Spin and Wave Function as Attributes of Ideal Fluid

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Abstract

An ideal fluid whose internal energy depends on density, density gradient, and entropy is considered. Dynamic eqautions are integrated, and a description in terms of hydrodynamic (Clebsch) potentials arises. All essential information on the fluid flow (including initial and boundary conditions) appears to be carried by the dynamic equations for hydrodynamic potentials. Information on initial values of the fluid flow is carried by arbitrary integration functions. Initial and boundary conditions for potentials contain only unessential information concerning the fluid particle labeling. It is shown that a description in terms of n-component complex wave function is a kind of such a description in terms of hydrodinamic potentials. Spin determined by the irreducible number n_m of the wave function components appears to be an attribute of the fluid flow. Classification of fluid flows by the spin appears to be connected with invariant subspaces of the relabeling group.

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I Introduction

Ideal (nondissipative) fluid with the internal energy E of a very general form is considered. The internal energy E is supposed to depend on the fluid density ρ, density gradient $\nabla\rho$, and entropy per unit mass S. The stress tensor for such a fluid has the form

$$P^{\alpha\beta}=\delta_{\alpha\beta}[\rho^2\frac{\partial E}{\partial\rho}+\frac{\partial(\rho E)}{\partial\rho_\gamma}\rho_\gamma]-\rho\partial_\alpha\frac{\partial(\rho E)}{\partial\rho_\beta},\qquad \alpha,\beta=1,2,3 \tag{1}$$

$$\rho_\alpha\equiv\partial_\alpha\rho,\qquad \partial_i\equiv\frac{\partial}{\partial x^i},\qquad i=0,1,2,3$$

If $E=E(\rho,S)$ does not depend on $\nabla\rho$, the stress tensor has the form

$$P^{\alpha\beta}=p\delta_\beta^\alpha$$

where $p=\rho^2\partial E/\partial\rho$ is the pressure. Conventionally the dependence of the internal energy on the $\nabla\rho$ is not considered. There are two motives for consideration of such an unusual fluid.

First, a proper dependence of E on $\nabla\rho$ prevents sound waves from tilting. Indeed, let the internal energy have the form

$$E=E_0(\rho,S)+a(\nabla\rho/\rho)^2 \tag{2}$$

where a is a small positive quantity. For usual laminar flows, where $\nabla\rho/\rho$ is small, the last term of (2) does not give a significant contribution in the stress tensor (1), and it is of no importance whether or not there is the last term in (2). In the case of the wave tilting the last term in (2) becomes to be principal. On the front of the tilted sound wave E tends to ∞, and the tilting of the wave may be stopped.

Second. Fluid models with the internal energy of a very general form are used for a description of statistical ensembles of stochastic particles. By definition a statistical ensemble $\mathcal{E}[\mathcal{S}_{\rm st}]$ of stochastic particles $\mathcal{S}_{\rm st}$ is a set of many independent identical stochastic particles $\mathcal{S}_{\rm st}$. Usually the term "statistical ensemble" associates with some tool for calculation of average values of physical quantities. But this tool is effective, provided the statistical ensemble is a set of deterministic (non-stochastic) particles. In reality the principal property of the statistical ensemble is formulated as follows. *The statistical ensemble of many stochastic (or deterministic) particles is a deterministic dynamic system.* This statement sounds rather unexpected, because for a statistical ensebmle of deterministic systems this property looks as a trivial one. In the statistical physics the statistical ensembles of deterministic systems are considered mainly (the only exception statistical ensemble of Brownian particles), and the property of the statistical ensemble of being a deterministic dynamic system needs some explanations [1, 2].

A result of an experiment with a single stochastic particle $\mathcal{S}_{\rm st}$ is irreproducible. But distributions of results of similar experiments with many independent stochastic particles are reproducible. Projecting many independent identical stochastic particles $\mathcal{S}_{\rm st}$ in the same space-time region, one obtains a cloud $\mathcal{E}[N,\mathcal{S}_{\rm st}]$ of N independent identical particles $\mathcal{S}_{\rm st}$ moving randomly. With the number N of particles tending to ∞, this cloud $\mathcal{E}[\infty,\mathcal{S}_{\rm st}]$ may be considered as a continuous medium, or as a fluid. This fluid is a deterministic dynamic system, because experiments with the fluid $\mathcal{E}[\infty,\mathcal{S}_{\rm st}]$ are reproducible. Besides any reproducible experiments with the stochastic particle can be described in terms of the fluid $\mathcal{E}[\infty,\mathcal{S}_{\rm st}]$ without a reference to any probabilistic construction (i.e. without a reference to the property of the statistical ensemble of being a tool for calculation of average values). The probabilistic constructions are effective only, if the statistical ensemble $\mathcal{E}[\infty,\mathcal{S}_{\rm d}]$ consists of deterministic particles $\mathcal{S}_{\rm d}$ whose properties can be determined independently of $\mathcal{E}[\infty,\mathcal{S}_{\rm d}]$. In the case of $\mathcal{E}[\infty,\mathcal{S}_{\rm st}]$ these probabilistic constructions (probability density, or probability amplitude) are needed only for interpretation of the fluid in terms of a single stochastic particle. (See for details [3]).

For instance, let us consider a single electron $\mathcal{S}_{\rm st}$, flying from an electron gun, passing through a narrow slit in a diaphragm and hitting a screen at a point x_1. Another electron $\mathcal{S}_{\rm st}$, prepared in the same way, hits the screen at other point x_2 which does not coincide with x_1. In other words, an experiment with single electron is irreproducible in general. It means that a single electron is a stochastic particle. Let us consider a series of $N \quad (N\to\infty)$ experiments with identically prepared independent electrons. Distribution of N impact points over the screen is reproducible, i.e. it is approximately the same for other series of N experiments. It means that a set $\mathcal{E}[N,\mathcal{S}_{\rm st}]$ of $N \quad (N\to\infty)$ independent identical electrons $\mathcal{S}_{\rm st}$ is a deterministic dynamic system, although a single electron $\mathcal{S}_{\rm st}$ is a stochastic system. If $\mathcal{E}[\infty,\mathcal{S}_{\rm st}]$ can be considered as a fluid, then solving dynamic equations for this fluid and calculating flux of the fluid $\mathcal{E}[\infty,\mathcal{S}_{\rm st}]$ through the screen, one can calculate the diffraction picture (distribution of the impact points over the screen). For such a calculation one needs only characteristics of the dynamic system $\mathcal{E}[\infty,\mathcal{S}_{\rm st}]$ (dynamic equations, expressions for the particle flux and the energy-momentum tensor). Any quantum axiomatics and corresponding probabilistic constructions (wave function, linear operators, commutation relations, etc.) are not needed. It means that quantum effects can be explained and calculated as purely dynamical effects [3].

On the other hand, quantum particles are described conventionally in terms of wave functions. The wave function is considered as a fundamental object which cannot be defined via other more fundamental objects. As a result, as any fundamental object, the wave function and its properties are defined by a system of axioms (quantum axiomatics, or quantum principles). Some connection of the wave function with the irrotational flow of some quantum (Madelung) fluid [4]-[10] is known for a long time. (Connection of the wave function with the irrotational flow was discovered comparatively recently [11]. But all the time the wave function is considered as a fundamental object, whereas the quantum fluid is considered as a derivative object. In this paper the fluid is considered as a fundamental object, connected directly with the statistical description of stochastic particles, whereas the wave function is considered as a derivative construction whose properties can be expressed via properties of the fluid.

In general, the wave function as a property of the fluid satisfies the quantum principles (linearity of dynamic equations, etc.) only in some special cases. For instance in the case, when the internal energy (2) depends only on $\mathbf{v}_{\mathrm{dif}} = -\hbar(2m)^{-1}\nabla\rho/\rho$ and has the form

$$E = E(\rho, \nabla\rho) = \frac{\mathbf{v}_{\mathrm{dif}}^2}{2}, \qquad \mathbf{v}_{\mathrm{dif}} = -\frac{\hbar\nabla\rho}{2m\rho} \tag{3}$$

where m is the particle mass, and $\hbar$ is the Planck constant. To avoid misunderstandings and to differ between the wave function as a fundamental object, satisfying the quantum axiomatics, and the wave function as a property of a fluid, we shall use two different terms "wave function" in the first case and "ψ-function" in the second one.

It is very important that the quantum phenomena are connected directly with the fluid model, i.e. such a connection does not contain any reference to the quantum principles. There is a hope that quantum superfluids like the liquid Helium may be described as an ideal fluid with the internal energy depending on $\nabla\rho$.

In the present paper some mathematical properties of conservative dynamic systems are investigated. Such a system $\mathcal{S}$ is a continuous set of particles, interacting via some self-consistent potential force field V. The dynamic system $\mathcal{S}$ is described by the action of the form.

$$\mathcal{A}_{\mathrm{L}}[\mathbf{x}] = \int \{\frac{m}{2}(\frac{d\mathbf{x}}{dt})^2 - V\}\rho_0(\xi)\mathrm{d}t\mathrm{d}\xi \tag{4}$$

where $\mathbf{x} = \{x^\alpha(t,\xi)\}$, $\alpha = 1,2,3$ are functions of time t and of particle labels $\xi = \{\xi_1, \xi_2, \xi_3\}$. $\rho_0(\xi)$ is some non-negative weight function, and $m =$const is some mass of the fluid particle. V is a self-consistent potential depending on $\mathbf{x}$ and derivatives of $\mathbf{x}$ with respect to ξ. This function is supposed to have such a form that the potential V is a given function of variables ρ, $\nabla\rho$, and S. Here

$$\rho \equiv m\rho_0(\xi)\frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(x^1, x^2, x^3)}, \tag{5}$$

and $S = S_0(\xi)$ is some fixed function of variables ξ. In this case the dynamic system $\mathcal{S}$ may be considered as some ideal fluid. It will be shown that in the Euler description, where $x = \{t, \mathbf{x}\}$ are independent variables, and ξ, ρ, $\mathbf{v} \equiv \mathrm{d}\mathbf{x}/\mathrm{d}t$, S are dependent variables, the action (4) generates dynamic equations of the form

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{v}) = 0 \tag{6}$$

$$\frac{\partial v^\alpha}{\partial t} + (\mathbf{v}\nabla)v^\alpha = -\frac{1}{\rho}\partial_\beta P^{\alpha\beta}, \qquad \alpha = 1,2,3, \tag{7}$$

$$\frac{\partial S}{\partial t} + (\mathbf{v}\nabla)S = 0 \tag{8}$$

where $x^0 = t$ is the time, $\mathbf{x} = \{x^1, x^2, x^3\}$ is the position vector, ρ and $\mathbf{v} = \{v^1, v^2, v^3\}$ are respectively the fluid mass density and the fluid velocity considered as functions of $x = \{t, \mathbf{x}\}$. $P^{\alpha\beta}$ is a stress tensor, defined by (1), and $E(\rho, \nabla\rho, S) =$

$V(\rho, \nabla\rho, S)/m$ is an internal energy of an unite mass. The E depends on the density ρ, on the density gradient $\nabla\rho$, and on the entropy S per unit mass.

In the case of an usual fluid, when the V does not depend on $\nabla\rho$, the stress tensor $P^{\alpha\beta}$ is isotropic, and the equation (7) turns to the Euler equation for the ideal fluid

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p, \qquad p = \rho^2\frac{\partial E}{\partial\rho} \tag{9}$$

where p is the pressure, and $E = E(\rho, S)$ is an internal energy of an unite mass considered as a function of ρ and S. Thus, if V depends only on variables $\rho, \nabla\rho, S$ the dynamic system $\mathcal{S}$, described by the action (4) will be referred to as nondissipative (ideal) fluid.

