Statement of problem on vortical inviscid flow of barotropic and incompressible fluids.

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Abstract

The question what information is necessary for determination of a unique solution of hydrodynamic equations for ideal fluid is investigated. Arbitrary inviscid flows of the barotropic fluid and of incompressible fluid are considered. After integrating hydrodynamic equations, all information on the fluid flow is concentrated in dynamic equations in the form of indefinite functions, whereas the initial and boundary conditions contain information on the fluid particle labeling. It is shown that for determination of the stationary flow of the incompressible fluid the vorticity on any stream line must be given. Giving the velocity on the boundary, one does not determine the vorticity, in general. If there are closed stream lines, the vorticity cannot be given on them via boundary conditions. This circumstance explains existence of different stationary vortical flows under the same boundary conditions.

Key words: ideal fluid, Clebsch potentials, vorticity

1 Introduction

In the present paper we consider the statement of the flow problem of barotropic fluid. If compressibility of the fluid tends to zero, we obtain an incompressible fluid. We consider the incompressible fluid as a special case of the barotropic fluid, when its compressibility tends to zero. In the passage to limit the dynamic equations, describing evolution of density ρ and that of the velocity potential φ, lose temporal derivatives and turn into constraints on the state of the incompressible fluid.

$$\partial_0\rho+\boldsymbol{\nabla}(\rho\mathbf{v})=0\quad\rightarrow\quad\boldsymbol{\nabla}\mathbf{v}=0\tag{1.1}$$

As a result the statement of the flow problem appears to be different for barotropic fluid and for the incompressible one. On one hand, the description of the incompressible fluid is simpler, than that of barotropic one. On the other hand, the incompressible fluid is a nonphysical fluid, because the speed of sound is infinite, and constraints on the state of the incompressible fluid appear to be nonphysical constraints. As a result the statement of the flow problem for the incompressible fluid appears to be complicated, than for the barotropic one.

Nonstationary flows are too difficult for calculations, and as a rule one considers stationary flows, which do not contain temporal derivatives. This fact complicates statement of the flow problem, because the problem cannot be considered to be an evolutional problem. Finally, the rotational stationary flows are too difficult for calculation also, and one considers usually stationary irrotational flows of the incompressible fluid. Statement of the problem for rotational flows of the incompressible fluid and for the irrotational ones appear to be quite different. In particular, the stationary irrotational flow of incompressible fluid is determined uniquely by the boundary conditions. The rotational stationary flow may contain stream lines, which do not cross the boundaries, and one cannot set the flow problem, using only boundary conditions.

We consider the statement of the flow problem, starting from the simple case of the arbitrary flow of the barotropic fluid, when the statement of the problem is very simple. Imposing in series the constraints of incompressibility and of stationary, we follow the evolution of the flow problem statement.

In our investigation we use essentially the fact that dynamic equations for the ideal barotropic fluid can be integrated [1]. Indefinite functions appear in dynanic equations as a result of this integration. As a rule, the investigation of integrals of differential equations is simpler and more effective, than the investigation of differential equations themselves, and we use this circumstance. Integration of hydrodynamic equations is connected closely with a use of generalized stream function (GSF) [2] and with GSF-technique which allows one to realize this integration.

We obtain hydrodynamic equations for barotropic fluid from the variational principle, which can be written in the form [2]

$$\mathcal{A}_{\mathrm{E}}[\rho,\mathbf{j},\boldsymbol{\xi},p]=\int\left\{\frac{\mathbf{j}^2}{2\rho}-\rho E-p_k\left(j^k-\rho_0\left(\boldsymbol{\xi}\right)\frac{\partial J}{\partial \xi_{0,k}}\right)\right\}dtd\mathbf{x} \tag{1.2}$$

where $j^k=\{j^0,\mathbf{j}\}=\{\rho,\rho\mathbf{v}\}$ is the 4-flux of the fluid, ρ is the density and $\mathbf{v}$ is its velocity. The quantity $E=E\left(\rho\right)$ is the fluid internal energy per unit mass, which depends only on the density ρ. The quantity $\rho_0=\rho_0\left(\boldsymbol{\xi}\right)$ is a given weight function of $\boldsymbol{\xi}$. Variables $\boldsymbol{\xi}=\{\xi_1,\xi_2,\xi_3\}$ are Lagrangian coordinates, labeling the fluid particles. Usually the Lagrangian coordinates are considered to be independent variables. Here they are considered to be dependent variables $\boldsymbol{\xi}=\boldsymbol{\xi}\left\{t,\mathbf{x}\right\}$. Considering $\boldsymbol{\xi}$ as dependent variables, we shall refer to them as Clebsch potentials. These potentials have been used by Clebsch for description of the incompressible fluid [3, 4]. In (1.2) and in what follows a summation over repeated Latin indices is produced $(0-3)$. All dependent dynamic variables j, $\boldsymbol{\xi}$, p are considered to be functions of $x=\{x^0,\mathbf{x}\}=\{t,\mathbf{x}\}$.

The quantities $\partial J/\partial \xi_{0,k}$, $k=0,1,2,3$ are derivatives of the Jacobian

$$J\equiv \frac{\partial (\xi_0,\xi_1,\xi_2,\xi_3)}{\partial (x^0,x^1,x^2,x^3)}\equiv \det ||\xi_{i,k}||\,, \qquad \xi_{i,k}\equiv \partial_k\xi_i\equiv \frac{\partial \xi_i}{\partial x^k}, \qquad i,k=0,1,2,3 \quad (1.3)$$

with respect to variables $\xi_{0,k}\equiv \partial_k\xi_0$. Here $\xi=\{\xi_0,\boldsymbol{\xi}\}=\{\xi_0,\xi_1,\xi_2,\xi_3\}$ are four scalars considered to be functions of $x=\{x^0,\mathbf{x}\}=\{t,\mathbf{x}\}$, $\xi=\xi(x)$. The functions $\{\xi_0,\xi_1,\xi_2,\xi_3\}$ are supposed to be independent in the sense that $J\neq 0$. It is useful to consider the Jacobian J as 4-linear function of variables $\xi_{i,k}\equiv \partial_k\xi_i$, $i,k=0,1,2,3$. Then one can introduce derivatives of J with respect to $\xi_{i,k}$. The derivative $\partial J/\partial \xi_{i,k}$ appears as a result of a replacement of ξ_i by x^k in the relation (1.3).

$$\frac{\partial J}{\partial \xi_{i,k}}\equiv \frac{\partial (\xi_0,...\xi_{i-1},x^k,\xi_{i+1},...\xi_3)}{\partial (x^0,x^1,x^2,x^3)}, \qquad i,k=0,1,2,3 \qquad (1.4)$$

Variables $\boldsymbol{\xi}=\{\xi_1,\xi_2,\xi_3\}$ are spatial Lagrangian coordinates of the fluid particles, whereas ξ_0 is the temporal Lagrangian coordinate. It is fictitious in the action (1.2). The quantities $\partial J/\partial \xi_{i,k}$ are useful, because they satisfy identically to the relations

$$\partial_k\frac{\partial J}{\partial \xi_{i,k}}\equiv 0, \qquad \frac{\partial J}{\partial \xi_{k,i}}\xi_{l,i}\equiv J\delta_l^k, \qquad \frac{\partial J}{\partial \xi_{i,k}}\xi_{i,l}\equiv J\delta_l^k, \qquad l,k=0,1,2,3 \qquad (1.5)$$

Identifying the fluid 4-flux j^k with the quantity $\rho_0(\boldsymbol{\xi})\,\partial J/\partial \xi_{0,k}$

$$j^k=\rho_0(\boldsymbol{\xi})\,\frac{\partial J}{\partial \xi_{0,k}}, \qquad k=0,1,2,3, \qquad (1.6)$$

we obtain from two first equations (1.5) that the 4-flux j^k satisfies the continuity equation

$$\partial_k j^k=\partial_k\left(\rho_0(\boldsymbol{\xi})\,\frac{\partial J}{\partial \xi_{0,k}}\right)=\rho_0(\boldsymbol{\xi})\,\partial_k\frac{\partial J}{\partial \xi_{0,k}}+\frac{\partial \rho_0(\boldsymbol{\xi})}{\partial \xi_\alpha}\xi_{\alpha,k}\frac{\partial J}{\partial \xi_{0,k}}\equiv 0 \qquad (1.7)$$

Here and in what follows a summation over two repeated Greek indices is produced $(1-3)$. It follows from the second identity (1.5) that the quantities $\boldsymbol{\xi}$ are labels of the fluid particles, and their substantial derivatives vanish

$$\left(\frac{\partial J}{\partial \xi_{0,0}}\right)^{-1}\frac{\partial J}{\partial \xi_{0,k}}\partial_k\xi_\alpha=\frac{j^k}{\rho}\partial_k\xi_\alpha=(\partial_0\xi_\alpha+\mathbf{v}\boldsymbol{\nabla}\xi_\alpha)=0, \qquad \alpha=1,2,3 \qquad (1.8)$$

A use of designation (1.6) is very useful, and we have introduced this designation in the variational principle (1.2) by means of the Lagrange multipliers $p_k=p_k(x)$, $k=0,1,2,3$.

