Some subtleties of Riemannian geometry

Yuri A. Rylov

Institute for Problems in Mechanics, Russian Academy of Sciences
101-1 ,Vernadskii Ave., Moscow, 117526, Russia
email: rylov@ipmnet.ru
Web site: $http://rsfq1.physics.sunysb.edu/\tilde{}rylov/yrylov.htm$
or mirror Web site:
$http://gasdyn-ipm.ipmnet.ru/\tilde{}rylov/yrylov.htm$

Abstract

It is shown, that the conventional presentation of the Maxwell equations for the electromagnetic field in the Riemannian space-time appears to be problematic. The reason of hesitations is the fact, that a solution of the Maxwell equations in the space-time of Minkowski do not turn into solution of the Maxwell equations in the Riemannian space-time after replacement of Minkowskian world function $\sigma_{\rm M}$ by the world function $\sigma_{\rm R}$ of the Riemannian space-time in the solution.

1 Introduction

The considered problem appeared at an attempt of generalization of dynamics in the Riemannian space-time on the case of non-Riemannian space-time geometry. The physical geometry $\mathcal{G}$ is a geometry described in terms of the world function σ and only in terms of σ [1, 2]. The physical geometry $\mathcal{G}$ and all its relations can be formulated without a reference to a coordinate system and other means of description (manifold, dimension, linear vector space). At the conventional approach to space-time geometries one states that the Riemannian geometry is the most general type of the space-time geometry. This statement is an unfounded constraint, because many physical geometries may be considered as possible space-time geometries. In general, the physical geometry is multivariant and nonaxiomatizable, whereas the Riemannian geometry pretends to be an axiomatizable geometry.

The multivariance of a geometry means that at a point P_0 there are many vectors $\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2, ..$, which are equivalent to the vector $\mathbf{Q}_0\mathbf{Q}_1$ at the point Q_0, but vectors $\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2, ..$are not equivalent between themselves. Vectors are equivalent ($\mathbf{P}_0\mathbf{P}_1\text{eqv}\mathbf{Q}_0\mathbf{Q}_1$), if vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ are parallel and their lengths $|\mathbf{P}_0\mathbf{P}_1|$

and $|\mathbf{Q}_0\mathbf{Q}_1|$ are equal

$$\mathbf{P}_0\mathbf{P}_1 \uparrow\uparrow \mathbf{Q}_0\mathbf{Q}_1: \qquad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = |\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1| \tag{1.1}$$

$$|\mathbf{P}_0\mathbf{P}_1| = |\mathbf{Q}_0\mathbf{Q}_1| \tag{1.2}$$

where the scalar product $(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1)$ and the length $|\mathbf{P}_0\mathbf{P}_1|$ of the vector $\mathbf{P}_0\mathbf{P}_1$ are defined by the relations

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \sigma(P_0, Q_1) + \sigma(P_1, Q_0) - \sigma(P_0, Q_0) - \sigma(P_1, Q_1) \tag{1.3}$$

$$|\mathbf{P}_0\mathbf{P}_1|^2 = (\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_0\mathbf{P}_1) = 2\sigma(P_0, P_1) \tag{1.4}$$

Let us stress, that the equivalence (1.1), (1.2) of two vectors is defined in terms of the world function σ and only in terms σ, which is defined as follows

$$\sigma: \qquad \Omega \times \Omega \to \mathbb{R}, \qquad \sigma(P, P) = 0, \qquad \forall P \in \Omega \tag{1.5}$$

Here Ω is the set of points, where the physical geometry $\mathcal{G}$ is given. The world function is interpreted in the form $\sigma(P, Q) = \frac{1}{2}\rho^2(P, Q)$, where $\rho(P, Q)$ is the distance between the points P and Q.

In the proper Euclidean geometry the equivalence relation (1.1), (1.2) coincides with the conventional definition of two vector equivalence, which is formulated as equality of the vector components in the Cartesian coordinate system

$$\mathbf{p} = \mathbf{q}, \text{ if} \qquad p_i = q_i, \qquad i = 1, 2, ...n \tag{1.6}$$

where p_i and q_i are coordinates of vectors $\mathbf{p}$ and $\mathbf{q}$ in some Cartesian coordinate system, and n is the dimension of the proper Euclidean geometry.

