Pair Production Problem and Canonical Quantization of Nonlinear Scalar Field in Terms of World Lines

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Abstract

A new quantization scheme (WL-scheme), using world lines as objects of quantization, is proposed. Applying to nonlinear scalar field, the WL-scheme is investigated and compared with the conventional PA-scheme of quantization. In the PA-scheme objects of quantization are particles and antiparticles, which are fragments of the total physical object – world line (WL). Applying to the nonlinear field, the PA-scheme of quantization leads to such difficulties as nonstationary vacuum, obligatory use of perturbation theory technique, normal ordering and cut-off at $t \to \infty$ in the scattering problem. These difficulties are corollaries of inconsistency of PA-scheme. The WL-scheme is free of these difficulties. These difficulties are connected with the reconstruction problem of the total world lines from their fragments (particles and antiparticles). In the case, when these fragments interact between themselves, such a reconstruction is very complicated problem. The new WL-scheme of quantization is free of all these problems, because it does not cut the total world line into fragments (particles and antiparticles). Formally appearance of fragments in the conventional quantization PA-scheme is a corollary of a forced identification of the energy with the Hamiltonian. In fact such an identification is not necessary. It leads only to difficulties. The new WL-scheme of quantization does not use this identification and enables to go around all these problems. The WL-scheme enables not to use additional (to nonrelativistic QM) quantization rules, used in the relativistic QFT (normal ordering, perturbation technique, renormalization).

1 Introduction

The problem of pair production is the crucial problem of relativistic quantum field theory (QFT), as well as that of the elementary particles theory. In this paper one investigates, if it is possible to describe the pair production in the scope of conventional quantum mechanics principles, i.e. without using additional quantum rules of QFT such as normal ordering, perturbation theory technique, manipulations with nonstationary vacuum, renormalization and interaction cut-off in the scattering problem. The main result of the investigation is formulated as follows. The secondary quantization of the nonlinear relativistic scalar field with the self-action term $\lambda\varphi^{*2}\varphi^2$ and its description without the perturbation theory is possible. The vacuum state appears to be stationary and the normal ordering is not used. In other words, one succeeds to overcome many problems of relativistic QFT, but the problem of pair production remains to be unsolved. This problem appears to be more subtle and complicated, than it is common practice to think[1].

The fact is that the secondary quantization of a relativistic field is accompanied usually by fragmentation of world lines into particles and antiparticles. Some fragments of the particle world line (WL) describe particles, other one describe antiparticles. Appearance of the perturbation theory in QFT is essentially a result of this fragmentation. Description of the fragmentation process is rather simple, whereas description of reciprocal process of defragmentation is more complicated and description of defragmentation is imperfect. In some cases the pair production arises as a result of the defragmentation process, but not as a result of turn of the world line in time.

Application of the world line fragmentation (perturbation theory technique) in QFT is connected closely with identification of Hamiltonian $p_0 \equiv -H$ with the energy E. The energy is defined by the relation

$$E = P^0 = \int T^{00} d\mathbf{x} \tag{1.1}$$

where T^{ik} is the energy-momentum tensor, whereas $p_0 \equiv -H$ is defined as the quantity canonically conjugate to time. In general, the energy E and Hamiltonian H are different quantities, but in the nonrelativistic case they coincide for free particle, i.e. $E = -H$, and it is common practice to think that this relation takes place in QFT, where it has the form of the relation

$$[u, P_k]_- = -i\hbar \frac{\partial u}{\partial x^k}, \tag{1.2}$$

[1]This investigation has long been carried out (before 1990), but we failed to publish it, because the editors of physical journals believed that one may not publish the paper, where the secondary qunatization of nonlinear field do not produce pair production, even if the procedure of secondary quantization is completely consistent. Nevertheless results of this investigation were a motivation for a search of an alternative approach to the pair production problem. Now, when alternative description of pair production has appeared [1], publication of this investigation may be interesting as an argument in favour of this alternative.

Here P_k is the energy-momentum operator, $[u, P_k]_-$ is commutator, and u is operator of arbitrary dynamic variable. $\hbar$ is the Planck constant which is set to be equal to 1 further. It is common practice to think [2] that the relation (1.2) is necessary for determination of relativistic commutation relations. Use of the relation (1.2) is the ground for the statement that the field operator φ cannot contain only annihilation operators (as in nonrelativistic case) and that φ contains both creation and annihilation operators. The last generates a necessity of a use of the perturbation theory methods in relativistic QFT.

Eliminating relation (1.2), one can use such a secondary quantization of relativistic field, where operator φ contains only annihilation operators, and Hermitian conjugate operator φ^* contains only creation operators. Commutation relations for operators φ and φ^* can be determined without a reference to relation (1.2). They are determined from their relativistic covariance and dynamic equation for φ [3].

It was shown [4] that the relation $E = -H$, (or $E = H$) takes place in the relativistic case only if particles and antiparticles are considered to be independent physical objects, (but not different states of a physical object WL, described by world line). Thus, the identification of E and H (relation $E = -H$) means fragmentation of WL into fragments, describing independent physical objects particles and antiparticles. The idea of using the world line as a physical object is the old idea. It goes back to Stueckelberg [5] and Feynman [6]. Unfortunately, it was developed somewhat inconsistently, and origin of this inconsistency is the relation (1.2). In general, the energy E and the Hamiltonian H are defined independently, and there is no reason for their forced identification. If they coincide for some reason (for instance, in force of dynamic equation), they will coincide independently of the relation (1.2). If they do not coincide, there is no reason for making them to be equal.

Farther we shall use and compare two schemes of canonical quantization. Conventional quantization scheme, using relation (1.2), will be referred to as PA-scheme. The quantization scheme which does not use the relation (1.2) and considers the world line as a physical object will be referred to as WL-scheme. In general, the special term "WL" will be used for the world line, considered to be a physical object.

Before consideration of different modifications of canonical quantization one should understand distinction between WL-description and PA-description on the classical level.

Let x^i be coordinates of a point of the space-time in some coordinate system, and

$$\mathcal{L}: \qquad x^i = q^i(\tau), \qquad i = 0, 1, 2, 3 \tag{1.3}$$

be a continuous world line, τ being a parameter along it. Let $\mathcal{L}$ describe a history of particles and antiparticles produced under influence of some given external field $f(q)$ which can produce pairs. WL $\mathcal{L}$ is described by the following parameters: the mass m, and the "charge" e. The mass is non-negative constant which describes interaction with the gravitational field. The "charge" e is a constant, describing interaction with the electromagnetic field. The orientation $\boldsymbol{\varepsilon}$ is one more Lorentz invariant quantity, describing WL. It is a discrete dynamic variable, describing the state of WL. Changing the orientation $\boldsymbol{\varepsilon}$ (with fixed m and e), one turns a particle to an antiparticle and vice versa. Orientation $\boldsymbol{\varepsilon}$ describes one of two possible directions of motion along the world line. If the parametrization $\mathcal{P}$ of $\mathcal{L}$ is realized by the parameter τ, then the orientation $\boldsymbol{\varepsilon}$ is determined by the component ε at this parametrization $\mathcal{P}$. The component ε takes values ± 1. At transformation of the WL $\mathcal{L}$ parametrization

$$\mathcal{P} \rightarrow \mathcal{P}', \qquad \tau \rightarrow \tau' = \tau'(\tau), \qquad \partial\tau'/\partial\tau \not\equiv 0 \tag{1.4}$$

the orientation component transforms according to the law

$$\varepsilon \rightarrow \varepsilon' = \varepsilon \ \mathrm{sgn} \frac{\partial\tau'(\tau)}{\partial\tau} \tag{1.5}$$

For WL $\mathcal{L}$ the action has the form

$$S[q] = \int\limits_{\min(\tau',\tau'')}^{\max(\tau',\tau'')} L d\tau, \tag{1.6}$$

$$L = -\sqrt{m^2c^2\dot{q}^i g_{ik}\dot{q}^k - \alpha f(q)} - \frac{\varepsilon e}{c}\dot{q}^i A_i(q), \qquad \dot{q}^i \equiv dq^i/d\tau, \tag{1.7}$$

where τ' and τ'' are values of the parameter τ at the integration interval boundaries. α is a non-negative constant. Here $f(q)$ is a given external field which can turn the world line in the time direction, i.e. it can create or annihilate particle-antiparticle pairs. The fact is that the Lagrangian (1.6) with $\alpha = 0$ admits only timelike WLs (1.3), ($\dot{q}^i\dot{q}_i > 0$ takes place everywhere). Introduction of the term $\alpha f(q)$ removes this constraint. This capacity of pair production remains in the limit $\alpha \rightarrow +0$, when the action (1.6) becomes to be invariant with respect to arbitrary transformation (1.4) of parametrization (see details in ref. [4]). The radical in Eq.(1.6) is supposed to be non-negative. The action (1.6), (1.7) is invariant with respect to arbitrary coordinate transformation. In the limit $\alpha \rightarrow +0$, it becomes to be invariant also with respect to arbitrary parametrization transformation (1.4).

The electric current $J^i(x)$ and energy-momentum tensor T^{ik} are defined as sources of electromagnetic field and gravitational one respectively. In the Galilean coordinate system, where g_{ik}= const and $A_i = 0$, they have the form

$$J^i(x) = -c\frac{\delta S}{\delta A_i(x)} = \varepsilon e \sum_l \frac{\dot{q}^i(\tau_l)}{|\dot{q}^0(\tau_l)|}\delta(\mathbf{x} - \mathbf{q}(\tau)), \qquad i = 0, 1, 2, 3 \tag{1.8}$$

$$T^{ik}(x) = -\frac{2c}{\sqrt{|g|}}\frac{\delta S}{\delta g_{ik}} = \sum_l \frac{mc^2\dot{q}^i(\tau_l)\dot{q}^k(\tau_l)}{\sqrt{\dot{q}^l(\tau_l)g_{lj}\dot{q}^j(\tau_j)}}\frac{\delta(\mathbf{x} - \mathbf{q}(\tau_l))}{|\dot{q}^0(\tau_l)|} \tag{1.9}$$

$$i, k = 0, 1, 2, 3$$

where $\tau_l = \tau_l\left(x^0\right)$ are roots of the equation

$$q^0(\tau_l) - x^0 = 0 \tag{1.10}$$

and

$$g = \det ||g_{ik}|| , \qquad i,k = 0,1,...n.$$

Intersection of the WL with the plane $x^0 =$ const consists of one or several points. The following quantities can be attributed to such a point: canonical momentum p_i ($i = 0,1,2,3$), electric charge Q, energy E, and momentum P_β ($\beta = 1,2,3$). They are defined as follows.

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = -\frac{mc\dot{q}_i}{\sqrt{\dot{q}^l g_{lk} \dot{q}^k}}, \qquad i = 0,1,2,3 \tag{1.11}$$

$$Q = \int J^0(x) d\mathbf{x} = \varepsilon e \ \mathrm{sgn}(\dot{q}^0) = -\varepsilon e \ \mathrm{sgn}(p_0) \tag{1.12}$$

$$P^0 = E = \int T^{00} d\mathbf{x} = \frac{mc\,|\dot{q}^0|}{\sqrt{\dot{q}^l g_{lk} \dot{q}^k}} = |p_0| \tag{1.13}$$

$$P^\alpha = \int T^{0\alpha} d\mathbf{x} = \frac{mc\dot{q}^\alpha}{\sqrt{\dot{q}^l g_{lk} \dot{q}^k}} \mathrm{sgn}(\dot{q}^0) = -p^\alpha \mathrm{sgn}(p_0), \qquad \alpha = 1,2,3 \tag{1.14}$$

One can attribute two invariants: $m = \sqrt{p_i p^i}$, $Q = \pm e$ to any point of intersection of $\mathcal{L}$ with $x^0 =$ const. These points will be referred to as SWLs. (Abbreviation (SWL) of the term "section of world line". SWLs can be of two kinds: (m, e) and $(m, -e)$. One of them is a particle, and other one is an antiparticle. Thus, a SWL is a collective concept with respect to concepts of a particle and an antiparticle.

The SWL state is described either by dynamic variables

$$x^\alpha, \varepsilon_p, p_\alpha, \qquad \alpha = 1,2,3, \tag{1.15}$$

or by observable physical variables

$$x^\alpha, Q, P^\alpha, \qquad \alpha = 1,2,3 \tag{1.16}$$

where ε_p takes values ± 1, which label particle and antiparticle. Dynamic variables $x^\alpha, \varepsilon_p, p_\alpha, \quad \alpha = 1,2,3$ are connected with physical ones by relations

$$\varepsilon_p = \ \mathrm{sgn}(p_0), \qquad p_0 = \varepsilon_p E, \qquad P_\alpha = -\varepsilon_p p_\alpha, \tag{1.17}$$
$$Q = -\varepsilon \varepsilon_p e, \qquad E = \left| \sqrt{\mathbf{p}^2 + m^2} \right|,$$

Let us consider such transformations of the way of the WL description which do not change conservative quantitites Q, E, P_α.

$$I_\tau : \qquad \tau \to -\tau, \qquad x^i \to x^i, \qquad \varepsilon \to -\varepsilon, \qquad \varepsilon_p \to -\varepsilon_p,$$
$$p_\alpha \to -p_\alpha, \qquad Q \to Q, \qquad P_\alpha \to P_\alpha, \tag{1.18}$$

$$I_{ey} : \qquad \tau \to \tau, \qquad x^i \to -(x^i - 2y^i), \qquad e \to -e, \qquad \varepsilon_p \to -\varepsilon_p, \qquad \varepsilon \to \varepsilon,$$
$$p_\alpha \to -p_\alpha, \qquad Q \to Q, \qquad P_\alpha \to P_\alpha, \tag{1.19}$$

where $y = y^i$ is a transformation parameter. Both transformations I_τ and I_{ey} change the sign of the canonical momentum p_i but do not change the energy-momentum vector P_i and the charge Q. Transformation I_τ changes parametrization of WL (parameter τ), but does not change parameters (m, e) of dynamic system WL. Vice versa, transformation (1.19) does not change parametrization of WL, but changes parameters $(m, e) \to (m, -e)$ of dynamic system WL, described by the action (1.6), (1.7) and its state $(x, \varepsilon_p, p_\alpha) \to (-x + 2y, -\varepsilon_p, -p_\alpha)$.

Two WLs (m, e) and $(m, -e)$ are two different dynamic systems, whereas $(x, \varepsilon_p, p_\alpha)$ and $(x, -\varepsilon_p, -p_\alpha)$ with the same (m, e) are two different states of the same dynamic system WL. In other words, transformation (1.18) does not change the dynamic system WL, described by the action (1.6), (1.7), whereas the transformation (1.19) does change dynamic system WL in itself.

When there is no pair production and WL does not make zigzag in the time direction, one can achieve coincidence of vectors p_i and $-P_i$. In this case the energy-momentum vector $-P_i$ is canonically conjugate to the vector x^i.

If WL describes the pair production (see Fig.1), the coincidence of p_i and $-P_i$ can be achieved by means of transformation (1.18) performed only on those WL interval, where p_i does not coincide with $-P_i$. For instance, one can choose $\tau = x^0$, then $p_i = -P_i$ everywhere, but the parameter τ will change non-monotonically along the WL.

The description, where $P_i = -p_i$, will be called PA-description. The approach, using such a non-monotone parametrization, will be referred to as the PA-approach (the approach from the standpoint of particles and antiparticles). The approach, where a monotone parametrization of WLs is used, will be referred to as WL-approach. Thus, the WL-approach distinguishes between the canonical momentum and energy-momentum, the PA-approach does not. A criterion of the PA-approach is the condition.

$$p_i = -P_i, \qquad i = 0, 1, 2, 3, \tag{1.20}$$

that the energy-momentum vector $-P_i$ defined by Eqs. (1.13), (1.14) were canonically conjugate to x^i.