The system of hydrodynamic equations (6)–(8), as well as the system (6), (9), (8) is a closed system of differential equations which has an unique solution inside some space-time region Ω, provided dependent dynamic variables ρ and $\mathbf{v} = \{v^1, v^2, v^3\}$, S are given as functions of three arguments on the space-time boundary Γ of the region Ω. Nevertheless, being closed, the system (6)–(8) is incomplete, because it describes only momentum-energetic characteristics of the fluid. The action (4) generates additional dynamic equations

$$\frac{\partial\xi}{\partial t} + (\mathbf{v}\nabla)\xi = 0, \tag{10}$$

known as Lin constraints [12]. These equations describe motion of fluid particles along their trajectories.

If the equations (10) are solved and ξ is determined as a function of $(t, \mathbf{x})$, the finite relations

$$\xi(t, \mathbf{x}) = \xi_{\text{in}} = \text{const}$$

describe implicitly a fluid particle trajectory and a motion along it.

The system of eight equations (6)–(8), (10) forms a complete system of dynamic equations describing a fluid, whereas the system of five equations (6)–(8) forms a curtailed system of dynamic equations. The last system is closed, but to be a complete system, it must be supplemented by the kinematic equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(t, \mathbf{x}), \qquad \mathbf{x} = \mathbf{x}(t, \xi), \tag{11}$$

or by the Lin constraints (10) which are equivalent to (11).

The fact that the complete system (6)–(8), (10) of dynamic equations admits a closed subsystem (6)–(8) is connected with the invariance of the system (6) –(8), (10) with respect to the group of relabeling transformations (relabeling group)

$$\xi_\alpha \rightarrow \tilde{\xi}_\alpha = \tilde{\xi}_\alpha(\xi), \qquad D = \det \parallel \partial\tilde{\xi}_\alpha/\partial\xi_\beta \parallel \neq 0, \qquad \alpha, \beta = 1, 2, 3 \tag{12}$$

$$\varphi = \xi_0 \rightarrow \tilde{\xi}_0 = \tilde{\varphi} = \tilde{\xi}_0(\xi_0) + a_0(\xi), \qquad \partial\tilde{\xi}_0/\partial\xi_0 > 0 \tag{13}$$

where $\xi = \{\xi_0, \xi\}$ are curvilinear Lagrangian coordinates in the space-time, $\tilde{\xi} = \{\tilde{\xi}_0, \tilde{\xi}\}$ is another system of curvilinear Lagrangian coordinates. $\tilde{\xi}$ and a_0 are arbitrary functions of ξ. $\tilde{\xi}_0$ is arbitrary function of ξ_0. ξ_0 is a temporal Lagrangian coordinate, and ξ are spatial ones.

The relabeling group properties are used in hydrodynamics comparatively recently [13, 14, 15, 16, 17, 18, 19, 20]. The action (4) is invariant with respect to the relabeling group (12), (13), provided the weight function $\rho_0(\xi)$ transforms as a scalar density

$$\rho_0(\xi) \rightarrow \tilde{\rho}_0(\tilde{\xi}) = D^{-1}\rho_0(\xi), \qquad D = \frac{\partial(\tilde{\xi})}{\partial(\xi)} \equiv \frac{\partial(\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3)}{\partial(\xi_1, \xi_2, \xi_3)} \tag{14}$$

The group of relabeling transformations appears to be a symmetry group of the dynamic system (fluid). Any special particle labeling is unessential from physical viewpoint. It is a reason why several equations (6) –(8) of the complete system form a closed system describing conservation laws. This symmetry group admits also to integrate the complete system (6) –(8), (10) in the form (see the proof below Sec.3)

$$S(t, \mathbf{x}) = S_0(\xi) \tag{15}$$

$$\rho(t, \mathbf{x}) = \rho_0(\xi) \frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(x^1, x^2, x^3)} \equiv \rho_0(\xi) \frac{\partial(\xi)}{\partial(\mathbf{x})} \tag{16}$$

$$\mathbf{v}(t, \mathbf{x}) = \pi(\varphi, \xi, \eta, S) \equiv \nabla\varphi + g^\alpha(\xi)\nabla\xi_\alpha - \eta\nabla S, \tag{17}$$

where $S_0(\xi)$, $\rho_0(\xi)$, $g(\xi) = \{g^\alpha(\xi)\}$, $\alpha = 1, 2, 3$ are arbitrary integration functions of argument ξ, and φ, η are new dependent variables, satisfying dynamic equations

$$\frac{\partial\varphi}{\partial t} + \pi(\varphi, \xi, \eta, S)\nabla\varphi - \frac{1}{2}[\pi(\varphi, \xi, \eta, S)]^2 + \frac{\partial(\rho E)}{\partial\rho} - \partial_\alpha \frac{\partial(\rho E)}{\partial\rho_\alpha} = 0 \tag{18}$$

$$\frac{\partial\eta}{\partial t} + \pi(\varphi, \xi, \eta, S)\nabla\eta = -\frac{\partial E}{\partial S}. \tag{19}$$

If five dependent variables φ, ξ, η satisfy the system of equations (10), (18), (19), the five dynamic variables S, ρ, $\mathbf{v}$ (15)–(17) satisfy dynamic equations (6)–(8). Indefinite functions $S_0(\xi)$, $\rho_0(\xi)$, $\mathbf{g}(\xi)$ can be determined from initial and boundary conditions in such a way that the initial and boundary conditions for variables φ, ξ, η were universal in the sense that they do not depend on the fluid flow.

The integration of the complete system (6)–(8), (10) and some corollaries of this integration correlate with the Hamilton properties of the ideal fluid [24, 14, 27, 23, 19, 20]. It is connected with the fact that the curtailed system (6)–(8) is not a Hamiltonian system in itself, whereas the complete system (6)–(8), (10) is a Hamiltonian one. Constructing Hamiltonian mechanics of the ideal fluid, one uses (implicitly or explicitly) the Lin constraints (or part of them). It is this expansion of the curtailed system (but not Hamiltonian properties) that is important for integration and derivation of other useful results. To show this, the Hamiltonian technique and Hamiltonian properties of the ideal fluid will not be used at all.

According to (16), (17) the physical quantities ρ, $\mathbf{v}$ are obtained as a result of differentiation of the variables φ, ξ, S, and the variables φ, ξ, η can be regarded as hydrodynamic potentials. These potentials appear in the Hamilton fluid dynamics [23] as dependent variables. They associate with the name of Clebsch [21, 22] who introduced these quantities for the incompressible fluid. Such quantities as $g^\alpha(\xi)$ also appear in the Hamilton fluid mechanics, [23] but they appear as dependent variables (Lagrange invariants) satisfying dynamic equations of the type (10). They also are regarded as hydrodynamic potentials. Note that in the Hamilton fluid mechanics [23] the quantities g^{α} are considered simply as dependent variables, but not as indefinite functions of ξ arising as a result of integration, although corresponding dynamic equations for g^{α} can be integrated easily.

Integration of the dynamic equations admits a description of any ideal fluid in terms of hydrodynamic potentials $\xi = \{\xi_0, \boldsymbol{\xi}\}$. The hydrodynamic potentials ξ are Lagrangian coordinates considered as functions of independent Eulerian coordinates $x = \{t, \mathbf{x}\}$. Spatial Lagrangian coordinates $\boldsymbol{\xi} = \{\xi_\alpha\}$, $\alpha = 1, 2, 3$ label fluid particles, whereas the temporal Lagrangian coordinate $\xi_0 = \xi_0(t, \mathbf{x})$ means some generalized time for the fluid particle placed at the space-time point $x = \{t, \mathbf{x}\}$.

The description of any ideal fluid in terms of hydrodynamic potentials ξ can transform into a description in terms of a complex n-component hydrodynamic potential $\psi = \{\psi_\alpha\}$, $\alpha = 1, 2, \dots n$ which associates with the wave function, used in the quantum mechanics, whereas the irreducible (minimally possible) number n_m of the ψ-function components, associates with the spin of the flow (not of the particle).

In the presented paper it is shown that the wave function is a way of a description of any ideal fluid. The spin is a natural property of any flow of the ideal fluid. Appearance of these enigmatic quantities at the description of quantum particles may be explained merely as a result of a quantum particle description in terms of an ideal fluid (statistical ensemble). Note that the curtailed system (6)–(8) has the same order as the integrated system (10), (18), (19), but it takes into account neither initial conditions, nor kinematic equations (11). The fact that the ideal fluid considered as a dynamic system admits both the curtailed system (6)–(8) and the integrated system (10), (18), (19) is connected closely with the group of the relabeling transformation (12).

The second section is devoted to presentation of the space-time symmetric Jacobian technique which is needed for integration of hydrodynamic equations. Use of Jacobians in hydrodynamics has had a long history, dating back to the time of Clebsch [21, 22]. It was the use of Jacobians that allowed to introduce the Clebsch potentials and integrate hydrodynamic equations. The Jacobian technique was used in [24, 14, 25, 23, 20] and many other papers). It seems that the progress in the integration of hydrodynamic equations is connected mainly with the developed Jacobian technique.

Further it will be proved (Sec.3) that the complete system of hydrodynamic equations (6)–(8), (10) can be integrated in the form (10), (15), (19) that leads to a special form of a description in terms of hydrodynamic potentials (DTHP). In the fourth section the initial and boundary conditions are used for determination of function $\mathbf{g}$. In the fifth section a special type of a complex hydrodynamic potentials is considered and the fluid flows are classified on the irreducible number of the wave function components which appears to be an invariant of the relabeling group.

II Jacobian technique

Let us consider such a space-time symmetric mathematical object as the Jacobian

$$J\equiv \frac{\partial(\xi_0,\xi_1,\xi_2,\xi_3)}{\partial(x^0,x^1,x^2,x^3)}\equiv \det\parallel \xi_{i,k}\parallel, \qquad \xi_{i,k}\equiv\partial_k\xi_i\equiv\frac{\partial\xi_i}{\partial x^k}, \qquad i,k=0,1,2,3 \quad (20)$$

Here $\xi=\{\xi_0,\boldsymbol{\xi}\}=\{\xi_0,\xi_1,\xi_2,\xi_3\}$ are four scalars considered as functions $\xi=\xi(x)$ of $x=\{x^0,\mathbf{x}\}$. The functions $\{\xi_0,\xi_1,\xi_2,\xi_3\}$ are supposed to be independent in the sense that $J\neq 0$. It is useful to consider the Jacobian J as 4-linear function of variables $\xi_{i,k}\equiv\partial_k\xi_i$, $i,k=0,1,2,3$. Then one can introduce derivatives of J with respect to $\xi_{i,k}$. The derivative $\partial J/\partial\xi_{i,k}$ appears as a result of a substitution of ξ_i by x^k in the relation (20).

$$\frac{\partial J}{\partial\xi_{i,k}}\equiv\frac{\partial(\xi_0,...\xi_{i-1},x^k,\xi_{i+1},...\xi_3)}{\partial(x^0,x^1,x^2,x^3)}, \qquad i,k=0,1,2,3 \quad (21)$$

For instance

$$\frac{\partial J}{\partial\xi_{0,i}}\equiv\frac{\partial(x^i,\xi_1,\xi_2,\xi_3)}{\partial(x^0,x^1,x^2,x^3)}, \qquad i=0,1,2,3 \quad (22)$$

This rule is valid for higher derivatives of J also.

