The way to skeleton conception of elementary particles

Yuri A. Rylov

Institute for Problems in Mechanics, Russian Academy of Sciences,
101-1, Vernadskii Ave., Moscow, 119526, Russia.
e-mail: rylov@ipmnet.ru
Web site: $http://rsfq1.physics.sunysb.edu/\tilde{}rylov/yrylov.htm$
or mirror Web site:
$http://gasdyn-ipm.ipmnet.ru/\tilde{}rylov/yrylov.htm$

Abstract

The tachyon model of neutrino is constructed, basing on the statement that quantum description is a statistical description of stochastically moving particles. Besides, the tachyon model contains two conceptual points: (1) universal formalism of particle dynamics, describing uniformly all particles: deterministic, stochastic and quantum, (2) discrete space-time geometry and skeleton conception of particle dynamics. The universal formalism is a result of a logical reloading, when the statistical ensemble becomes to be the basic object of particle dynamics instead of a single particle. Such a reloading admits one to describe uniformly the quantum, stochastic and deterministic particles in terms of a statistical ensemble without a reference to principles of quantum mechanics. Besides, one uses a relativistic state of a particle, when the state is described by the particle skeleton (several space-time points) instead of the point in the phase space, what is nonrelativistic concept of the particle state. Representing the Dirac equation in terms of the statistical ensemble, one concludes that in the deterministic approximation the world line of the Dirac particle may be a spacelike helix with timelike axis. The rotational component of the relativistic Dirac particle is described nonrelativistically. It shows that the world line may be spacelike, and the Dirac particle may be a tachyon. Neutrino is a Dirac particle, and it is a tachyon. Free quantum particles appear to move stochastically, and this bring up the question, what is the reason of stochastic motion of free quantum particles. It appears, that the discrete space-time geometry is a multivariant geometry. It is a reason of stochastic particle motion. If the elementary length λ_0 of the discrete space-time geometry is connected with the quantum constant $\hbar$ by the relation $\lambda_0^2 = \hbar/bc$, where b is some universal constant, then statistical description of the free particle motion coincides with the quantum description in terms of the Schrödinger equation.

Key words: structural approach; united formalism of dynamics; multivariant geometry, dynamic disquantization; tachyon, tachyon gas, tachyon dynamics, dark matter

1 Introduction

The particles moving with the velocity, which is greater than the speed of the light, are called tachyons [1]-[4]. We shall use this name for particle, whose world line is spacelike. Both definitions mean the same, if the world line is smooth, and one can define a derivative along the world line. This derivative is known as a velocity. We shall show that the world line of tachyons is not smooth. This property differ tachyons from tardions which are particles moving with velocity less, than the speed of the light, and the world line of a tardion is smooth.

Neutrino is a tachyon, whose world line is a spacelike helix with timelike axis. However, most physicists believe that tachyons do not exist, in particular, tachyons with helical world line do not exist. Our model of neutrino is based on the skeleton conception of elementary particles [5]. The skeleton conception is a new conception based on such unusual fundamentals as (1) refuse from quantum principles, which are replaced by a discrete space-time geometry, (2) description of the particle state by its skeleton (several space-time points) instead of a point in the phase space of coordinates and momenta.

It is useful to describe characteristic features of these fundamentals, using method of the book by Lee Smolin [6]. It is an excellent book, where all problems of the elementary particle theory are presented without any formula. Lee Smolin distinguishes principle theories and constructive theories. The principle theory is to be valid for all physical phenomena, whereas the constructive theory is valid only for some class of physical phenomena. The constructive theory is created on the basis of some experimental data, and it is valid for the class of the physical phenomena close to phenomena verified by experiment. For instance, the special relativity and the general relativity are principle theories. The elementary particle theory is a constructive theory.

Lee Smolin formulated five unsolved important problems of contemporary theoretical physics:

Problem 1: Unification of general relativity and quantum theory (quantum gravitation)

Problem 2: Rationale of quantum mechanics.

Problem 3: Unification of particles and fields.

Problem 4: Explanation how to choose free constants in the standard model of elementary particle physics.

Problem 5: Explanation of the phenomenon of dark matter and dark energy.

Besides, Lee Smolin describes new and old fundamental conceptions as unifications. For instance, he formulates the special relativity theory as an unification of space and time. The inertia law is formulated as an unification of the rest and motion. The general relativity is formulated as unification of space-time and gravitation.

The tachyon model of neutrino is constructed on the basis of a principle theory (skeleton conception). This principle theory is formulated on the basis of two unifications:

1. Unification of the deterministic particle motion with the stochastic particle motion

2. Unification of the continuous space-time geometry with the discrete space-time geometry.

The two unifications concern space-time geometry and the particle dynamics. These disciplines relate to all physical phenomena. The two unifications are more fundamental, than problems formulated by Lee Smolin. They solve four of five Smolin's problems (the fourth problem is not solved, because it a specific problem of the standard model of elementary particles). The first problem (the quantum gravitation) is solved in the sense, that the gravitation field does not need to be quantized, as well as other geometrical fields.