From the standpoint of WL-approach the non-monotone parametrization of the WL is not consistent, as far as such non-monotony is absent in non-relativistic description. Vice versa, from the standpoint of PA-approach the separation of concepts of the energy-momentum P_i and the canonical momentum p_i is not satisfactory, because such a separation is absent in non-relativistic mechanics.

From the standpoint of Stueckelberg-Feynman idea that the world line is a physical object, WL-approach is more consistent, than PA-approach.

Distinction between the two approaches can be manifested only at pair production, i.e. only in the quantum field theory (QFT). The canonical quantization in QFT uses PA-approach, i.e. Eq.(1.20) is fulfilled always.

In QFT the condition (1.20) takes the form (1.2). From the non-relativistic viewpoint the PA-approach is more clear, than WL-approach. But a use of PA-approach at a quantization of interacting fields leads to a set of difficulties. The principal difficulty is absence of a stationary vacuum state.

In the present paper one investigates application of the WL-approach to the Boson field quantization. In Sec.2 the equivalency of WL-approach and PA-approach at the free field quantization is shown. In section 3 the concept of quantization model is introduced. Quantization of nonlinear field is investigated in Section 4. Section 5 is devoted to description of nonlinear field in terms of free fields. In section 6 the scattering problem without interaction cut-off at $t \to \infty$ is investigated. In section 7 some peculiarities of introducing physical quantities for nonlinear field are considered.

2 Quantization of the Free Scalar Field

Let us consider a charged scalar field φ in the $(n+1)$-dimensional space-time. $x^i = (x^0, \mathbf{x}) = (x^0, x^\alpha)$ are Galilean coordinates. Latin indices take values $0, 1, \ldots n$, Greek ones do $1, 2, \ldots n$. As usually a summation is made on like super- and subscripts. Let the operator $\varphi(x) = \varphi(\mathbf{x}, t)$ satisfy the equation

$$(\partial_i \partial^i + m^2)\varphi = 0 \tag{2.1}$$

$\varphi^*(x)$ be a corresponding Hermitian conjugate operator.

It is convenient to use variables

$$b(\varepsilon_k, \mathbf{k}, t) = b(K, t) = i(2\pi)^{-n/2} \int\limits_{-\infty}^{\infty} \exp(-i\mathbf{kx}) \left[\frac{\sqrt{\beta(\mathbf{k})}}{2} \varphi(\mathbf{x}, t) + \frac{i\varepsilon_k}{\sqrt{\beta(\mathbf{k})}} \dot{\varphi}(\mathbf{x}, t) \right] d\mathbf{x}$$

$$K = \{\varepsilon_k, \mathbf{k}\}, \qquad \dot{\varphi} \equiv \partial\varphi/\partial t, \qquad d\mathbf{x} \equiv dx^1 dx^2 \ldots dx^n, \tag{2.2}$$

where ε_k=sgn(k_0) describes orientation.

$$k_0 = \varepsilon_k E(\mathbf{k}), \qquad E(\mathbf{k}) = \left| \sqrt{m^2 + \mathbf{k}^2} \right|, \qquad \beta(\mathbf{k}) = 2E(\mathbf{k}),$$

$$\mathbf{kx} = -k_\alpha x^\alpha = k^\alpha x^\alpha, \qquad k = (k_0, \mathbf{k}) \tag{2.3}$$

The reciprocal transformation has the form

$$\varphi(\mathbf{x}, t) = (2\pi)^{-n/2} \int\limits_{-\infty}^{\infty} e^{i\mathbf{kx}} \frac{b(K, t)}{\sqrt{\beta(\mathbf{k})}} dK \tag{2.4}$$

$$\dot{\varphi}(\mathbf{x}, t) = \frac{\partial \varphi}{\partial t} = -i(2\pi)^{-n/2} \int\limits_{-\infty}^{\infty} \frac{\varepsilon_k}{2} \sqrt{\beta(\mathbf{k})} e^{i\mathbf{kx}} b(K, t) dK \tag{2.5}$$

where

$$\int (.) dK \equiv \sum_{\varepsilon_k = \pm 1} \int (.) d\mathbf{k}, \qquad d\mathbf{k} = dk_1 dk_2 \ldots dk_n, \tag{2.6}$$

The dynamic equation (2.1) in terms of $b(K,t)$ takes the form

$$\dot{b}(K,t) = -i\varepsilon_k E(\mathbf{k}) b(K,t) \tag{2.7}$$

or

$$b(K,t) = e^{-i\varepsilon_k E(\mathbf{k})t} b(K), \qquad b(K) = b(K,0) \tag{2.8}$$

Connection between $\varphi^*(\mathbf{x},t)$, $\dot{\varphi}^*(\mathbf{x},t)$ and $b^*(K,t)$ is obtained as a result of Hermitian conjugation of Eqs.(2.2), (2.4), (2.5), (2.7), (2.8).

Definition 2.1. Quantization scheme is a totality of three relations: (1) dynamic equation, (2) definition of a vacuum vector, (3) commutation relation between the dynamic variable operators.

The Fock's representation is supposed to be used. In this representation there is only one vacuum vector $|0\rangle$. Any state Φ can be obtained as a result of acting a dynamic variable operator upon vacuum vector $|0\rangle$.

The scheme of quantization in terms of particles and antiparticles (PA-scheme) is defined as follows

$$1: \qquad \dot{c}(\mathbf{k},t) = -iE(\mathbf{k})c(\mathbf{k},t), \qquad \dot{d}(\mathbf{k},t) = -iE(\mathbf{k})d(\mathbf{k},t) \tag{2.9}$$

$$2: \quad \begin{array}{lll} c(\mathbf{k})|0\rangle_{\mathrm{PA}} = 0, & d(\mathbf{k})|0\rangle_{\mathrm{PA}} = 0, & c(\mathbf{k}) = c(\mathbf{k},0) \\ {}_{\mathrm{PA}}\langle 0|c^*(\mathbf{k}) = 0, & {}_{\mathrm{PA}}\langle 0|d^*(\mathbf{k}) = 0, & d(\mathbf{k}) = d(\mathbf{k},0) \end{array} \tag{2.10}$$

$$\begin{array}{ll} 3a: & [c(\mathbf{k}), c^*(\mathbf{k}')]_- = [d(\mathbf{k}), d^*(\mathbf{k}')]_- = \delta(\mathbf{k}-\mathbf{k}') \\ 3b: & [c(\mathbf{k}), c(\mathbf{k}')]_- = [d(\mathbf{k}), d(\mathbf{k}')]_- = [c(\mathbf{k}), d(\mathbf{k}')]_- = 0 \end{array} \tag{2.11}$$

where

$$c(\mathbf{k},t) = b(1,\mathbf{k},t), \qquad d(\mathbf{k},t) = b^*(-1,\mathbf{k},t) \tag{2.12}$$

and $|0\rangle_{\mathrm{PA}}$ is the vacuum state vector.

The quantization scheme in terms of WLs (WL-scheme) has the form

$$1 \ : \qquad \dot{b}(K,t) = -i\varepsilon_k E(\mathbf{k}) b(K,t) \tag{2.13}$$

$$2 \ : \qquad b(K)|0\rangle_{\mathrm{WL}} = 0, \qquad {}_{\mathrm{WL}}\langle 0|b^*(K) = 0 \tag{2.14}$$

$$\begin{array}{ll} 3a: & [b(K), b^*(K')]_- = \delta(K-K') \equiv \delta_{\varepsilon_k,\varepsilon_{k'}}\delta(\mathbf{k}-\mathbf{k}') \\ 3b: & [b(K), b(K')]_- = 0 \end{array} \tag{2.15}$$

Substitution of Eq. (2.12) into Eqs. (2.9) – (2.11) do not lead to relations (2.13) – (2.15), because the vacuum state vectors are defined differently. Indeed, substitution of Eq. (2.12) in Eq. (2.9) gives (2.10

$$b(1,\mathbf{k})|0\rangle_{\mathrm{PA}} = 0, \qquad b^*(-1,\mathbf{k})|0\rangle_{\mathrm{PA}} = 0, \tag{2.16}$$

That differs from the vacuum state, defined by the relation (2.14). Nevertheless both quantization schemes are equivalent, because the PA-scheme can be transformed into WL-scheme by means of another change of variables

$$c(\mathbf{k},t) = b_{\mathrm{E}}(1,\mathbf{k},t), \qquad d(\mathbf{k},t) = b_{\mathrm{E}}(-1,\mathbf{k},-t). \tag{2.17}$$

Substitution of relations (2.17) into Eqs. (2.9) – (2.11) leads to relations

$$\begin{array}{rcl} 1 & : & \dot{b}_{\rm E}(K,t) = -i\varepsilon_k E(\mathbf{k}) b_{\rm E}(K,t) \\ 2 & : & b_{\rm E}(K)|0\rangle_{\rm PA} = 0, \qquad {}_{\rm PA}\langle 0| b_{\rm E}^*(K) = 0 \\ 3a & : & [b_{\rm E}(K), b_{\rm E}^*(K')]_- = \delta(K-K') \equiv \delta_{\varepsilon_k,\varepsilon_{k'}}\delta(\mathbf{k}-\mathbf{k}') \\ 3b & : & [b_{\rm E}(K), b_{\rm E}(K')]_- = 0 \end{array} \tag{2.18}$$

Comparing Eq. (2.18) with Eqs. (2.13) – (2.15), one can see that the PA-scheme is distinguished from the WL-scheme only with designations. They are equivalent. The change of variables (2.17) which transforms PA-scheme into WL-scheme is not unique. There are as many such transformations as there are Galilean coordinate systems.

Let us construct the field $\varphi_{\rm E}(\mathbf{x},t)$, expressing it through variables $b_{\rm E}(K,t)$ by means of Eqs.(2.4), (2.5).

$$\varphi_{\rm E}(\mathbf{x},t) = \frac{1}{(2\pi)^{n/2}} \int e^{i\mathbf{kx}} \frac{b_{\rm E}(K,t)}{\sqrt{\beta(\mathbf{k})}} dK \tag{2.19}$$

Using Eqs.(2.12), (2.17), (2.2), (2.19), one obtains

$$\varphi_{\rm E}(x) = \hat{\Pi}_+\varphi(x) + \hat{\Pi}_-\hat{I}_0\varphi^*(x), \qquad \hat{I}_0 \equiv \hat{I}_y\Big|_{y=0} \tag{2.20}$$

where $\hat{\Pi}_+, \hat{\Pi}_+, \hat{I}_y$ are operators acting on functions of x.

$$\hat{\Pi}_+ = \frac{1}{2\hat{E}}(\hat{E} + i\hat{\partial}_0), \qquad \hat{\Pi}_- = \frac{1}{2\hat{E}}(\hat{E} - i\hat{\partial}_0), \qquad \hat{E} \equiv |(m^2 + \partial_\alpha\partial^\alpha)^{1/2}| \tag{2.21}$$

$$\hat{E}\varphi(t,\mathbf{x}) = \frac{1}{(2\pi)^{2n}} \int\int e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} E(\mathbf{k})\varphi(t,\mathbf{x}') d\mathbf{k} d\mathbf{x}'$$

and $\hat{I}_y$ is an operator of the coordinates x^i reflection with respect to the point y^i.

$$\hat{I}_y\varphi(x) = \varphi(2y - x) \tag{2.22}$$

If the function $\varphi(x)$ satisfies Eq. (2.1) then operators $\hat{\Pi}_+$ and $\hat{\Pi}_-$ are projection operators, having the properties

$$\hat{\Pi}_+\hat{\Pi}_+\varphi = \hat{\Pi}_+\varphi, \qquad \hat{\Pi}_-\hat{\Pi}_-\varphi = \hat{\Pi}_-\varphi, \qquad \hat{\Pi}_+\hat{\Pi}_-\varphi = \hat{\Pi}_-\hat{\Pi}_+\varphi = 0,$$

$$(\hat{\Pi}_+ + \hat{\Pi}_-)\varphi = \varphi, \qquad \hat{\Pi}_+\hat{I}_y = \hat{I}_y\hat{\Pi}_-, \qquad (\hat{\Pi}_+\varphi)^* = \hat{\Pi}_-\varphi^* \tag{2.23}$$

It follows from Eqs. (2.20) – (2.23)

$$\varphi(x) = \hat{\Pi}_+\varphi_{\rm E}(x) + \hat{\Pi}_-\hat{I}_0\varphi_{\rm E}^*(x), \tag{2.24}$$

Thus, if $\varphi(x)$ is quantized according to PA-scheme, then $\varphi_{\rm E}(x)$ is quantized according to WL-scheme.

$$[\varphi(x), \varphi^*(x')]_- = -iD(x - x') \tag{2.25}$$

where $D(x-x')$ is the Pauli-Jordan commutation function in the $(n+1)$-dimensional space-time.

$$D(x) = \frac{1}{(2\pi)^n i} \int \frac{d\mathbf{k}}{\beta(\mathbf{k})} \left(e^{iE(\mathbf{k})x^0 - i\mathbf{kx}} - e^{-iE(\mathbf{k})x^0 - i\mathbf{kx}} \right) = D^+(x) + D^-(x)$$

$$D^+(x) = \frac{1}{(2\pi)^n i} \int \frac{d\mathbf{k}}{\beta(\mathbf{k})} e^{iE(\mathbf{k})x^0 - i\mathbf{kx}}$$

$$D^-(x) = \frac{i}{(2\pi)^n} \int \frac{d\mathbf{k}}{\beta(\mathbf{k})} e^{-iE(\mathbf{k})x^0 - i\mathbf{kx}} = -D^+(-x)$$

In this case $\varphi_{\mathrm{E}}(x)$ is quantized according to WL-scheme. The commutation relation has the form

$$[\varphi_{\mathrm{E}}(x), \varphi_{\mathrm{E}}^*(x')]_- = D_1(x-x') = i[D^+(x-x') - D^-(x-x')] \tag{2.26}$$

For odd n and $(x-x')^2 < 0$ the condition $D(x-x') = 0$ is fulfilled. But the function $D_1(x-x')$ has not this property. It is a common practice to believe [7] that a violation of this property leads to a causality violation. Producing a transformation from PA-scheme to WL-scheme, the change of designation (2.17) cannot violate causality, as well other physical properties. Nevertheless it leads to relation (2.26) which does not vanish at $(x-x')^2 < 0$. It means only that the this property is an attribute of the PA-scheme of quantization (see also discussion of this question in ref.[3]).

Let $\varphi_1(x)$ and $\varphi_2(x)$ be two scalars. At the translation

$$x^i \to \tilde{x}^i = x^i + a^i, \qquad \tilde{x} = x + a, \qquad a^i = \mathrm{const} \tag{2.27}$$

each of them transforms according to the law

$$\varphi(x) \to \tilde{\varphi}(\tilde{x}) = \varphi(x) = \varphi(\tilde{x} + a) \tag{2.28}$$

At the proper Lorentz transformation

$$x^i \to \tilde{x}^i = \Lambda^{i.}_{.k} x^k, \qquad \tilde{x} = \Lambda x, \qquad \Lambda^{i.}_{.k} g^{kl} \Lambda^{j.}_{.l} = g^{ij}, \qquad \Lambda^{0.}_{.0} > 0 \tag{2.29}$$

each of them transforms

$$\varphi(x) \to \tilde{\varphi}(\tilde{x}) = \varphi(x) = \varphi(\Lambda^{-1}\tilde{x}), \qquad (\Lambda\Lambda^{-1})^{i.}_{.k} = \delta^i_k. \tag{2.30}$$

Then the quantity

$$f_y(x) = \varphi_1(x) + \hat{I}_y \varphi_2(x) = \varphi_1(x) + \varphi_2(2y - x), \tag{2.31}$$

considered to be a function of argument x and y is a scalar

$$f_y(x) \to \tilde{f}_{\tilde{y}}(\tilde{x}) = f_y(x). \tag{2.32}$$

At the same time the quantity $f_y(x)$ considered to be a function of only x is not a scalar. The quantity (2.31) considered to be a function of only x will be referred to as a scalar centaur with the contact point y.