$$\frac{\partial^2 J}{\partial\xi_{i,k}\partial\xi_{s,l}}\equiv\frac{\partial(\xi_0,...\xi_{i-1},x^k,\xi_{i+1},...\xi_{s-1},x^l,\xi_{s+1},...\xi_3)}{\partial(x^0,x^1,x^2,x^3)}\equiv$$

$$\frac{\partial(x^k,x^l)}{\partial(\xi_i,\xi_s)}\frac{\partial(\xi_0,\xi_1,\xi_2,\xi_3)}{\partial(x^0,x^1,x^2,x^3)}\equiv J(\frac{\partial x^k}{\partial\xi_i}\frac{\partial x^l}{\partial\xi_s}-\frac{\partial x^k}{\partial\xi_s}\frac{\partial x^l}{\partial\xi_i}), \qquad i,k,l,s=0,1,2,3 \quad (23)$$

It follows from (20), (21) that

$$\frac{\partial x^k}{\partial\xi_i}\equiv\frac{\partial(\xi_0,...\xi_{i-1},x^k,\xi_{i+1},...\xi_3)}{\partial(\xi_0,\xi_1,\xi_2,\xi_3)}\equiv\frac{\partial(\xi_0,...\xi_{i-1},x^k,\xi_{i+1},...\xi_3)}{\partial(x^0,x^1,x^2,x^3)}\times$$

$$\frac{\partial(x^0,x^1,x^2,x^3)}{\partial(\xi_0,\xi_1,\xi_2,\xi_3)}\equiv\frac{1}{J}\frac{\partial J}{\partial\xi_{i,k}}, \qquad i,k=0,1,2,3 \quad (24)$$

and (23) may be written in the form

$$\frac{\partial^2 J}{\partial\xi_{i,k}\partial\xi_{s,l}}\equiv\frac{1}{J}(\frac{\partial J}{\partial\xi_{i,k}}\frac{\partial J}{\partial\xi_{s,l}}-\frac{\partial J}{\partial\xi_{i,l}}\frac{\partial J}{\partial\xi_{s,k}}), \qquad i,k,l,s=0,1,2,3 \quad (25)$$

The derivative $\partial J/\partial\xi_{i,k}$ is a cofactor to the element $\xi_{i,k}$ of the determinant (20). Then one has the following identities

$$\xi_{l,k}\frac{\partial J}{\partial\xi_{s,k}}\equiv\delta_l^s J, \qquad \xi_{k,l}\frac{\partial J}{\partial\xi_{k,s}}\equiv\delta_l^s J, \qquad l,s=0,1,2,3 \quad (26)$$

$$\partial_k\frac{\partial J}{\partial\xi_{i,k}}\equiv\frac{\partial^2 J}{\partial\xi_{i,k}\partial\xi_{s,l}}\partial_k\partial_l\xi_s\equiv 0, \qquad i=0,1,2,3. \quad (27)$$

Here and further a summation on two repeated indices is produced (0-3) for Latin indices and (1-3) for the Greek ones. The identity (27) can be considered as a corollary of the identity (25) and a symmetry of $\partial_k\partial_l\xi_s$ with respect to permutation of indices k, l. Convolution of (25) with ∂_k, or ∂_l vanishes also.

Relations (20) –(25) are written for four independent variables x, but they are valid in an evident way for arbitrary number $n+1$ of variables $x=\{x^0,x^1,\ldots x^n\}$ and $\xi=\{\xi_0,\boldsymbol{\xi}\}$, $\quad \boldsymbol{\xi}=\{\xi_1,\xi_2,\ldots\xi_n\}$.

Application of the Jacobian J to hydrodynamics is founded on the property, that the fluid flux

$$j^i=m\frac{\partial J}{\partial\xi_{0,i}},\qquad j=\{j^i\}=\{\rho,\rho\mathbf{v}\},\qquad i=0,1,2,3 \tag{28}$$

constructed on the basis of the variables $\boldsymbol{\xi}=\{\xi_1,\xi_2,\xi_3\}$ satisfies Lin constraints (10) and the continuity equation

$$\partial_i j^i=0 \tag{29}$$

identically for any choice of variables $\boldsymbol{\xi}$, as it follows from the identity (27) for $i=0$. The continuity equation (29) is used without approximations in all hydrodynamic models, and the change of variables $\{\rho,\rho\mathbf{v}\}\leftrightarrow\boldsymbol{\xi}$ described by (28) is very important.

In particular, in the case of two-dimensional established flow of incompressible fluid the variables $\boldsymbol{\xi}$ reduce to one variable $\xi_1=\psi$, known as the stream function. In this case there are only two essential dependent variables $x^0=x$, $x^1=y$, and the relations (28), (29) reduce to relations

$$\rho^{-1}j_x=v_x=\frac{\partial\psi}{\partial y},\qquad \rho^{-1}j_y=v_y=-\frac{\partial\psi}{\partial x},\qquad \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=0 \tag{30}$$

Defining the stream line as a line tangent to the flux j

$$\frac{dx}{j_x}=\frac{dy}{j_y}, \tag{31}$$

one obtains that the stream function is constant along the stream line, because according to two first equations (30), $\psi=\psi(x,y)$ is an integral of the equation (31).

In the general case, when the space dimensionality is n and $x=\{x^0,x^1,\ldots x^n\}$, $\xi=\{\xi_0,\boldsymbol{\xi}\}$, $\boldsymbol{\xi}=\{\xi_1,\xi_2,\ldots\xi_n\}$, the quantities $\boldsymbol{\xi}=\{\xi_\alpha\}$, $\alpha=1,2,\ldots n$ are constant along the line $\mathcal{L}$ tangent to the flux vector $j=\{j^i\}$, $i=0,1,\ldots n$

$$\mathcal{L}:\qquad \frac{dx^i}{d\tau}=j^i(x),\qquad i=0,1,\ldots n \tag{32}$$

where τ is a parameter along the line $\mathcal{L}$ which is described parametrically by the equation $x=x(\tau)$. This statement is formulated mathematically in the form

$$\frac{d\xi_\alpha}{d\tau}=j^i\partial_i\xi_\alpha=m\frac{\partial J}{\partial\xi_{0,i}}\partial_i\xi_\alpha=0,\qquad \alpha=1,2,\ldots n$$

The last equality follows from the first identity (26) taken for $s=0$, $l=1,2,\ldots n$

Interpretation of the line (32) tangent to the flux is different for different cases. If $x = \{x^0, x^1, \ldots x^n\}$ contains only spatial coordinates, the line (32) is a line in the usual space. It is regarded as a stream line, and ξ can be interpreted as quantities which are constant along the stream line (i.e. as a generalized stream function). If x^0 is the time coordinate, the equation (32) describes a line in the space-time. This line (known as a world line of a fluid particle) determines a motion of the fluid particle. Variables $\xi = \{\xi_1, \xi_2, \ldots \xi_n\}$ which are constant along the world line are different, generally, for different particles. If ξ_α, $\alpha = 1, 2, \ldots n$ are independent, they may be used for the fluid particle labeling.

Thus, although interpretation of the relation (28) considered as a replacement of dependent variables j by ξ may be different, from the mathematical viewpoint this transformation means a replacement of the continuity equation by some equations for the labeling (or generalized stream function) ξ. Difference of the interpretation is of no importance in this context.

Note that the expressions

$$j^i = m\rho_0(\xi)\frac{\partial J}{\partial \xi_{0,i}} \equiv m\rho_0(\xi)\frac{\partial(x^i, \xi_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)}, \qquad i = 0, 1, 2, 3, \tag{33}$$

can be also considered as four-flux satisfying the continuity equation (29). Here m is a constant and $\rho_0(\xi)$ is an arbitrary function of ξ. It follows from the identity

$$m\rho_0(\xi)\frac{\partial(x^i, \xi_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)} \equiv m\frac{\partial(x^i, \tilde{\xi}_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)}, \qquad \tilde{\xi}_1 = \int\limits_0^{\xi_1} \rho_0(\xi_1', \xi_2, \xi_3) d\xi_1'.$$

As an example of application of the Jacobian technique, let us show that (5) satisfies (6) in virtue of (10). Let us multiply (10) by (5) and introduce new variables $\mathbf{j} = \rho\mathbf{v} = \{j^1, j^2, j^3\}$. One obtains three equations

$$m\rho_0(\xi)\frac{\partial J}{\partial \xi_{0,0}}\xi_{\beta,0} + j^\alpha \xi_{\beta,\alpha} = 0, \qquad \beta = 1, 2, 3. \tag{34}$$

Considering (34) as a system of three linear equations for j^α, $\alpha = 1, 2, 3$ and resolving it with respect to j^α, one obtains

$$j^\alpha = m\rho_0(\xi)\frac{\partial J}{\partial \xi_{0,\alpha}}, \qquad \alpha = 1, 2, 3 \tag{35}$$

It is easy to verify this, substituting (35) into (34) and using (26). One obtains that $j = \{j^0, \mathbf{j}\} = \{\rho, \rho\mathbf{v}\}$ is described by the relations (33) which satisfy the continuity equation (29) identically. Thus, (6) is satisfied by (16) in virtue of (10).

III Variational principle

In general, equivalency of the system (10), (18), (19) and the system (6)–(8), (10) can be verified by a direct substitution of variables ρ, S, $\mathbf{v}$, defined by the relations

(15)–(17), into the equations (6)–(8). Using equations (10), (18), (19), one obtains identities after subsequent calculations. But such computations do not display a connection between the integration and the invariance with respect to the relabeling group (12). Besides a meaning of new variables φ, η is not clear. We shall use for our investigations a variational principle. Note that for a long time a derivation of a variational principle for hydrodynamic equations (6)–(8) was existing as a self-dependent problem [26], [24], [14], [12], [27], [16], [23]. Existence of this problem was connected with a lack of understanding that the system of hydrodynamic equations (6)–(8) is a curtailed system, and the full system of dynamic equations (6)-(8), (10) includes equations (10) describing a motion of the fluid particles in the given velocity field. The variational principle can generate only the complete system of dynamic variables (but not its closed subsystem). Without understanding this one tried to form the Lagrangian for the system (6)–(8) as a sum of some quantities taken with Lagrange multipliers. lhs of dynamic equations (6)–(8) and some other constraints were taken as such quantities.

Now this problem has been solved (see review by Salmon [23]) on the basis of the Eulerian version of the variational principle for the Lagrangian description (4), where equations (10) appear automatically and cannot be ignored. In our version of the variational principle we follow [23] with some modifications which underline a curtailed character of hydrodynamic equations (6)–(8), because the understanding of the curtailed character of the system (6)–(8) removes the problem of derivation of the variational principle for the hydrodynamic equations (6)–(8).

We consider the ideal fluid as a conservative dynamic system whose dynamic equations can be derived from the variational principle. This dynamic system is a continuous set of many identical particles moving in some self-consistent potential force field. The action functional has the form (4). Variation of the action with respect to $\mathbf{x}$ generates six first order dynamic equations for six dependent variables $\mathbf{x}$, $\mathbf{v} = d\mathbf{x}/dt$, considered as functions of t and of independent curvilinear Lagrangian coordinates ξ. It is a Lagrangian representation of hydrodynamic equations.

We prefer to work with Eulerian representation, when Lagrangian coordinates (particle labeling) $\xi = \{\xi_0, \xi\}$, $\xi = \{\xi_1, \xi_2, \xi_3\}$ are considered as dependent variables, and Eulerian coordinates $x = \{x^0, \mathbf{x}\} = \{t, \mathbf{x}\}$, $\mathbf{x} = \{x^1, x^2, x^3\}$ are considered as independent variables. Here ξ_0 is a temporal Lagrangian coordinate which evolves along the particle trajectory in an arbitrary way. Now the ξ_0 is a fictitious variable, but after integration of equations the ξ_0 stops to be fictitious and turns to the variable φ, appearing in the integrated system (10), (18), (19).