To obtain hydrodynamic equations we should vary the action (1.2) with respect to variables ξ_k,j^k,p_k, $\;k=0,1,2,3$. The variable ξ_0 is fictitious, and a variation with respect to ξ_0 gives identity.

Dynamic equations have the form

$$\frac{\delta\mathcal{A}}{\delta\xi_i}=-\partial_l\left(\rho_0\left(\boldsymbol{\xi}\right)p_k\frac{\partial^2 J}{\partial\xi_{0,k}\partial\xi_{i,l}}\right)+\frac{\partial\rho_0\left(\boldsymbol{\xi}\right)}{\partial\xi_i}p_k\frac{\partial J}{\partial\xi_{0,k}}=0,\qquad i=0,1,2,3 \tag{1.9}$$

As far as the variable ξ_0 is fictitious, dynamic equation (1.9) with $i=0$ is to be an identity in force of other dynamic equations. Another dynamic equations have the form

$$\frac{\delta\mathcal{A}}{\delta j^\alpha}=\frac{j^\alpha}{\rho}-p_\alpha=0,\qquad \alpha=1,2,3 \tag{1.10}$$

$$\frac{\delta\mathcal{A}}{\delta\rho}=-\frac{\mathbf{j}^2}{2\rho^2}-\frac{\partial\left(\rho E\right)}{\partial\rho}-p_0=0 \tag{1.11}$$

$$\frac{\delta\mathcal{A}}{\delta p_k}=-j^k+\rho_0\left(\boldsymbol{\xi}\right)\frac{\partial J}{\partial\xi_{0,k}}=0,\qquad k=0,1,2,3 \tag{1.12}$$

Let us transform (1.9), using identities

$$\partial_l\frac{\partial^2 J}{\partial\xi_{0,k}\partial\xi_{i,l}}\equiv 0,\qquad \frac{\partial^2 J}{\partial\xi_{0,k}\partial\xi_{i,l}}\equiv J^{-1}\left(\frac{\partial J}{\partial\xi_{0,k}}\frac{\partial J}{\partial\xi_{i,l}}-\frac{\partial J}{\partial\xi_{0,l}}\frac{\partial J}{\partial\xi_{i,k}}\right),\qquad i,k,l=0,1,2,3, \tag{1.13}$$

By means of the first identity (1.13) the equations (1.9) can be written in the form

$$-\frac{\partial^2 J}{\partial\xi_{0,k}\partial\xi_{i,l}}\rho_0\left(\boldsymbol{\xi}\right)\partial_l p_k-\frac{\partial^2 J}{\partial\xi_{0,k}\partial\xi_{i,l}}p_k\partial_l\rho_0\left(\boldsymbol{\xi}\right)+\frac{\partial\rho_0\left(\boldsymbol{\xi}\right)}{\partial\xi_i}p_k\frac{\partial J}{\partial\xi_{0,k}}=0,\qquad i=0,1,2,3 \tag{1.14}$$

Two last terms of (1.14) compensate each other. Indeed, using the second identity (1.13) we rewrite two last terms of (1.14) in the form

$$-J^{-1}\left(\frac{\partial J}{\partial\xi_{0,k}}\frac{\partial J}{\partial\xi_{i,l}}-\frac{\partial J}{\partial\xi_{0,l}}\frac{\partial J}{\partial\xi_{i,k}}\right)p_k\frac{\partial\rho_0\left(\boldsymbol{\xi}\right)}{\partial\xi_\beta}\xi_{\beta,l}+\frac{\partial\rho_0\left(\boldsymbol{\xi}\right)}{\partial\xi_i}p_k\frac{\partial J}{\partial\xi_{0,k}},\qquad i=0,1,2,3 \tag{1.15}$$

In (1.15) and in what follows a summation over two repeated Greek indices is produced $(1-3)$. Using the second identity (1.5), the expression (1.15) is transformed to the form

$$-\frac{\partial J}{\partial\xi_{0,k}}p_k\frac{\partial\rho_0\left(\boldsymbol{\xi}\right)}{\partial\xi_\beta}\delta_\beta^i+\frac{\partial\rho_0\left(\boldsymbol{\xi}\right)}{\partial\xi_i}p_k\frac{\partial J}{\partial\xi_{0,k}}=0,\qquad i=0,1,2,3 \tag{1.16}$$

where two terms are compensated for $i=\beta=1,2,3$. For $i=0$ the first term of (1.16) vanishes because of the multiplier δ_β^i, whereas the second term vanishes because $\partial\rho_0\left(\boldsymbol{\xi}\right)/\partial\xi_0=0$.

Thus, two last terms of (1.14) vanish, and using the second identity (1.13), the equation (1.14) takes the form

$$-J^{-1}\left(\frac{\partial J}{\partial\xi_{0,k}}\frac{\partial J}{\partial\xi_{i,l}}-\frac{\partial J}{\partial\xi_{0,l}}\frac{\partial J}{\partial\xi_{i,k}}\right)\rho_0\left(\boldsymbol{\xi}\right)p_{k,l}=0,\qquad i=0,1,2,3 \tag{1.17}$$

Let us convolve (1.17) with $\xi_{i,s}$. Using the last identity (1.5) and the equation (1.12), we obtain from (1.17)

$$\frac{\partial J}{\partial \xi_{0,k}}\left(p_{k,s}-p_{s,k}\right)=0, \qquad s=0,1,2,3, \qquad p_{k,s}\equiv\partial_s p_k \tag{1.18}$$

It follows from (1.10) - (1.12) that

$$\frac{\partial J}{\partial \xi_{0,0}}=\frac{\rho}{\rho_0\left(\boldsymbol{\xi}\right)}, \qquad p_0=-\frac{\mathbf{v}^2}{2}-\frac{\partial\left(\rho E\right)}{\partial\rho}, \qquad p_\alpha=v^\alpha, \qquad \frac{\partial J}{\partial \xi_{0,\alpha}}=\frac{\rho v^\alpha}{\rho_0\left(\boldsymbol{\xi}\right)}, \qquad \alpha=1,2,3 \tag{1.19}$$

Substituting (1.19) in (1.18), we obtain after transformations for $s=\beta=1,2,3$

$$v^\alpha{}_{,0}+v^\beta v^\alpha{}_{,\beta}=-\partial_\alpha\frac{\partial\left(\rho E\right)}{\partial\rho}=-\frac{1}{\rho}\partial_\alpha\left(\rho^2\frac{\partial E}{\partial\rho}\right), \qquad \beta=1,2,3 \tag{1.20}$$

and for $s=0$

$$v^\beta v^\beta{}_{,0}+v^\beta\partial_\beta\left(\frac{\mathbf{v}^2}{2}+\frac{\partial\left(\rho E\right)}{\partial\rho}\right)=0 \tag{1.21}$$

Here comma before index k means differentiation with respect to x^k. It is easy to see that (1.21) is a result of convolution of (1.20) with v^α. It is connected with that the equation (1.21) appeared as a result of variation with respect to fictitious variable ξ_0.

Equations (1.7), (1.20) and (1.8) form the complete system of dynamic equations, which consists of seven first order differential equations for seven dependent variables $\rho,\mathbf{v},\boldsymbol{\xi}$. This system may be written in the vector form

$$\partial_0\rho+\boldsymbol{\nabla}\left(\rho\mathbf{v}\right)=0 \tag{1.22}$$

$$\partial_0\mathbf{v}+\left(\mathbf{v}\boldsymbol{\nabla}\right)\mathbf{v}=-\frac{\boldsymbol{\nabla}p}{\rho}, \qquad p=\rho^2\frac{\partial E}{\partial\rho} \tag{1.23}$$

$$\partial_0\boldsymbol{\xi}+\left(\mathbf{v}\boldsymbol{\nabla}\right)\boldsymbol{\xi}=0 \tag{1.24}$$

Four equations (1.22), (1.23) form a closed subsystem (Euler equations) of dynamic equations. These equations can be solved independently of dynamic equations (1.24), which describe labeling of the fluid particles and the character of the fluid particle motion along its trajectory.

Indeed, if three quantities $\xi_1\left(t,\mathbf{x}\right)$, $\xi_2\left(t,\mathbf{x}\right)$, $\xi_3\left(t,\mathbf{x}\right)$ are three independent solution of equations (1.24) known as Lin constraints [5]. They are three independent integrals of the system of ordinary dynamic equations

$$\frac{d\mathbf{x}}{dt}=\mathbf{v}\left(t,x\right) \tag{1.25}$$

If three independent integrals $\xi_1\left(t,\mathbf{x}\right)$, $\xi_2\left(t,\mathbf{x}\right)$, $\xi_3\left(t,\mathbf{x}\right)$ of the system (1.5) are known, the world lines (trajectories) of the fluid particles $\mathbf{x}=\mathbf{x}\left(t,\boldsymbol{\xi}_{\rm in}\right)$ are determined implicitly by the algebraic equations

$$\xi_\alpha\left(t,\mathbf{x}\right)=\left(\boldsymbol{\xi}_{\rm in}\right)_\alpha=\text{const}, \qquad \alpha=1,2,3 \tag{1.26}$$

Three quantities $\boldsymbol{\xi}_{\rm in}=\boldsymbol{\xi}$ label the fluid particles and their world lines.