Let the scalar φ satisfy Eq. (2.1). Then $\hat{\Pi}_+\varphi(x)$ and $\hat{\Pi}_+\varphi^*(x)$ are scalars satisfying Eq.(2.1). The quantity $\varphi_{\rm E}(x)$ defined by Eq.(2.20) is a scalar centaur with the contact point $y = 0$. This scalar centaur $\varphi_{\rm E}(x)$ satisfies Eq.(2.1) also.

If a scalar field $\varphi(x)$ is quantized according to PA-scheme, then $\varphi_{\rm E}(x)$ is a scalar centaur which is quantized according to WL-scheme. If $\varphi_{\rm E}(x)$ is a scalar quantized according to WL-scheme, then $\varphi(x)$ defined by Eq.(2.24) is a scalar centaur quantized according to PA-scheme.

Thus, PA-scheme and WL-scheme of quantization are equivalent in the sense that each of them can be transformed into another by means of a change of variables, but at such a transformation a scalar transforms into a scalar centaur and vice versa.

Note that the transformation (2.22) associates with the transformation (1.19), but not with the transformation (1.18), because description of the field φ does not contain a quantity analogical to parameter τ in the action (1.6). Taking into account the constraint (1.2), one is forced to use transformation analogical to (1.19) and use different dynamic systems for description of the field φ. In the absence of electromagnetic field ($A_i = 0$) the action (1.6), (1.7) is invariant with respect to transformation (1.19), and one can use this transformation as well as the transformation (2.12) in the case of free field φ. In the case of interaction, when $A_i \neq 0$, the action (1.6), (1.7) is not invariant with respect to transformation (1.19), and one cannot use this transformation. The same relates to the non-linear field $\varphi(x)$, as we shall see in the fourth section.

3 Quantization Model

Definition 3.1. The quantization model is a totality of the quantization scheme and observable quantities represented as functions of dynamic variables in this quantization scheme.

Different quantization models can exist at the same quantization scheme. Let us introduce observable quantities by means of corresponding classic quantities. The Lagrangian density has the form

$$L = \varphi_i^*\varphi^i - m^2\varphi^*\varphi, \qquad \varphi_i \equiv \partial_i\varphi, \qquad \varphi^i \equiv \partial^i\varphi = g^{ik}\varphi_k \tag{3.1}$$

Let us define the energy E and momentum P_α by relations

$$P_0 = E = \int T_{0.}^{.0}d\mathbf{x}, \qquad P_\alpha = \int T_{\alpha.}^{.0}d\mathbf{x},$$

$$T_{i.}^{.k} = \varphi_i^*\varphi^k + \varphi^{*k}\varphi_i - \delta_i^k L \tag{3.2}$$

The number N of SWLs and the electric charge Q are defined by relations

$$N = -i\int(\dot{\varphi}^*\varphi - \varphi^*\dot{\varphi})d\mathbf{x}, \qquad Q = eN \tag{3.3}$$

In the expression for the energy-momentum P_i and for the charge Q the normal ordering should be used, i.e. the creation operators should be placed to the left of the annihilation ones. In the WL-scheme the operator φ contains only annihilation operators, and φ^* contains only creation operators. Then the expressions (3.1)-(3.3) appear to be normally ordered automatically.

Let $\varphi(x)$ satisfy Eq.(2.1) and be quantized according to PA-scheme. Let $\varphi_E(x)$ be connected with $\varphi(x)$ by transformation (2.20). Then a calculation leads to the following expressions for P_i and Q

$$P_0 = E = \int E(\mathbf{k})[c^*(\mathbf{k})c(\mathbf{k}) + d^*(\mathbf{k})d(\mathbf{k})]dk = \int E(\mathbf{k})b_{\mathrm{E}}^*(K)b_{\mathrm{E}}(K)dK \qquad (3.4)$$

$$P_\alpha = \int k_\alpha[c^*(\mathbf{k})c(\mathbf{k}) - d^*(\mathbf{k})d(\mathbf{k})]d\mathbf{k} = \int \varepsilon_k k_\alpha b_{\mathrm{E}}^*(K)b_{\mathrm{E}}(K)dK \qquad (3.5)$$

$$Q = \int e[c^*(\mathbf{k})c(\mathbf{k}) - d^*(\mathbf{k})d(\mathbf{k})]d\mathbf{k} = \int e\varepsilon_k b_{\mathrm{E}}^*(K)b_{\mathrm{E}}(K)dK \qquad (3.6)$$

Let $\varphi(x)$ satisfy Eq.(2.1) and be quantized according to WL-scheme. Then

$$P_i = \int \varepsilon_k k_i b^*(K)b(K)dK, \qquad k_0 = \varepsilon_k E(\mathbf{k}), \qquad i = 0, 1, \dots n \qquad (3.7)$$

$$Q = \int e\varepsilon_k b^*(K)b(K)dK \qquad (3.8)$$

Comparison of Eqs. (3.4)-(3.6) with Eqs. (3.7, (3.8) shows that at the quantization of φ according to PA-scheme the expressions of integral quantities P_i, Q through operators b_E^*, b_E have the same form as the expressions of the same quantities P_i, Q through the operators b^*, b at quantization of φ according to WL-scheme. Let us formulate this circumstance as follows. The PA-model of the field φ quantization is equivalent to the WL-model of the field φ quantiazation with respect to integral quantities P_i, Q.

But PA-model and WL-model are not equivalent with respect to local quantities of the type of the current density j^i or the energy momentum tensor T^{ik}. For instance, at the quantization of φ according to PA-scheme one obtains

$$PA: \qquad j^0 = -ie : \dot{\varphi}^*(x)\varphi(x) - \varphi^*(x)\dot{\varphi}(x) :$$

$$= \frac{e}{(2\pi)} \int\limits_{\infty}^{\infty} \frac{d\mathbf{k}d\mathbf{k}'}{\sqrt{\beta(\mathbf{k})\beta(\mathbf{k}')}} \exp[-i(\mathbf{k} - \mathbf{k}')\mathbf{x}]$$

$$\times\{[E(\mathbf{k}) + E(\mathbf{k}')][b_{\mathrm{E}}^*(1, \mathbf{k}, x^0)b_{\mathrm{E}}(1, \mathbf{k}', x^0) - b_{\mathrm{E}}^*(-1, \mathbf{k}', x^0)b_{\mathrm{E}}(-1, \mathbf{k}, x^0)]$$

$$+[E(\mathbf{k}) - E(\mathbf{k}')][b_{\mathrm{E}}^*(1, \mathbf{k}, x^0)b_{\mathrm{E}}^*(-1, \mathbf{k}', x^0) - b_{\mathrm{E}}(-1, \mathbf{k}, x^0)b_{\mathrm{E}}(1, \mathbf{k}', x^0)]\} \qquad (3.9)$$

Here colon ":" denotes the normal ordering.

At the quantization of the field φ according to WL-scheme one obtains

$$WL: \qquad j^0 = -ie[\dot{\varphi}^*(x)\varphi(x) - \varphi^*(x)\dot{\varphi}(x)]$$

$$= \frac{e}{(2\pi)^n} \int\limits_{-\infty}^{\infty} \exp\{i[\varepsilon_k E(\mathbf{k}) - \varepsilon_{k'} E(\mathbf{k}')]x^0 - i(\mathbf{k}-\mathbf{k}')\mathbf{x}\}$$

$$\times \frac{\varepsilon_k E(\mathbf{k}) + \varepsilon_{k'} E(\mathbf{k}')}{\sqrt{\beta(\mathbf{k})\beta(\mathbf{k}')}} b^*(K) b(K') dK dK' \qquad (3.10)$$

Integration of Eqs. (3.9) and (3.10) over $\mathbf{x}$ leads to the same result (3.6), (3.8).

Let us define canonical momentum $p_i = (-H, p_\alpha) = (-H, -\pi_\alpha)$ by means of relation

$$\partial_k \varphi(x) = i\,[\varphi(x), p_k]_- \qquad (3.11)$$

For PA-scheme one obtains from Eqs. (3.11), (2.9), (2.11) and (2.17)

$$PA: \qquad H = -p_0 = \int E(\mathbf{k})[c^*(\mathbf{k})c(\mathbf{k}) + d^*(\mathbf{k})d(\mathbf{k})]d\mathbf{k}$$

$$\pi_\alpha = -p_\alpha = \int k_\alpha [c^*(\mathbf{k})c(\mathbf{k}) - d^*(\mathbf{k})d(\mathbf{k})]d\mathbf{k}, \qquad \alpha = 1, 2, ...n \qquad (3.12)$$

At quantization according to WL-scheme it follows from Eqs. (3.11), (2.13), (2.15)

$$WL: \qquad \pi_i = -p_i = \int k_i b^*(K) b(K) dK, \qquad i = 0, 1, ...n \qquad (3.13)$$

Comparing Eq. (3.12) with Eqs. (3.4), (3.5), one can see that Eq.(1.20) is fulfilled. It is this condition that is a criterion of PA-approach.

One can see from (3.13) and (3.8) that the condition (1.20) is fulfilled for contribution with $\varepsilon_k = 1$, and it is not fulfilled for contribution with $\varepsilon_k = -1$.

In the classical case a transformation from WL-approach to PA-approach can be realized by means of transformation (1.18), where the canonical momentum reflection is produced without the coordinate reflection. In the quantum theory for a transition from WL-approach to PA-approach one has to use the transformation (1.19), where the canonical momentum reflection is accompanied with the coordinate reflection. It leads to transformation of the scalar into the centaur.

Neither consideration of the quantization scheme, nor that of the quantization model of the linear scalar field answers the question which of the two approaches is valid. But such a question should be put, because both approaches cannot be valid simultaneously. If $\varphi(x)$ is a scalar and the quantization according to WL-scheme is valid, then the quantization according to PA-scheme cannot be valid, because in this case $\varphi(x)$ is a centaur. Vice versa, if the quantization of the scalar field $\varphi(x)$ according to PA-scheme is valid, then its quantization according to WL-scheme cannot be valid, because in this case $\varphi(x)$ is a centaur. Essentially, the problem is reduced to the question how to distinguish between a scalar and a centaur, as far as the centaur is a "spoiled" scalar.

4 Quantization of Nonlinear Scalar Field

Let us consider a quantization of the charged scalar field $\varphi(x)$ described by the Lagrangian density

$$L =: \varphi_i^* \varphi^i - m^2 \varphi^* \varphi + \frac{\lambda}{2} \varphi^* \varphi^* \varphi \varphi : \tag{4.1}$$

$$\varphi = \varphi(x), \qquad \varphi_i \equiv \partial_i \varphi, \qquad \varphi^i \equiv \partial^i \varphi, \qquad x = (t, x).$$

Here λ is a self-action constant.

Let us introduce variables $b(K,t), b^*(K,t)$ by means of relations (2.2) - (2.6). Then dynamic equation for the field φ.

$$(\partial_i \partial^i + m^2)\varphi =: \lambda \varphi^* \varphi \varphi : \tag{4.2}$$

transforms into the equation

$$\dot{b}(K,t) = -i\varepsilon_k E(\mathbf{k}) b(K,t) + : i\lambda \frac{\varepsilon_k g(\mathbf{k},t)}{\sqrt{\beta(\mathbf{k})}} : \tag{4.3}$$

where

$$g(\mathbf{k},t) = \frac{1}{(2\pi)^n} \int\int\int \frac{\delta(\mathbf{k}+\mathbf{p}-\mathbf{p}'-\mathbf{k}')}{\sqrt{\beta(\mathbf{p})\beta(\mathbf{k}')\beta(\mathbf{p}')}} b^*(P,t) b(P',t) b(K',t) dP dP' dK'. \tag{4.4}$$

The colon denotes the normal ordering. An expression between two colons is considered as normally ordered. The field is scalar with respect to a transformation of the Poincaré group, i.e. it transforms according to Eqs. (2.27) - (2.30)

Let the field φ be quantized according to WL-scheme. In this case the colon can be omitted in Eq.(4.3). The unique vacuum state vector $\Phi_0 = |0\rangle, \Phi_0^* = \langle 0|$ is supposed to exist. It is defined by relations

$$b(K)|0\rangle = 0, \qquad \langle 0|b^*(K) = 0, \qquad b(K) = b(K,0). \tag{4.5}$$

It follows from Eqs.(4.3), (4.5)

$$\dot{b}(K)|0\rangle = 0, \qquad \langle 0|\dot{b}^*(K) = 0 \tag{4.6}$$

and, hence

$$b(K)|0\rangle = 0, \qquad \varphi(x)|0\rangle = 0, \qquad \forall x \in \mathbb{R}^{n+1}, \tag{4.7}$$

$$\partial_k \varphi(x)|0\rangle = 0, \qquad \forall x \in \mathbb{R}^{n+1}, \tag{4.8}$$

From Eqs.(4.7), (4.8) and definition (3.11) of the canonical momentum operator it follows

$$\partial_k \varphi(x)|0\rangle = i\varphi(x) p_k |0\rangle = 0, \qquad \forall x \in \mathbb{R}^{n+1}, \tag{4.9}$$

As far as only one vacuum state defined by Eq.(4.5), or by (4.8) exists, it follows from Eq.(4.9) that $p_k|0\rangle$ distinguishes from $|0\rangle$ by a factor only. It means the vacuum $|0\rangle$ is an eigenvector of the operator p_k

$$p_k|0\rangle = p'_k|0\rangle \tag{4.10}$$

where p'_k, $(k = 0, 1, ...n)$ are c-numbers.

The vacuum state $|0\rangle$ is supposed to be invariant with respect to Poincarè group transformations (2.27)-(2.30). This supposition is compatible with Eq.(4.7), (4.8) and dynamic equations. In particular, the vacuum state $|0\rangle$ is stationary, as it follows from Eq.(4.10).

Now let the field $\varphi(x)$ be quantized according to PA-scheme. Then according to Eqs. (2.12), (2.17)

$$b(\varepsilon_k, \mathbf{k}, t) = b_{\mathrm{E}}^{-\varepsilon_k}(\varepsilon_k, \mathbf{k}, \varepsilon_k t) \tag{4.11}$$

where the following designation is used

$$b_{\mathrm{E}}^{\varepsilon}(K, t) = \begin{cases} b_{\mathrm{E}}^{*}(K, t), & \text{if} \quad \varepsilon = 1 \\ b_{\mathrm{E}}(K, t), & \text{if} \quad \varepsilon = -1 \end{cases} \tag{4.12}$$

In terms of variables b_{E} Eq.(4.4) takes the form

$$\dot{b}_{\mathrm{E}}(K, t) = -i\varepsilon_k E(\mathbf{k}) b_{\mathrm{E}}(K, t) + \frac{i\lambda\varepsilon_k}{(2\pi)^n} \int\int\int \frac{\delta(\mathbf{k} + \mathbf{p} - \mathbf{q} - \mathbf{r})}{\sqrt{\beta(\mathbf{k})\beta(\mathbf{p})\beta(\mathbf{q})\beta(\mathbf{r})}}$$

$$\times : b_{\mathrm{E}}^{-\varepsilon_k\varepsilon_p}(\varepsilon_p, \mathbf{p}, \varepsilon_k\varepsilon_p t) b_{\mathrm{E}}^{\varepsilon_k\varepsilon_q}(\varepsilon_q, \mathbf{q}, \varepsilon_k\varepsilon_q t) b_{\mathrm{E}}^{\varepsilon_k\varepsilon_r}(\varepsilon_r, \mathbf{r}, \varepsilon_k\varepsilon_r t) : dPdQdR \tag{4.13}$$

Definition of the vacuum vector $|0\rangle$ has the form of the second relation of Eq.(2.18). At $t = 0$ it follows from Eq. (4.13) and

$$\dot{b}_{\mathrm{E}}(K)|0\rangle_{\mathrm{PA}} = \frac{i\lambda\varepsilon_k}{(2\pi)^n} \int\int\int \frac{b_{\mathrm{E}}^{*}(-\varepsilon_k, \mathbf{p}) b_{\mathrm{E}}^{*}(\varepsilon_k, \mathbf{q}) b_{\mathrm{E}}^{*}(\varepsilon_k, \mathbf{k} + \mathbf{p} - \mathbf{q})}{\sqrt{\beta(\mathbf{k})\beta(\mathbf{p})\beta(\mathbf{q})\beta(\mathbf{k} + \mathbf{p} - \mathbf{q})}} |0\rangle_{\mathrm{PA}} d\mathbf{p} d\mathbf{q} \tag{4.14}$$

Thus, generally,

$$\dot{b}_{\mathrm{E}}(K)|0\rangle_{\mathrm{PA}} \neq 0, \tag{4.15}$$

It means the vacuum vector $|0\rangle$ cannot be an eigenvector of the Hamiltonian $H = -p_0$, i.e. $|0\rangle_{\mathrm{PA}}$ is not a stationary vector. The fact that the vacuum $|0\rangle_{\mathrm{PA}}$ is nonstationary excites a disappointment and dissatisfaction, because it means a translation non-invariance of vacuum $|0\rangle_{PA}$ (see discussion of this question in sec. 6 of ref.[7]) and that of the PA-scheme of quantization. In application to Eq.(2.7) the PA- and WL-scheme are equivalent, but in application to Eqs. (4.3), (4.4) they are not equivalent, because the form of Eq.(4.3) is not invariant with respect to transformation (4.11). If one needs to select between the two schemes, then it is reasonable to select the scheme, where the vacuum state is well defined.