The definition (1.1), (1.2) distinguishes from the conventional definition (1.6) in the relation, that the definition (1.1), (1.2) does not contain such auxiliary means of description as dimension, coordinate system and concept of the linear vector space. Besides, the definition (1.1), (1.2) contains two equations for the proper Euclidean geometry of any dimension, whereas in the conventional definition the number of equations depends on the dimension of the space. All this means that the definition (1.1), (1.2) is more fundamental, than the definition (1.6), which can be used, only if in the geometry one can introduce concept of the linear vector space with the scalar product, given on it. The relation of equivalence (1.1), (1.2) can be used in any physical geometry, whereas the equivalence relation (1.6) can be used only in the space-time geometry, where the linear vector space can be introduced.

In general, the physical geometry is multivariant, because the definition (1.1), (1.2) admits an existence of many vectors $\mathbf{P}_0\mathbf{P}_1$, which are equivalent to the given vector $\mathbf{Q}_0\mathbf{Q}_1$. If the physical geometry $\mathcal{G}$ is multivariant, the equivalence relation (1.1), (1.2) is intransitive. In this case the physical geometry $\mathcal{G}$ cannot be axiomatizable, because in any axiomatizable geometry the equivalence relation is transitive.

In the Riemannian geometry the world function σ_{R} is defined by the relation

$$\sigma_{\mathrm{R}}(P,Q)=\frac{1}{2}\left(\int\limits_{\mathcal{L}_{[PQ]}}\sqrt{g_{ik}dx^{i}dx^{k}}\right)^{2} \qquad (1.7)$$

where $\mathcal{L}_{[PQ]}$ is a segment of the geodesic, connecting the point P and Q. One can construct the Riemannian geometry as a physical geometry $\mathcal{G}_{\sigma\mathrm{R}}$, using the relation (1.7), as a definition of the world function. We shall refer to the geometry $\mathcal{G}_{\sigma\mathrm{R}}$ as the σ-Riemannian geometry, which distinguishes from the conventional construction of the Riemannian geometry.

The conventional Riemannian geometry is constructed as a set of infinitesimal Euclidean geometries, "glued" between themselves in some manner. The manner of conglutination determines the peculiar properties of the Riemannian geometry. This manner of conglutination is described by the character of the dependence of the metric tensor on the coordinates. In general, the Riemannian geometry appears to be multivariant in the sense, that equivalence of remote vectors depends on the path of their parallel transport. To remove multivariance of the Riemannian geometry, the equivalence relation of the remote vectors is removed. As a result the conventional Riemannian geometry pretends to be single-variant and axiomatizable. However, such an approach is not consecutive, because the multivariant geometry is nonaxiomatizable, and one cannot turn the nonaxiomatizable geometry into axiomatizable one by the prohibition of the remote vector equivalence.

Thus, the σ-Riemannian geometry is multivariant, in general, and consistent. The σ-Riemannian geometry cannot be inconsistent in principle, because it is not deduced from axiomatics. Inconsistency of a geometry is an attribute of the geometry construction method, when the geometry is deduced from a system of axiom. The σ-Riemannian geometry is constructed as a deformation of the proper Euclidean geometry. All propositions $\mathcal{P}_{\mathrm{E}}$ of the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ are presented in terms of the Euclidean world function σ_{E} in the form $\mathcal{P}_{\mathrm{E}}(\sigma_{\mathrm{E}})$. Replacing σ_{E} by the world function $\sigma_{\sigma\mathrm{R}}$ of the σ-Riemannian geometry, one obtains all propositions $\mathcal{P}_{\mathrm{E}}(\sigma_{\sigma\mathrm{R}})$ of the σ-Riemannian geometry. Procedure of deformation does not use logical reasonings, and it cannot be inconsistent in principle.

The conventional Riemannian geometry is single-variant, but inconsistent. This inconsistency manifests itself, in particular, in the problem of generalization of dynamics in the Riemannian space-time on the case of arbitrary space-time physical geometry.

2 Generalization of dynamics in the Riemannian space-time on the case of arbitrary space-time

If the space-time geometry may be an arbitrary physical geometry, we are to generalize the dynamics in the Riemannian space-time on the case of arbitrary physical space-time geometry. The first part of this generalization (motion of a pointlike particle in the given classical fields: gravitational and electromagnetic) was made successfully in [3]. This generalization leads to the difference dynamic equations. It is quite reasonable, because the space-time geometry may be discrete, and differential dynamic equations in the discrete space-time geometry are not natural, whereas the difference dynamic equations are suitable in both continuous and discrete space-time geometry. Such a generalization gives rather unexpected results. It appears, that the proper choice of the space-time geometry, which is free of unfounded constraints of the Riemannian geometry, admits one to explain quantum effects as a statistical description of multivariant motion of particles, generated by the multivariance of the space-time geometry. Besides, arrangement of elementary particle is determined by the structure of its skeleton. The skeleton is a set of several points in the space-time. The mutual displacement of these points determines structure of the skeleton [3]. The quantum properties (wave function, quantization, renormalization) appear to be needless. In particular, the Dirac particle is composite [4]. Its skeleton consists of three points. World chain of such a particle is a spacelike helix with the timelike axis. Thus, generalization, suggested in [3], realize the generalization of the special relativity on the case of the arbitrary physical space-time geometry.