2 Integration of dynamic equations for barotropic fluid

There exists another form of hydrodynamic equations. The fact is that the equations (1.18) can be integrated. Note that equations (1.18) are linear partial differential equations for the variables p_k, $k = 0, 1, 2, 3$. They can be solved exactly in the form

$$p_k = g^0 \left(\xi_0\right) \xi_{0,k} + g^\alpha \left(\boldsymbol{\xi}\right) \xi_{\alpha,k}, \qquad k = 0, 1, 2, 3 \tag{2.1}$$

where ξ_0 ceases to be fictitious and becomes to be a new dynamic variable. The quantities g^0 and g^α, $\alpha = 1, 2, 3$ are indefinite functions of arguments ξ_0 and $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}$ respectively.

Substituting (2.1) in (1.18) and using identities (1.5), one can verify that (2.1) is a solution of equations (1.18) for any functions g^i, $i = 0, 1, 2, 3$. It means that expression (2.1) gives the general solution of (1.18). Taking into account that the first term in rhs of (2.1) is a gradient of some quantity φ, we may write (2.1) in the form

$$p_k = \partial_k \varphi + g^\alpha \left(\boldsymbol{\xi}\right) \xi_{\alpha,k}, \qquad k = 0, 1, 2, 3 \tag{2.2}$$

The first equation (1.19) takes the form

$$\rho = \rho_0 \left(\boldsymbol{\xi}\right) \frac{\partial J}{\partial \xi_{0,0}} \equiv \rho_0 \left(\boldsymbol{\xi}\right) \frac{\partial \left(\xi_1, \xi_2, \xi_3\right)}{\partial \left(x, y, z\right)} \equiv \rho_0 \left(\boldsymbol{\xi}\right) \frac{\partial \left(\xi_1, \xi_2, \xi_3\right)}{\partial \left(x^1, x^2, x^3\right)} \tag{2.3}$$

It follows from (2.2) and (1.19)

$$v^\mu = \partial_\mu \varphi + g^\alpha \left(\boldsymbol{\xi}\right) \xi_{\alpha,\mu}, \qquad \mu = 1, 2, 3 \tag{2.4}$$

Then equations (1.24) are transformed to the form

$$\partial_0 \xi_\mu + \left(\boldsymbol{\nabla}\varphi + g^\alpha \left(\boldsymbol{\xi}\right) \boldsymbol{\nabla}\xi_\alpha\right) \boldsymbol{\nabla}\xi_\mu = 0, \qquad \mu = 1, 2, 3 \tag{2.5}$$

Let us set $k = 0$ in (2.2). Eliminating $\xi_{\alpha,0}$ by means of (2.5), we obtain

$$\partial_0 \varphi - g^\alpha \left(\boldsymbol{\xi}\right) \left(\boldsymbol{\nabla}\varphi + g^\beta \left(\boldsymbol{\xi}\right) \boldsymbol{\nabla}\xi_\beta\right) \boldsymbol{\nabla}\xi_\alpha + \frac{1}{2} \left(\boldsymbol{\nabla}\varphi + g^\alpha \left(\boldsymbol{\xi}\right) \boldsymbol{\nabla}\xi_\alpha\right)^2 + P = 0 \tag{2.6}$$

$$P = \left[\frac{\partial \left(\rho E\right)}{\partial \rho}\right]_{\rho = \rho_0 \left(\boldsymbol{\xi}\right) \partial J / \partial \xi_{0,0}}$$

Equation (2.6) can be written in the form

$$\partial_0 \varphi + \frac{1}{2} \left(\boldsymbol{\nabla}\varphi\right)^2 - \frac{1}{2} g^\alpha \left(\boldsymbol{\xi}\right) g^\beta \left(\boldsymbol{\xi}\right) \boldsymbol{\nabla}\xi_\beta \boldsymbol{\nabla}\xi_\alpha + \left[\frac{\partial \left(\rho E\right)}{\partial \rho}\right]_{\rho = \rho_0 \left(\boldsymbol{\xi}\right) \partial J / \partial \xi_{0,0}} = 0 \tag{2.7}$$

which allows one to interpret the variable φ. On one hand, the variable φ is a function of the temporal Lagrange coordinate ξ_0. On the other hand, in the case, when $g^\alpha = 0$ and φ is the velocity potential, the equation (2.7) may be considered to be the Hamilton–Jacobi equation with the Hamilton function

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2}\mathbf{p}^2 + U(t, \mathbf{x}), \qquad U(t, \mathbf{x}) = \left[\frac{\partial(\rho E)}{\partial \rho}\right]_{\rho=\rho_0(\boldsymbol{\xi})\partial J/\partial \xi_{0,0}}$$

In this case the Clebsch potential φ may be regarded as the action variable.

Thus, we have the system of four equations (2.5), (2.6) for four dependent variables $\boldsymbol{\xi}$, φ. If solution of this system (2.5), (2.6) has been obtained, the variables ρ, $\mathbf{v}$, are expressed via this solution by means of relations (2.3), (2.4).

If we are interested in determination of the fluid flow, i.e. in determination of variables ρ, $\mathbf{v}$ as functions of variables $t, \mathbf{x}$, we must solve either four Euler equations (1.22), (1.23) with proper initial and boundary conditions, or four equations (2.5), (2.6) with properly given functions g^α, $\alpha = 1, 2, 3$ and properly given initial and boundary conditions for variables $\boldsymbol{\xi}$, φ.

Before comparative analyses of the two different systems of dynamic equations we consider transition to the case of the incompressible fluid. To pass to the incompressible fluid, we consider the slightly compressible fluid with the internal energy of the form

$$E(\rho) = E_0\left(\frac{\rho}{\rho_1}\right)^{1/\varepsilon}, \qquad E_0, \rho_1 = \text{const}, \qquad \varepsilon \ll 1 \tag{2.8}$$

The incompressible fluid appears in the limit $\varepsilon \to 0$.

Let us substitute (2.8) in (2.6) and resolve the obtained relation with respect to the term, containing the constant E_0. We obtain

$$\left(\frac{1+\varepsilon}{\varepsilon}E_0\right)^{\varepsilon}\frac{\rho}{\rho_1} = \left|\partial_0\varphi + \frac{1}{2}(\boldsymbol{\nabla}\varphi)^2 - \frac{1}{2}g^\alpha(\boldsymbol{\xi})\, g^\beta(\boldsymbol{\xi})\, \boldsymbol{\nabla}\xi_\beta \boldsymbol{\nabla}\xi_\alpha\right|^{\varepsilon} \tag{2.9}$$

In the limit $\varepsilon \to 0$ the equations (2.9) and (2.3) turn respectively into

$$\rho = \rho_1 = \text{const}, \qquad \rho_0(\boldsymbol{\xi})\frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(x^1, x^2, x^3)} = \rho_1 = \text{const} \tag{2.10}$$

Other dynamic equations of the system (2.3) - (2.6) do not depend on ρ.

Conventional procedure of passage to the incompressible fluid in (1.22), (1.23) is an addition of the constraint

$$\rho = \rho_1 = \text{const} \tag{2.11}$$

to the Eulerian equations and elimination of connection between the density ρ and the pressure p. As a result the pressure p in (1.23) appears to be indefinite.

Taking into account (2.11), the Euler system of hydrodynamic equations for the incompressible fluid takes the form

$$\boldsymbol{\nabla}\mathbf{v} = 0, \qquad \partial_0\mathbf{v} + (\mathbf{v}\boldsymbol{\nabla})\mathbf{v} = -\frac{\boldsymbol{\nabla}p}{\rho_0}, \qquad \rho_0 = \text{const} \tag{2.12}$$

The equations (2.3) - (2.6) have the form

$$\frac{\partial J}{\partial \xi_{0,0}} \equiv \frac{\partial \left(\xi_1, \xi_2, \xi_3\right)}{\partial \left(x^1, x^2, x^3\right)} = \frac{\rho_1}{\rho_0 \left(\boldsymbol{\xi}\right)}, \qquad \rho_1 = \text{const} \tag{2.13}$$

$$\partial_0 \xi_\mu + \left(\boldsymbol{\nabla} \varphi + g^\alpha \left(\boldsymbol{\xi}\right) \boldsymbol{\nabla} \xi_\alpha\right) \boldsymbol{\nabla} \xi_\mu = 0, \qquad \mu = 1, 2, 3 \tag{2.14}$$

$$\mathbf{v} = \boldsymbol{\nabla} \varphi + g^\alpha \left(\boldsymbol{\xi}\right) \boldsymbol{\nabla} \xi_\alpha \tag{2.15}$$

In the system of integrated equations the condition

$$\boldsymbol{\nabla} \mathbf{v} = \boldsymbol{\nabla}^2 \varphi + \boldsymbol{\nabla} \left(g^\alpha \left(\boldsymbol{\xi}\right) \boldsymbol{\nabla} \xi_\alpha\right) = 0 \tag{2.16}$$

takes place also, but it is not an independent relation. It is a corollary of dynamic equations (2.13), (2.14), (2.15).