Thus, the WL-scheme is selected. Further we shall deal with the scalar field quantized according to WL-scheme.

The commutation relation between dynamic variables $b(K, t)$ and $b^*(K', t)$ are necessary for a calculation of matrix elements of an operator $\mathcal{R}$ between two states Φ and Φ'. Any state Φ can be represented as follows

$$\Phi_l = \sum_{l=0} \Phi_l, \qquad \Phi_l^* = \sum_{l=0} \Phi_l^*, \tag{4.16}$$

where Φ_l is an l-WL state defined by the relations

$$\Phi_l = \frac{1}{\sqrt{l!}}\int f_l(\mathcal{K}^l)B_l^*(\mathcal{K}^l)|0\rangle d\mathcal{K}^l \equiv \int f_l(\mathcal{K}^l)|\mathcal{K}^l\rangle d\mathcal{K}^l$$

$$\Phi_l^* = \frac{1}{\sqrt{l!}}\int \langle 0|f_l^*(\mathcal{K}^l)B_l(\mathcal{K}^l)d\mathcal{K}^l \equiv \int \langle\mathcal{K}^l|f_l^*(\mathcal{K}^l)d\mathcal{K}^l \tag{4.17}$$

where

$$\mathcal{K}^l \equiv \{K_1, K_2, \dots K_l\}, \qquad d\mathcal{K}^l \equiv dK_1 dK_2 \dots dK_l,$$
$$B_l^*(\mathcal{K}^l) \equiv b^*(K_1)b^*(K_2)\dots b^*(K_l), \qquad B_l(\mathcal{K}^l) \equiv b(K_l)b(K_{l-1})\dots b_1(K), \tag{4.18}$$
$$|\mathcal{K}^l\rangle \equiv B_l^*(\mathcal{K}^l)|0\rangle, \qquad \langle\mathcal{K}^l| \equiv \langle 0|B_l(\mathcal{K}^l)$$

and $f_l(\mathcal{K}^l)$ is a complex function of arguments $K_1, K_2, \dots K_l$. The wave function $f_l(\mathcal{K}^l)$ is symmetric, if it does not change at transposition of any two arguments. The wave function is antisymmetric, if it changes sign at transposition of any two arguments.

For calculation of $(\Phi, \mathcal{R}\Phi)$ it is sufficient to know commutation relations

$$[b(K,t), b^*(K',t')]_- = D(t,t';K,K') \tag{4.19}$$

$$b(K,t)b^*(K',t') = F(t,t';K,K') \tag{4.20}$$

where $D(t,t';K,K')$ and $F(t,t';K,K')$ are operators depending on parameters K, K', t, t'. They are supposed to be functions of dynamic variables $b(K), b^*(K)$ disposed in the normal order, when in each term any creation operator $b^*(K)$ is placed to the left of all annihilation operators $b(K)$. $D(t,t';K,K')$ and $F(t,t';K,K')$ are connected by the evident relation

$$F(t,t';K,K') = D(t,t';K,K') + b^*(K',t')b(K,t) \tag{4.21}$$

Determination of the operators D and F is equivalent to a solution of the equation (4.3). For calculating matrix elements the commutation relations $[b(K,t), b(K,t')]_-$, $[b^*(K,t), b^*(K',t')]_-$ are not necessary. These commutation relations are important at the WL identity consideration.

For determination of operators D and F the following propositions are supposed to be fulfilled.

I. Relations (4.19), (4.20) have all symmetries that the field φ has.

II. Relations (4.19), (4.20) are compatible with the dynamic equation (4.3).

III. Relations (4.19), (4.20) are invariant with respect to Poincare group.

IV. For any l-WL state (4.17) $l = 1, 2, \dots$ the scalar product $(\Phi, \Phi) > 0$, if the symmetrical part of the wave function $f_l(\mathcal{K}^l)$ does not vanish, and $(\Phi, \Phi) = 0$, if the wave function $f_l(\mathcal{K}^l)$ is antisymmetric.

V. The operator (3.8) has only e-fold eigenvalues. At the l-WL state the eigenvalues of Q/e are equal to $-l, -l+2, \dots l$.

VI. At $t = t'$ in the limit $\lambda \to 0$ the commutation relations (4.19) turn to Eq.(2.15). It follows from Eq. (4.20)

$$F(t,t';K,K') = F^*(t',t;K',K) \tag{4.22}$$

As far as lhs of Eq.(4.20) is invariant with respect to transformation

$$b(K,t) \to \tilde{b}(K,t) = b(K,t)e^{i\alpha}, \qquad \alpha = \text{const}$$
$$b^*(K,t) \to \tilde{b}^*(K,t) = b^*(K,t)e^{-i\alpha}, \tag{4.23}$$

the operator F has the form

$$F(t,t';K,K') = F_0(t,t';K,K') +$$
$$\sum_{l=1}^{\infty} \int\int F_l(t,t';K,K';\mathcal{P}^l,\mathcal{P}'^l)B_l^*(\mathcal{P}^l)B_l(\mathcal{P}'^l)d\mathcal{P}d\mathcal{P}'^l \tag{4.24}$$

where $F_l(t,t';K,K';\mathcal{P}^l,\mathcal{P}'^l)$ $\quad l = 0,1,2,\dots$ are some c-numerical functions of their arguments. The form of functions F_l is determined by the conditions II-VI. The operator D expansion has a like form.

Differentiating Eq.(4.20) with respect to t and using Eq. (4.3), one obtains an integro-differential equation for $F(t,t';K,K')$

$$\frac{\partial F}{\partial t}(t,t';K,K') = -i\varepsilon_k E(k)F(t,t';K,K')$$
$$+\frac{i\lambda\varepsilon_k}{(2\pi)^n}\int\int\int \frac{\delta(\mathbf{k}+\mathbf{p}-\mathbf{k}''-\mathbf{p}'')}{\sqrt{\beta(\mathbf{k})\beta(\mathbf{p})\beta(\mathbf{k}'')\beta(\mathbf{p}'')}} b^*(P,t)b(P'',t)F(t,t';K'',K')dK''dP''dP \tag{4.25}$$

The expression for $\partial F/\partial t'$ is obtained from Eq.(4.25) by means of Hermitian conjugation and the substitution $t \leftrightarrow t'$, $K \leftrightarrow K'$.

Let us take the matrix element $\langle 0|\dots|0\rangle$ from Eq.(4.25) and a like equation for $\partial F/\partial t'$. One obtains

$$\frac{\partial F_0}{\partial t}(t,t';K,K') = -i\varepsilon_k E(k)F_0(t,t';K,K')$$
$$\frac{\partial F_0}{\partial t'}(t,t';K,K') = i\varepsilon_{k'} E(k)F_0(t,t';K,K') \tag{4.26}$$

Taking into account conditions III$-V$, the solution of these equations are unique and has the form [3]

$$D_0(t,t';K,K') = F_0(t,t';K,K') = \delta(K-K')e^{-i\varepsilon_k E(k)(t-t')} \tag{4.27}$$

It agrees with the commutation relation (2.15) for the free field, quantized according to WL-scheme.

Let us take matrix element $\langle P|\dots|P'\rangle$ from Eq. (4.25). Then using Eq.(4.27), one obtains

$$\frac{\partial F_1}{\partial t}(t,t';K,K';P,P') = -i\varepsilon_k E(k)F_1(t,t';K,K';P,P')$$

$$+\frac{i\lambda\varepsilon_k}{(2\pi)^n}\int\int\int \frac{\delta(\mathbf{k}+\mathbf{p}-\mathbf{k}''-\mathbf{p}'')}{\sqrt{\beta(\mathbf{k})\beta(\mathbf{p})\beta(\mathbf{k}'')\beta(\mathbf{p}'')}} e^{i[\varepsilon_p E(\mathbf{p})-\varepsilon_{p''}E(\mathbf{p}'')]t}$$
$$\times F_1(t,t';K'',K';P'',P')dK''dP'' \tag{4.28}$$

Calculating matrix elements $\langle\mathcal{P}^l|\ldots|\mathcal{P}'^l\rangle$ $(l=2,3,\ldots)$ from Eq.(4.25), one obtains a chain of integro-differential equations for determination of the functions $F_2, F_3, \ldots$ The equation of the lth order is linear with respect to functions F_l. Coefficients of this equation contain the functions F_{l-1}, F_{l-2},... which can be determined from the preceding equations of the chain.

To solve Eq. (4.28), it is convenient to use the following designations

$$\begin{aligned} -w &= \{P,K\}\equiv\{\mathbf{s},-u\}, \qquad u=\{\varepsilon_p,\varepsilon_k,-\mathbf{q}/2\}, \qquad \mathbf{q}=\mathbf{k}-\mathbf{p} \\ w &= \{K,P\}\equiv\{\mathbf{s},u\}, \qquad u=\{\varepsilon_k,\varepsilon_p,\mathbf{q}/2\}, \qquad \mathbf{s}=\mathbf{k}+\mathbf{p} \end{aligned} \tag{4.29}$$

$$\begin{aligned} -w' &= \{P',K'\}\equiv\{\mathbf{s}',-u'\}, \qquad u'=\{\varepsilon_{p'},\varepsilon_{k'},-\mathbf{q}'/2\}, \qquad \mathbf{q}'=\mathbf{k}'-\mathbf{p}' \\ w' &= \{K',P'\}\equiv\{\mathbf{s}',u'\}, \qquad u'=\{\varepsilon_{k'},\varepsilon_{p'},\mathbf{q}'/2\}, \qquad \mathbf{s}'=\mathbf{k}'+\mathbf{p}' \end{aligned} \tag{4.30}$$

$$\begin{aligned} \int(.)dw &\equiv \int\int(.)dKdP\equiv\int\int(.)d\mathbf{s}du, \\ \int(.)du &\equiv 2^{-n}\sum_{\varepsilon_k,\varepsilon_p=\mp1}\int\limits_{-\infty}^{\infty}(.)d\mathbf{q} \end{aligned} \tag{4.31}$$

$$\begin{aligned} \delta(w-w') &\equiv \delta(K-K')\delta(P-P')\equiv\delta(\mathbf{s}-\mathbf{s}')\delta(u-u') \\ \delta(w+w') &\equiv \delta(K-P')\delta(P-K')\equiv\delta(\mathbf{s}-\mathbf{s}')\delta(u+u') \end{aligned} \tag{4.32}$$

Supposing

$$\begin{aligned} F_1(t,t';K,K',P,P') &= e^{-i\varepsilon_k E(\mathbf{k})t+i\varepsilon_{k'}E(\mathbf{k}')t'}D_1'(t,t';w,w') \\ &\quad -e^{-i\varepsilon_p E(\mathbf{p})(t-t')}\delta(w-w') \end{aligned} \tag{4.33}$$

one obtains from Eq.(4.28)

$$\frac{\partial D_1'}{\partial t}(t,t';w,w')=i\lambda\int\mathcal{K}(w,w'')e^{i[\omega(w)-\omega(w''))]t}D_1'(t,t';w'',w')dw''$$

$$\frac{\partial D_1'}{\partial t'}(t,t';w,w')=-i\lambda\int\mathcal{K}(w',w'')e^{i[\omega(w')-\omega(w'')]t}D_1'(t,t';w,w'')dw'' \tag{4.34}$$

where the following designations are used

$$\mathcal{K}(w,w')=(2\pi)^{-n}\eta_1(w)\eta_2(w')\delta(\mathbf{s}-\mathbf{s}'), \qquad \eta_1(w)\equiv\zeta_1(u,\mathbf{s})=\frac{\varepsilon_k}{\sqrt{\beta(\mathbf{k})\beta(\mathbf{p})}}, \tag{4.35}$$

$$\eta_2(w)\equiv\zeta_2(u,\mathbf{s})=\frac{1}{\sqrt{\beta(\mathbf{k})\beta(\mathbf{p})}}, \qquad \omega(w)=\omega_1(u,\mathbf{s})\equiv\varepsilon_k E(\mathbf{k})+\varepsilon_p E(\mathbf{p}) \tag{4.36}$$

The solution of Eq.(4.34) can be obtained in the form

$$D_1'(t,t';w,w') = \int\int \kappa_{\overline{w}}^{(+)}(w) e^{i[\omega(\overline{w})-\omega(w)]t} B_{\overline{w}\overline{w}'} \kappa_{\overline{w}}^{(-)}(w') e^{-i[\omega(\overline{w}')-\omega(w')]t'} d\overline{w} d\overline{w}' \tag{4.37}$$

Here $\overline{w}$ and $\overline{w}'$ label the eigenfunctions $\kappa_{\overline{w}}^{(\pm)}(w)$ of the equation

$$[\omega(\overline{w}) - \omega(w)]\kappa_{\overline{w}}^{(\pm)}(w) + \lambda \int \mathcal{K}(w,w')\kappa_{\overline{w}}^{(\pm)}(w')dw' = 0 \tag{4.38}$$

and $B_{\overline{w}\overline{w}'}$ is some function of arguments $\overline{w}, \overline{w}'$. Integral over $\overline{w}$ includes summation over discrete spectrum and integration over continuous one.

For determination of eigenfunctions $\kappa^{(\pm)}$ the following integrals are important

$$I^{(\pm)}(\omega, \mathbf{s}) = \frac{1}{(2\pi)^n} \int \frac{\eta_1(w)\eta_2(w)\delta(\mathbf{s}-\mathbf{k}-\mathbf{p})}{\omega - \omega(w) \pm i0} dw \tag{4.39}$$

Here the symbol ± 0 determines the round way of the integrand poles. It means $\pm i\varepsilon$, $\varepsilon \to +0$.

As it was shown in [3], the integrals $I^{(\pm)}(\omega, \mathbf{s})$ depend on the only argument $z = \gamma^2 = \omega^2 - s^2$. They are analytical functions of z on the complex plane with a cut along the real axis $[4m^2, +\infty)$. $I^{(\pm)}$ are values of the integral $I(\gamma)$ at the different cut edges.