Further mainly space-time symmetric designations will be used, that simplifies considerably all computations. In the Eulerian description the action functional (4) is to be represented as an integral over independent variables $x = \{x^0, \mathbf{x}\} = \{t, \mathbf{x}\}$. One uses the Jacobian technique for such a transformation of the action (4),

Let us note that according to (22) the derivative $\mathrm{d}\mathbf{x}/\mathrm{d}t$ can be written in the form

$$v^\alpha = \frac{dx^\alpha}{dt} \equiv \frac{\partial J}{\partial \xi_{0,\alpha}} \left(\frac{\partial J}{\partial \xi_{0,0}} \right)^{-1}, \qquad \alpha = 1, 2, 3.$$

Then components of the 4-flux $j = \{j^0, \mathbf{j}\} \equiv \{\rho, \rho\mathbf{v}\}$ can be written in the form

(33), provided the designation (5)

$$j^0 = \rho = m\rho_0(\xi)\frac{\partial J}{\partial \xi_{0,0}} \equiv m\rho_0(\xi)\frac{\partial(x^0,\xi_1,\xi_2,\xi_3)}{\partial(x^0,x^1,x^2,x^3)} \tag{36}$$

is used.

At such a form of the mass density ρ the four-flux $j = \{j^i\}$, $i = 0, 1, 2, 3$ satisfies identically the continuity equation (29) which takes place in virtue of identities (26), (27). Besides in virtue of identities (26), (27) the Lin constraints (10) are fulfilled identically

$$j^i \partial_i \xi_\alpha = 0, \qquad \alpha = 1, 2, 3. \tag{37}$$

Components j^i are invariant with respect to the relabeling group (12), provided the function $\rho_0(\xi)$ transforms according to (14).

One has

$$\rho_0(\xi)\mathrm{d}t\mathrm{d}\xi = \rho_0(\xi)\frac{\partial J}{\partial \xi_{0,0}}\mathrm{d}t\mathrm{d}\mathbf{x} = \frac{\rho}{m}\mathrm{d}t\mathrm{d}\mathbf{x}$$

$$\frac{m}{2}\left(\frac{dx^\alpha}{dt}\right)^2 = \frac{m}{2}\left(\frac{\partial J}{\partial \xi_{0,\alpha}}\right)^2\left(\frac{\partial J}{\partial \xi_{0,0}}\right)^{-2} = \frac{m}{2}\left(\frac{j^\alpha}{\rho}\right)^2,$$

and the variational problem with the action functional (4) is written as a variational problem with the action functional

$$\mathcal{A}_{\mathrm{E}}[\xi] = \int (\frac{\mathbf{j}^2}{2\rho} - \rho E)\mathrm{d}t\mathrm{d}\mathbf{x}, \qquad E = \frac{V}{m} \tag{38}$$

where $\rho = j^0$ and $\mathbf{j} = \{j^1, j^2, j^3\}$ are fixed functions of $\xi = \{\xi_0, \xi\}$ and of $\xi_{\alpha,i} \equiv \partial_i \xi_\alpha$, $\alpha = 1, 2, 3$, $i = 0, 1, 2, 3$, defined by the relations (33). E is the internal energy of the fluid which is supposed to be a fixed function of ρ, $\nabla\rho$, $S_0(\xi)$

$$E = E(\rho, \nabla\rho, S_0(\xi)), \tag{39}$$

where ρ is defined by (36) and $S_0(\xi)$ is some fixed function of ξ, describing initial distribution of the entropy over the fluid.

The action (38) is invariant with respect to subgroup $\mathcal{G}_{S_0}$ of the relabeling group (12). The subgroup $\mathcal{G}_{S_0}$ is determined in such a way that any surface $S_0(\xi)$ =const is invariant with respect to $\mathcal{G}_{S_0}$. In general, the subgroup $\mathcal{G}_{S_0}$ is determined by two arbitrary functions of ξ.

The action (38) generates the six order system of dynamic equations, consisting of three second order equations for three dependent variables ξ. Invariance of the action (38) with respect to the subgroup $\mathcal{G}_{S_0}$ admits one to integrate the system of dynamic equations. The order of the system reduces, and two arbitrary integration functions appear. The order of the system reduces to five (but not to four), because the fictitious dependent variable ξ_0 stops to be fictitious as a result of the integration.

Unfortunately, the subgroup $\mathcal{G}_{S_0}$ depends on the form of the function $S_0(\xi)$ and cannot be obtained in a general form. In the special case, when $S_0(\xi)$ does not depend on ξ, the subgroup $\mathcal{G}_{S_0}$ coincides with the whole relabeling group $\mathcal{G}$, and the order of the integrated system reduces to four.

In the general case it is convenient to introduce a new dependent variable

$$S = S_0(\xi).$$

According to (37) the variable S satisfies the dynamic equation (8)

$$j^i \partial_i S = 0. \tag{40}$$

In virtue of designations (28) and identities (26), (27) the equations (40), (37) are fulfilled identically. Hence, they can be added to the action functional (38) as side constraints without a change of the variational problem. Adding (40) to the Lagrangian of the action (38) by means of a Lagrange multiplier η, one obtains

$$\mathcal{A}_{\mathrm{E}}[\xi, \eta, S] = \int \{ \frac{\mathbf{j}^2}{2\rho} - \rho E + \eta j^k \partial_k S \} \mathrm{d}t \mathrm{d}\mathbf{x} \tag{41}$$

where the quantities $j = \{\rho, \mathbf{j}\}$ are determined by (33), and $E = E(\rho, \nabla\rho, S)$. The action (41) is invariant with respect to the relabeling group $\mathcal{G}$ which is determined by three arbitrary functions of ξ.

To obtain the dynamic equations, it is convenient to introduce new dependent variables j^i, defined by (33). Let us introduce the new variables j^i by means of designations (33) taken with the Lagrange multipliers p_i, $i = 0, 1, 2, 3$. Then the action (41) takes the form

$$\mathcal{A}_{\mathrm{E}}[\rho, \mathbf{j}, \xi, p, \eta, S] = \int \{ \frac{\mathbf{j}^2}{2\rho} - \rho E - p_k [j^k - m\rho_0(\xi) \frac{\partial J}{\partial \xi_{0,k}}] + \eta j^k \partial_k S \} \mathrm{d}t \mathrm{d}\mathbf{x} \tag{42}$$

It is useful to keep in mind that four designations (33), introducing variables ρ, $\mathbf{j} = \rho\mathbf{v}$ via variables ξ, are equivalent to three Lin constraints (10) together with the designation (36), as it was shown in the end of sec.2. Addition of relations (33) to the action (41) as side constraints is equivalent to the addition of relations (10), (36) considered as side constraints.

For obtaining dynamic equations, the variables $\rho, \mathbf{j}, \xi, p, \eta, S$ are to be varied. Let us eliminate the variables p_i from the action (42). Dynamic equations arising as a result of a variation with respect to ξ_α have the form

$$\frac{\delta \mathcal{A}_{\mathrm{E}}}{\delta \xi_\alpha} \equiv \hat{\mathcal{L}}_\alpha p = -m\partial_k [\rho_0(\xi) \frac{\partial^2 J}{\partial \xi_{0,i} \partial \xi_{\alpha,k}} p_i] + m \frac{\partial \rho_0(\xi)}{\partial \xi_\alpha} \frac{\partial J}{\partial \xi_{0,k}} p_k = 0, \qquad \alpha = 1, 2, 3 \tag{43}$$

where $\hat{\mathcal{L}}_\alpha$ are linear operators acting on variables $p = \{p_i\}$, $i = 0, 1, 2, 3$. These equations can be integrated in the form

$$p_i = bg^0(\xi_0)\partial_i \xi_0 + bg^\alpha(\xi)\partial_i \xi_\alpha, \qquad i = 0, 1, 2, 3, \tag{44}$$

where b is an arbitrary scale constant, ξ_0 is some new variable (temporal Lagrangian coordinate), $g^\alpha(\xi)$, $\alpha = 1, 2, 3$ are arbitrary functions of the labels ξ, $g^0(\xi_0)$ is an arbitrary function of ξ_0. The relations (44) satisfy equations (43) identically. Indeed, substituting (44) into (43) and using identities (25), (26), one obtains

$$-m\partial_k\left\{\rho_0(\xi)\left[\frac{\partial J}{\partial\xi_{\alpha,k}}g^0(\xi_0)-\frac{\partial J}{\partial\xi_{0,k}}g^\alpha(\xi)\right]\right\}+m\frac{\partial\rho_0(\xi)}{\partial\xi_\alpha}Jg^0(\xi_0)=0,\qquad \alpha=1,2,3, \tag{45}$$

Differentiating braces and using identities (27), (26), one concludes that (45) is an identity.

Setting for simplicity

$$\partial_k\varphi=g^0(\xi_0)\partial_k\xi_0,\qquad k=0,1,2,3$$

one obtains

$$p_k=b\partial_k\varphi+bg^\alpha(\xi)\partial_k\xi_\alpha,\qquad k=0,1,2,3 \tag{46}$$

Substituting (46) in (42), one can eliminate variables p_i, $i=0,1,2,3$ from the functional (42). The term $g^\alpha(\xi)\partial_k\xi_\alpha\partial J/\partial\xi_{0,k}$ vanishes, the term $\partial_k\varphi\partial J/\partial\xi_{0,k}$ gives no contribution into dynamic equations. The action functional takes the form

$$\mathcal{A}_{\mathbf{g}}[\rho,\mathbf{j},\varphi,\xi,\eta,S]=\int\{\frac{\mathbf{j}^2}{2\rho}-\rho E-j^k[b\partial_k\varphi+bg^\alpha(\xi)\partial_k\xi_\alpha-\eta\partial_kS]\}\mathrm{d}t\mathrm{d}\mathbf{x} \tag{47}$$

where $g^\alpha(\xi)$ are considered as fixed functions of ξ which are determined from initial conditions. The action (47) is a functional of indefinite fixed functions $\mathbf{g}(\mathbf{x})$. Varying the action (47) with respect to φ, ξ, η, S, $\mathbf{j}$, ρ, one obtains dynamic equations

$$\delta\varphi:\qquad \partial_kj^k=0, \tag{48}$$

$$\delta\xi_\alpha:\qquad \Omega^{\alpha\beta}j^k\partial_k\xi_\beta=0,\qquad \alpha=1,2,3, \tag{49}$$

$$\delta\eta:\qquad j^k\partial_kS=0, \tag{50}$$

$$\delta S:\qquad j^k\partial_k\eta=-\rho\frac{\partial E}{\partial S}, \tag{51}$$

$$\delta\mathbf{j}:\qquad \mathbf{v}\equiv\mathbf{j}/\rho=b\nabla\varphi+bg^\alpha(\xi)\nabla\xi_\alpha-\eta\nabla S, \tag{52}$$

$$\delta\rho:\qquad -\frac{\mathbf{j}^2}{2\rho^2}-\frac{\partial(\rho E)}{\partial\rho}+\partial_\alpha\frac{\partial(\rho E)}{\partial\rho_\alpha}-b\partial_0\varphi-bg^\alpha(\xi)\partial_0\xi_\alpha+\eta\partial_0S=0, \tag{53}$$

Here $\Omega^{\alpha\beta}$ is defined by

$$\Omega^{\alpha\beta}=b\left(\frac{\partial g^\alpha(\xi)}{\partial\xi_\beta}-\frac{\partial g^\beta(\xi)}{\partial\xi_\alpha}\right),\qquad \alpha,\beta=1,2,3 \tag{54}$$

Deriving relations (49), (51), the continuity equation (48) was used. It is easy to see that (49) is equivalent to (10), provided

$$\det\parallel\Omega^{\alpha\beta}\parallel\neq0 \tag{55}$$

Then the equations (50) and (48) can be integrated in the form of (15) and (16) respectively. Equations (51) and (52) are equivalent to (19) and (17). Finally, eliminating $\partial_0\xi_\alpha$ and $\partial_0 S$ from (53) by means of (49) and (50), one obtains the equation (18) and, hence, the system of dynamic equations (10), (18), (19), where designations (15)–(17) are used.