Both unifications are produced on the basis of a logical reloading, which means a change of basic statements of a theory.

In the unification of the deterministic particle motion with the stochastic particle motion it means as follows. A deterministic particle is a dynamical system $\mathcal{S}_{\rm d}$, and one can write dynamic equations for the deterministic particle $\mathcal{S}_{\rm d}$. Stochastic particle $\mathcal{S}_{\rm st}$ is not a dynamic system. It is a stochastic system $\mathcal{S}_{\rm st}$, and there are no dynamic equations for a single stochastic particle $\mathcal{S}_{\rm st}$. One can describe only mean motion of a stochastic particle $\mathcal{S}_{\rm st}$. To describe the mean motion, one considers a statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\rm st}\right]$ of stochastic particles $\mathcal{S}_{\rm st}$. The statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\rm st}\right]$ is a dynamic system of the type of continuous medium, and one can write dynamic equations for $\mathcal{E}\left[\mathcal{S}_{\rm st}\right]$. Statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$ of deterministic dynamic systems $\mathcal{S}_{\rm d}$ can be constructed also. Ensemble $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$ is also a dynamic system of the type of continuous medium. Any statistical ensemble ($\mathcal{E}\left[\mathcal{S}_{\rm st}\right]$ and $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$) may be considered as some fluid (continuous medium). In the Lagrange representation dynamic equations for $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$ coincide with dynamic equations for $\mathcal{S}_{\rm d}$. Only the number of dynamic equations is different. If the number of degrees of freedom for $\mathcal{S}_{\rm d}$ is equal to n, the number of the freedom degrees for $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$ is equal to nN, where N is the number of dynamic system $\mathcal{S}_{\rm d}$ in $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$, $N\rightarrow\infty$.

If the dynamical equations for $\mathcal{S}_{\rm d}$ are known, the dynamical equations for $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$ are also known. Vice versa, if dynamic equations for $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$ are known, one can write dynamic equations for a single particle $\mathcal{S}_{\rm d}$. In other words, it is not essential, what is the basic object ($\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$ or $\mathcal{S}_{\rm d}$) at description of the deterministic particle $\mathcal{S}_{\rm d}$. However, at description of stochastic particle $\mathcal{S}_{\rm st}$ it is essential, because there are dynamic equations for $\mathcal{E}\left[\mathcal{S}_{\rm st}\right]$, whereas there are no dynamic equations for $\mathcal{S}_{\rm st}$. If the statistical ensemble is a basic object of the particle dynamics, then dynamic equations exist for all sorts of basic objects $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$ and $\mathcal{E}\left[\mathcal{S}_{\rm st}\right]$. The difference between $\mathcal{S}_{\rm d}$ and $\mathcal{S}_{\rm st}$ consists in the circumstance, that dynamic equations for $\mathcal{S}_{\rm d}$ can be obtained from dynamic equations for $\mathcal{E}\left[\mathcal{S}_{\rm d}\right]$, but dynamic equations for $\mathcal{S}_{\rm st}$ cannot be obtained from dynamic equations for $\mathcal{E}\left[\mathcal{S}_{\rm st}\right]$. How it may be possible, will be shown later in a simple example.

The logical reloading leads to creation of united formalism for description of deterministic, stochastic and quantum particles in terms of the statistical ensemble [7]. It appears, that quantum particles are stochastic particles, described in terms of the statistical ensemble. The wave function ψ appears at such a description, because it is simply a way of the ideal fluid description [8]. The wave function ψ is used, because describing statistical ensemble of quantum particles, the internal energy of the "quantum fluid" appears to be such one, that dynamic equations in terms of ψ are linear (Schrödinger equation) for non-rotational flow of the "quantum fluid", describing this ensemble.

Thus, one does not need quantum principles, if the basic object of particle dynamics is a statistical ensemble. Quantum particles are simply stochastic particles. It appears that the quantum principles are not fundamental principles of nature, and there is no necessity to quantize the gravitational field, especially if one takes into account that the dynamic equations of the gravitational field do not contain the quantum constant.

Explanation of quantum theory as a statistical description of stochastic particles brings up the question: "Why free elementary particles move stochastically?" The answer to this question is as follows. The real space-time geometry is discrete. There is a minimal space-time distance between the events (points) of space-time. This distance is called elementary length λ_0. Condition of the space-time discreteness is written in the form

$$\left|\rho\left(P,Q\right)\right| \notin \left(0,\lambda_0\right), \quad \forall P,Q\in\Omega \tag{1.1}$$

where Ω is the set of the space-time points and $\rho\left(P,Q\right)$ is the space-time interval between points P and Q. Note that $\rho\left(P,Q\right)=0$ is possible, and it is compatible with (1.1), for instance, if $P=Q$.