Let us set

$$\Delta(z) = 1 + \lambda I(\omega, \mathbf{s}), \qquad z = \omega^2 - s^2 = \gamma^2 \tag{4.40}$$

The calculation [3] gives the following expression of $\Delta(z)$ at different values of the dimension n of the configurational space.

$$n = 1, \qquad \Delta_1(z) = 1 - \frac{\lambda}{\pi\gamma\sqrt{4m^2-\gamma^2}} \arctan \frac{\gamma}{\sqrt{4m^2-\gamma^2}}, \qquad \gamma = \sqrt{z} \tag{4.41}$$

$$n = 2, \qquad \Delta_2(z) = 1 - \frac{\lambda}{8\pi\gamma} \log \frac{2m+\gamma}{2m-\gamma}, \qquad \gamma = \sqrt{z} \tag{4.42}$$

For $n = 3$ the integral (4.39) diverges. It can be represented in the form

$$\Delta_3(z) = 1 - \frac{\lambda}{8\pi^2} \left\{ \frac{\sqrt{4m^2-\gamma^2}}{\gamma} \arctan \frac{\gamma}{\sqrt{4m^2-\gamma^2}} + \lim_{\mu\to+\infty} \log \frac{\mu + \sqrt{4m^2+\mu^2}}{2m} \right\} \tag{4.43}$$

The values of frequencies $\omega = \pm\sqrt{4m^2+\mathbf{s}^2} \geq 0$ at the cut edges determine the continuous spectrum of Eq.(4.38). The roots $z = M^2$ of the function $\Delta(z)$ determine the discrete spectrum $\omega = \pm\sqrt{4m^2+\mathbf{s}^2}$. The analysis shows [3] $\Delta(z)$ has no roots, or one root depending on the value of λ. The root lies in the interval $0 \leq M^2 < 4m^2$.

Parameter $\overline{w}$ labelling the functions $\kappa_{\overline{w}}^{(\pm)}(w)$ can be represented as follows

$$\overline{w} = \{(\overline{K}, \overline{P}), \overline{S}\}, \qquad \overline{S} = \{\varepsilon_{\overline{s}}, \overline{\mathbf{s}}\}, \qquad \varepsilon_{\overline{s}} = \pm 1 \tag{4.44}$$

$$\omega(\overline{w}) = \{\omega(\overline{K},\overline{P}), \omega(\overline{S})\}, \qquad \omega(\overline{K},\overline{P}) = \varepsilon_{\overline{k}}E(\overline{\mathbf{k}})+\varepsilon_{\overline{p}}E(\overline{\mathbf{p}}) \quad \text{for } \overline{w} = (\overline{K},\overline{P}) \tag{4.45}$$

$$\omega(\overline{w}) = \omega(\overline{S}) = \varepsilon_{\overline{s}}E_M(\overline{\mathbf{s}}) \quad \text{for} \quad \overline{w} = \overline{S} = \{\varepsilon_{\overline{s}}, \overline{\mathbf{s}}\} \tag{4.46}$$

where

$$\beta_M(\mathbf{s}) = 2E_M(\mathbf{s}) = 2\left|\sqrt{M^2+\mathbf{s}^2}\right|, \qquad \varepsilon_s = \pm 1 \tag{4.47}$$

M^2 is a root of the function $\Delta(z)$. Values $\varepsilon_s = \pm 1$ label the states of the discrete spectrum.

For $\overline{w} = \{\overline{K},\overline{P}\}$ the eigenfunctions of Eq.(4.38) have the form

$$\kappa_{\overline{w}}^{(\pm)}(w) = \kappa_{\overline{KP}}^{(\pm)}(K,P) = \delta(w-\overline{w}) - \frac{\lambda}{(2\pi)^n}\frac{\eta_1(w)\eta_2(\overline{w})\delta(\mathbf{s}-\overline{\mathbf{s}})}{[\omega(\overline{w})-\omega(w)\pm i0]\Delta_{\pm}(\overline{w})} \tag{4.48}$$

where

$$\Delta_{\pm}(\overline{w}) = 1+\lambda I^{(\pm)}(\omega(\overline{w}),\overline{\mathbf{s}}), \qquad \overline{\mathbf{s}} = \overline{\mathbf{k}}+\overline{\mathbf{p}} \tag{4.49}$$

For discrete spectrum $\overline{w} = \overline{S}$, one has

$$\kappa_{\overline{w}}^{(\pm)}(w) = \kappa_{\overline{S}}^{(\pm)}(K,P) = \frac{1}{\sqrt{(2\pi)^n B}}\frac{\varepsilon_s\eta_2(w)\delta(\mathbf{s}-\overline{\mathbf{s}})}{\sqrt{\beta_M(\mathbf{s})}[\omega(\overline{S})-\omega(w)]} \tag{4.50}$$

$$B = \left|\frac{1}{\lambda}\frac{\partial\Delta(z)}{\partial z}\right|_{z=M^2} \tag{4.51}$$

Equation (4.38) is not self-adjoint. The adjoint equation has the form

$$[\omega(\overline{w})-\omega(w)]\overline{\kappa}_{\overline{w}}^{(\pm)}(w) + \lambda\int\mathcal{K}(w',w)\overline{\kappa}_{\overline{w}}^{(\pm)}(w')dw' = 0 \tag{4.52}$$

Here $\overline{\kappa}_{\overline{w}}^{(\pm)}(w)$ are the functions adjoint to $\kappa_{\overline{w}}^{(\pm)}(w)$. $\overline{\kappa}_{\overline{w}}^{(\pm)}(w)$ are obtained from $\kappa_{\overline{w}}^{(\pm)}(w)$ by replacing $\eta_1 \leftrightarrow \eta_2$, i.e.

$$\overline{\kappa}_{\overline{w}}^{(\pm)}(w) = \varepsilon_{\overline{k}}\varepsilon_k\kappa_{\overline{w}}^{(\pm)}(w) \quad \text{for} \quad \overline{w} = (\overline{K},\overline{P})$$

$$\overline{\kappa}_{\overline{S}}^{(\pm)}(w) = \varepsilon_{\overline{S}}\varepsilon_k\kappa_{\overline{w}}^{(\pm)}(w) \quad \text{for} \quad \overline{w} = \overline{S} \tag{4.53}$$

Each of the sets $\kappa_{\overline{w}}^{(+)}$ and $\kappa_{\overline{w}}^{(-)}$ is complete. They satisfy the relations

$$\int\overline{\kappa}_{\overline{w}}^{(\pm)}(w)\kappa_{\overline{w}'}^{(\mp)}(w)dw = \delta(\overline{w}-\overline{w}') \tag{4.54}$$

$$\int\overline{\kappa}_{\overline{w}}^{(\pm)}(w)\kappa_{\overline{w}}^{(\mp)}(w')d\overline{w} = \delta(w-w') \tag{4.55}$$

where

$$\delta(\overline{w}-\overline{w}') = \{\delta(\overline{K}-\overline{K}')\delta(\overline{P}-\overline{P}'), \delta(\overline{S}-\overline{S}')\} \tag{4.56}$$

Integral over $\overline{w}$ concludes integration over all continuous and discrete states

$$\int(.)d\overline{w} = \int(.)d\overline{K}d\overline{P} + \int(.)d\overline{S} \tag{4.57}$$

Besides the following designations like Eq.(4.29) will be used

$$\overline{w} = \{(\overline{K}, \overline{P}), \overline{S}\}, \qquad -\overline{w} = \{(\overline{P}, \overline{K}), \overline{S}\},$$

$$\delta(\overline{w} + \overline{w}') = \{\delta(\overline{K} - \overline{P}')\delta(\overline{P} - \overline{K}'), \delta(\overline{S} - \overline{S}')\} \tag{4.58}$$

The bound states arise at definite values of λ only. The result depends on the dimension n. At $n = 1$ there is a root of $\Delta_1(z)$, if $0 < \lambda < 4\pi m^2$. There is no root, if $\lambda < 0 \vee \lambda > 4\pi m^2$. At $n = 2$ there is a root of $\Delta_2(z)$, if $0 < \lambda < 8\pi m$. There is no root, if $\lambda \leq 0 \vee \lambda > 8\pi m$. At $n = 3$ λ is dimensionless quantity. There are no roots at any value of λ except for the case, when λ is an infinitesimal quantity and $\lambda > 0$. If λ depends on μ in such a way that

$$\lim_{\mu \to +\infty} \left[\frac{\lambda}{8\pi} \log \left(\frac{\mu}{8\pi} + \sqrt{\frac{\mu^2}{4m^2} + 1} \right) - 1 \right] = +0 \tag{4.59}$$

then the bound state described by Eq.(4.50) can exist. This depends neither on λ, nor on the diverging part of Eq.(4.43). But the mass M of the bound state cannot be determined from Eq. (4.43). It should be given independently.

The solution (4.37) of Eqs. (4.34) which satisfies the conditions III-V has the form

$$D_1'(t, t'; w, w') = \int \kappa_{\overline{w}}^{(+)}(w) e^{i[\omega(\overline{w}) - \omega(w)]t} [\kappa_{\overline{w}}^{(-)}(w') + \kappa_{-\overline{w}}^{(-)}(w')] e^{-i[\omega(\overline{w}) - \omega(w')]t'} d\overline{w} \tag{4.60}$$

Then it follows from Eqs. (4.21), (4.33)

$$D_1(t, t'; w, w') = [D_1'(t, t'; w, w') - \delta(w - w') - \delta(w + w')] e^{-i\varepsilon_k E(\mathbf{k})t + i\varepsilon_{k'} E(\mathbf{k}')t'} \tag{4.61}$$

At $t = t' = 0$ and $\lambda = 0$ the simultaneous commutation relation (4.19) takes the form

$$[b(K), b^*(K')]_- = \delta(K - K') + O_{2,2} \tag{4.62}$$

where the designations are used

$$O_{k,l} = \sum_{s=0}^{\infty} \int \int f_{k+s,l+s}(\mathcal{K}^{k+s}, \mathcal{K}'^{l+s}) B_{k+s}^*(\mathcal{K}^{k+s}) B_{l+s}(\mathcal{K}'^{l+s}) d\mathcal{K}^{k+s} d\mathcal{K}'^{l+s} \tag{4.63}$$

$f_{k+s,l+s}$ is some function of arguments $\mathcal{K}^{k+s}, \mathcal{K}'^{l+s}$.

In the two-WL case, where Eq.(4.62) was obtained, the relations (4.62) coincides with Eq.(2.15) at $\lambda = 0$. But at $\lambda \neq 0$ the simultaneous commutation relation does not coincide with the free simultaneous relation (2.15).

An attempt of using the simultaneous commutation relation (2.15) as initial conditions of solution of the system (4.34) leads to the result (4.37) with $B_{\overline{w}\overline{w}'}$ defined by the expression

$$B_{\overline{w}\overline{w}'} = \int \overline{\kappa}_{\overline{w}}^{(-)}(w) [\overline{\kappa}_{\overline{w}'}^{(+)}(w) + \overline{\kappa}_{\overline{w}'}^{(+)}(-w)] dw \tag{4.64}$$

In explicit form the expression (4.64) is written for $\overline{w} = \{\overline{K}, \overline{P}\}, \quad \overline{w}' = \{\overline{K}', \overline{P}'\}$

$$B_{\overline{w}\overline{w}'} = \delta(\overline{w} - \overline{w}') + \delta(\overline{w} + \overline{w}')\frac{\lambda}{(2\pi)^n}\frac{\eta_2(\overline{w})\eta_2(\overline{w}')\delta(\overline{\mathbf{s}} - \overline{\mathbf{s}}')}{[\omega(\overline{w}) - \omega(\overline{w}') \pm i0]\Delta_+(\overline{w}')\Delta_-(\overline{w})}$$

$$\times\{2\varepsilon_{\overline{k}}\Delta_+(\overline{w}') - 2\varepsilon_{\overline{k}'}\Delta_+(\overline{w}) - (\varepsilon_{\overline{k}'} + \varepsilon_{\overline{k}})[\tilde{\Delta}_+(\overline{w}') - \tilde{\Delta}_+(\overline{w})]\} \tag{4.65}$$

where

$$\tilde{\Delta}_\pm(\overline{w}) = 1 + \lambda\tilde{I}^\pm(\omega(\overline{w}), \mathbf{s})$$

$$\tilde{I}^{(\pm)}(\omega, \mathbf{s}) = \frac{1}{(2\pi)^n}\int\frac{\eta_2^2(w)\delta(\mathbf{s} - \mathbf{k} - \mathbf{p})}{\omega - \omega(w) \pm i0}dw \tag{4.66}$$

The functions η_2, $\omega(\overline{w})$ are defined by Eq.(4.36). The functions $\Delta_\pm$ are defined by Eqs. (4.39), (4.49). The commutation relation is supposed to be translation invariant. If this condition is fulfilled (condition III), then the function F_1 determined by Eqs.(4.33), (4.37), (4.65) has to depend on the difference $t - t'$, and $B_{\overline{w}\overline{w}'}$ has to contain the factor $\delta[\omega(\overline{w}) - \omega(\overline{w}')]$. It is easy to verify that it is possible in the only case, when $\lambda = 0$, i.e. for the free field.

Thus, at the quantization according to WL-scheme the conventional form of simultaneous commutation relation occurs to be incompatible with the translation invariance. It cannot be used at the nonlinear field quantization according to WL-scheme.

At the quantization *according to WL-scheme the simultaneous commutation relation depends on* λ in general. If for determination of commutation relations one uses the free simultaneous commutation relations

$$b(K,0)\,b^*(K',0) - b^*(K',0)\,b(K,0) = \delta(K - K')$$

instead of conditions I -IV, one obtains the nontranslation-invariant commutation relations which asssociate with the Haag's theorem [8].

Remark. Expression for the function D_1' can be taken in the form

$$D_1'(t,t';w,w') = \int\kappa_{\overline{w}}^{(-)}(w)e^{i[\omega(\overline{w})-\omega(w)]t}[\kappa_{\overline{w}}^{(+)}(w') + \kappa_{-\overline{w}}^{(+)}(w')]e^{-i[\omega(\overline{w})-\omega(w')]t'}d\overline{w} \tag{4.67}$$

The expression for D_1', obtained from Eq.(4.60) by the substitution $\kappa^{(+)} \to \kappa^{(-)}$, $\kappa^{(-)} \to \kappa^{(+)}$, satisfies the conditions II-VI.

5 Free Fields

Let $\mathcal{H}_2$ be a Hilbert space of states, containing not more than two WLs, i.e. $\mathcal{H}_2$ consists of vectors of the form (4.17) with $l \leq 2$. Two-WL states $|K, P\rangle$ are not orthonormal. Indeed, due to Eqs. (4.33), (4.60) the simultaneous commutation relation (4.20) takes the form

$$b(K)b^*(K') = \delta(K - K')[1 - \int b^*(P)b(P)dP$$

$$+\int\int D_1'(K,P;K',P')b^*(P)b(P')dPdP'] + O_{2,2} \tag{5.1}$$

where

$$D_1'(K,P;K'P') = D_1'(0,0;w,w') \tag{5.2}$$

and $O_{2,2}$ is defined by Eq. (4.63). Then

$$\langle w'|w\rangle \equiv \langle P',K'|K,P\rangle \equiv \frac{1}{2}\langle 0|b(P')b(K')b^*(K)b^*(P)|0\rangle = \frac{1}{2}D_1'(K,P;K'P') \tag{5.3}$$

that is distinguished from $[\delta(w'-w)+\delta(w+w')]/2$, if $\lambda \neq 0$.