Indeed, resolving (2.14) with respect to $v^\alpha = \partial_\alpha \varphi + g^\beta \left(\boldsymbol{\xi}\right) \xi_{\beta,\alpha}$, we obtain in accordance with (1.5)

$$v^\alpha = \partial_\alpha \varphi + g^\beta \left(\boldsymbol{\xi}\right) \xi_{\beta,\alpha} = \left(\frac{\partial J}{\partial \xi_{0,0}}\right)^{-1} \frac{\partial J}{\partial \xi_{0,\alpha}}, \qquad \alpha = 1, 2, 3 \tag{2.17}$$

Then (2.17) satisfies the relation (2.16), as it follows from (1.7) and (2.13).

Thus, in the case of incompressible fluid we have three evolutional dynamic equations, containing temporal derivatives, and one dynamic equation, which does not contain temporal derivative (the first equation (2.12) and the equation (2.13)). This equation is a constraint, imposed on the state of incompressible fluid.

3 Cauchy problem for barotropic fluid flow in infinite volume

To obtain an unique solution for the barotropic fluid flow in the infinite volume, one should give initial state $\rho, \mathbf{v}$ of the fluid at the time moment $t = 0$.

$$\rho \left(0, \mathbf{x}\right) = \rho_{\text{in}} \left(\mathbf{x}\right), \qquad \mathbf{v} \left(0, \mathbf{x}\right) = \mathbf{v}_{\text{in}} \left(\mathbf{x}\right) \tag{3.1}$$

Evolution of the fluid state $\rho, \mathbf{v}$ is determined by evolutional dynamic equations (2.12).

In the case of the integrated system (2.3) - (2.6) the initial conditions (3.1) are to be given, but these conditions are not sufficient for determination of unique solution of equations (2.3) - (2.6). One needs to give initial values for the Clebsch potentials φ, $\boldsymbol{\xi}$. We choose the simplest initial conditions for the quantities φ, $\boldsymbol{\xi}$

$$\varphi \left(0, \mathbf{x}\right) = \varphi_{\text{in}} \left(\mathbf{x}\right) = 0, \qquad \boldsymbol{\xi} \left(0, \mathbf{x}\right) = \boldsymbol{\xi}_{\text{in}} \left(\mathbf{x}\right) = \mathbf{x} \tag{3.2}$$

Substituting (3.1) and (3.2) in (2.4) we obtain

$$\mathbf{g} \left(\mathbf{x}\right) = \mathbf{v}_{\text{in}} \left(\mathbf{x}\right), \qquad \mathbf{g} \left(\mathbf{x}\right) = \left\{g^1 \left(\mathbf{x}\right), g^2 \left(\mathbf{x}\right), g^3 \left(\mathbf{x}\right)\right\} \tag{3.3}$$

Substituting (3.1) and (3.2) in dynamic equations (2.3) - (2.7), we obtain

$$\rho_0(\boldsymbol{\xi}) = \rho_{\mathrm{in}}(\mathbf{x}) \left(\frac{\partial (\xi_{\mathrm{in}1}, \xi_{\mathrm{in}2}, \xi_{\mathrm{in}3})}{\partial (x, y, z)} \right)^{-1} = \rho_{\mathrm{in}}(\mathbf{x}) = \rho_{\mathrm{in}}(\boldsymbol{\xi}) \tag{3.4}$$

$$\partial_0 \xi_\mu + \left(\boldsymbol{\nabla} \varphi + v_{\mathrm{in}}^\alpha (\boldsymbol{\xi}) \boldsymbol{\nabla} \xi_\alpha \right) \boldsymbol{\nabla} \xi_\mu = 0, \qquad \mu = 1,2,3 \tag{3.5}$$

$$\partial_0 \varphi + \frac{1}{2} (\boldsymbol{\nabla} \varphi)^2 - \frac{1}{2} v_{\mathrm{in}}^\alpha (\boldsymbol{\xi}) v_{\mathrm{in}}^\beta (\boldsymbol{\xi}) \boldsymbol{\nabla} \xi_\beta \boldsymbol{\nabla} \xi_\alpha + \left[\frac{\partial (\rho E)}{\partial \rho} \right]_{\rho = \rho_0(\boldsymbol{\xi}) \partial J / \partial \xi_{0,0}} = 0 \tag{3.6}$$

Then relations (2.3), (2.4) take the form

$$\rho = \rho_{\mathrm{in}}(\boldsymbol{\xi}) \frac{\partial (\xi_1, \xi_2, \xi_3)}{\partial (x, y, z)}, \qquad \mathbf{v} = \boldsymbol{\nabla} \varphi + v_{\mathrm{in}}^\alpha (\boldsymbol{\xi}) \boldsymbol{\nabla} \xi_\alpha \tag{3.7}$$

where φ, $\boldsymbol{\xi}$ are solutions of (3.6), (3.5) with initial conditions (3.2)

Choice of initial conditions for the Clebsch potentials φ, $\boldsymbol{\xi}$ in the form (3.3) is unessential. Variables $\boldsymbol{\xi}$ label the fluid particles, and one can use any single-valued method of labeling. It means that equations (3.5), (3.6) are invariant with respect to relabeling transformation.

$$\xi_\alpha \rightarrow \tilde{\xi}_\alpha = \tilde{\xi}_\alpha (\boldsymbol{\xi}), \qquad v_{\mathrm{in}}^\alpha (\boldsymbol{\xi}) \rightarrow \tilde{v}_{\mathrm{in}}^\alpha \left(\tilde{\boldsymbol{\xi}} \right) = \frac{\partial \xi_\beta}{\partial \tilde{\xi}_\alpha} v_{\mathrm{in}}^\beta (\boldsymbol{\xi}), \qquad \alpha = 1,2,3 \tag{3.8}$$

Choice of initial condition φ_{in} in the form (3.2) is also unessential. Let us choose the initial conditions (2.3), (2.4) in the general form

$$\varphi(0, \mathbf{x}) = \varphi_{\mathrm{in}}(\mathbf{x}), \qquad \boldsymbol{\xi}(0, \mathbf{x}) = \boldsymbol{\xi}_{\mathrm{in}}(\mathbf{x}) = \{ \xi_{\mathrm{in}1}(\mathbf{x}), \xi_{\mathrm{in}2}(\mathbf{x}), \xi_{\mathrm{in}3}(\mathbf{x}) \} \tag{3.9}$$

Substituting (3.1) and (3.9) in (2.3) and (2.4), we obtain

$$\rho_{\mathrm{in}}(\mathbf{x}) = \rho_0 (\boldsymbol{\xi}_{\mathrm{in}}(\mathbf{x})) D_{\mathrm{in}}(\mathbf{x}), \qquad g^\alpha (\mathbf{x}) = \left(v_{\mathrm{in}}^\mu (\mathbf{x}) - \partial_\mu \varphi_{\mathrm{in}}(\mathbf{x}) \right) \frac{\partial D_{\mathrm{in}}(\mathbf{x})}{\partial \zeta_{\alpha,\mu}}, \tag{3.10}$$

$$D_{\mathrm{in}}(\mathbf{x}) \equiv \det ||\zeta_{\alpha,\beta}|| \equiv \frac{\partial (\xi_{\mathrm{in}1}, \xi_{\mathrm{in}2}, \xi_{\mathrm{in}3})}{\partial (x^1, x^2, x^3)}, \qquad \zeta_{\alpha,\beta} \equiv \frac{\partial \xi_{\mathrm{in}\alpha}}{\partial x^\beta}, \qquad \alpha, \beta = 1,2,3 \tag{3.11}$$

Then we have instead of (3.7)

$$\rho = \left[\frac{\rho_{\mathrm{in}}(\mathbf{x})}{D_{\mathrm{in}}(\mathbf{x})} \right]_{\mathbf{x} = \boldsymbol{\xi}} \frac{\partial (\xi_1, \xi_2, \xi_3)}{\partial (x^1, x^2, x^3)}, \tag{3.12}$$

$$\mathbf{v} = \boldsymbol{\nabla} \varphi + \left(v_{\mathrm{in}}^\mu (\boldsymbol{\xi}) - \partial_\mu \varphi_{\mathrm{in}}(\boldsymbol{\xi}) \right) \left[\frac{\partial D_{\mathrm{in}}}{\partial \zeta_{\alpha,\mu}} (\mathbf{x}) \right]_{\mathbf{x} = \boldsymbol{\xi}} \boldsymbol{\nabla} \xi_\alpha \tag{3.13}$$

where φ, $\boldsymbol{\xi}$ are solutions of equations (2.3), (2.7) with the initial conditions (3.9),(3.1). These equations have the form

$$\partial_0 \xi_\mu + \left(\boldsymbol{\nabla} \varphi + \left(v_{\mathrm{in}}^\nu (\boldsymbol{\xi}) - \partial_\nu \varphi_{\mathrm{in}}(\boldsymbol{\xi}) \right) \left[\frac{\partial D_{\mathrm{in}}}{\partial \zeta_{\alpha,\nu}} (\mathbf{x}) \right]_{\mathbf{x} = \boldsymbol{\xi}} \boldsymbol{\nabla} \xi_\alpha \right) \boldsymbol{\nabla} \xi_\mu = 0, \qquad \mu = 1,2,3 \tag{3.14}$$