The second part of the generalization is a consideration of the influence of the matter distribution on the space-time geometry. The general relativity considers this influence in the framework of the Riemannian space-time geometry. One needs to generalize the general relativity on the case of the arbitrary space-time geometry. In principle this problem is solved by representation of the Maxwell equations and the gravitation equations in terms of the world function $\sigma_{\sigma\mathrm{R}}$ of the Riemannian space-time geometry. Thereafter the world function $\sigma_{\sigma\mathrm{R}}$ is replaced by the world function σ of arbitrary physical space-time geometry. Then one needs to choose such a space-time geometry, which agrees with the experimental data.

However, an attempt of generalization of Maxwell equations for the electromagnetic field meets a difficulty. In the space-time geometry of Minkowski the dynamic equations for the electromagnetic potential A_k have the form

$$g_{\mathrm{M}}^{ik}\partial_i\partial_k A_l = \frac{4\pi}{c}j_l, \qquad F_{ik} = \partial_i A_k - \partial_k A_i \tag{2.1}$$

where F_{ik} is the tensor of the electromagnetic field, and $j_l(x)$ is the 4-vector of the electric current, generating the electromagnetic field. The world function between the points x and x' has the form

$$\sigma_{\mathrm{M}}(x, x') = \frac{1}{2}g_{\mathrm{M}ik}\left(x^i - x'^i\right)\left(x^k - x'^k\right) \tag{2.2}$$

The first equation (2.1) can be integrated in the form

$$A_l(x) = -\frac{4\pi}{c}\int G_{\mathrm{ret}}(x - x')\, j_l(x')\, d^4x' \tag{2.3}$$

where the retarded Green function $G_{\rm ret}(x-x')$ satisfies the equation

$$g_{\rm M}^{ik}\partial_i\partial_k G_{\rm ret}(x-x') = -\delta^{(4)}(x-x') = -\prod_{i=0}^{i=3}\delta\left(x^i - x'^i\right) \tag{2.4}$$

and has the form

$$G_{\rm ret}(x-x') = \frac{1}{2\pi}\theta\left(x^0 - x^{0\prime}\right)\delta\left(2\sigma_{\rm M}(x,x')\right) \tag{2.5}$$

$$\theta(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases}$$

Here for simplicity we consider the case, when the nonvanishing 4-current density is concentrated inside small spatial region.

To write dynamic equations (2.1) in the Riemannian space-time geometry, conventionally one replaces the partial derivatives by covariant derivatives. One obtains instead of (2.1)

$$g^{ik}\nabla_i\nabla_k A_l = \frac{4\pi}{c}j_l, \qquad F_{ik} = \nabla_i A_k - \nabla_k A_i = \partial_i A_k - \partial_k A_i \tag{2.6}$$

where ∇_k means the covariant derivative.

It is rather difficult to express differential equations in terms of the world function $\sigma_{\sigma\rm R} = \sigma_{\rm R}$ of the σ-Riemannian geometry, because the world function is an integral (two-point) quantity. The finite relations and integral relations are expressed in terms of the world function more effective. Let us replace the world function $\sigma_{\rm M}$ by the world function $\sigma_{\rm R}$ in the expression (2.5) and substitute the obtained expression in the relation of the type of (2.4). We omit the first factor $\frac{1}{2\pi}\theta\left(x^0 - x^{0\prime}\right)$, because it gives a contribution to dynamic equation only at $x^0 = x^{0\prime}$. We obtain

$$\begin{aligned} & g^{ik}\nabla_i\nabla_k\left(\delta\left(2\sigma_{\rm R}(x,x')\right)\right) \\ = \; & g^{ik}\nabla_i\left(\delta'\left(2\sigma_{\rm R}(x,x')\right)2\sigma_{{\rm R}|k}\right) \\ = \; & g^{ik}\left(\delta''\left(2\sigma_{\rm R}(x,x')\right)4\sigma_{{\rm R}|k}\sigma_{{\rm R}|i} + \delta'\left(2\sigma_{\rm R}(x,x')\right)2\sigma_{{\rm R}|k|i}\right) \end{aligned} \tag{2.7}$$

where the vertical stroke means the covariant derivative.

$$\sigma_{{\rm R}|k} = \nabla_k\sigma_{\rm R}(x,x') = \frac{\partial\sigma_{\rm R}(x,x')}{\partial x^k}$$

We use the world function $\sigma_{\rm R}$ instead $\sigma_{\sigma\rm R}$, because these quantities coincide.