Let us introduce orthonormal states $|\overline{w}\rangle_+$, $|\overline{w}\rangle_-$, defining them by relations

$$|\overline{w}\rangle_\alpha = \{|\overline{K},\overline{P}\rangle_\alpha, |\overline{S}\rangle_\alpha\}, \qquad \alpha = \pm \tag{5.4}$$

$$|\overline{K},\overline{P}\rangle_\alpha = \int\int \overline{\kappa}^{(\alpha)}_{\overline{K},\overline{P}}(K,P)|K,P\rangle dKdP = \int \kappa^{(\alpha)}_{\overline{w}}(w)|w\rangle dw, \qquad \alpha = \pm \tag{5.5}$$

$$|S\rangle_\alpha = \int\int \overline{\kappa}_{\overline{S}}(K,P)|K,P\rangle dKdP, \qquad \alpha = \pm \tag{5.6}$$

The wave functions $f(K,P)$,

$$\psi_{(+)}(\overline{w}) = \{\psi_{(+)}(\overline{K},\overline{P}),\psi(S)\}, \quad \psi_{(-)}(\overline{w}) = \{\psi_{(-)}(\overline{K},\overline{P}),\psi(S)\}$$

of the state Φ_2 in these three different representations are defined by the relations

$$\Phi_2 = \int f(w)|w\rangle dw = \int \psi_{(+)}(\overline{w})|\overline{w}\rangle_+ d\overline{w} = \int \psi_{(-)}(\overline{w})|\overline{w}\rangle_- d\overline{w}. \tag{5.7}$$

They are connected by means of relations

$$\begin{gathered}
\psi_{(\pm)}(\overline{w}) = \int \kappa^{(\mp)}_{\overline{w}}(w)f(w)d\overline{w}, \\
f(w) = \int \overline{\kappa}^{(\pm)}_{\overline{w}}(w)\psi_{(\pm)}(\overline{w})d\overline{w}, \\
\psi_{(\alpha)}(\overline{w}) = \int \Omega^{(-\alpha)}(w)\psi_{(-\alpha)}(\overline{w}')d\overline{w}', \qquad \alpha = \pm,
\end{gathered} \tag{5.8}$$

where

$$\begin{gathered}
\Omega^{(\pm)}(\overline{w},\overline{w}') = \int \overline{\kappa}^{(\pm)}_{\overline{w}}(w)\kappa^{(\pm)}_{\overline{w}'}(w)dw, \\
\Omega^{(\pm)}(\overline{K},\overline{P};\overline{K}'\overline{P}') = \delta(\overline{w}-\overline{w}') \pm \frac{i\lambda}{(2\pi)^{n-1}}\frac{\varepsilon_{\overline{k}}\eta_2(\overline{w})\eta_2(\overline{w}')}{\Delta_\pm(\overline{w})}\delta[\omega(\overline{w})-\omega(\overline{w}')] \\
\times\delta(\overline{\mathbf{k}}+\overline{\mathbf{p}}-\overline{\mathbf{k}}'-\overline{\mathbf{p}}'), \\
\Omega^{(\pm)}(\overline{S};\overline{S}') = \delta(\overline{S}-\overline{S}'), \\
\Omega^{(\pm)}(\overline{K},\overline{P};\overline{S}') = \Omega^{(\pm)}(\overline{S};\overline{K},\overline{P}) = 0.
\end{gathered} \tag{5.9}$$

Here the upper signs, or the lower ones are taken simultaneously.

One can see from Eq. (5.8) that a symmetrization of anyone of functions $f(w)$, $\psi_{(\pm)}(\overline{w})$, $\psi_{(-)}(\overline{w})$ leads to symmetrization of remaining functions, i.e. a fulfillment of anyone of equalities

$$f(w) = f(-w), \qquad \psi_{(+)}(\overline{w}) = \psi_{(+)}(-\overline{w}),$$

$$\psi_{(-)}(\overline{w}) = \psi_{(-)}(-\overline{w}) \tag{5.10}$$

leads to fulfillment of two others.

The wave function of identical WLs has to be symmetric. In QFT the symmetric wave functions arise as a result of all creation operators commutativity (2.15). It leads to identification of the states $|K,P\rangle$ and $|P,K\rangle$ in the form

$$|K,P\rangle - |P,K\rangle = \frac{1}{\sqrt{2}}[b^*(K)b^*(P) - b^*(P)b^*(K)]|0\rangle = 0 \tag{5.11}$$

But it is possible only for a free field. Indeed, convoluting Eq. (5.11) with the state vector $\langle P',K'|$, one obtains by means of Eqs.(5.3) and (4.60)

$$D_1'(w',w) - D_1'(w',-w) = 0 \tag{5.12}$$

For $\varepsilon_k = -\varepsilon_p$ Eq. (5.12) is fulfilled only at $\lambda = 0$.

Let us introduce operators of free field $b_0(K,t)$, $b_1(S,t)$, $b_0^*(K,t)$, $b_1^*(S,t)$, defining them by relations

$$b_0(K,t)|0\rangle = b_1(K,t)|0\rangle = 0 \tag{5.13}$$

$$b_0^*(K,t)|0\rangle = b^*(K,t)|0\rangle \tag{5.14}$$

$$b_0^*(\overline{K},t)b_0^*(\overline{P},t)|0\rangle = \int\int \overline{\kappa}^{(+)}_{\overline{K},\overline{P}}(K,P)b^*(K,t)b^*(P,t)|0\rangle dKdP \tag{5.15}$$

$$b_1^*(\overline{S},t)|0\rangle = \int\int \overline{\kappa}_{\overline{S}}(K,P)b^*(K,t)b^*(P,t)|0\rangle dKdP \tag{5.16}$$

The solution of Eqs. (5.13)-(5.16) can be represented in the form

$$b_0(K,t) = b(K,t) + \int\int\int \overline{\mu}^{(-)}_{K,P}(K',P')b^*(P,t)b(P',t)b(K',t)dK'dP'dP + O_{2,3} \tag{5.17}$$

$$b_1(S,t) = \int\int \overline{\mu}_S(K',P')b(P',t)b(K',t)dK'dP' + O_{1,3} \tag{5.18}$$

where

$$\overline{\mu}^{(\pm)}_{\overline{K},\overline{P}}(K,P) = \overline{\kappa}^{(\pm)}_{\overline{K},\overline{P}}(K,P) - \delta(\overline{K}-K)\delta(\overline{P}-P) =$$
$$-\frac{\lambda}{(2\pi)^n}\frac{\eta_1(\overline{w})\eta_2(w)\delta(\mathbf{s}-\overline{\mathbf{s}})}{[\omega(\overline{w})-\omega(w)\pm i0]\Delta_\pm(\overline{w})} \tag{5.19}$$

$$\overline{\mu}_{\overline{S}}(K,P) = \frac{1}{\sqrt{2}}\overline{\kappa}_{\overline{S}}(K,P) = \frac{\varepsilon_s\eta_2(w)\delta(\mathbf{s}-\overline{\mathbf{s}})}{\sqrt{(2\pi)^nB}\sqrt{\beta_{\mathrm{M}}(\mathbf{s})}[\omega(\overline{S})-\omega(w)]} \tag{5.20}$$

where B is defined by Eq.(4.51)

The reciprocal relation has the form

$$
\begin{aligned}
b(K,t) &= b_0(K,t) + \int\int \mu^{(+)}_{K',P'}(K,P) b_0^*(P,t) b_0(P',t) b_0(K',t) dK' dP' dP \\
&\quad + \int\int \mu_S(K,P) b_0^*(P,t) b_1(K,t) dS dP + O_{2,3} \qquad (5.21)
\end{aligned}
$$

where

$$\mu^{(\pm)}_{\overline{K},\overline{P}}(K,P) = \kappa^{(\pm)}_{\overline{K},\overline{P}}(K,P) - \delta(\overline{K}-K)\delta(\overline{P}-P) \qquad (5.22)$$

$$\mu_{\overline{S}}(K,P) = \sqrt{2}\kappa_{\overline{S}}(K,P) \qquad (5.23)$$

the relations for b_0^*, b_1^*, b are obtained from Eqs.(5.17), (5.18), (5.21) by means of a Hermitian conjugation.

Let the commutation relation have the form (5.1), where D_1' is defined by Eq. (4.60). Then a calculation gives

$$[b_0(K,t), b_0^*(K',t)]_- = \delta(K-K') + O_{2,2} \qquad (5.24)$$

$$b_0(K,t) b_1^*(K',t) = O_{2,1} \qquad (5.25)$$

$$b_1(K,t) b_0^*(K',t) = O_{1,2} \qquad (5.26)$$

$$b_1(K,t) b_1^*(K',t) = \delta(K-K') + O_{2,2} \qquad (5.27)$$

Differentiating Eqs.(5.17), (5.18) with respect to t and using Eqs.(4.3), (4.4), (5.1), one obtains

$$\dot{b}_A(K,t) = -i\varepsilon_k E_A(\mathbf{k}) b_A(K,t) + O_{2-A,3}, \qquad A = 0,1 \qquad (5.28)$$

$$E_A(k) = |\sqrt{m_A^2 + \mathbf{k}^2}|, \qquad m_0 = m, \qquad m_1 = M, \qquad A = 0,1 \qquad (5.29)$$

Let us introduce the fields $\varphi_A(x) = \varphi_A(\mathbf{x},t)$, $A = 0,1$, defining them by relations of the type (2.4), (2.5)

$$\varphi_A(\mathbf{x},t) = (2\pi)^{-n/2} \int e^{i\mathbf{kx}} \frac{b_A(K,t)}{\sqrt{\beta_A(\mathbf{k})}} dK, \qquad \beta_A(\mathbf{k}) = 2E_A(\mathbf{k}), \qquad A = 0,1 \quad (5.30)$$

$$\dot{\varphi}_A(\mathbf{x},t) = \frac{\partial \varphi_A}{\partial t} = -i(2\pi)^{-n/2} \int \frac{\varepsilon_k}{2} \sqrt{\beta_A(\mathbf{k})} e^{i\mathbf{kx}} b_A(K,t) dK, \qquad A = 0,1 \quad (5.31)$$

Then according to Eq.(5.23) the fields $\varphi_A(x)$ satisfy the dynamic equations

$$(\partial_i \partial^i + m_A^2)\varphi_A = O_{2-A,3}, \qquad A = 0,1 \qquad (5.32)$$

In other words, inside the Hilbert space $\mathcal{H}_2$ the fields $\varphi_A(x)$, $A = 0,1$ are free non-interacting fields of the mass m_A.

Using the transformation properties of φ and relations (5.17), (5.18), one can show the fields φ_0, φ_1 are scalars. Thus, nonlinear field φ can be described by means of two free scalar fields φ_0, φ_1 to within $O_{2,3}$.

In the case of the commutation relations (5.24) – (5.27) the Hamiltonian $H = -p_0$ and the canonical momentum $\pi_\alpha = -p_\alpha$ defined by Eq. (3.11) have the following form

$$H = \sum_{A=0,1} \int \varepsilon_k E_A(\mathbf{k}) b_A^*(K) b_A(K) dK + O_{3,3} \tag{5.33}$$

$$\boldsymbol{\pi} = \sum_{A=0,1} \int \mathbf{k} b_A^*(K) b_A(K) dK + O_{3,3} \tag{5.34}$$

Vectors $|0\rangle$, $|K\rangle$, $|K,P\rangle_+ + |P,K\rangle_+$, $|K,P\rangle_- + |P,K\rangle_-$, $b_1^*(S)|0\rangle$ are eigenvectors of operators H, $\boldsymbol{\pi}$ with eigenvalues $\{0, \mathbf{0}\}$, $\{\varepsilon_k E(\mathbf{k}), \mathbf{k}\}$, $\{\omega(K,P) + i0, \mathbf{k} + \mathbf{p}\}$, $\{\omega(K,P) - i0, \mathbf{k} + \mathbf{p}\}$, $\{\varepsilon_s E_M(\mathbf{k}), \mathbf{s}\}$ respectively. All of them describe stationary states. Vectors $|K,P\rangle_+$, $|P,K\rangle_+$, $|K,P\rangle_-$, $|P,K\rangle_-$ are not stationary states at $K \neq P$, generally.

The commutation relations (5.24) – (5.27) permit only to commutate the creation operator b^* with the annihilation operator b. They do not permit to commutate two creation operators b^*. For this reason the vector $|K,P\rangle_+ = 2^{-1/2} b_0^*(K) b_0^*(P)|0\rangle$ does not connect with the vector $|P,K\rangle_+ = 2^{-1/2} b_0^*(P) b_0^*(K)|0\rangle$, if $K \neq P$.

Consideration of the WL identity is realized by the symmetric wave functions (5.8). Using the condition (5.11) is impossible, as we have seen. But the states $|K,P\rangle_+$ and $|P,K\rangle_+$ can be identified , as far as according to Eq. (5.24) the vector $|K,P\rangle_+ - |P,K\rangle_+$ is orthogonal to all vectors of $\mathcal{H}_2$.

$$\langle P', K' | K, P\rangle_+ - \langle P', K' | P, K\rangle_+ = 0 \tag{5.35}$$

Let us identify the vector $|K,P\rangle_+ - |P,K\rangle_+$ with the zero vector. It is equivalent to the commutation relations

$$[b_0^*(K), b_0^*(K')]_- = O_{3,1} \tag{5.36}$$

$$[b^*(K), b^*(P)]_- = \frac{i\lambda}{(2\pi)^n} \int \frac{(\varepsilon_k - \varepsilon_p)\eta_2(w)\eta_2(\overline{w})\delta(\mathbf{s} - \overline{\mathbf{s}})}{[\omega(\overline{w}) - \omega(w) - i0]\Delta_-(\overline{w})} b_0^*(\overline{K}) b_0^*(\overline{P}) d\overline{K} d\overline{P} + O_{3,1} \tag{5.37}$$

If the condition (5.36) takes place, then the Hilbert space $\mathcal{H}_2$ turns to the Hilbert space $\mathcal{H}_{2,s}$ of identical WLs.

Let us introduce operators $S^{(+)}$ and $S^{(-)}$ by means of relations

$$S^{(\pm)} = 1 + \int\int \sigma^{(\pm)}(w, w') b_0^*(P) b_0^*(K) b_0(K') b_0(P') dw dw' + O_{3,3} \tag{5.38}$$

where

$$\sigma^{(\pm)}(w, w') = \frac{1}{2}\{\Omega^{(\pm)}(K, P; K', P') - \delta(w - w')\} + (K \leftrightarrow P)$$

$$\pm\frac{i\lambda}{(2\pi)^{n-1}}\frac{(\varepsilon_k+\varepsilon_p)\eta_2(w)\eta_2(w')}{2\Delta_\pm(w)}\delta[\omega(w)-\omega(w')]\delta(\mathbf{k}+\mathbf{p}-\mathbf{k}'-\mathbf{p}') \qquad (5.39)$$

where $\Omega^{(\pm)}$ is defined by Eqs. (5.9), and $(K\leftrightarrow P)$ means the term, obtained from the preceding one by the substitution $K\leftrightarrow P$.

One can verify that operators $S^{(+)}$ and $S^{(-)}$ are unitary in $\mathcal{H}_{2,s}$

$$(S^{(+)})^*=S^{(-)}+O_{3,3},\qquad S^{(+)}S^{(-)}=S^{(-)}S^{(+)}=1+O_{3,3} \qquad (5.40)$$

and commutate with operators H and $\boldsymbol{\pi}$

$$[S^{(\pm)},H]_-=O_{3,3},\qquad [S^{(\pm)},\boldsymbol{\pi}]_-=O_{3,3} \qquad (5.41)$$

Let us introduce operators

$$c_0^*(K,t)=S^{(-)}b_0^*(K,t)S^{(+)},\qquad c_0(K,t)=S^{(-)}b_0(K,t)S^{(+)} \qquad (5.42)$$

From unitary property of $S^{(+)}$ and the relation

$$c_1(K,t)=S^{(-)}b_1(K,t)S^{(+)}=b_1(K,t)+O_{1,3}. \qquad (5.43)$$

it follows that operators c_0, c_1 satisfy the same commutation relations (5.24)-(5.27), (5.36), as operators b_0, b_1. From relations (5.41) it follows that operators $c_0(K,t)$, $c_1(K,t)$ satisfy the same equations (5.28), as operators $b_0(K,t)$, $b_1(K,t)$ do.