$$
\begin{aligned}
\partial_0 \varphi &= -\frac{1}{2} (\nabla \varphi)^2 - \left[\frac{\partial (\rho E)}{\partial \rho} \right]_{\rho = \rho_0 (\boldsymbol{\xi}) \partial J / \partial \xi_{0,0}} \\
&+ \frac{1}{2} \left(v_{\text{in}}^{\mu} (\boldsymbol{\xi}) - \partial_{\mu} \varphi_{\text{in}} (\boldsymbol{\xi}) \right) \left[\frac{\partial D_{\text{in}}}{\partial \zeta_{\alpha,\mu}} (\mathbf{x}) \right]_{\mathbf{x} = \boldsymbol{\xi}} \\
&\times \left(v_{\text{in}}^{\nu} (\boldsymbol{\xi}) - \partial_{\nu} \varphi_{\text{in}} (\boldsymbol{\xi}) \right) \left[\frac{\partial D_{\text{in}}}{\partial \zeta_{\beta,\nu}} (\mathbf{x}) \right]_{\mathbf{x} = \boldsymbol{\xi}} \nabla \xi_{\alpha} \nabla \xi_{\beta}
\end{aligned} \tag{3.15}
$$

where

$$
\rho_0 (\boldsymbol{\xi}) = \left[\frac{\rho_{\text{in}} (\mathbf{x})}{D_{\text{in}} (\mathbf{x})} \right]_{\mathbf{x} = \boldsymbol{\xi}} \tag{3.16}
$$

Equations (3.14), (3.15) should be solved at the initial conditions (3.9). Instead we can also solve equations (3.5), (3.6) at the initial conditions (3.2).

We see that the integrated dynamic equations (2.5), (2.7) (or in expanded form (3.14), (3.15), (3.16)) contain full information on the fluid flow. Initial conditions (3.9), which are necessary for determination of the unique solution of dynamic equations (2.5), (2.7), concern only physically unessential information on the fluid particles labeling and separation of the velocity into potential and vortical components.

If we consider the Lagrangian coordinates $\boldsymbol{\xi}$ as independent variables the dynamic equations (3.5), (3.6) and (3.7) are reduced to the form

$$
\varphi^{,0} - \frac{1}{2} \left(\varphi^{,\alpha} + v_{\text{in}}^{\alpha} (\boldsymbol{\xi}) \right) \left(\varphi^{,\alpha} + v_{\text{in}}^{\alpha} (\boldsymbol{\xi}) \right) X^{-2} \frac{\partial X}{\partial x^{\mu,\alpha}} \frac{\partial X}{\partial x^{\mu,\alpha}} + P (\rho) = 0, \tag{3.17}
$$

$$
x^{\beta,0} = \left(\frac{\partial \varphi}{\partial \xi_{\alpha}} + v_{\text{in}}^{\alpha} (\boldsymbol{\xi}) \right) X^{-1} \frac{\partial X}{\partial x^{\beta,\alpha}}, \qquad \beta = 1,2,3 \tag{3.18}
$$

$$
P (\rho) = \left[\frac{\partial (\rho E (\rho))}{\partial \rho} \right]_{\rho = X^{-1} \rho_{\text{in}} (\boldsymbol{\xi})} \tag{3.19}
$$

where $\mathbf{x} = \{ x^{\alpha} (t, \boldsymbol{\xi}) \}$, $\alpha = 1,2,3$, and $\varphi = \varphi (t, \boldsymbol{\xi})$. Jacobian

$$
X = \frac{\partial (x^1, x^2, x^3)}{\partial (\xi_1, \xi_2, \xi_3)} = \det \left\| x^{\alpha,\beta} \right\|, \qquad x^{\alpha,\beta} \equiv \frac{\partial x^{\alpha}}{\partial \xi_{\beta}}, \qquad \alpha, \beta = 1,2,3 \tag{3.20}
$$

is considered to be a function of variables $x^{\alpha,\beta}$. The quantities of the type $u^{,0}$ mean the time derivative of u with constant $\boldsymbol{\xi}$

$$
x^{\beta,0} \equiv \frac{dx^{\beta}}{dt} = \frac{\partial \left(x^{\beta}, \xi_1, \xi_2, \xi_3 \right)}{\partial (t, \xi_1, \xi_2, \xi_3)}, \qquad \varphi^{,0} \equiv \frac{d\varphi}{dt} = \frac{\partial (\varphi, \xi_1, \xi_2, \xi_3)}{\partial (t, \xi_1, \xi_2, \xi_3)} \tag{3.21}
$$

Hydrodynamic equations (3.17), (3.18) are rather bulky, but they contain arbitrary initial conditions as functions of independent variables $\boldsymbol{\xi}$.

4 Cauchy problem for the incompressible fluid flow in infinite volume

The main difference between the barotropic and incompressible fluids consists in the constraint imposed on the state of the incompressible fluid by the first equation (2.12). This condition does not contain temporal derivative and it is to be satisfied at the initial moment $t=0$

$$\nabla \mathbf{v}_{\text{in}}(\mathbf{x})=0 \tag{4.1}$$

It means that the initial state of the incompressible fluid $\mathbf{v}_{\text{in}}$ cannot be given arbitrarily. But the main property of initial state is the possibility of giving it arbitrarily. To conserve this property, we are forced to redefine the concept of initial state of the incompressible fluid. Let us consider the generalized stream function (GSF) $\{\psi_2,\psi_3\}$ [2] to be the quantity describing the state of the incompressible fluid. The velocity $\mathbf{v}$, defined via GSF by the relation

$$v^\mu=\frac{\partial(x^\mu,\psi_2,\psi_3)}{\partial(x^1,x^2,x^3)},\qquad \mu=1,2,3, \tag{4.2}$$

satisfies the first equation (2.12) for any choice of functions $\{\psi_2,\psi_3\}$. Considering GSF as a state of the incompressible fluid, we use two quantities $\{\psi_2,\psi_3\}$ instead of three components of velocity, but these quantities $\{\psi_2,\psi_3\}$ may be given arbitrarily at the initial time $t=0$. It is very useful and important.

On the other hand, the variables $j^0=\rho$ and $\mathbf{j}=\rho\mathbf{v}$ are described by relations (1.12). Choosing $\rho_0(\boldsymbol{\xi})=\rho_1=$const, we obtain in the case of the incompressible fluid from (1.12)

$$\rho=\rho_1\frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(x^1,x^2,x^3)}=\rho_1=\text{const},\qquad j^\mu=\rho_1 v^\mu=\rho_1\frac{\partial(x^\mu,\xi_1,\xi_2,\xi_3)}{\partial(x^0,x^1,x^2,x^3)},\qquad \mu=1,2,3 \tag{4.3}$$

The velocity $\mathbf{v}$, defined by the second equation (4.3), satisfies the first equation (2.12) identically, because

$$\frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(x^1,x^2,x^3)}=1 \tag{4.4}$$

Let us choose initial conditions in the form

$$\boldsymbol{\xi}(0,\mathbf{x})=\boldsymbol{\xi}_{\text{in}}(\mathbf{x})=\mathbf{x},\qquad \varphi(0,\mathbf{x})=\varphi_{\text{in}}(\mathbf{x})=0 \tag{4.5}$$

$$v^\mu(0,\mathbf{x})=v^\mu_{\text{in}}(\mathbf{x})=\frac{\partial(x^\mu,\psi_{\text{in}2}(\mathbf{x}),\psi_{\text{in}3}(\mathbf{x}))}{\partial(x^1,x^2,x^3)},\qquad \mu=1,2,3 \tag{4.6}$$

where $\psi_{\text{in}2}$, $\psi_{\text{in}3}$ are given functions of $\mathbf{x}$ (initial values of GSF). Then it follows from (2.15) written at $t=0$

$$g^\mu(\mathbf{x})=\frac{\partial(x^\mu,\ \psi_{\text{in}2}(\mathbf{x}),\psi_{\text{in}3}(\mathbf{x}))}{\partial(x^1,x^2,x^3)},\qquad \mu=1,2,3 \tag{4.7}$$

Substituting (4.7) in dynamic equations (2.14) and setting $\rho_1/\rho_0(\mathbf{x})$ =const, we obtain three dynamic equations for variables $\boldsymbol{\xi}$

$$\partial_0\xi_\alpha + \left(\varphi_{,\nu} + \frac{\partial\left(\xi_\mu, \psi_{\text{in2}}(\boldsymbol{\xi}), \psi_{\text{in3}}(\boldsymbol{\xi})\right)}{\partial\left(\xi_1, \xi_2, \xi_3\right)}\xi_{\mu,\nu}\right)\xi_{\alpha,\nu} = 0, \qquad \alpha = 1,2,3 \tag{4.8}$$

and a constraint (4.4), imposed on the values of the quantities $\boldsymbol{\xi}$.

