.Bagrov, V.V.Belov, A.Yu.Trifonov, A.A.Yevsijevich, *J. Phys.* **A27**, 5273 (1994).