In other words, the operators $c_0(K,t)$, $b_0(K,t)$ describe the free fields and satisfy the same commutation relation. Operators b_0, c_0 have many properties common with *in*- and *out*-operators in QFT [9]. But there are differences. For instance, the simultaneous commutation relations are the same for operators $b_{\rm in}, b_{\rm out}$ and b, whereas they are different for b_0 and b. For this reason we do not identify operators b_0, c_0 with $b_{\rm in}$ and $b_{\rm out}$.

From Eqs.(5.5), (5.6), (5.9), (5.38), (5.39) it follows

$$|K,P\rangle_-=S^{(-)}|K,P\rangle_+,\qquad |K,P\rangle_+=S^{(+)}|K,P\rangle_- \qquad (5.44)$$

According to Eqs. (5.42), (5.15) it can be written in the form

$$\frac{1}{\sqrt{2}}c_0^*(K)c_0^*(P)|0\rangle=S^{(-)}|K,P\rangle_+=|K,P\rangle_- \qquad (5.45)$$

The operator $S^{(-)}$ is usually referred to as the scattering matrix or S-matrix. Further it will be shown that, indeed, the operator $S^{(-)}$ describes the SWL scattering. But it is impossible to make *in the scope of the quantization scheme.* In addition the operator, describing the spatial SWL distribution, has to be introduced, for instance, the SWL current density.

6 The Scattering Problem

At the conventional approach [1] to the relativistic scattering problem the interaction cut-off at $t \to \infty$ is used. In other words, at the scattering problem statement one uses two Hamiltonians: the Hamiltonian H defined by Eq.(5.33) and unperturbed Hamiltonian H_0. But the real interaction cannot be cut off. There is only one Hamiltonian H, and *the scattering problem should be stated in terms of only this Hamiltonian* H.

Let us state the scattering problem as follows. Let the state $\Phi_2 \in \mathcal{H}_{2,s}$ describe two wave packets of particles, moving one through another at the time moment $t = 0$ at the origin of the coordinate system. Each of two wave packets is supposed to be almost monomomentum and to have the size $a \gg m^{-1}$, where m^{-1} is the Compton wave length of the particle. It is necessary, for the spread of the wave packets could be neglected. Let the particles of the wave packets be recorded by detectors of the size $L \gg a$. The detectors are placed at such a large distance $d \gg L$ from the frame origin, that the detector angular size $\Omega = L/d \ll 1$. In this case the particles of one of the wave packets are recorded by a detector α_1 and those of the other one are recorded by a detector α_2. The scattered particle, (if there are such ones) are recorded by detectors $\beta_1, \beta_2, \ldots$

Let $\Phi_2' \in \mathcal{H}_{2,s}$ be another state, which is distinguished from Φ_2 only by some displacement of the second wave packet. It is displaced at the distance l, $(L \gg l \gg a)$ in such a way that it does not go through the first wave packet. In this case the particles of the first wave packet are recorded by the same detector α_1, those of the second one are recorded by the detector α_2. There are no scattered particles in this case. Thus, one can investigate the scattering, comparing the particle densities in the states Φ_1 and Φ_2'.

Let $d\nu$ be the number of particles scattered in the direction $\mathbf{l}$ inside the solid angle $d\omega$. Let $n_1^i(\mathbf{x}, t), n_2^i(\mathbf{x}, t)$ be the flux densities of particles in the first wave packet and in the second one respectively. Then the section $d\sigma$ of the scattering in the direction $\mathbf{l}$ is described by the relation

$$d\nu = J d\sigma \tag{6.1}$$

$$J = \int\int \sqrt{(n_1^i, n_{2i})^2 - (n_1^i, n_{1i})^2 (n_2^k, n_{2k})^2} d\mathbf{x} dt \tag{6.2}$$

Here the invariant integral J describes a degree of overlapping of the wave packets. In particular, in the coordinate frame, where the first wave packet is at rest and the second one moves with the velocity $\mathbf{v}_{\mathrm{rel}}$, i.e.

$$n_1^0 = n_1, \quad \mathbf{n}_1 = 0, \qquad n_2^0 = n_2, \quad \mathbf{n}_2 = n_2 \mathbf{v}_{\mathrm{rel}} \tag{6.3}$$

Eq. (6.1) transforms into well known expression

$$d\nu = d\sigma \int n_1 n_2 \left|\mathbf{v}_{\mathrm{rel}}\right| d\mathbf{x} dt. \tag{6.4}$$

Let us choose the state Φ_2 in the form

$$\Phi_2 = \int f_W(K,P)|K,P\rangle dKdP \tag{6.5}$$

where

$$f_W(K,P) = \frac{1}{2}F(K-R)F(P-Q)e^{-i\mathbf{ky}-i\mathbf{pz}} + (K \leftrightarrow P) \tag{6.6}$$

Here and further $(K \leftrightarrow P)$ means the term obtained from the preceding one by the substitution $K \leftrightarrow P$. $W = \{R, Q, \mathbf{y}, \mathbf{z}\}$, $R = \{\varepsilon_r, \mathbf{r}\}$, $Q = \{\varepsilon_q, \mathbf{q}\}$ is a set of parameters describing the wave function ψ_W.

$$F(K-R) = A\delta_{\varepsilon_k,\varepsilon_r}\exp\{-\frac{1}{2}(\mathbf{k}-\mathbf{r})^2a^2\} \tag{6.7}$$

is a real function which is distinguished from zero essentially only in the region

$$|\mathbf{k}-\mathbf{r}| < a^{-1} \ll m \tag{6.8}$$

A is a normalization constant.

Remaining in the scope of the quantization scheme one cannot understand the meaning of parameters W of the wave function ψ_W. Only passing to the quantization model and introducing operators of physical quantities, on can understand what the parameters W mean. But the physical quantities can be introduced in different ways.

The charge density $j^0(\mathbf{x},t)$ can be introduced at least by two ways. The first way

$$j^0(\mathbf{x},t) = e^{iHt-i\boldsymbol{\pi}\mathbf{x}}\frac{e}{(2\pi)^n}\int \frac{\varepsilon_k}{2}\sqrt{\beta(\mathbf{k})/\beta(\mathbf{k}')}b^*(K)b(K')dKdK'e^{-iHt+i\boldsymbol{\pi}\mathbf{x}} + (\text{h.c.}) \tag{6.9}$$

The second one

$$\begin{aligned} j^0(\mathbf{x},t) &= e^{iHt-i\boldsymbol{\pi}\mathbf{x}}\int \frac{\varepsilon_k}{2(2\pi)^n}\sum_{A=0,1} e_A\sqrt{\beta(\mathbf{k})/\beta(\mathbf{k}')} \\ &\quad \times b_A^*(K)b_A(K')dKdK'e^{-iHt+i\boldsymbol{\pi}\mathbf{x}} + (\text{h.c.}) \end{aligned} \tag{6.10}$$

Here (h.c.) means an addition of Hermitian conjugate expression. $e_0 = e$, $e_1 = 2e$.

Eq.(6.9) corresponds to expression (3.10) for the nonlinear field φ. Eq.(6.10) corresponds to the same expression (3.10) for linear noninteracting fields φ_0, φ_1. The total charge Q is the same in both cases.

$$Q = e\int \varepsilon_k b^*((K)b(K)dK = \sum_{A=0,1} e\int \varepsilon_k b_A^*((K)b_A(K)dK + O_{3,3} \tag{6.11}$$

In the second model of quantization j^0 is defined by Eq. (6.10), the state (6.5), (6.6) describes two wave packets of the size $\cong a$. They move with velocities

$$\mathbf{v}_1 = \frac{\varepsilon_r\mathbf{r}}{E(\mathbf{r})}, \qquad \mathbf{v}_2 = \frac{\varepsilon_q\mathbf{q}}{E(\mathbf{q})} \tag{6.12}$$

and are placed at the moment $t=0$ at the points $\mathbf{y}$ and $\mathbf{z}$ respectively. There are *no scattered particles* in the second quantization model.

In the first quantization model, where j^0 is defined by Eq. (6.9), the state (6.5), (6.6) describes the same wave packets as in the second model. But in this case there are *"scattered charges", if the wave packets overlap.*

Let us consider the first model in details. Averaging the operator (6.9) over Φ_2 one obtains the following expression

$$\langle j^0(\mathbf{x},t)\rangle_{\Phi_2} = (\Phi_2, j^0(\mathbf{x},t)\Phi_2)$$

$$= \frac{e}{(2\pi)^n} \sum_{\varepsilon_{k'},\varepsilon_{k''}=\pm 1} \int \varepsilon_{k'} \Psi^{(+)}(\mathbf{x},t;\varepsilon_{k'},P)\Psi^{(-)}(\mathbf{x},t;\varepsilon_{k''},P)dP + (\text{c.c.}) \tag{6.13}$$

where

$$\Psi^{(\pm)}(\mathbf{x},t;\varepsilon_{k''},P) = \Psi^{(\pm)}_{\text{fr}}(\mathbf{x},t;\varepsilon_{k''},P) + \Psi^{(\pm)}_{\text{sc}}(\mathbf{x},t;\varepsilon_{k''},P) \tag{6.14}$$

$$\Psi^{(\pm)}_{\text{fr}}(\mathbf{x},t;\varepsilon_{k''},P) = \frac{1}{2}\int F(K''-R)F(P''-Q)e^{\pm i\zeta_{\text{fr}}(K'',P'')}d\mathbf{k}'' + (W\to -W) \tag{6.15}$$

$$\Psi^{(\pm)}_{\text{sc}}(\mathbf{x},t;\varepsilon_{k''},P'') = \frac{1}{2}\int F(K-R)F(P-Q)\mu^{(\mp)}_{K,P}(K'',P'')$$
$$\times e^{\pm i\zeta_{\text{fr}}(K,P)}dKdPd\mathbf{k}'' + (W\to -W) \tag{6.16}$$

Here and further $(W\to -W)$ means the term obtained from the preceding one by means of transposition $(R\leftrightarrow Q,\ \mathbf{y}\leftrightarrow\mathbf{z})$.

$$\zeta_{\text{fr}}(K,P) = \omega(K,P)t - \mathbf{k}(\mathbf{x}-\mathbf{y}) - \mathbf{p}(\mathbf{x}-\mathbf{z}) \tag{6.17}$$

where $\omega(K,P)$ is defined by Eq. (4.36), or (4.45).

The functions $\mu^{(\pm)}$ are defined by Eqs.(4.48), (5.22). Obtaining the expression (6.13), it was taken into account that the factor $\sqrt{\beta(\mathbf{k})/\beta(\mathbf{k}')}$ in Eq.(6.9) changes slowly in the region of unvanishing contribution. $\Psi^{(-)}(\mathbf{x},t;\varepsilon_{k''},P)$ can be treated as a wave function in the mixed momentum-coordinate representation. $\Psi^{(+)}$ is a complex conjugate function. $\Psi^{(-)}_{fr}, \Psi^{(-)}_{sc}$ are responsible respectively for the free motion and for the scattering.

Using expressions (4.48), (5.22) for $\mu^{(\pm)}$, Eq. (6.16) can be represented in the form

$$\Psi^{(\pm)}_{\text{sc}}(\mathbf{x},t;\varepsilon_{k''},P'') = \pm\frac{i\lambda}{(2\pi)^n}\int\int dKdP\frac{\varepsilon_{k''}\eta_2(\mathbf{k}+\mathbf{p}-\mathbf{p}'',\mathbf{p})\eta_2(K,P)}{\Delta_\pm(K,P)}$$

$$\times F(K-R)F(P-Q)\int_0^\infty d\alpha e^{\pm i\zeta_{\text{sc}}(K,P;\varepsilon_{k''},P'',\alpha)} + (W\to -W) \tag{6.18}$$

$$\zeta_{\text{sc}}(K,P;\varepsilon_{k''},P'',\alpha) = \zeta_{\text{fr}}(K,P) + \alpha[\omega(K,P) - \varepsilon_{k''}E(\mathbf{k}+\mathbf{p}-\mathbf{p}'') - \varepsilon_{p''}E(\mathbf{p}'')] \tag{6.19}$$

The wave functions (6.15), (6.18) are integrals of quickly oscillating functions. They are distinguished from zero essentially only for those values of parameters $\mathbf{x}, t, W$, for which the phases $\zeta_{\rm fr}$ and $\zeta_{\rm sc}$ are stationary (i.e. the phases have extrema for all arguments which are integrated over). The phase $\zeta_{\rm fr}$ is stationary, if the following conditions are fulfilled

$$\mathbf{x} = \mathbf{y} + \mathbf{v}_1 t, \qquad \mathbf{x} = \mathbf{z} + \mathbf{v}_2 t \tag{6.20}$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are defined by Eq. (6.12).

Equations (6.19) describe the free motion of the wave packet centres. In their vicinity the $\Psi_{fr}^{(\pm)}$ is distinguished essentially from zero.

The phase ζ_{sc} is stationary, if the following relations are fulfilled

$$\mathbf{x} = \mathbf{X} + \frac{\varepsilon_{k''}(\mathbf{r}+\mathbf{q}-\mathbf{p}'')}{E(\mathbf{r}+\mathbf{q}-\mathbf{p}'')}(t-T), \qquad t > T \tag{6.21}$$

$$\mathbf{X} = \mathbf{y} + \mathbf{v}_1 T = \mathbf{z} + \mathbf{v}_2 T \tag{6.22}$$

$$\omega(R,Q) - \varepsilon_{k''}E(\mathbf{r}+\mathbf{q}-\mathbf{p}'') - \varepsilon_{p''}E(\mathbf{p}'') = 0 \tag{6.23}$$

Conditions (6.21), (6.22) can be fulfilled only in that case, when the wave packets, moving according to Eq. (6.20), pass one through another, and their centres coincide at the time moment T at the point $\mathbf{X}$. If it takes place, then the "scattered charges" arise. They move with the velocity

$$\mathbf{v}_{\rm sc} = \frac{\varepsilon_{k''}(\mathbf{r}+\mathbf{q}-\mathbf{p}'')}{E(r+q-p'')} \tag{6.24}$$

in the direction from the collision point $\mathbf{X}$.

The admissible values $\mathbf{p}''$ are determined from Eq.(6.23). For $\varepsilon_r = \varepsilon_q$ Eq. (6.23) is the energy conservation law. In this case the scattering is elastic. Indeed, in the coordinate system, where $\mathbf{q}+\mathbf{r}=0$, it follows from Eq. (6.23)

$$\varepsilon_{k''} = \varepsilon_{p''} = \varepsilon_r = \varepsilon_q, \qquad |\mathbf{p}''| = |\mathbf{r}| = |\mathbf{q}| \tag{6.25}$$

$$\mathbf{v}_{\rm sc} = \varepsilon_{p''}\frac{\mathbf{p}''}{E(\mathbf{p}'')} \qquad |\mathbf{v}_{\rm sc}| = |\mathbf{v}_1| = |\mathbf{v}_2| \tag{6.26}$$

In the case $\varepsilon_r = -\varepsilon_q$, when a particle collides with an antiparticle, the scattering is absent entirely. Indeed, $\langle j^0(\mathbf{x},t)\rangle_{\Phi_2}$ can be separated into three parts

$$\langle j^0(\mathbf{x},t)\rangle_{\Phi_2} = \langle j^0(\mathbf{x},t)\rangle_{\rm fr} + \langle j^0(\mathbf{x},t)\rangle_{\rm sc} + \langle j^0(\mathbf{x},t)\rangle_{-} \tag{6.27}$$

where

$$\langle j^0(\mathbf{x},t)\rangle_{\rm fr} = \frac{e}{(2\pi)^n}\sum_{\varepsilon_{k''},\varepsilon_{k'}=\pm1}\int \varepsilon_{k'}\Psi_{\rm fr}^{(+)}(\mathbf{x},t;\varepsilon_{k''},P'')\Psi_{\rm fr}^{(-)}(\mathbf{x},t;\varepsilon_{k'},P'')dP'' + ({\rm c.c}) \tag{6.28}$$

$$\langle j^0(\mathbf{x},t)\rangle_{\rm sc} = \frac{e}{(2\pi)^n}\sum_{\varepsilon_{k''},\varepsilon_{k'}=\pm1}\int \varepsilon_{k'}\Psi_{\rm sc}^{(+)}(\mathbf{x},t;\varepsilon_{k''},P'')\Psi_{\rm sc}^{(-)}(\mathbf{x},t;\varepsilon_{k'},P'')dP'' + ({\rm c.c}) \tag{6.29}$$

$$\langle j^0(\mathbf{x},t)\rangle_- = \frac{e}{(2\pi)^n} \sum_{\varepsilon_{k''},\varepsilon_{k'}=\pm 1} \int \Psi_{\rm fr}^{(+)}(\mathbf{x},t;\varepsilon_{k''},P')(\varepsilon_{k'}+\varepsilon_{k''})$$
$$\Psi_{\rm sc}^{(-)}(\mathbf{x},t;\varepsilon_{k'},P'')dP'' + ({\rm c.c}) \tag{6.30}$$

Here $\langle j^0\rangle_{\rm fr}$, $\langle j^0\rangle_{\rm sc}$ and $\langle j^0\rangle_-$ are respectively the charge of noninteracting wave packets, scattering charge and the charge escaping from the wave packets as a result of the scattering.