Usually one considers the restriction (1.1) as a restriction on the set Ω, and the distance function ρ is defined as follows

$$\rho\left(P,Q\right)=\sqrt{2\sigma\left(P,Q\right)} \tag{1.2}$$

where σ is the world function $\sigma_{\rm M}$ of the space-time of Minkowski. In the inertial coordinate system the world function $\sigma_{\rm M}$ has the form

$$\sigma_{\rm M}\left(x,x'\right)=\frac{1}{2}g_{ik}\left(x^i-x'^i\right)\left(x^k-x'^k\right), \quad g_{ik}=\operatorname{diag}\left(c^2,-1,-1,-1\right) \tag{1.3}$$

Consideration of (1.1) as restriction on Ω leads to a geometry on a lattice. Geometry of Minkowski on a lattice is not uniform and isotropic. It is not invariant with respect to Lorentz transformations. Nevertheless it is used by theorists for approximate calculations.

It is more correct to consider (1.1) as a restriction on the form of the distance function ρ and the world function $\sigma=\frac{1}{2}\rho^2$. A use of the world function is more convenient, than a use of distance, because it is always real (σ is positive for timelike distances, and it is negative for spacelike ones). World function σ_{d} for a discrete geometry $\mathcal{G}_{\mathrm{d}}$ can be taken in the form

$$\sigma_{\mathrm{d}}(P,Q) = \sigma_{\mathrm{M}}(P,Q) + \frac{\lambda_0^2}{2}\operatorname{sgn}\left(\sigma_{\mathrm{M}}(P,Q)\right) \quad \forall P,Q \in \Omega \tag{1.4}$$

where Ω is the same point set (continuum) which is used in the space-time geometry of Minkowski. The relation (1.4) is compatible with restriction (1.1).

Multivariance is the most unexpected and important property of the discrete geometry $\mathcal{G}_{\mathrm{d}}$. Multivariance of a geometry means that a vector $\mathbf{AB}$ at the point A has many equivalent vectors $\mathbf{CD}$, $\mathbf{CD}'$, $\mathbf{CD}''$,...at the point C, but these vectors are not equivalent between themselves. Contemporary theorists do not accept the property of multivariance in a geometry and try to remove it, if it appears by accident in geometry. For instance, when it appears in the Riemannian geometry, one removes this property, connecting any of numerous vectors $\mathbf{CD}$, $\mathbf{CD}'$, $\mathbf{CD}''$,...at the point C with the path of its parallel transport from the point A and asserting absence of absolute parallelism in the Riemannian geometry.

Multivariance for spacelike vectors takes place in the space-time geometry of Minkowski. Ignoring this multivariance, one cannot describe motion of tachyons. Conventional viewpoint, that tachyons do not exist is connected with disregard of the spacelike vectors multivariance.

Such a relation to multivariance is connected with the fact that beginning from Euclid one studied only proper Euclidean geometry, assuming that the space-time geometry cannot have any additional properties which are absent in the Euclidean geometry. The multivariance is denied in the Riemannian geometry, because one considers absence of absolute parallelism as a less defect of the geometry, than multivariance of the vector equivalence. Besides, the multivariance is incompatible with contemporary methods of differential geometry. The operations of the linear vector space $\mathcal{L}_n$ (summation of vectors $u \in \mathcal{L}_n$ and multiplication of a vector $u \in \mathcal{L}_n$ by a real number) are not adequate in the multivariant geometry. The fact is that any linvector $u \in \mathcal{L}_n$ exist in one copy. We use the name linear vector (linvector) for vectors $u \in \mathcal{L}_n$, in order to distinguish it from the geometric vector (g-vector) $\mathbf{AB}$, which is defined as the ordered set of two points $\mathbf{AB} = \{A, B\} \in \Omega \times \Omega$. There are many equivalent g-vectors in any space-time geometry. In the Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ the set $\Omega_{\mathbf{AB}}$ of g-vectors $\mathbf{CD} \in \Omega \times \Omega$ form the equivalence class $[\mathbf{AB}]$ of the g-vector $\mathbf{AB}$. All the g-vectors of $[\mathbf{AB}]$ are equivalent between themselves, and all $[\mathbf{AB}]$ may be set in correspondence with all linvectors of $\mathcal{L}_n$. As a result linear operations of $\mathcal{L}_n$ can be used for g-vectors of Euclidean geometry $\mathcal{G}_{\mathrm{E}}$. In the multivariant geometry the set of g-vectors $\Omega_{\mathbf{AB}}$ does not form the equivalence class, because $\Omega_{\mathbf{AB}}$ contains the g-vectors, which are not equivalent between themselves (they are equivalent only to g-vector $\mathbf{AB}$). As a result operations of the linear space $\mathcal{L}_n$ are not adequate in the multivariant geometry. They can be introduced, but these operations appear to be ambiguous.