The curtailed system (6)–(8) can be obtained from equations (48)–(53) as follows. Equations (48), (50) coincide with (6), (8). For deriving (7) let us note that the vorticity $\omega \equiv \nabla \times \mathbf{v}$ and $\mathbf{v} \times \omega$ are obtained from (52) in the form

$$\omega = \nabla \times \mathbf{v} = \frac{1}{2}\Omega^{\alpha\beta}\nabla\xi_\beta \times \nabla\xi_\alpha - \nabla\eta \times \nabla S \tag{56}$$

$$\mathbf{v} \times \omega = \Omega^{\alpha\beta}\nabla\xi_\beta(\mathbf{v}\nabla)\xi_\alpha + \nabla S(\mathbf{v}\nabla)\eta - \nabla\eta(\mathbf{v}\nabla)S \tag{57}$$

Let us form a difference between the time derivative of (52) and the gradient of (53). Eliminating $\Omega^{\alpha\beta}\partial_0\xi_\alpha$, $\partial_0 S$ and $\partial_0\eta$ by means of equations (49), (50), (51), one obtains

$$\partial_0\mathbf{v} + \nabla\frac{\mathbf{v}^2}{2} + \frac{\partial^2(\rho E)}{\partial\rho^2}\nabla\rho + \frac{\partial^2(\rho E)}{\partial\rho\partial S}\nabla S + \nabla\rho_\beta\frac{\partial^2(\rho E)}{\partial\rho_\beta\partial\rho} - \nabla\partial_\beta\frac{\partial^2(\rho E)}{\partial\rho_\beta}$$

$$-\frac{\partial E}{\partial S}\nabla S - \Omega^{\alpha\beta}\nabla\xi_\beta(\mathbf{v}\nabla)\xi_\alpha + \nabla\eta(\mathbf{v}\nabla)S - \nabla S(\mathbf{v}\nabla)\eta = 0 \tag{58}$$

Using (56), (57) the expression (58) reduces to

$$\partial_0\mathbf{v} + \nabla\frac{\mathbf{v}^2}{2} - \mathbf{v} \times (\nabla \times \mathbf{v}) + \frac{1}{\rho}\nabla(\rho^2\frac{\partial E}{\partial\rho}) - \frac{1}{\rho}\partial_\beta\left[\rho\nabla\frac{\partial^2(\rho E)}{\partial\rho_\beta}\right] = 0 \tag{59}$$

In virtue of the identity

$$\mathbf{v} \times (\nabla \times \mathbf{v}) \equiv \nabla\frac{\mathbf{v}^2}{2} - (\mathbf{v}\nabla)\mathbf{v}$$

the last equation is equivalent to (7). The form of the stress tensor (1) appears as a result of transformations of the relation (59) to the form (7). The stress tensor (1) is determined to within the tensor with a vanishing divergence.

Thus, differentiating equations (52), (53) and eliminating the variables φ, ξ, η, one obtains the curtailed system (6)–(8), whereas the system (10), (18), (19) follows from the system (48)–(53) directly (i.e. without differentiating). It means that the system (10), (18), (19) is an integrated system, whereas the curtailed system (6)–(8) is not, although formally they have the same order.

The action of the form (47), or close to this form was obtained by some authors [27], [23], but the quantities g^α, $\alpha = 1, 2, 3$ are always considered as additional dependent variables (but not as indefinite functions of ξ which can be expressed via initial conditions). The action was not considered as a functional of fixed indefinite functions $g^\alpha(\xi)$.

The variable η was introduced, for making the action invariant with respect to the transformations of the whole relabeling group (12). To understand what the η means from the mathematical viewpoint, let us return to the action (38), where the internal energy E has the form (39). Adding new variables j by means of designations (33), one obtains instead of (42)

$$\mathcal{A}_{\mathrm{E}}[\rho, \mathbf{j}, \xi, p] = \int \{ \frac{\mathbf{j}^2}{2\rho} - \rho E - p_k [j^k - m\rho_0(\xi) \frac{\partial J}{\partial \xi_{0,k}}]\} \mathrm{d}t \mathrm{d}\mathbf{x} \tag{60}$$

where E has the form (39).

Variation of (60) with respect to ξ_α leads to the equation

$$\hat{\mathcal{L}}_\alpha p = \rho \frac{\partial E(\rho, S_0(\xi))}{\partial S_0} \frac{\partial S_0}{\partial \xi_\alpha}, \qquad \alpha = 1, 2, 3 \tag{61}$$

where linear operators $\hat{\mathcal{L}}_\alpha$ are defined by (43). Equations (61) are linear non-uniform equations for the variables p. A solution of (61) is a sum of the general solution (46) of the uniform equations (43) and of a particular solution the non-uniform equations (61). This particular solution depends on the form of the function S_0 and cannot be found in a general form. Adding an extraterm $-\eta j^k \partial_k S$ with η satisfying (51) to (41), a reduction of non-uniform equations (61) to uniform equations (43) appears to be possible. Thus, the extravariable η is responsible for the particular solution of (61).

From the viewpoint of the action (60) a dependence of the internal energy E on the entropy means simply a dependence of E on the labels ξ via a function $S(\xi)$. If such a dependence cannot be expressed through one function (for instance $E = E[\rho, S_1(\xi), S_2(\xi)]$) the ideal fluid is described by two entropies S_1 and S_2 and by two temperatures $T_1 = \partial E/\partial S_1$, $\quad T_2 = \partial E/\partial S_2$. Such a situation may appear for a conducting fluid in a strong magnetic field, where there are two temperatures – longitudinal and transversal.

Thus five equations (10), (18), (19) with S, ρ and $\mathbf{v}$, defined respectively by (15), (16) and (17), constitute the fifth order system for five dependent variables $\xi = \{\xi_0, \xi\}$, η. Equations (6), (8), (10), (18), (19) constitute the seventh order system for seven variables ρ, ξ, φ, η, S.

IV Initial and Boundary Conditions

Boundary conditions describing vessel walls can be taken into account by means of a proper choice of the internal energy $E(x, \rho, \nabla\rho, S)$ which can include the energy of the fluid in an external potential U.

$$E = E_0(\rho, \nabla\rho, S) + U(t, \mathbf{x}),$$

where U is some given external potential. For instance, let the fluid move inside a volume $\mathcal{V}$. Then

$$U(\mathbf{x}) = \begin{cases} 0, & \text{inside} \quad \mathcal{V} \\ \infty, & \text{outside} \quad \mathcal{V} \end{cases}$$

Such a choice of the energy E provides that the fluid does not escape the volume $\mathcal{V}$.

In this section let us set for simplicity the scale constant $b=1$, and consider the case, when E does not depend on $\nabla\rho$, and the fluid flow is considered in the space-time region Ω defined by inequalities

$$\Omega: \qquad t\geq 0, \qquad x^3\geq 0$$

The region Ω has two boundaries: $\mathcal{I}$ defined by the relations $t=0$, $x^3\geq 0$, and $\mathcal{B}$ defined by the relations $x^3=0$, $t\geq 0$. The initial conditions for the system of equations (6)–(8), (10) have the form

$$\rho(0,\mathbf{x})=\rho_{\mathrm{in}}(\mathbf{x}), \qquad v^\alpha(0,\mathbf{x})=v_{\mathrm{in}}^\alpha(\mathbf{x}), \qquad \alpha=1,2,3 \tag{62}$$

$$S(0,\mathbf{x})=S_{\mathrm{in}}(\mathbf{x}), \qquad \xi_\alpha(0,\mathbf{x})=\xi_{\mathrm{in}}^\alpha(\mathbf{x}), \qquad \alpha=1,2,3 \tag{63}$$

at $\mathbf{x}\in\mathcal{I}$ ($t=0$, $x^3\geq 0$). Here ρ_{in}, $\mathbf{v}_{\mathrm{in}}$, S_{in}, ξ_{in} are given functions of argument $\mathbf{x}$. The boundary conditions on the boundary $\mathcal{B}$ of Ω have the form:

$$\rho(x)|_{x^3=0}=\rho_{\mathrm{b}}(t,\mathbf{y}), \qquad S(x)|_{x^3=0}=S_{\mathrm{b}}(t,\mathbf{y}), \qquad \{t,\mathbf{y}\}\in\mathcal{B} \tag{64}$$

$$v^\alpha(x)|_{x^3=0}=v_{\mathrm{b}}^\alpha(t,\mathbf{y}), \qquad \alpha=1,2,3, \qquad \{t,\mathbf{y}\}\in\mathcal{B} \tag{65}$$

$$\xi_\alpha(x)|_{x^3=0}=\xi_{\mathrm{b}}^\alpha(t,\mathbf{y}), \qquad \alpha=1,2,3, \qquad \{t,\mathbf{y}\}\in\mathcal{B} \tag{66}$$

where

$$\mathbf{y}\equiv\{x^1,x^2\} \tag{67}$$

Here ρ_{b}, S_{b}, $\mathbf{v}_{\mathrm{b}}$, ξ_{b} are given functions of the argument $\{t,\mathbf{y}\}$.

Let us show that indefinite functions $\mathbf{g}$, S_0, ρ_0 can be expressed via initial and boundary conditions (62)–(66). The initial conditions for the system (48)–(53) have the form

$$\xi_\alpha(0,\mathbf{x})=\xi_{\mathrm{in}}^\alpha(\mathbf{x}), \qquad \alpha=1,2,3 \tag{68}$$

$$\rho(0,\mathbf{x})=\rho_{\mathrm{in}}(\mathbf{x}), \qquad S(0,\mathbf{x})=S_0[\xi_{\mathrm{in}}(\mathbf{x})], \tag{69}$$

$$\varphi(0,\mathbf{x})=\varphi_{\mathrm{in}}(\mathbf{x}), \qquad \eta(0,\mathbf{x})=\eta_{\mathrm{in}}(\mathbf{x}), \tag{70}$$

(68)-(70) take place at $\mathbf{x}\in\mathcal{I}$. The functions $\varphi_{\mathrm{in}}(\mathbf{x}),\eta_{\mathrm{in}}(\mathbf{x})$ as well $g^\alpha(\xi)$ are to be determined from the relations

$$\partial_\alpha\varphi_{\mathrm{in}}(\mathbf{x})+g^\beta[\xi_{\mathrm{in}}(\mathbf{x})]\partial_\alpha\xi_{\mathrm{in}}^\beta(\mathbf{x})-\eta_{\mathrm{in}}(\mathbf{x})\frac{\partial S_0[\xi_{\mathrm{in}}(\mathbf{x})]}{\partial\xi_{\mathrm{in}}^\beta}\partial_\alpha\xi_{\mathrm{in}}^\beta(\mathbf{x})=$$

$$=v_{\mathrm{in}}^\alpha(\mathbf{x}), \qquad \alpha=1,2,3; \qquad \mathbf{x}\in\mathcal{I}. \tag{71}$$

It is clear that five functions $\mathbf{g},\varphi_{\mathrm{in}},\eta_{\mathrm{in}}$ cannot be determined unambiguously from three relations (71).

There are at least two different approaches to determination of functions $\xi_{\mathrm{in}}(\mathbf{x})$ and $\mathbf{g}(\xi)$.