Four equations (4.8), (4.4) form a system of dynamic equations for four dynamic variables $\boldsymbol{\xi}$,φ. Equation (4.4) may be replaced by the equation (2.16), which after substitution of (4.7) takes the form

$$\varphi_{,\nu\nu} + \partial_\nu\left(\frac{\partial\left(\xi_\mu, \psi_{\text{in2}}(\boldsymbol{\xi}), \psi_{\text{in3}}(\boldsymbol{\xi})\right)}{\partial\left(\xi_1, \xi_2, \xi_3\right)}\xi_{\mu,\nu}\right) = 0 \tag{4.9}$$

where $\psi_{\text{in2}}(\boldsymbol{\xi})$, $\psi_{\text{in3}}(\boldsymbol{\xi})$ are given functions of argument $\boldsymbol{\xi}$. We stress that the equations (4.8), (4.9) are to be solved at initial conditions (4.5). May we set $\varphi = 0$ in the equation (4.9)? In general, no. If we set $\varphi = 0$, the equation (4.9) turns into the equation

$$\partial_\nu\left(\frac{\partial\left(\xi_\mu, \psi_{\text{in2}}(\boldsymbol{\xi}), \psi_{\text{in3}}(\boldsymbol{\xi})\right)}{\partial\left(\xi_1, \xi_2, \xi_3\right)}\xi_{\mu,\nu}\right) = 0 \tag{4.10}$$

which is valid at $t = 0$, when $\boldsymbol{\xi} = \mathbf{x}$. If $\boldsymbol{\xi} \neq \mathbf{x}$, equation (4.10) is not valid, in general.

Thus, four equations (4.8), (4.9) form a system of dynamic equations for four dynamic variables $\boldsymbol{\xi}$, φ. Dynamic equations (4.8) are evolutional equations, describing evolution of $\boldsymbol{\xi}$ in the sense that they contain time derivatives of $\boldsymbol{\xi}$. If the state $\boldsymbol{\xi}(t,\mathbf{x})$, $\varphi(t,\mathbf{x})$ of the fluid is given at the time t, the dynamic equations (4.8) determine the quantities $\boldsymbol{\xi}(t+dt,\mathbf{x})$ at the next time moment uniquely.

The equation (4.9) as well as the equation (4.4) is not an evolutional equation, because it does not contain temporal derivatives. The value $\varphi(t+dt,\mathbf{x})$ is not connected with the value $\varphi(t,\mathbf{x})$ directly. One can determine the unique solution of (4.9), if there is some additional information about the variable φ on the boundary ∂V of the volume V, where the fluid flow is considered. Formally it follows from the fact that the Poisson equation (4.9) has an unique solution in the region V, provided a proper information is given on the boundary ∂V of the volume V.

In the case of the barotropic fluid the corresponding equation (3.15) for φ is evolutional, and one does not need such an information. At least at the point $\mathbf{x}$, which is far enough from the boundary ∂V, the values of $\varphi(t,\mathbf{x})$, $0 < t < T$ can be determined uniquely for a time T, which is necessary for passage of the signal from the nearest point of boundary ∂V to the point $\mathbf{x}$. The speed of the signal is equal to the sound speed in the fluid. In the incompressible fluid the sound speed is infinite, the time interval $T = 0$, and dynamic equation (4.9) is not evolutional. In accordance with this fact the value $\varphi(t+dt,\mathbf{x})$ depends on the boundary conditions at the time $t+dt$, but not on the values of $\varphi(t,\mathbf{x})$ and $\boldsymbol{\xi}(t,\mathbf{x})$.

One can see two parts in dynamic equations (4.8): nonlocal potential term $\varphi_{,\nu}\xi_{\alpha,\nu}$ and local vortical term $v_{\text{in}}^\mu\xi_{\mu,\nu}\xi_{\alpha,\nu}$. The vortical term describes influence of the "frozen" vorticity on the fluid flow. The potential term describes nonlocal interaction in the fluid connected with the potential φ and with the infinite speed of sound. In the barotropic fluid the speed of sound is finite, and the potential term describes a local interaction.

5 Stationary flow of incompressible fluid.

Let us imagine that setting the Cauchy problem, we choose the initial conditions in such a way that the incompressible fluid flow appears to be stationary, i.e.

$$\partial_0 \rho = 0, \qquad \partial_0 \mathbf{v} = 0 \tag{5.1}$$

In this case the stream lines are stationary. Let Clebsch potentials ξ_2 and ξ_3 label these stream lines. The Clebsch potentials ξ_2 and ξ_3 can be chosen independent of time. Dynamic equations (2.13), (2.14), (2.15), (2.17) take the form

$$\frac{\partial J}{\partial \xi_{0,0}} \equiv \rho_1 \frac{\partial (\xi_1, \xi_2, \xi_3)}{\partial (x^1, x^2, x^3)} = \rho_1 = \text{const} \tag{5.2}$$

$$v^\mu = \frac{\partial J}{\partial \xi_{0,\mu}} \equiv \frac{\partial (x^\mu, \xi_1, \xi_2, \xi_3)}{\partial (x^0, x^1, x^2, x^3)} = -\xi_{1,0} \frac{\partial (x^\mu, \xi_2, \xi_3)}{\partial (x^1, x^2, x^3)}, \qquad \mu = 1, 2, 3 \tag{5.3}$$

$$\partial_0 \xi_1 + (\mathbf{v}\boldsymbol{\nabla}) \xi_1 = 0, \qquad (\mathbf{v}\boldsymbol{\nabla}) \xi_2 = 0, \qquad (\mathbf{v}\boldsymbol{\nabla}) \xi_3 = 0 \tag{5.4}$$

$$\mathbf{v} = \boldsymbol{\nabla} \varphi + g^\alpha (\boldsymbol{\xi}) \boldsymbol{\nabla} \xi_\alpha \tag{5.5}$$

Let us write equations (5.3), (5.2) respectively in the form

$$\mathbf{v} = -\xi_{1,0} (\boldsymbol{\nabla} \xi_2 \times \boldsymbol{\nabla} \xi_3), \qquad \boldsymbol{\nabla} \xi_1 (\boldsymbol{\nabla} \xi_2 \times \boldsymbol{\nabla} \xi_3) = 1 \tag{5.6}$$

Let us set $\xi_{1,0} = -1$. We obtain

$$\mathbf{v} = (\boldsymbol{\nabla} \xi_2 \times \boldsymbol{\nabla} \xi_3), \qquad \xi_{1,0} = -1 \tag{5.7}$$

The first equation (5.4) takes the form

$$(\mathbf{v}\boldsymbol{\nabla}) \xi_1 = 1, \tag{5.8}$$

We integrate the second equation (5.6), considering variables ξ_2, ξ_3 as given functions of argument $\mathbf{x}$. It can be written in the form

$$\frac{\partial (\xi_1, \xi_2, \xi_3)}{\partial (x, y, z)} = \frac{\partial (\xi_1, \xi_2, \xi_3)}{\partial (s, \xi_2, \xi_3)} D(\mathbf{x}) = 1, \qquad D(\mathbf{x}) = \frac{\partial (s, \xi_2, \xi_3)}{\partial (x, y, z)} \tag{5.9}$$

where s is some function of $\mathbf{x}$, which is chosen in such a way, that

$$D(\mathbf{x}) = \frac{\partial (s, \xi_2, \xi_3)}{\partial (x, y, z)} \neq 0 \tag{5.10}$$

Let us resolve equations

$$\xi_2 = \xi_2 (x, y, z), \qquad \xi_3 = \xi_3 (x, y, z), \qquad s = s (x, y, z) \tag{5.11}$$

in the form

$$x = F_1 (s, \xi_2, \xi_3), \qquad y = F_2 (s, \xi_2, \xi_3), \qquad z = F_3 (s, \xi_2, \xi_3) \tag{5.12}$$

and calculate

$$\Delta (s, \xi_2, \xi_3) = D (F_1 (s, \xi_2, \xi_3), F_2 (s, \xi_2, \xi_3), F_3 (s, \xi_2, \xi_3)) \tag{5.13}$$

The equation (5.9) can be written in the form

$$\frac{\partial (\xi_1, \xi_2, \xi_3)}{\partial (s, \xi_2, \xi_3)} = \frac{1}{\Delta (s, \xi_2, \xi_3)} \tag{5.14}$$

It is integrated in the form

$$\xi_1 = \int \frac{ds}{\Delta (s, \xi_2, \xi_3)} \tag{5.15}$$

where integration is produced at fixed ξ_2, ξ_3.