Multivariance of the space-time geometry generates a random wobbling of particle world lines. In the geometry of Minkowski $\mathcal{G}_{\mathrm{M}}$ the equivalence of timelike vectors is not multivariant, whereas the equality (equivalence) of spacelike vectors is multivariant. Wobbling of spacelike world line has infinite amplitude, and a single tachyon cannot be detected due to infinite random wobbling of its world line. As a result it is used to think that tachyons do not exist.

In the discrete geometry $\mathcal{G}_{\rm d}$ both timelike and spacelike world lines wobble. However, the wobbling amplitude of timelike world lines is restricted by the elementary length λ_0, and it vanishes in the geometry of Minkowski $\mathcal{G}_{\rm M}$, where $\lambda_0 = 0$. The wobbling amplitude of spacelike world lines of tachyons is infinite in $\mathcal{G}_{\rm M}$ and in $\mathcal{G}_{\rm d}$. In the real space-time geometry $\mathcal{G}_{\rm d}$ the restricted wobbling of timelike world lines is a reason of stochastic motion of particles. If λ_0^2 is proportional to the quantum constant $\hbar$, the statistical description of wobbling world lines (dynamic equations for the statistical ensemble) leads to the Schrödinger equation [10].

In the discrete geometry $\mathcal{G}_{\rm d}$ all geometric quantities are functions of the world function $\sigma_{\rm d}$. In particular, the dimension n of a geometry is determined by its world function σ. In $\mathcal{G}_{\rm d}$ the dimension has no definite value. It is rather unusual, because in the Riemannian geometry the dimension of the geometry is a definite natural number.

If the quantum particles $\mathcal{S}_{\rm q}$ are stochastic particles, described in terms of statistical ensemble $\mathcal{E}[\mathcal{S}_{\rm st}]$, dynamic equations for any quantum particle can be reduced to dynamic equations for a statistical ensemble $\mathcal{E}[\mathcal{S}_{\rm st}]$ of stochastic particles $\mathcal{S}_{\rm st}$. In particular, the Dirac equation is to be reduced to the dynamic equations for some statistical ensemble $\mathcal{E}[\mathcal{S}_{\rm Dst}]$ of stochastic particles $\mathcal{S}_{\rm Dst}$. One can introduce a "deterministic model" $\mathcal{S}_{\rm d}$ of a quantum particle $\mathcal{S}_{\rm st}$ by means of dynamic disquantization (D-disquantization) of the statistical ensemble $\mathcal{E}[\mathcal{S}_{\rm st}]$ of the stochastic particle $\mathcal{S}_{\rm st}$ [9]. Dynamic disquantization is a dynamic operation which does not use quantum principles. As a result of the dynamic disquantization all derivatives $\partial_k \equiv \partial/\partial x^k$ in the dynamic equations are replaced by derivatives which are in parallel with 4-vector j^k of the particle current

$$\partial_k \rightarrow \frac{j_k j^l}{j_s j^s}\partial_l \qquad (1.5)$$

As a result of the dynamic disquantization the dynamic equations for the statistical ensemble $\mathcal{E}[\mathcal{S}_{\rm st}]$ of stochastic particles $\mathcal{S}_{\rm st}$ turn to dynamic equations for the statistical ensemble $\mathcal{E}[\mathcal{S}_{\rm d}]$ of deterministic particles $\mathcal{S}_{\rm d}$. In the Lagrange representation the dynamic equations for $\mathcal{S}_{\rm d}$ are ordinary differential equations, because they contain derivatives only along the direction of the vector j^k. The dynamic system $\mathcal{S}_{\rm d}$ has finite number of the freedom degrees. The dynamic system $\mathcal{S}_{\rm d}$ can be interpreted as a deterministic model of the stochastic particle $\mathcal{S}_{\rm st}$. Dynamic equations for $\mathcal{S}_{\rm d}$ may contain the quantum constant $\hbar$, because in the dynamic disquantization one uses only procedure (1.5), but one does not use the limit $\hbar \rightarrow 0$. In particular, if the stochastic particle $\mathcal{S}_{\rm st}$ is the Dirac particle $\mathcal{S}_{\rm D}$ described by the Dirac equation, the deterministic model of the Dirac particle is a dynamic system having ten degrees of freedom [11]. It may be interpreted as a rotator (two rigidly connected pointlike particles). If one follows only one particle of the rotator, one concludes that the world line of the deterministic Dirac particle appears to be a helix (spacelike or timelike) with timelike axis. Neutrino is believed to be a Dirac particle. As a result neutrino appears to be a tachyon moving along the spacelike helix with timelike axis. (Timelike world line of neutrino is improbable, because in this case the regular velocity of neutrino appears to be essentially less, than the speed of the light). Note that conventionally the deterministic model $\mathcal{S}_{\rm Dd}$ of the Dirac particle $\mathcal{S}_{\rm Dst}$ is considered as a pointlike tardion equipped with spin (angular momentum). The term "tardion" means a particle having timelike world line, whereas the term "tachyon" means a particle having spacelike world line.