Let us take into account the identity

$$x\delta''(x) + 2\delta'(x) = 0 \tag{2.8}$$

and the fact, that the Riemannian world function satisfies the relation [5]

$$\sigma_{{\rm R}|k}g^{ik}\sigma_{{\rm R}|i} = 2\sigma_{\rm R} \tag{2.9}$$

We obtain from (2.7)

$$g^{ik}\nabla_i\nabla_k\left(\delta\left(2\sigma_{\mathrm{R}}\left(x,x'\right)\right)\right)=2\delta'\left(2\sigma_{\mathrm{R}}\left(x,x'\right)\right)\left(g^{ik}\sigma_{\mathrm{R}|k|i}-4\right)\tag{2.10}$$

If the Riemannian space-time coincides with the 4-dimensional space-time of Minkowski, the rhs of (2.10) vanishes, because the quantity $g^{ik}\sigma_{\mathrm{M}|k|i}$ is a scalar, which in the inertial coordinate system has the form

$$g^{ik}\sigma_{\mathrm{M}|k|i}=4\tag{2.11}$$

Note, that the relation (2.11) takes place only in the 4-dimensional space-time of Minkowski.

In the case of arbitrary Riemannian space-time the equation is not valid, in general, and rhs of (2.10) does not vanish, in general. It means, that the transition from the space-time of Minkowski to the Riemannian space-time by means of replacement of the world function of Minkowski in the relations (2.3) - (2.5) and the replacement procedure of partial derivatives by the covariant ones in the dynamic equations (2.1) are different procedures.

Thus, writing the Maxwell equations in terms of the world function, we are to choose between two alternatives: (1) use of conventional representation of the Riemannian geometry, which pretends to a single–variance, which is not single-variant in reality and (2) use of σ-Riemannian geometry, which multivariant and nonaxiomatizable, in general.

The σ-Riemannian geometry has the advantage of the Riemannian geometry in the sense, that it is consistent, whereas the Riemannian geometry is inconsistent.

Procedure of deduction of dynamic equations (2.6) in the Riemannian geometry is founded on a use of curvilinear coordinate system. Dynamic equations (2.1) are written in the curvilinear coordinate system of the space-time of Minkowski in the form (2.6). Thereafter one declares, that the form (2.6) of dynamic equations is valid in arbitrary Riemannian space-time. However, it appears, that this declaration is incompatible with the replacement of the world function σ_{M} by the world function σ_{R} in solutions of dynamic equations (2.6) in the Minkowski space-time. A use of the coordinate system in deduction of dynamic equations (2.6) in the case of the Riemannian geometry seems to be problematic (compare the role of coordinate system (1.6) in the definition of equivalence (1.1), (1.2)).

3 Concluding remarks

Thus, trying to generalize the Maxwell equation on the case of non-Riemannian geometry, we meet unexpected problem, that the conventional presentation of the Maxwell equations in the Riemannian geometry appears to be problematic. It is possible, that, the same problem will appear at an attempt of generalization of the gravitation equation. One should look for the way around these problems.

References

[1] Yu.A.Rylov, Geometry without topology as a new conception of geometry. *Int. Jour. Mat. & Mat. Sci.* **30**, iss. 12, 733-760, (2002), (Available at http://arXiv.org/abs/math.MG/0103002).

[2] Yu. A.Rylov, Tubular geometry construction as a reason for new revision of the space-time conception. Printed in *What is Geometry?* polimetrica Publisher, Italy, pp.201-235, 2005.

[3] Yu. A. Rylov, Generalization of relativistic particle dynamics on the case of non-Riemannian space-time geometry *e-print http://arXiv.org/abs/0811.4562*

[4] Yu. A. Rylov, Geometrical dynamics: spin as a result of rotation with superluminal speed. *e-print http://arXiv.org/abs/0801.1913*

[5] J.L.Synge, *Relativity: the general theory,* North-Holland Publishing company, Amsterdam, 1960.