We eliminate the variable φ from the equation (5.5), taking its curl. We obtain

$$\boldsymbol{\nabla} \times \mathbf{v} = \frac{1}{2} \Omega^{\alpha\beta} (\boldsymbol{\nabla} \xi_\alpha \times \boldsymbol{\nabla} \xi_\beta) = \Omega^{23} (\boldsymbol{\nabla} \xi_2 \times \boldsymbol{\nabla} \xi_3) + \Omega^{31} (\boldsymbol{\nabla} \xi_3 \times \boldsymbol{\nabla} \xi_1) + \Omega^{12} (\boldsymbol{\nabla} \xi_1 \times \boldsymbol{\nabla} \xi_2) \tag{5.16}$$

where

$$\Omega^{\alpha\beta} = \Omega^{\alpha\beta} (\boldsymbol{\xi}) \equiv \frac{\partial g^a}{\partial \xi_\beta} (\boldsymbol{\xi}) - \frac{\partial g^\beta}{\partial \xi_\alpha} (\boldsymbol{\xi}) = g^{a,\beta} (\boldsymbol{\xi}) - g^{\beta,\alpha} (\boldsymbol{\xi}) \tag{5.17}$$

The quantities $\Omega^{\alpha\beta}$ describe the fluid flow vorticity in the coordinates ξ_1, ξ_2, ξ_3, as it follows from the relation (5.16), written in the form

$$\partial_\mu v^\nu (\mathbf{x}) - \partial_\nu v^\mu (\mathbf{x}) = \frac{\partial \xi_\alpha}{\partial x^\mu} \frac{\partial \xi_\beta}{\partial x^\nu} \Omega^{\alpha\beta} (\boldsymbol{\xi}) \tag{5.18}$$

This relation may interpreted as a transformation of the tensor $\partial_\mu v^\nu - \partial_\nu v^\mu$ from the coordinates $\boldsymbol{\xi}$ to coordinates $\mathbf{x}$.

The variables ξ_2 and ξ_3 do not depend on t, whereas ξ_1 is a linear function of t. Left hand side of (5.16) does not depend on t. It means that the quantities $\Omega^{\alpha\beta}$ do not depend on t also. They depend only on ξ_2, ξ_3, but not on ξ_1, because ξ_1 depends on t. Differentiating (5.15), we obtain

$$\boldsymbol{\nabla} \xi_1 \times \boldsymbol{\nabla} \xi_2 = \frac{\boldsymbol{\nabla} s \times \boldsymbol{\nabla} \xi_2}{\Delta (s, \xi_2, \xi_3)} + (\boldsymbol{\nabla} \xi_3 \times \boldsymbol{\nabla} \xi_2) \frac{\partial}{\partial \xi_3} \int \frac{ds}{\Delta (s, \xi_2, \xi_3)} \tag{5.19}$$

$$\boldsymbol{\nabla} \xi_1 \times \boldsymbol{\nabla} \xi_3 = \frac{\boldsymbol{\nabla} s \times \boldsymbol{\nabla} \xi_3}{\Delta (s, \xi_2, \xi_3)} + (\boldsymbol{\nabla} \xi_2 \times \boldsymbol{\nabla} \xi_3) \frac{\partial}{\partial \xi_2} \int \frac{ds}{\Delta (s, \xi_2, \xi_3)} \tag{5.20}$$

Substituting relations (5.19) and (5.20) in (5.16) and using (5.7), we obtain

$$\begin{aligned}\nabla\times\mathbf{v} &= \left(\Omega^{23}-\Omega^{31}\frac{\partial}{\partial\xi_2}\int\frac{ds}{\Delta(s,\xi_2,\xi_3)}-\Omega^{12}\frac{\partial}{\partial\xi_3}\int\frac{ds}{\Delta(s,\xi_2,\xi_3)}\right)\mathbf{v}\\ &+\frac{\nabla s\times\left(\Omega^{12}\nabla\xi_2-\Omega^{31}\nabla\xi_3\right)}{\Delta(s,\xi_2,\xi_3)},\qquad \Omega^{\alpha\beta}=\Omega^{\alpha\beta}(\xi_2,\xi_3)\end{aligned}\tag{5.21}$$

Three independent equations (5.21) carry out the description of the stationary flow of incompressible fluid in terms of the generalized stream function (GSF) $\{\xi_2,\xi_3\}$. The velocity $\mathbf{v}$ is expressed via GSF by means of the relation (5.7). The variable s is chosen in such a way, that the inequality (5.10) takes place. For instance, one can set $s=x$. Then we obtain from (5.10)

$$\Delta=\frac{\partial(x,\xi_2,\xi_3)}{\partial(x,y,z)}=v^1(x,\xi_2,\xi_3)$$

In this case the equations (5.21) has a singular point, when the velocity component v^1 vanishes. At another choice of the variable s the singular point does not appear, or it appears in other place. It means that the possible singular point is a result of unsuccessful description, which can be eliminated by a proper choice of the variable s.

The conventional statement of the stationary flow problem of the incompressible fluid can be obtained from the dynamic equations (2.12) and (5.1). It has the form

$$\nabla\mathbf{v}=0,\qquad \nabla\times(\mathbf{v}\nabla)\,\mathbf{v}=0\tag{5.22}$$

The first equation (5.22) can be solved by introduction of GSF $\{\xi_2,\xi_3\}$. Then the velocity is expressed by means of (5.7), and we obtain the dynamic equations for the stationary flow of the incompressible fluid

$$\nabla\times(\mathbf{v}\nabla)\,\mathbf{v}=0,\qquad \mathbf{v}=(\nabla\xi_2\times\nabla\xi_3)\tag{5.23}$$

Let us compare two different statements of the problem of the stationary flow of the incompressible fluid. We shall compare equations (5.21) and (5.22). At first, we consider equations (5.22). In the case of the irrotational flow, when

$$\mathbf{v}=\nabla\varphi\tag{5.24}$$

the second equation (5.22) is satisfied identically. The first equation (5.22) leads to the Laplace equation

$$\nabla^2\varphi=0\tag{5.25}$$

which has a unique solution, provided that the normal derivative $(\mathbf{n}\nabla)\,\varphi$ is given on the boundary ∂V of the volume V, where the fluid flow is considered. In other words, for determination of an unique irrotational flow it is sufficient to give the velocity $\mathbf{v}$ on the boundary ∂V. What information is necessary for determination of the unique rotational flow? It is believed that in the case of the rotational flow the same information is sufficient as in the case of the irrotational flow. Why? Because in the case of the nonstationary flow it is sufficient to give the initial velocity $\mathbf{v}_{\text{in}}(\mathbf{x})$ in the whole volume V and the velocity $\mathbf{v}_{\text{b}}(t,,\mathbf{x})$, $\mathbf{x}\in\partial V$ on the boundary ∂V. This information is the same for both rotational and irrotational flow. In the stationary case the determination of the rotational flow needs additional information as compared with the irrotational flow. It is necessary to give the vorticity $\Omega^{\alpha\beta}$ on each stream line. In reality, the vorticity is given also in the case of the stationary irrotational flow. It is simply equal to zero, and this information is perceived usually as an absence of information.

Why is it necessary to give the vorticity in addition to the velocity in the stationary case? Why may we not give the vorticity in the nonstationary case? In reality, we need to give the vorticity in both cases, but in the nonstationary case the vorticity is determined by the initial velocity field $\mathbf{v}_{\text{in}}(\mathbf{x})$, given in the whole volume V. In the stationary case the velocity is given only on the boundary ∂V, and the vorticity cannot be determined on the basis of this information. In this case we can determine only one component of the vorticity vector $\boldsymbol{\omega}=\boldsymbol{\nabla}\times\mathbf{v}$. As a result the vorticity must be given in addition.

Remark. Usually the problem of motion of a body in the fluid is considered to be equivalent to the problem of the fluid flow around the immovable body. Solution of the first problem depends on the body velocity. Transition from the first problem to the second one is carried out by means of the transition to the coordinate system, where the body is at rest. This coordinate transformation depends only on the body velocity, and it does not contain any fluid characteristics (for instance, the fluid vorticity). Motion of the body in the immovable fluid with velocity $\mathbf{v}=0$ distinguishes, in general, from the body motion in the practically immovable fluid, where the mean velocity $\langle\mathbf{v}\rangle=0$, but the fluid flow contains vortexes. In the last case one cannot give the fluid flow with arbitrary vortexes, because this flow is to be compatible with hydrodynamic equations. To give the flow of the "practically immovable" fluid, one should solve dynamic equations. There is a lot of fluid flows with the vanishing mean velocity $\langle\mathbf{v}\rangle=0$ and vortexes located in different places of the fluid flow. The set of these flows is described by the vorticity $\Omega^{\alpha\beta}(\boldsymbol{\xi})$ given on all stream lines.

Now we consider the equations (5.21), which contain indefinite functions $\Omega^{\alpha\beta}(\xi_2,\xi_3)$. As we have seen in section 4, the functions $g^{\alpha}(\boldsymbol{\xi})$, $\alpha=1,2,3$ contain the complete information, which is necessary for determination of the unique fluid flow, whereas initial and boundary conditions for variables $\boldsymbol{\xi}$ do not influence the fluid flow and may be taken arbitrarily. In the case of the irrotational flow all functions $\Omega^{\alpha\beta}(\xi_2,\xi_3)=0$, and one obtains from (5.21)

$$\boldsymbol{\nabla}\times\mathbf{v}=0,\qquad \mathbf{v}=(\boldsymbol{\nabla}\xi_2\times\boldsymbol{\nabla}\xi_3)\tag{5.26}$$

Replacing the second equation (5.26) by the equivalent equation $\boldsymbol{\nabla}\mathbf{v}=0$, we reduce the problem to equations (5.24), (5.25).