As we shall see, the Dirac particle is described by three-point skeleton $\mathcal{P}^2 = \{P_0, P_1, P_2\}$, or by three connected vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_0\mathbf{P}_2$, $\mathbf{P}_1\mathbf{P}_2$. One of these vectors is spacelike and two of them are timelike. The timelike vector wobbling amplitude is restricted by the elementary length λ_0, whereas the spacelike vector wobbling amplitude is not restricted (infinite). All points of the skeleton $\mathcal{P}^2 = \{P_0, P_1, P_2\}$ are connected rigidly. The world chain wobbling of such a particle is a mixture of the unrestricted tachyon wobbling and of the tardion wobbling, restricted by value of λ_0 (or by $\hbar$). As a result of dynamical disquantization of $\mathcal{S}_{\rm Dst}$ one obtains the dynamic system $\mathcal{S}_{\rm Dd}$, which is described in the space-time geometry of Minkowski by a helical world line (spacelike or timelike) with the timelike axis. The dynamic equations for $\mathcal{S}_{\rm Dd}$ contain the quantum constant $\hbar$ (in the expression for spin). It means that the dynamic system $\mathcal{S}_{\rm Dd}$ is not a classical approximation of the stochastic Dirac particle $\mathcal{S}_{\rm Dst}$.

The deterministic model $\mathcal{S}_{\rm Dd}$ of the stochastic particle $\mathcal{S}_{\rm Dst}$ is a dynamic system $\mathcal{S}_{\rm Dd}$, which can be considered in the space-time geometry of Minkowski. The dynamic system $\mathcal{S}_{\rm Dd}$ has finite number of the freedom degrees. The deterministic model $\mathcal{S}_{\rm Dd}$ describes the arrangement of the particle, described by the Dirac equation. One should take into account that usually one does not consider the deterministic model $\mathcal{S}_{\rm Dd}$, which contains the quantum constant $\hbar$ and explains the particle spin by a rotation of a particle due to its helical world line. Instead, one considers the dynamic system $\mathcal{S}_{\rm Dcl}$, which is described by a straight world line (not helix), and spin is introduced axiomatically. From the dynamic system $\mathcal{S}_{\rm cl}$ one cannot obtain any information on arrangement of $\mathcal{S}_{\rm Dst}$.

Attempts of obtaining information on arrangement of $\mathcal{S}_{\rm Dst}$ from the contemporary elementary particle theory remind attempts of investigating the atom arrangement on the basis of the periodical system of chemical elements and of chemical reactions between chemical elements. One obtains a lot of information on the atom properties of different chemical elements and no information on the planetary model of atoms.

Replacement of quantum theory by a statistical description together with the dynamic disquantiazation admit one to construct a new approach to description of elementary particles. We qualify this approach as structural approach. The conventional approach to the elementary particles description (standard model) is qualified as empirical approach. Empirical approach to theory of elementary particles is based on the quantum theory. The empirical approach labels elementary particles by quantum numbers. It cannot determine the connection of quantum numbers with the elementary particles arrangement. The structural approach admits one to determine arrangement (structure) of the elementary particle.

The difference between the structural approach and the empirical one can be seen in the theory of chemical elements. The empirical approach is realized by chemical methods, when chemical elements are systematized by the periodic system of chemical elements, and one investigates reactions between different chemical substances. Empirical approach cannot determine the atom structure (nucleus, electronic envelope). On the contrary, the ctructural approach uses methods of quantum mechanics and of atomic physics. It admits one to discover the atom arrangement.

Thus, we have presented briefly the way to the tachyon model of neutrino, which is based on two logical reloadings (in dynamics and in the space-time geometry). It is the most short way, but in reality we went to the deterministic model of neutrino by another way. We search defects and mistakes in the existing theory of microcosm physics and eliminate them step by step. Such an investigation strategy is the best one in the case, when a theory continues to be in crisis. As far as I know, nobody uses such a strategy. Furthermore I was criticized for such a strategy, because nobody believes that there may be mistakes in the existing theory of microcosm physics. All researchers dreamed about new happy ideas, which should help us to go out of crisis. Further we shall present the way to the deterministic model of neutrino. It was a long way, which took thirty years. Happily, it was the way not only to the deterministic model of neutrino. It was the way to the skeleton conception of elementary particles [12].