(1) One fixes the functions $\xi_{\mathrm{in}}^\alpha(\mathbf{x})$ in some conventional way, sets

$$\varphi_{\mathrm{in}}(\mathbf{x})=0, \qquad \eta_{\mathrm{in}}(\mathbf{x})=0, \qquad \mathbf{x}\in\mathcal{I} \tag{72}$$

and determines functions $\mathbf{g}$ from three relations (71).

(2) Functions $\mathbf{g}$ are fixed in some conventional way, and remaining functions are determined from relations (71)

The first way. Let the condition (68) be given in the form

$$\xi_\alpha(0,\mathbf{x}) = \xi_{\mathrm{in}}^\alpha(\mathbf{x}) = x^\alpha, \qquad \alpha = 1,2,3, \qquad \mathbf{x} \in \mathcal{I}. \tag{73}$$

In other words, at $t = 0$ the labels ξ coincide with the Eulerian coordinates. The relations (71) take the form

$$g^\beta[\xi_{\mathrm{in}}(\mathbf{x})] = v_{\mathrm{in}}^\beta(\mathbf{x}), \qquad \alpha = 1,2,3; \qquad \mathbf{x} \in \mathcal{I}, \tag{74}$$

which are resolved in the form

$$g^\alpha(\xi) = v_{\mathrm{in}}^\alpha(\xi), \qquad \alpha = 1,2,3, \qquad \xi_3 > 0, \tag{75}$$

Thus, the functions $\mathbf{g}$ are expressed through initial conditions (62).

The boundary conditions for the system of equations (48)-(53) have the form

$$\xi_\alpha(x)|_{x^3=0} = \xi_{\mathrm{b}}^\alpha(t,\mathbf{y}), \qquad \alpha = 1,2,3, \qquad \{t,\mathbf{y}\} \in \mathcal{B} \tag{76}$$

$$S(x)|_{x^3=0} = S_0[\xi_{\mathrm{b}}(t,\mathbf{y})] = S_{\mathrm{b}}(t,\mathbf{y}), \qquad \{t,\mathbf{y}\} \in \mathcal{B}, \tag{77}$$

$$\rho(x)|_{x^3=0} = \rho_{\mathrm{b}}(x)|_{x^3=0}, \qquad \mathbf{v}(x)|_{x^3=0} = \mathbf{v}_{\mathrm{b}}(t,\mathbf{y}), \qquad \{t,\mathbf{y}\} \in \mathcal{B}, \tag{78}$$

$$\varphi(x)|_{x^3=0} = \eta(x)|_{x^3=0} = 0, \qquad \{t,\mathbf{y}\} \in \mathcal{B}, \tag{79}$$

Let us set

$$\xi_{\mathrm{b}}^\alpha(t,\mathbf{y}) = x^\alpha, \qquad \alpha = 1,2; \qquad \xi_{\mathrm{b}}^3(t,\mathbf{y}) = -ct, \qquad (t,\mathbf{y}) \in \mathcal{B}, \tag{80}$$

where c is a constant.

Writing relations (10) and (53) for $\xi_3 < 0$ on the boundary $\mathcal{B}$ and using (79), (80), one obtains constraints for the functions $\mathbf{g}(\xi)$

$$g^\beta[\xi_{\mathrm{b}}(t,\mathbf{y})]\partial_\alpha \xi_{\mathrm{b}}^\beta(t,\mathbf{y}) = v_{\mathrm{b}}^\alpha(t,\mathbf{y}), \qquad \alpha = 1,2, \qquad \{t,\mathbf{y}\} \in \mathcal{B} \tag{81}$$

$$g^\beta[\xi_{\mathrm{b}}(t,\mathbf{y})]\partial_0 \xi_{\mathrm{b}}^\beta(t,\mathbf{y}) = -K_{\mathrm{b}}(t,\mathbf{y}), \qquad \{t,\mathbf{y}\} \in \mathcal{B}, \tag{82}$$

where

$$K_{\mathrm{b}}(t,\mathbf{y}) \equiv \frac{\mathbf{v}_{\mathrm{b}}^2(t,\mathbf{y})}{2} + \frac{\partial\{\rho_{\mathrm{b}}(t,\mathbf{y})E[\rho_{\mathrm{b}}(t,\mathbf{y}), S_{\mathrm{b}}(t,\mathbf{y})]\}}{\partial\rho_{\mathrm{b}}(t,\mathbf{y})}, \qquad \{t,\mathbf{y}\} \in \mathcal{B}, \tag{83}$$

Substituting relations (80) into (81), (82), one obtains three equations for determination of functions $\mathbf{g}(\xi)$. Resolving this system of equations with respect to $\mathbf{g}$, one obtains

$$g^\alpha(\xi) = v_{\mathrm{b}}^\alpha(-\xi_3/c, \xi_1, \xi_2), \qquad \alpha = 1,2; \qquad \xi_3 < 0$$

$$g^3(\xi) = c^{-1}K_{\mathrm{b}}(-\xi_3/c, \xi_1, \xi_2), \qquad \xi_3 < 0 \tag{84}$$

Thus, $\mathbf{g}(\xi)$ is determined by (75) for $\xi_3 > 0$ and by (84) for $\xi_3 < 0$. In other words, the boundary conditions and the initial conditions determine the vector field $\mathbf{g}(\xi)$ in different regions of the argument ξ. The field $\mathbf{g}(\xi)$ can describe both initial and boundary conditions. For any fluid flow the system (10), (18), (19) of dynamic equations for variables φ, η, ξ is to be solved under universal initial conditions (72), (73) and under universal boundary conditions (79), (80). All essential information on the fluid flow is found in the dynamic equations (10), (18), (19), where the quantities S, ρ, $\mathbf{v}$ are determined by (15)-(17).

The second way. Let us choose the functions $\mathbf{g}$ in a simple form. Let for instance,

$$g^1(\xi) = \xi_2, \qquad g^2(\xi) = 0, \qquad g^3(\xi) = 0$$

Let us set

$$\chi = \varphi, \qquad \lambda = \xi_2, \qquad \mu = \xi_1$$

Then the expression (17) takes the form

$$\mathbf{u}(\chi, \lambda, \mu, \eta, S) \equiv \nabla\chi + \lambda\nabla\mu - \eta\nabla S = \mathbf{v} \tag{85}$$

where χ, λ, μ, are Clebsch potentials [21, 22]. Now six equations (6), (8), (49)-(53), (55) [(49) for $\alpha = 3$ is of no importance] for six dependent variables $\rho, \chi, \lambda, \mu, \eta, S$ do not contain indefinite functions and have an unambiguous form.

$$\begin{gathered}\partial_0\rho + \nabla(\rho\mathbf{u}) = 0, \qquad \partial_0\lambda + (\mathbf{u}\nabla)\lambda = 0\\ \partial_0\mu + (\mathbf{u}\nabla)\mu = 0, \qquad \partial_0 S + (\mathbf{u}\nabla)S = 0\\ \partial_0\eta + (\mathbf{u}\nabla)\eta = -\frac{\partial E}{\partial S}, \qquad \partial_0\chi + \lambda\partial_0\mu - \eta\partial_0 S + \frac{1}{2}\mathbf{u}^2 + \frac{\partial(\rho E)}{\partial\rho} = 0\end{gathered} \tag{86}$$

where $\mathbf{u}$ is defined by (85).

The initial conditions for variables $\rho, \chi, \lambda, \mu, \eta, S$ are determined by relations

$$\rho(0, \mathbf{x}) = \rho_{\text{in}}(0, \mathbf{x}), \qquad S(0, \mathbf{x}) = S_{\text{in}}(0, \mathbf{x}), \tag{87}$$

$$\nabla\chi_{\text{in}} + \lambda_{\text{in}}\nabla\mu_{\text{in}} - \eta_{\text{in}}\nabla S_{\text{in}} = \mathbf{v}_{\text{in}} \tag{88}$$

Three equations (87), (88) do not determine the initial conditions

$$\chi(0, \mathbf{x}) = \chi_{\text{in}}(\mathbf{x}), \qquad \lambda(0, \mathbf{x}) = \lambda_{\text{in}}(\mathbf{x}), \tag{89}$$

$$\mu(0, \mathbf{x}) = \mu_{\text{in}}(\mathbf{x}), \qquad \eta(0, \mathbf{x}) = \eta_{\text{in}}(\mathbf{x}), \tag{90}$$

unambiguously.

If the fluid is described in terms of Clebsch potentials, the dynamic equations contain neither arbitrary functions, nor information about the initial conditions. It should be interpreted in the sense that the description (85)-(86) in terms of the Clebsch potentials is a result of a change of variables in dynamic equations (6)-(8), whereas the description (48)-(53) is a result of integration of the dynamic equations (6)-(8). (10). In other words, the description (85)-(86) in terms of Clebsch potentials relates to the description (48)-(53) in the same way, as a particular solution of a system of differential equations relates to a general solution of the same system. Let us note that there are many other ways for determination of indefinite functions $\mathbf{g}(\xi)$.

V Wave Function and Spin

The equations (6), (8), (10), (18), (19) can be derived from the action functional

$$\mathcal{A}[\rho,\varphi,\xi,\eta,S]=\int\rho[-\pi_0(\varphi,\xi,\eta,S)-\frac{1}{2}\pi^2(\varphi,\xi,\eta,S)-E(x,\rho,\nabla\rho,S)]\mathrm{d}^4x \qquad (91)$$

where $\pi=\{\pi_1,\pi_2,\pi_3\}$, and $\pi_k=p_k-\eta\partial_kS$, $\quad k=0,1,2,3$ are determined by relations (46)

$$\pi_k(\varphi,\xi,\eta,S)\equiv b[\partial_k\varphi+g^\alpha(\xi)\partial_k\xi_\alpha]-\eta\partial_kS,\qquad k=0,1,2,3. \qquad (92)$$

The action (91) results from the action (47) after elimination of the variable $\mathbf{j}$ from the relations (47), (52). The functions $\mathbf{g}=\{g^\beta(\xi)\}$, $\beta=1,2,3$ are considered as fixed functions of their arguments. Equations (6), (8), (10), (18), (19) can be obtained as a result of variation with respect to φ, η, ξ, S, ρ respectively. Equation (10) is obtained, provided the field $\mathbf{g}$ is non-potential. If the field $\mathbf{g}$ is potential $g^\alpha(\xi)=\partial\Phi/\partial\xi_\alpha$, it can be included in the variable φ by mean of the substitution

$$\varphi+\Phi\rightarrow\varphi,$$

In this case the action (91) does not depend on ξ, and (10) may be omitted.

Let us introduce n-component complex function $\psi=\{\psi_\alpha\},\alpha=1,2,\dots n$, defining it by the relations

$$\psi_\alpha=\sqrt{\rho}e^{i\varphi}u_\alpha(\xi),\qquad \psi^*_\alpha=\sqrt{\rho}e^{-i\varphi}u^*_\alpha(\xi),\qquad \alpha=1,2,\dots n$$

$$\psi^*\psi\equiv\sum_{\alpha=1}^n\psi^*_\alpha\psi_\alpha$$

where (*) means the complex conjugate, $u_\alpha(\xi)$, $\alpha=1,2,\dots n$ are functions of only variables ξ, and satisfy the relations

$$-\frac{i}{2}\sum_{\alpha=1}^n(u^*_\alpha\frac{\partial u_\alpha}{\partial\xi_\beta}-\frac{\partial u^*_\alpha}{\partial\xi_\beta}u_\alpha)=g^\beta(\xi),\qquad \beta=1,2,3,\qquad \sum_{\alpha=1}^n u^*_\alpha u_\alpha=1 \qquad (93)$$

n is such a natural number that equations (93) admit a solution. In general n may depend on the form of the arbitrary integration functions $\mathbf{g}=\{g^\beta(\xi)\}$, $\beta=1,2,3$.