In the case of the rotational flow the vorticity $\Omega^{\alpha\beta}$ is to be given on any stream line, labeled by ξ_2, ξ_3. If the fluid flow contains closed stream lines, which do not cross the boundary ∂V, one cannot determine the vorticity $\Omega^{\alpha\beta}$ on these stream lines, giving boundary conditions. In this case one needs an additional information other than the boundary condition. We can obtain the stationary flow as a result of establishing process. In this case we are to give initial velocity $\mathbf{v}_{\text{in}}(\mathbf{x})$ in the whole volume V, and the vorticity on the closed stream lines is determined by the initial velocity $\mathbf{v}_{\text{in}}(\mathbf{x})$. Using different establishing processes, we obtain different stationary flows at the same stationary boundary conditions. Experimenters dealing with rotating fluid flows know this fact very well.

Equations (5.21) are equations for the generalized stream function (GSF) $\{\xi_2, \xi_3\}$. These equations are nonlinear. In terms of the variables $\mathbf{v}, \xi_2, \xi_3$ equations (5.21) contain terms linear with respect $\mathbf{v}$. Equations (5.21) became to be simple in the special case, when Ω^{23} =const and $\Omega^{13} = \Omega^{12} = 0$. In this case we obtain linear equations

$$\boldsymbol{\nabla} \times \mathbf{v} = \Omega^{23}\mathbf{v}, \qquad \boldsymbol{\nabla}\mathbf{v} = 0, \qquad \Omega^{23} = \text{const} \tag{5.27}$$

which can be solved without consideration of variables ξ_2, ξ_3.

6 Stationary flow of incompressible fluid described in Lagrangian coordinates

Dynamic equations (5.21) contain indefinte functions $\Omega^{\alpha\beta}(\boldsymbol{\xi})$ as functions of dependent variables. Sometimes it is useful to have dynamic equations, where indefinite functions be those of independent variables. Let us transform dynamic equations (5.4), (5.9) and (5.16) to independent variables $t, \boldsymbol{\xi}$.

We introduce designations (3.20)

$$x^\alpha = x^\alpha(t, \xi_1, \xi_2, \xi_3), \qquad x^{\alpha,\beta} \equiv \frac{\partial x^\alpha}{\partial \xi_\beta}, \qquad \alpha, \beta = 1,2,3 \tag{6.1}$$

$$X \equiv \frac{\partial(x^1, x^2, x^3)}{\partial(\xi_1, \xi_2, \xi_3)} = \det\left|\left|x^{\alpha,\beta}\right|\right|, \qquad \alpha, \beta = 1,2,3 \tag{6.2}$$

We consider the Jacobian X as the 3-linear function of arguments $x^{\alpha,\beta}$. We obtain

$$\xi_{\alpha,\beta} = X^{-1}\frac{\partial X}{\partial x^{\beta,\alpha}}, \qquad \alpha, \beta = 1,2,3 \tag{6.3}$$

$$v^\alpha = \frac{\partial(x^\alpha, \xi_2, \xi_3)}{\partial(x^1, x^2, x^3)} = \frac{\partial(x^\alpha, \xi_2, \xi_3)}{\partial(\xi_1, \xi_2, \xi_3)}\frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(x^1, x^2, x^3)} = x^{\alpha,1}, \qquad \alpha = 1,2,3 \tag{6.4}$$

We write dynamic equations (5.4), (5.9) and (5.16) in the form

$$X \equiv \frac{\partial(x^1, x^2, x^3)}{\partial(\xi_1, \xi_2, \xi_3)} = 1 \tag{6.5}$$

$$v^\alpha \xi_{1,\alpha} = 1, \qquad v^\alpha \xi_{2,\alpha} = 0, \qquad v^\alpha \xi_{3,\alpha} = 0, \tag{6.6}$$

$$\varepsilon_{\mu\alpha\beta}\partial_{\alpha}v^{\beta}=\frac{1}{2}\varepsilon_{\mu\alpha\beta}\Omega^{\rho\sigma}\left(\xi_2,\xi_3\right)\xi_{\rho,\alpha}\xi_{\sigma,\beta} \tag{6.7}$$

where $\varepsilon_{\mu\alpha\beta}$ is the Levi-Chivita pseudotensor.

Substituting $\xi_{\alpha,\beta}$ and v^{α}, taken from (6.3), (6.4) in (6.6), we obtain equations

$$x^{\alpha,1}X^{-1}\frac{\partial X}{\partial x^{\alpha,1}}=1,\qquad x^{\alpha,1}X^{-1}\frac{\partial X}{\partial x^{\alpha,2}}=0,\qquad x^{\alpha,1}X^{-1}\frac{\partial X}{\partial x^{\alpha,3}}=0, \tag{6.8}$$

which are satisfied identically. We substitute $\xi_{\alpha,\beta}$ and v^{α}, taken from (6.3), (6.4) in (6.7). Taking into account (6.5), we obtain

$$x^{\beta,1\gamma}\frac{\partial X}{\partial x^{\alpha,\gamma}}-x^{\alpha,1\gamma}\frac{\partial X}{\partial x^{\beta,\gamma}}=\Omega^{\rho\sigma}\left(\xi_2,\xi_3\right)\frac{\partial X}{\partial x^{\alpha,\rho}}\frac{\partial X}{\partial x^{\beta,\sigma}} \tag{6.9}$$

Convolving (6.9) with $x^{\alpha,\mu}$ and $x^{\beta,\nu}$and taking into account (6.5), we obtain

$$x^{\alpha,1\mu}x^{\alpha,\nu}-x^{\alpha,1\nu}x^{\alpha,\mu}=\Omega^{\mu\nu}\left(\xi_2,\xi_3\right),\qquad \mu,\nu=1,2,3 \tag{6.10}$$

where summation is made over $\alpha=1,2,3$.

In the three-dimensional space of coordinates ξ_1,ξ_2,ξ_3 the equations (6.10) can be written in the vector form

$$\sum_{\alpha=1}^{\alpha=3}\boldsymbol{\nabla}_{\xi}v^{\alpha}\times\boldsymbol{\nabla}_{\xi}x^{\alpha}=\boldsymbol{\omega}\left(\xi_2,\xi_3\right),\qquad \boldsymbol{\nabla}_{\xi}=\left\{\frac{\partial}{\partial\xi_1},\frac{\partial}{\partial\xi_2},\frac{\partial}{\partial\xi_3}\right\},\qquad v^{\alpha}=x^{\alpha,1},\qquad \alpha=1,2,3 \tag{6.11}$$

$$\boldsymbol{\omega}=\left\{\omega_1,\omega_2,\omega_3\right\},\qquad \omega_{\mu}\left(\xi_2,\xi_3\right)=\frac{1}{2}\varepsilon_{\mu\alpha\beta}\Omega^{\alpha\beta}\left(\xi_2,\xi_3\right),\qquad \mu=1,2,3 \tag{6.12}$$

where $\varepsilon_{\mu\alpha\beta}$ is the Levi-Chivita pseudotensor.

Finally, equation (6.5) can be resolved with respect to x^1 in the same way, as the equation (5.9) has been resolved with respect to ξ_1. We obtain

$$x^1=\int\frac{ds}{\Delta_1\left(s,x^2,x^3\right)},\qquad \Delta_1=\frac{\partial\left(s,x^2,x^3\right)}{\partial\left(\xi_1,\xi_2,\xi_3\right)} \tag{6.13}$$

where s is some function of arguments ξ_1,ξ_2,ξ_3 and the Jacobian Δ_1 is considered as a function of arguments s,x^2,x^3. In particular, if $s=\xi_1$,

$$\Delta_1=\frac{\partial\left(x^2,x^3\right)}{\partial\left(\xi_2,\xi_3\right)}=\frac{1}{v^1\left(\xi_1,x^2,x^3\right)},\qquad x^1=\int v^1\left(\xi_1,x^2,x^3\right)d\xi_1 \tag{6.14}$$

Using relation (6.13) or (6.14), we can eliminate variables x^1 and $v^1=x^{1,1}$ from equations (6.11). We obtain the system of dynamic equations for variables x^2,x^3 in the two-dimensional space of coordinates ξ_2,ξ_3 with variable ξ_1, considered as an evolutional variable (time).

Equations (6.11), (6.14) are rather complicated, especially because of the equation (6.14), which contains the operation of transition from independent variables $\{\xi_1,\xi_2,\xi_3\}$ to independent variables $\{\xi_1,x^2,x^3\}$. Properties of this operation are investigated slightly. Apparently, this operation is an attribute of incompressible fluid, because this operation is present in dynamic equations (5.21), written in the Eulerian coordinates. Indefinite functions $\Omega^{\alpha\beta}$ describing vorticity are functions of independent variables in the equations (6.11), whereas they are functions of dependent variables in equations (5.21). From this viewpoint the equations (6.11) are more convenient for investigation of vortical flows.

7 Concluding remarks

We have seen that the statement of the flow problem is more complicated for stationary flows, than for nonstationary ones, although dynamic equations for nonstationary flows are more complicated. Stationary boundary conditions do not destine a unique stationary flow, even if they determine vorticity on any stream line crossing the boundary of the considered flow region. There is a hope that the description of stationary flows of the ideal fluid may appear to be more effective in Lagrangian coordinates.

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