2 United formalism for particle dynamics

After explanation of heat phenomena by means of the kinetic gas theory it was reasonable to think, that quantum effects may be explained as some stochastic motion of microparticles. Some researchers [13, 14] tried to obtain quantum mechanics as a statistical description of stochastically moving microparticles. They failed to explain the quantum mechanics as a statistical description of stochastically moving particles. Moyal [13] tried to reduce quantum dynamic equations to the form, which is characteristic for dynamic equations of stochastic processes. Fenyes [14] tried to obtain statistical description, using similarity between the Schrödinger equation and the Fokker equation for diffusion processes. Both authors used the concept of the wave function without understanding, what it means. Explanation of quantum phenomena is hardly possible without understanding, what is the wave function. However, then nobody knew, what is the wave function.

The fact, that the Schrödinger equation may be reduced to irrotational flow of some quantum fluid was shown by Madelung [15]. However, representation of the hydrodynamic equations for ideal fluid in terms of the wave function needs a complete integration of hydrodynamic equations.

For transition from the Schrödinger equation to the system of four hydrodynamic equations, the complex Schrödinger equation for the wave function $\psi =$

$\sqrt{\rho}\exp(i\varphi/\hbar)$ is represented in the form of two real equations for amplitude $\sqrt{\rho}$ and for the phase φ. To obtain hydrodynamic equations, it is sufficient to take gradient from the equation for the phase φ. As a result one obtains four dynamic equations, which turn into hydrodynamic equations after introducing proper designations. In other words, for transition from dynamic equations in terms of the wave function to the hydrodynamic form of these equations, one needs to differentiate dynamic equations. On the contrary, to pass from hydrodynamic form of dynamic equations to their representation in terms of the wave function, one needs to integrate dynamic equations. In the case of the irrotational flow this integration is carried out rather simply, whereas in the case of vortical flow the way of integration became to be known only in the end of twentieth century [8].

Bohm [16] used the hydrodynamic representation of the Schrödinger equation for interpretation of quantum mechanics. He started from the wave function and quantum principles and interpreted them in hydrodynamic terms. However, he could not found quantum mechanics on the basis of hydrodynamics, because for such a foundation he would start from hydrodynamic concepts and equations, in order to obtain the wave function in hydrodynamic terms. He could not make this, because in this case he would be forced to integrate hydrodynamic equations in general case, but not only for irrotational flows. Integration of the hydrodynamic equations was not known almost during the whole twentieth century.

Information on other attempts of a statistical foundation of quantum mechanics can be found in the book by Holland [17]. All authors tried to found the nonrelativistic quantum phenomena on the basis of nonrelativistic statistical description. This circumstance was the main reason of failures. The nonrelativistic quantum mechanics describes a mean motion of particles, and the mean motion is nonrelativistic. However, the nonrelativistic character of the mean motion does not mean, that the exact particle motion is also nonrelativistic. Stochastic component of the particle motion may be relativistic, and this component disappear at the averaging. To obtain a correct description one should use a relativistic statistical description.

Nonrelativistic statistical description is produced usually in terms of the probability density. It uses nonrelativistic concept of particle state as a point in the phase space of coordinates and momenta. At proper normalization the nonnegative density ρ of particles in the phase space is used as a probability density.

In the relativistic physics the state of a particle is determined by its world line (not as a point in the phase space). As a result the state density of a statistical ensemble of relativistic particles at some space-time point x is determined by the vector $j^k(x)$ of the 4-current [18]. This vector cannot be described in terms of the probability density. As a result the statistical description of relativistic stochastic particle differs from that of the nonrelativistic particles. The relativistic statistical description of stochastically moving particles is a consideration of many stochastic particles (statistical ensemble), and it is the primary definition of the statistical description. Consideration of the statistical ensemble of stochastic particles is a consideration of some continuous medium, consisting of infinite number of independent stochastic particles [18, 19, 20]. Thus, a statistical ensemble of stochastic particles is a dynamic system, which is described by some dynamic equations, whereas a single stochastic particle is not a dynamic system, and there are no dynamic equations, describing a single stochastic particle.

Consideration of the statistical ensemble admits one to obtain a dynamic system, whose evolution can be investigated. Of course, the relativistic statistical description in terms of statistical ensemble and that in terms of a fluid are connected. However, one prefers to use nonrelativistic statistical description in terms of the probability density. The Brownian particles are described by means of the nonrelativistic statistical description. Such an approach is true, because the stochastic component of the Brownian particle motion is nonrelativistic, and the state of the Brownian particle may be described as a point in the usual space.

However, application of nonrelativistic statistical description to quantum particle is incorrect, because *the nonrelativistic quantum mechanics is in reality a relativistic conception*. This statement looks rather unexpected. But note, that if one knows nothing about the stochastic component of a particle motion, one should consider the general (relativistic) case. If one considers the nonrelativistic quantum mechanics as a relativistic conception, but the quantum mechanics appears to be a nonrelativistic conception, such a consideration of quantum mechanics as a relativistic conception will be true, because a nonrelativistic conception is a special case of a relativistic conception. However, if one considers the nonrelativistic quantum mechanics as a nonrelativistic conception, but it appears to be a relativistic conception, the nonrelativistic consideration will be incorrect, in general. The difference lies in the concept of the particle state.