It is easy to verify that

$$\rho\pi_k(\varphi,\xi,\eta,S)=-\frac{ib}{2}(\psi^*\partial_k\psi-\partial_k\psi^*\cdot\psi)-\eta\partial_kS\psi^*\psi,\qquad k=0,1,2,3 \qquad (94)$$

$$\rho=\psi^*\psi,\qquad \mathbf{j}=-\frac{ib}{2}(\psi^*\nabla\psi-\nabla\psi^*\cdot\psi)-\eta\nabla S\psi^*\psi \qquad (95)$$

The variational problem with the action (91) appears to be equivalent to the variational problem with the action functional

$$\mathcal{A}[\psi,\psi^*,\eta,S]=\int\{\frac{ib}{2}(\psi^*\partial_0\psi-\partial_0\psi^*\cdot\psi)+\eta\partial_0S\psi^*\psi$$

$$-\frac{1}{2\psi^*\psi}[\frac{ib}{2}(\psi^*\nabla\psi-\nabla\psi^*\cdot\psi)+\eta\nabla S\psi^*\psi]^2-E[x,\psi^*\psi,\nabla(\psi^*\psi),S]\psi^*\psi\}\mathrm{d}^4x \quad (96)$$

Note that the function ψ considered as a function of independent variables $\{t,\mathbf{x}\}$ is very indefinite in the sense that the same fluid flow may be described by different ψ-functions. There are two reasons for such an indefiniteness. First, the functions $u_\alpha(\xi)$ are not determined uniquely by differential equations (93). Second, their arguments ξ as functions of x are determined only to within the transformation (12). Description of a fluid in terms of the function ψ is more indefinite, than the description in terms of the hydrodynamic potentials ξ. Information about initial and boundary conditions containing in the functions $\mathbf{g}(\xi)$ is lost at the description in terms of the ψ-function. The ψ-function can be obtained from the Clebsch variables by means of a proper change of variables [11].

Let the function ψ have n components. Regrouping components of the function ψ in the action (96), one obtains the action in the form

$$\mathcal{A}_E[\psi,\psi^*,\eta,S]=\int\{\frac{1}{2}[\psi^*(ib\partial_0+A_0)\psi+(-ib\partial_0\psi^*+A_0\psi^*)\psi]-$$

$$-\frac{1}{2}(ib\nabla\psi^*-\mathbf{A}\psi^*)(-ib\nabla\psi-\mathbf{A}\psi)+$$

$$+\frac{b^2}{4}\sum_{\alpha,\beta=1}^{n}Q^*_{\alpha\beta,\gamma}Q_{\alpha\beta,\gamma}\rho+\frac{b^2}{8\rho}(\nabla\rho)^2-\rho E\}\mathrm{d}^4x, \qquad \rho\equiv\psi^*\psi \quad (97)$$

where

$$A=\{A_0,\mathbf{A}\}, \qquad A_0\equiv\eta\partial_0 S, \qquad \mathbf{A}\equiv\eta\nabla S,$$

$$Q_{\alpha\beta,\gamma}=\frac{1}{\psi^*\psi}\left|\begin{array}{cc}\psi_\alpha & \psi_\beta\\ \partial_\gamma\psi_\alpha & \partial_\gamma\psi_\beta\end{array}\right|, \qquad \alpha,\beta=1,2,\ldots n \qquad \gamma=1,2,3 \quad (98)$$

Corresponding dynamic equations have the form

$$\frac{\delta\mathcal{A}}{\delta\psi^*_\alpha}=(ib\partial_0+A_0)\psi_\alpha-\frac{1}{2}(ib\nabla+\mathbf{A})^2\psi_\alpha-\frac{b^2}{4}\sum_{\mu,\nu=1}^{n}Q^*_{\mu\nu,\gamma}Q_{\mu\nu,\gamma}\psi_\alpha$$

$$+\frac{b^2}{2}\sum_{\nu=1}^{n}Q_{\alpha\nu,\gamma}\partial_\gamma\psi^*_\nu+\frac{b^2}{2}\sum_{\nu=1}^{n}\partial_\gamma(Q_{\alpha\nu,\gamma}\psi^*_\nu)+\frac{\partial}{\partial\rho}[\frac{b^2}{8\rho}(\nabla\rho)^2-\rho E]\psi_\alpha$$

$$-\partial_\gamma\{\frac{\partial}{\partial\rho_\gamma}[\frac{b^2}{8\rho}(\nabla\rho)^2-\rho E]\}\psi_\alpha=0, \qquad \alpha=1,2,\ldots n \quad (99)$$

$$\frac{\delta\mathcal{A}}{\delta S}=\partial_i(j^i\eta)-\frac{\partial(\rho E)}{\partial S}=0, \quad (100)$$

$$\frac{\delta\mathcal{A}}{\delta\eta}=-\partial_i(j^iS)=0, \quad (101)$$

where $j=\{\rho,\mathbf{j}\}=\{j^k\}$, $k=0,1,2,3$ is defined by (95).

In the case of the irrotational flow, when $g^\alpha(\xi)=\partial\Phi(\xi)/\partial\xi_\alpha$ equations (93) have a solution for $n=1$, and the function ψ may have one component. Then all $Q_{\alpha\beta,\gamma}\equiv 0$, as it follows from Eq.(98).

Let us consider an irrotational flow of a fluid with the internal energy per unit mass defined by the relation (3), where m is the mass of a stochastic particle associated with the fluid. The internal energy does not depend on the entropy, and according to (3), and (100) the variable η is a function of only labels ξ. Then the expression $\eta\partial_k S$ has the form $f^\alpha(\xi)\partial_k\xi_\alpha$. It may be included in the term $g^\alpha(\xi)\xi_{\alpha,k}$. It means that without a loss of generality one may set $\eta\equiv 0, \quad S\equiv 0$. Then for an irrotational flow, when ψ-function is one-component, Eq.(99) takes the form

$$\frac{\delta\mathcal{A}}{\delta\psi^*}=ib\partial_0\psi-\frac{b}{2}^2\nabla^2\psi+(b^2-\frac{\hbar^2}{m^2})\{\frac{\partial}{\partial\rho}\frac{(\nabla\rho)^2}{8\rho}-\partial_\gamma[\frac{\partial}{\partial\rho_\gamma}\frac{(\nabla\rho)^2}{8\rho}]\}\psi=0 \qquad (102)$$

Choosing arbitrary constant b in the form $b=-\hbar/m$, one obtains instead of Eq.(102) the well known Schrödinger equation

$$i\hbar\partial_0\psi+\frac{\hbar^2}{2m}\nabla^2\psi=0$$

where the complex variable ψ is known as the wave function. The Schrödinger equation describes an irrotational flow of the Madelung fluid [4].

On this basis it is possible in general to identify the function ψ with the wave function and consider the wave function as a way of description of any ideal fluid. If the fluid flow is rotational, the dynamic equation in terms of the ψ-function is nonlinear, even in the case (3) and at $b=-\hbar/m$. In this case the ψ-function is not one-component, and the quantities $Q_{\alpha\beta,\gamma}$ do not vanish generally.

In general, the dynamic equation (97) for the ψ-function is nonlinear and rather complicated. But for special form (3) of the internal energy and for a special form of the arbitrary phase constant b the dynamic equation in terms of the ψ-function becomes linear and simple.

It is worth to note that the internal energy per unit mass (3) associates with the mean diffusion velocity $\mathbf{v}_{\rm dif}=-D\nabla\rho/\rho$ describing the mean motion of random wandering of stochastic particles (D is the diffusion coefficient). The diffusion velocity is characteristic for any stochastic particles (both Brownian and quantum). The Brownian fluid is dissipative, and the evolution of the fluid state ρ is described directly by $\mathbf{v}_{\rm dif}$ by means of the continuity equation

$$\partial_0\rho+\nabla(\rho\mathbf{v}_{\rm dif})=0$$

For the ideal Madelung fluid the diffusion velocity influences on the fluid flow via the internal fluid energy per unit mass determined by means of the relation (3). Besides the diffusion coefficients D are different for Brownian particles and for quantum ones, because the origin of the stochasticity is different in the two cases.

The number n of the ψ-function components in the actions (96) and (97) is arbitrary. A formal variation of the action with respect to ψ_α and $\psi_\alpha^*, \quad \alpha=1,2,\ldots n$ leads to $2n$ real dynamic equations, but not all of them are independent. There are such combinations of variations $\delta\psi_\alpha$, $\delta\psi_\alpha^*$, $\alpha=1,2,\ldots n$ which do not change expressions (94), (95). Such combinations of variations $\delta\psi_\alpha$, $\delta\psi_\alpha^*$, $\alpha=1,2,\ldots n$ do not change the action (96), and corresponding combinations of dynamic equations

$\delta\mathcal{A}/\delta\psi_\alpha = 0$, $\delta\mathcal{A}/\delta\psi_\alpha^* = 0$ are identities that associates with a correlation between dynamic equations. Thus, increasing the number n, one increases the number of dynamic equations, but the number of independent dynamic equations remains the same.

In such a situation it is important to determine the minimal number n_m of the ψ-function components, sufficient for a description of the given vector field $g^\beta(\xi)$ in the space V_ξ of the labels ξ.

Note that under the relabeling transformations (12) the quantity $\mathbf{g}(\xi)$ transforms as a vector

$$g^\beta(\xi) \to \tilde{g}^\beta(\tilde{\xi}) = \frac{\partial \xi_\alpha}{\partial \tilde{\xi}_\beta} g^\alpha(\xi), \qquad \beta = 1,2,3$$

It is necessary for the quantities (94), (95) and the action (91) to be invariant with respect to the transformation (12)

Let $\mathcal{G}$ be a set of all vector fields $g^\beta(\xi)$ in V_ξ, and $\mathcal{G}_n$ be a set of such vector fields $g^\beta(\xi)$ in V_ξ which can be presented in the form

$$g^\beta(\xi) = \sum_{k=1}^{n} \eta_k^2(\xi)\partial\zeta_k(\xi)/\partial\xi_\beta, \qquad \beta = 1,2,3, \qquad \eta_1 \equiv 1 \tag{103}$$

where n is a fixed natural number and the functions η_k, ζ_k, $k = 1,2,\dots n$ are scalars in V_ξ. Under the relabeling transformation (12) the functions (103) transform as follows

$$\eta_k(\xi) \to \tilde{\eta}_k(\tilde{\xi}) = \eta_k(\xi), \qquad \zeta_k(\xi) \to \tilde{\zeta}_k(\tilde{\xi}) = \zeta_k(\xi), \qquad k = 1,2,\dots n$$

$$g^\beta(\xi) \to \tilde{g}^\beta(\tilde{\xi}) = \frac{\partial \xi_\alpha}{\partial \tilde{\xi}_\beta} g^\alpha(\xi) = \frac{\partial \xi_\alpha}{\partial \tilde{\xi}_\beta} \sum_{k=1}^{n} \eta_k^2(\xi) \frac{\partial \zeta_k(\xi)}{\partial \xi_\alpha} = \sum_{k=1}^{n} \tilde{\eta}_k^2(\tilde{\xi}) \frac{\partial \tilde{\zeta}_k(\tilde{\xi})}{\partial \tilde{\xi}_\alpha}$$

In other words, a vector field $g^\beta(\xi)$ of the form (103) transforms into the vector field $\tilde{g}^\beta(\tilde{\xi})$ of the same form (103), and the set $\mathcal{G}_n$ is invariant with respect to the group (12) of the relabeling transformations.