Due to the factor $\varepsilon_{k'}$ in Eq. (6.29) the scattered charge can vanish even in the case, when Ψ_{sc} does not vanish. But in the case $\varepsilon_r = -\varepsilon_q$ the charge

$$Q_-(t) = \int \langle j^0(\mathbf{x},t)\rangle_- d\mathbf{x}, \tag{6.31}$$

escaping from each of the wave packets taken separately, vanishes. It means that in the two-WL case *the particle does not interact with antiparticle.*

Let us present results of calculations produced by means of the stationary phase technique.

$$\langle j^0(\mathbf{x},t)\rangle_{\rm fr} = \frac{e}{(\pi a^2)^{n/2}}\{\varepsilon_r \exp[-(x-y-v_1t)^2/a^2]+\varepsilon_q \exp[-(x-z-v_2t)^2/a^2]\} \tag{6.32}$$

$$Q_{\rm sc}(t) = -Q_-(t) = -\frac{e\lambda\Delta_1(R,Q)\varepsilon_q}{(2\pi a^2)^{(n-1)/2}E(\mathbf{r})E(\mathbf{q})\sqrt{(\mathbf{v}_1-\mathbf{v}_1)^2}|\Delta_+(R,Q)|^2} \tag{6.33}$$

where

$$\Delta_1(R,Q) = \frac{\Delta_+(R,Q)-\Delta_-(R,Q)}{2i} \tag{6.34}$$

Calculation of the integral (6.2) leads to the result

$$J = (2\pi a^2)^{(n-1)/2}\sqrt{1-[(\mathbf{v}_1,\mathbf{v}_2)^2-\mathbf{v}_1^2\mathbf{v}_2^2](\mathbf{v}_1-\mathbf{v}_2)^{-2}} \tag{6.35}$$

By means of Eqs. (6.1), (6.3), (6.35) one obtains for the total scattering section

$$\sigma_{\rm tot} = \frac{|Q_{\rm sc}|}{J} = \frac{\lambda|\Delta_1(R,Q)|}{2|\Delta_+(R,Q)|\sqrt{(r_iq^i)^2-m^2}}, \tag{6.36}$$

$$r^i = \{\varepsilon_r E(\mathbf{r}),\mathbf{r}\}, \qquad q^i = \{\varepsilon_q E(\mathbf{q}),\mathbf{q}\} \tag{6.37}$$

Using Eqs.(4.49), (4.42), one obtains in the case $n=2$

$$\sigma_{\rm tot} = \frac{\lambda^2\delta_{\varepsilon_r,\varepsilon_q}}{8\gamma\sqrt{2(\gamma^2-4m^2)}[(1-\frac{\lambda}{8\pi\gamma}\log|\frac{2m+\gamma}{2m-\gamma}|)^2+\frac{\lambda^2}{64\gamma^2}]},$$

$$\gamma^2 = (r^i+q^i)(r_i+q_i).$$

In the case $n=3$ Δ_1 is finite and Δ_+ diverges. Then $\sigma_{\rm tot}=0$, and there is no scattering.

The result (6.36) can be obtained also in the scope of conventional S-matrix approach. Let us identify the operator $S^{(-)}$ defined by Eq. (5.38) with S-matrix. The elements of S-matrix have the form

$$_+\langle P,K|S^{(-)}|K',P'\rangle_+ \equiv_+ \langle u,\mathbf{s}|S^{(-)}|\mathbf{s}',u'\rangle_+ =$$

$$\delta(\mathbf{s}-\mathbf{s}')\{\frac{1}{2}[\delta(u-u')+\delta(u+u')]-2\pi iT(\mathbf{s},u,u')\delta[\omega_1(u,\mathbf{s})-\omega_1(u',\mathbf{s})]\} \tag{6.38}$$

where according to Eqs. (5.39), (4.29), (4.36)

$$T(\mathbf{s},u,u')=\frac{\lambda}{(2\pi)^n}\frac{(\varepsilon_k+\varepsilon_p)\xi_2^2(u,\mathbf{s})}{2\Delta_-(w)} \tag{6.39}$$

According to optical theorem [9] (cpt 5, f-la (39))

$$\sigma_{\mathrm{tot}}=-\frac{2}{v}(2\pi)^n\mathrm{Im}(T(\mathbf{s},u,u)) \tag{6.40}$$

where v is the relative velocity of colliding particles. Substituting Eq.(6.39) into Eq.(6.40) and using Eq.(6.34), one obtains the result (6.36).

Thus, without interaction cut-off one has founded a use of the S-matrix. But one should bear in mind that the scattering model obtained is *a result of the quantization model, where the charge density operator is chosen in the form* (6.9). If the charge density operator has another form, then one has, generally, another scattering model even at the same quantization scheme.

7 Operators of Physical Quantities

Among physical quantities

$$E(t)=\int E(k)b^*(K,t)b(K,t)dK$$

$$+\frac{\lambda}{2(2\pi)^n}\int\int\int\frac{b^*(K,t)b^*(P,t)b(P',t)b(K',t)}{\sqrt{\beta(\mathbf{k})\beta(\mathbf{p})\beta(\mathbf{k}')\beta(\mathbf{p}')}}\delta(\mathbf{k}+\mathbf{p}-\mathbf{k}'-\mathbf{p}')dKdPdK'dP' \tag{7.1}$$

$$P_\alpha(t)=\int\varepsilon_k k_\alpha b^*(K,t)b(K,t)dK,\qquad \alpha=1,2,3 \tag{7.2}$$

$$Q(t)=\int\varepsilon_k b^*(K,t)b(K,t)dK \tag{7.3}$$

only the charge $Q(t)$ does not depend on t due to dynamic equation (4.3).

Really, calculating $\partial E/\partial t$ and using Eq. (4.3), one obtains

$$\left.\frac{\partial E}{\partial t}\right|_{t=0} = \frac{i\lambda}{2(2\pi)^n}\int\int\int\int\frac{\varepsilon_kE(\mathbf{k})-\varepsilon_pE(\mathbf{p})+\varepsilon_{k'}E(\mathbf{k}')-\varepsilon_{p'}E(\mathbf{p}')}{\sqrt{\beta(\mathbf{k})\beta(\mathbf{p})\beta(\mathbf{k}')\beta(\mathbf{k}')}}$$
$$\times b^*(K)b^*(P)b(K')b(P')\delta(\mathbf{k}+\mathbf{p}-\mathbf{k}'-\mathbf{p}')dKdPdK'dP'+O_{3,3} \tag{7.4}$$

$\partial E/\partial t$ vanishes, if operators $b(P')$ and $b(K')$ commute. In the classic case. when $b(K)$ is a c-number, $\partial E/\partial t = 0$ and E conserves. At the commutation relations (5.36), (5.37) $\partial E/\partial t \neq 0$, and $E(t)$ is not a conservative quantity. The matrix elements of the type ${}_+\langle P, K|\partial E/\partial t|K', P'\rangle_+$ diverge.

The operator $E(0)$ can be written in the form

$$E(0) = \int E(\mathbf{k})b_0^*(K)b_0(K)dK$$
$$+\int\int\{E(\mathbf{k})\mu_{w'}^{(-)}(w) + E(\mathbf{k}')\mu_w^{(+)}(w') + C_1(w,w')$$
$$-\frac{\lambda\eta_2(w)\eta_2(w')}{2(2\pi)^n\Delta_+(w)\Delta_-(w')}\delta(\mathbf{k}+\mathbf{p}-\mathbf{k}'-\mathbf{p}')\}b_0^*(K')b_0^*(P')b_0(K)b_0(P)dwdw'$$
$$+\int\int\{E(\mathbf{k})\mu_S(w) + C_2(w,S) + \frac{\eta_2(w)\delta(\mathbf{s}-\mathbf{k}-\mathbf{p})}{\Delta_-(w)\sqrt{2(2\pi)^n\beta(\mathbf{s})B}}\}b_0^*(K)b_0^*(P)b_1(S)dwdS$$
$$+(\text{h.c.}) + \int\int\{C_3(S,S') - \frac{\lambda\delta(\mathbf{s}-\mathbf{s}')}{2(2\pi)^n\beta_1(\mathbf{s})\,|B|}\}b_1^*(S)b_1(S')dSdS' + O_{3,3} \qquad (7.5)$$

where

$$C_1(w,w') = \int \mu_{w'}^{(-)}(w'')\mu_w^{(+)}(w'')E(\mathbf{k}'')dw''$$
$$C_2(w,S) = \int \mu_w^{(-)}(w'')\mu_S(w'')E(\mathbf{k}'')dw'' \qquad (7.6)$$
$$C_3(S,S') = \int \mu_S(w'')\mu_{S'}(w'')E(\mathbf{k}'')dw''$$

In order that $E(t)$ does not depend on t, it is sufficient to restrict the region of integration in the integrals (7.6), adding a factor $\delta_{\varepsilon_{k''},\varepsilon_{p''}}$ in each integrand. After such correction the expression (7.5) takes the form

$$E(t) = E(0) = \sum_{A=0,1}\int E_A(\mathbf{k})b_A^*(K)b_A(K)dK + O_{3,3} \qquad (7.7)$$

It is worth to note that addition of the factor $\delta_{\varepsilon_{k''},\varepsilon_{p''}}$ annihilates matrix elements between Hilbert spaces $\mathcal{H}_2$ and $\mathcal{H}_2\backslash\mathcal{H}_{2,s}$. It prevents from passing from $\mathcal{H}_{2,s}$ into $\mathcal{H}_2$. A like correction should be made in the expansion of $P_\alpha(t)$ over operators $b_A^*(K), b_A(K')$. After this correction $P_\alpha(t)$ takes the form

$$P_\alpha(t) = P_\alpha(0) = \int \varepsilon_k k_\alpha b_A^*(K)b_A(K)dK + O_{3,3}, \qquad \alpha = 1,2,3 \qquad (7.8)$$

A like restriction of the integration region should be made in the expansion of the energy-momentum tensor

$$T^{ik} = \varphi^{*i}\varphi^k + \varphi^{*k}\varphi^i - g^{ik}L \qquad (7.9)$$

over operators $b_A^*(K), b_A(K')$. Then it becomes to satisfy the conservation law

$$\partial_k T^{ik} = O_{3,3} \qquad (7.10)$$

8 Concluding Remarks

The problem of pair production is the principal problem of QFT. The relations obtained in $\mathcal{H}_{2,s}$ are exact. They are applicable at any energies of colliding particles. But nonlinear model (4.1), (4.3) of scalar field does not describe pair production. It does not describe even particle-antiparticle scattering. This circumstance excites a feeling of dissatisfaction.

Such a dissatisfaction is based to an extent on the way of thinking connected with the perturbation theory. In the relativistic QFT the dynamics of all nonlinear models is investigated by the perturbation theory technique. Nonlinear term generates a set of diagrams which describe both creation and annihilation of particles. According to this approach *each nonlinear interaction has to lead to pair production, if* energies of colliding particles are large enough.

But investigation of the classic description of the pair production leads to another conclusion. Let us return to the action (1.6) describing behavior of WL in some external field $f(q)$ which produces pairs. (For simplicity the gravitational field and electromagnetic one are absent). The Jacobi-Hamilton equation for the action (1.6) has the form

$$\sqrt{f(q)(\frac{\partial S}{\partial q^i}g^{ik}\frac{\partial S}{\partial q^k})} = \alpha b, \qquad b = \text{const} \tag{8.1}$$

Eq.(8.1) at $\alpha > 0$ permits spacelike 4-momentum $p_i = \partial S/\partial q^i$, provided $f(q) < 0$. Hence, it permits the WL turn in time [5]. In the limit $\alpha \to 0$ Eq.(8.1) turns to

$$f(q)(\frac{\partial S}{\partial q^i}g^{ik}\frac{\partial S}{\partial q^k} - m^2c^2) = 0 \tag{8.2}$$

This equation means that the point, where $f(q) = 0$, can be a break point. Remaining timelike, the WL turns in time direction here, i.e. the external field $f(q)$ produces or annihilates a pair at this point. In ref.[4] example of such solutions is constructed for the finite $\alpha > 0$. At finite $\alpha > 0$ there is a vicinity of the turning point, where WL is spacelike. At $\alpha \to +0$ this vicinity degenerates into a break point, and WL is timelike everywhere except for the break point. At this point WL changes its time direction.

It was shown also [4] that the pair production in the external field $f(q)$ takes place in the limit $\alpha \to +0$. In other words, the action (1.6) can describe pair production at infinitesimal $\alpha > 0$.

Let us use this circumstance and expand Eq. (1.6) over the parameter α.

$$S[q] = -\int\limits_{\min(\tau',\tau'')}^{\max(\tau',\tau'')} \left\{ mc\sqrt{\dot{q}^k g_{kl}\dot{q}^l} - \frac{\alpha f(q)}{2mc\sqrt{\dot{q}^k g_{kl}\dot{q}^l}} \right\} d\tau \tag{8.3}$$

In this case the Jacobi-Hamilton equation has the form

$$\frac{\partial S}{\partial q^i}g^{ik}\frac{\partial S}{\partial q^k} = \left[mc + \frac{\alpha b^2}{f(q)} \right]^2, \qquad b = \text{const} \tag{8.4}$$

Here the 4-momentum $p_i = \partial S/\partial q^i$ is always timelike. The WL turn in time is impossible. In the limit $\alpha \to 0$ the field $f(q)$ disappears, and the action (8.3) cannot describe pair production.

The action (1.6) permits spacelike $\dot{q}^i$ (at $\alpha f(q) < 0$), but the action (8.3) does not permit them in principle. It is a distinction between the two actions. This example shows that a special interaction is needed for pair production. It is possible that the self-action of the model (4.1), (4.2) has not this property.

It seems rather improbable that pair production can appear in the model (4.1) at consideration of situations in $\mathcal{H}_3$ and $\mathcal{H}_4$ where three and four WLs are considered, because the nonlinear term of the model associates with the expansion (8.3) over powers of the interaction constant.

But why does consideration of the model (4.1), using conventional quantization [11] - [14], describe pair production? The answer is very simple. Because the conventional PA-scheme of quantization is inconsistent. When one uses an inconsistent conception, then, exhibiting enough ingenuity, one may obtain any desired result. At the conventional quantization according to PA-scheme the total world line is separated into fragments (particles and antiparticles). But one fails to join these fragments in proper way. Some portion of fragments remains disconnected. They imitate pair production.

The most unexpected feature of the quantization in terms of WLs is different simultaneous commutation relations for the free field and for the self-acting field, (i.e. the simultaneous commutation relations depend on λ). As a whole this conception *does not contain any visible inconsistencies.*

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