Thus, if one tries to obtain a statistical foundation of quantum mechanics as a statistical description of stochastically moving particles, one should use adequate relativistic concepts. Formalism of nonrelativistic quantum mechanics is nonrelativistic. To produce a statistical foundation of quantum mechanics, one should carry out a logical reloading, i.e. a transition from inadequate (nonrelativistic) concepts to adequate (relativistic) concepts. It means that the probability density $\rho(x)$ should be replaced by the "probability vector" $j^k(x)$ (world lines density). Introduction of 4-vector $j^k(x)$ means a consideration of some "quantum fluid". The wave function ψ is a way of the fluid description [8], and it appears as a result of description of the "quantum fluid", which describes the state of the statistical ensemble. As a result the main concept of the quantum mechanics (the wave function) appears to be a secondary derivative concept. The wave function may be introduced and interpreted in terms of concepts of the statistical ensemble. This fact admits one to found the quantum mechanics as a statistical description of stochastically moving particles.

Relativistic character of the nonrelativistic quantum mechanics makes to be useless the construction of relativistic quantum theory as a result of uniting of quantum and relativistic principles. Such an uniting is inconsistent, because nonrelativistic quantum mechanics is already a nonrelativistic approximation of a relativistic conception. Such an uniting reminds an unification of axiomatic conception of thermodynamics with the model conception of the kinetic gas theory. Relativistic quantum theory should be obtained as a refuse from the nonrelativistic approximation of the relativistic statistical foundation of the quantum mechanics. It means that the conventional conception of the relativistic quantum theory is doomed to fitting instead of logical development of the existing relativistic statistical description.

The main difference between the quantum mechanics and statistical description of stochastic particles lies in a use of the von Neumann formula for calculations of mean values

$$\langle f\rangle = \int \psi^* f \psi d\mathbf{x} \tag{2.1}$$

According to statistical approach this formula is valid, if f is an arbitrary function of coordinate $\mathbf{x}$, or it is an additive quantity (energy, momentum, angular momentum). According to the von Neumann interpretation the formula (2.1) is valid for arbitrary function of coordinates and momentum $f(\mathbf{x},\mathbf{p})$, $\mathbf{p} = -i\hbar\boldsymbol{\nabla}$. The statement of the von Neumann theorem that one cannot introduce hidden variables in quantum mechanics is based on application of formula (2.1) to arbitrary functions $f(\mathbf{x},\mathbf{p})$ [21]. The statistical description of stochastic particles my be considered as an introduction of hidden variables, but in this case formula (2.1) is not valid for arbitrary functions $f(\mathbf{x},\mathbf{p})$, and there is no conflict with the theorem on hidden variables.

It is worth to note, that the logical reloading of statistical description to a relativistic conception does not need any new hypothesis. The probability density is not used simply, because it is an attribute of nonrelativistic description. As far as the quantum mechanics is a dynamics of a statistical ensemble of stochastic particles, it follows that the wave function ψ describes a state of the dynamic system $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ [8]. This dynamic system $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ is statistical ensemble of stochastic particles $\mathcal{S}_{\mathrm{st}}$. If the statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ is normalized to one particle, it can be interpreted as a statistically averaged particle $\langle\mathcal{S}_{\mathrm{st}}\rangle$. The statistically averaged particle $\langle\mathcal{S}_{\mathrm{st}}\rangle$ has energy, momentum and other total characteristics of a single particle $\mathcal{S}_{\mathrm{d}}$, but its motion is a motion of a statistical ensemble. For instance, $\langle\mathcal{S}_{\mathrm{st}}\rangle$ may move through two open slits at once, whereas a single deterministic particle $\mathcal{S}_{\mathrm{d}}$ may move only through one of two open slits.

The Copenhagen interpretation of quantum mechanics, where the wave function describes a single particle is incompatible with the formalism of quantum mechanics [21, 22]. As far as the quantum mechanics is a statistical theory (dynamics of a statistical ensemble), there are two different kinds of quantum measurements: (1) a massive measurement (M-measurement) which is produced over all elements (particles) $\mathcal{S}_{\mathrm{st}}$ of the statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$, and (2) a single measurement (S-measurement) which is produced over a single particle $\mathcal{S}_{\mathrm{st}}$ of the statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$. These measurements have different properties, and one may not mix up them.

S-measurement of a quantity R gives a random quantity R', which cannot be obtained, generally speaking, at a repeated S-measurement. In the S-measurement one deals with a single stochastic system $\mathcal{S}_{\mathrm{st}}$. The state of the stochastic system $\mathcal{S}_{\mathrm{st}}$ may be changed after S-measurement, which is a dynamical effect on $\mathcal{S}_{\mathrm{st}}$. However, the state of $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ cannot be changed by this effect, because $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ contains infinite number of stochastic particles $\mathcal{S}_{\mathrm{st}}$. A repeated S-measurement of the same quantity

R can be produced on other stochastic particle $\mathcal{S}_{\mathrm{st}}$. It gives, generally speaking, another value R'' of the measured quantity R.