It is easy to see that

$$\mathcal{G}_{n-1} \subseteq \mathcal{G}_n, \qquad \mathcal{G}_0 = \emptyset, \qquad n = 1,2,\dots$$

because the nth term of the sum (103) can be combined with the first one, if ζ_n is a function of η_n. Let

$$\mathcal{S}_n = \mathcal{G}_n \backslash \mathcal{G}_{n-1}, \qquad n = 1,2,\dots$$

Then

$$\mathcal{G} = \bigcup_{s=1}^{s=n_m} \mathcal{S}_s, \qquad \mathcal{S}_l = \emptyset, \qquad l = n_m+1, n_m+2, \dots$$

where n_m is the number of non-empty invariant subsets of the set $\mathcal{G}$. Each subset $\mathcal{S}_k$ contains only such vector fields $g^\beta(\xi)$ which associate with the k-component ψ-function $\psi = \{\psi_\alpha\}$, $\quad \alpha = 1,2,\dots k$, having the components

$$\psi_1 = \left\{(1 - \sum_{\alpha=2}^{k} \eta_\alpha^2)\rho\right\}^{1/2} \exp[i(\varphi + \zeta_1)],$$

$$\psi_\alpha = \eta_\alpha \sqrt{\rho} \exp[i(\varphi + \zeta_\alpha + \zeta_1)], \qquad \alpha = 2, 3, \dots k$$

In particular, the set $\mathcal{S}_1$ associates with a irrotational flow, described by a one-component ψ-function determined by one scalar ζ_1; and the set $\mathcal{S}_2$ associates with a rotational flow described by a two-component ψ-function, determined by three scalar functions ζ_1, η_2, ζ_2 (Clebsch variables).

In the conventional quantum mechanics the number n of the ψ-function components is connected with the spin s of the particle, described by the ψ, by means of the relation $s = (n-1)/2$. The spin is considered as an internal property of a quantum particle. Particles with different spins are considered as different physical objects, described by different dynamic equations.

In a like manner the irrotational flow, when $g^\beta(\xi)$ is described by one function ζ_1, associates with the kinematic spin (k-spin) $s = 0$, whereas the rotational flow, when $g^\beta(\xi)$ is described by three scalar functions ζ_1, η_2, ζ_2, associates with the kinematic spin $s = 1/2$. The term "kinematic spin" (instead of "spin" simply) is used now for the following reasons.

First, the kinematic spin (k-spin) is determined by the form of the vector field $g^\beta(\xi)$, which arises essentially as a result of an integration of the equations (43). The vector field $g^\beta(\xi)$ is a kinematic structure, because it does not depend on the form of the internal energy. At the same time the k-spin is not an internal property of a particle in itself, because the action (96) describes, at least, two different k-spins ($s = 0$ and $s = 1/2$) simultaneously, and the k-spin looks rather as an integration constant, than a property of single fluid particles.

Second, there is a distinction between the transformation properties of the spin and those of the k-spin under the space rotation group. Components of the n-component ψ-function are scalars under the space rotation group for any value of n (and for any value of the k-spin s). In the conventional quantum mechanics the wave function transforms according to a representation of the rotation group. In particular, a one-component wave function ψ ($s = 0$) is a scalar, whereas the two-component ψ-function ψ ($s = 1/2$) transforms as a spinor under the rotation group.

Taking into account the indefiniteness of the ψ-function, it is possible to change transformation properties of the ψ-function, accompanying any spatial rotation by a proper transformation of the group (12). The additional transformations (12) can be chosen in such a way, that the two-component ψ-function becomes a spinor under spatial rotations. Then the formal distinction between the "k-spin" and the "spin" vanishes, and one can identify them.

For instance, let the two-component ψ-function is written in the form

$$\psi = \sqrt{\rho} \exp[i(\varphi + \sigma\xi)]\chi,$$

where ρ, φ, ξ are scalar functions of x, $\sigma = \{\sigma_\alpha\}$, $\alpha = 1, 2, 3$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and χ is a two-component constant column ($\chi^*\chi = 1$).

Let any infinitesimal spatial rotation

$$x^0 \to \tilde{x}^0 = x^0, \qquad \mathbf{x} \to \tilde{\mathbf{x}} = \mathbf{x} + \omega \times \mathbf{x} + O(\omega^2), \qquad |\, \omega \,| \ll 1 \tag{104}$$

be accompanied by an infinitesimal transformation

$$\xi \to \tilde{\xi} = \xi - \omega/2 \tag{105}$$

Then the ψ-function transforms as a spinor with respect to the combined transformation (104), (105)

$$\psi(x) \to \tilde{\psi}(\tilde{x}) = \sqrt{\tilde{\rho}(\tilde{x})}\exp[i(\tilde{\varphi}(\tilde{x}) + \sigma\tilde{\xi}(\tilde{x}))]\chi = \exp(-i\omega\sigma/2)\psi(x) + O(\omega^2)$$

and as a scalar with respect to the space rotation (104) alone.

If the dynamic system is described in terms of a two-component ψ-function, the labels ξ are not mentioned at all, and the ψ-function can be considered equally readily as a scalar and as a spinor.

For the two-component ψ-function the following identity takes place

$$(\nabla\rho)^2 - (\psi^*\nabla\psi - \nabla\psi^*\psi)^2 \equiv 4\rho\nabla\psi^*\nabla\psi - \rho^2\mathbf{s}^2 \tag{106}$$

$$\rho \equiv \psi^*\psi, \qquad \mathbf{s} \equiv \psi^*\sigma\psi/(2\rho), \qquad \sigma = \{\sigma_\alpha\} \qquad \alpha = 1,2,3 \tag{107}$$

where σ_α are Pauli matrices. In virtue of the identity (106) the action (96) reduces to the form

$$\mathcal{A}[\psi,\psi^*,\eta,S] = \int\{\frac{1}{2}[\psi^*(ib\partial_0 + A_0)\psi + (-ib\partial_0 + A_0)\psi^*\psi] -$$

$$-\frac{1}{2}[ib\nabla\psi^* - \mathbf{A}\psi^*][-ib\nabla\psi - \mathbf{A}\psi] + \frac{b^2}{2}(\nabla s_\alpha)(\nabla s_\alpha)\rho + \frac{b^2}{8\rho}(\nabla\rho)^2 - \rho E\}\mathrm{d}^4x, \tag{108}$$

$$\rho \equiv \psi^*\psi, \qquad A_k \equiv \eta\partial_k S, \qquad k = 0,1,2,3$$

where s_α are defined by Eq.(107). The quantity $\mathbf{s} = \{s_\alpha\}, \quad \alpha = 1,2,3$ associates with the mean spin (especially, if $b = -\hbar/m$), because it is constructed on the base of the Pauli matrices. As one can see, in Eq.(108) s_α convolutes only with s_α, but not with ∇_α. As a result the action (108) is invariant with respect to space-time rotations and relabeling transformations (12) separately.

It is interesting to note in this connection that the action $\mathcal{A}_{\mathrm{P}}[\psi,\psi^*]$ for the dynamical system $\mathcal{S}_{\mathrm{P}}[\psi]$, described by the Pauli equation, implies the convolution between $\mathbf{s}$ and ∇. The action $\mathcal{A}_{\mathrm{P}}[\psi,\psi^*]$ is invariant only with respect to the combined transformation (104), (105), i.e. if the ψ is considered as a spinor, but it is not invariant with respect to the transformation (104), when ψ is considered as a scalar. In other words, the action $\mathcal{A}_{\mathrm{P}}[\psi,\psi^*]$ is not invariant with respect to the rotation group (104), if $\mathcal{S}_{\mathrm{P}}[\psi]$ is considered as a fluidlike dynamic system. The same action $\mathcal{A}_{\mathrm{P}}[\psi,\psi^*]$ can be made invariant with respect to the rotation group (104) alone, provided ψ is considered as a fundamental object (not as an attribute of a dynamic system). In the last case the ψ is declared as a spinor, but the mathematical object, described by the action $\mathcal{A}_{\mathrm{P}}[\psi,\psi^*]$, stops to be a dynamic system. It may be regarded, for instance, as a "*dynamic system restricted by quantum axiomatics*", but it is not a dynamic system in the conventional sense of this term, because a possibility of change of dynamic variables is restricted [any rotation (104) is accompanied by a proper relabeling (105)]. Of course, one may insist on the fundamental character of ψ and state that ψ is a spinor, but then $\mathcal{S}_{\mathrm{P}}[\psi]$ stops to be a dynamic system, and this fact may have far-reaching consequences (see details in ref. [28]). There is a similar problem with the relativistic invariance of the dynamic system $\mathcal{S}_{\mathrm{D}}[\psi]$ described by the Dirac equation [29].

Thus, the k-spin is an integral property of a fluid flow, connected with kinematic properties of a dynamic system. Locally any vector field $\mathbf{g}(\xi)$ in the 3-dimensional V_ξ can be written in the form

$$\mathbf{g}(\xi) = \nabla\zeta_1 + \eta_2\nabla\zeta_2 \tag{109}$$

(expressed via Clebsch potentials), and one should expect that $s = 1/2$ is a maximal k-spin of any flow in the 3-dimensional space. But possible singular points of the representation (109) may lead to the circumstance that the spin of the total flow appears to be higher, than $s = 1/2$. It is connected with so called helicity of a vector field. Examples and discussion of such a velocity field can be found in ref. [30, 16]

VI Concluding remarks

Taking into account dynamic equations for labels ξ (Lagrangian coordinates considered as dynamic variables), one succeeds to integrate the system of hydrodynamic equations for a ideal fluid. This integration leads to appearance of three arbitrary functions g^α, $\alpha = 1, 2, 3$ of labels ξ. The functions g^α form a vector $\mathbf{g}$ in the space V_ξ of labels ξ. The vector $\mathbf{g}$ can be expressed via initial and boundary values of the fluid velocity. Dynamic equations appear to carry all essential information about the fluid motion. This form of the fluid description may appear to be important in such problems, where character of the fluid motion depends essentially on the character of initial and boundary conditions, and one needs to investigate dynamic equations together with boundary and initial conditions. For instance, such a necessity arises at investigation of phenomena connected with a transition to irregular motion of a fluid (turbulence).

Appearance of the field $\mathbf{g}$ activates the relabeling group. Invariant subsets of this group can be used for a classification of the fluid flows. The field $\mathbf{g}$ admits to introduce such concepts as ψ-function and k-spin which are new for the fluid dynamics. In some special cases these new constructive concepts can be identified with the wave function and the spin. Concepts of the wave function and of the spin are fundamental concepts in the sense that they cannot be defined via other more fundamental concepts. In the quantum mechanics the concepts of the wave function and of the spin are defined by their properties, i.e. by a system of axioms (quantum axiomatics).

On one hand there are derivative constructions of ψ-function and k-spin, connected with stochastic electron via the construction of the statistical ensemble (ideal fluid). On the other hand, there are fundamental concepts of wave function and spin, connected with the stochastic electron via system of axioms (quantum axiomatics). Sometimes the ψ-function coincides with the wave function, but not always. Then such a question arises. Which of the two conception is valid? ψ-function, or wave function?

A like problem arose in the theory of thermal phenomena. On one hand there was an axiomatic thermodynamics with its fundamental concepts of thermodynamic potentials. On the other hand there was the statistical physics, where the thermodynamic potentials were constructive quantities derived from the conception of the heat as a chaotic motion of molecules. Then the constructive theory (statistical physics) appeared to be more successful, than the axiomatic one (thermodynamics). Now the question is yet open, although there is a series of arguments in favour of the constructive approach which seems to be more reasonable and less enigmatic.

The considered general approach to the fluid dynamics is interesting from the point of view that sometimes it permits to use advantages of the quantum technique in the dynamics of usual fluids, as well as the general technique of the fluid dynamics in application to quantum mechanics.

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