M-measurement is a set of N S-measurements ($N \to \infty$). M-measurement of a quantity R gives a distribution $F(R)$, which can be obtained at repeated M-measurement. M-measurement of the quantity R at the state ψ of the statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ can change the state of the statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$, because in this case one deals with N ($N \to \infty$) stochastic systems $\mathcal{S}_{\mathrm{st}}$. Any stochastic system changes after S-measurement produced over it. N changed system $\mathcal{S}'_{\mathrm{st}}$ form a statistical ensemble $\mathcal{E}[\mathcal{S}'_{\mathrm{st}}]$ at $N \to \infty$. As a result the state ψ of the statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ changes.

Is it possible to obtain a definite value R' at a M-measurement of the quantity R (instead of the distribution $F(R)$)? It is possible, provided the measurement is accompanied by a discriminating operation, which removes from the statistical ensemble $\mathcal{E}[\mathcal{S}'_{\mathrm{st}}]$ all stochastic systems $\mathcal{S}'_{\mathrm{st}}$, where the measured value of the quantity R is not equal to R'. The M-measurement accompanied by some discriminating operation will be referred to as a selective M-measurement (SM-measurement). The SM-measurement of the quantity R may give a definite value R' and change the state ψ of the statistical ensemble. In other words, the SM-measurement may have properties of the S-measurement and of M-measurement.

At the Copenhagen interpretation of quantum mechanics, where ψ is a state of a quantum particle $\mathcal{S}_{\mathrm{q}}$, there is only one kind of measurement. In some situation it is interpreted as M-measurement, in other situation it is interpreted as S-measurement. It is supposed that such a measurement of the quantity R can give a definite random value R' of the quantity R and simultaneously change the state $\psi \to \psi_{R'}$. In other words, in the Copenhagen interpretation a measurement is supposed to have properties of SM-measurement. A use of one term for different kinds of measurement (S-,M-,SM-) leads to numerous paradoxes.

We consider only one of paradoxes: "action of a measurement at a distance". Let us consider a system $\mathcal{S} = \mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ at the state ψ. Let $\mathcal{S}$ at the state ψ can decay into two systems (particles): $\mathcal{S}_1$ and $\mathcal{S}_2$. Let the two particles states $\mathcal{S}_1$ and $\mathcal{S}_2$ appear to be correlated in the sense, that if a S-measurement of the dichotomic quantity s (spin) in $\mathcal{S}_1$ gives the result $s' = 1/2$, the S-measurement of the same quantity s (spin) in $\mathcal{S}_2$ gives the result $s'' = -1/2$. Let these particles $\mathcal{S}_1$ and $\mathcal{S}_2$ move, and at some moment they appear to be divided by the distance L. According to Copenhagen viewpoint, when one measures the quantity s in $\mathcal{S}_1$ and obtains the result $s' = 1/2$, the state of $\mathcal{S}_1$ changes (SM-measurement) $\psi_1 \to \psi'_1$. At the same time the state of $\mathcal{S}_2$ is to be changed $\psi_2 \to \psi'_2$, because in $\mathcal{S}_2$ the quantity s takes the value $s'' = -1/2$. As a result a measurement of the quantity s in $\mathcal{S}_1$ changes instantly the state of the particle $\mathcal{S}_2$, although the distance L between the particles $\mathcal{S}_1$ and $\mathcal{S}_2$ may be large ("action of a measurement at a distance"). Such a situation is incompatible with the special relativity principles, and it is considered as a paradox.

The paradox is resolved by a reference, that in the given case there is a SM-measurement, which is accompanied by a discriminating operation, and information on this operation is to be transmitted from point A_1, , to the point A_2, where $\mathcal{S}_2$ is located. Indeed, if one speaks on influence of measurement in $\mathcal{S}_1$ on the wave function of $\mathcal{S}_2$, one should consider ensembles $\mathcal{E}[\mathcal{S}_1]$ and $\mathcal{E}[\mathcal{S}_2]$, because the wave function relates to the statistical ensemble $\mathcal{E}[\mathcal{S}_1]$, but not to the single particle $\mathcal{S}_1$ In the SM-measurement in one considers N $(N\rightarrow\infty)$ stochastic systems $S_1', S_2', ...S_N'$ of the statistical ensemble $\mathcal{E}[\mathcal{S}_1]$ and N $(N\rightarrow\infty)$ stochastic systems $S_1'', S_2'', ...S_N''$ of the statistical ensemble $\mathcal{E}[\mathcal{S}_2]$. The stochastic system S_k' of $\mathcal{E}[\mathcal{S}_1]$ correlates with stochastic system S_k'' of $\mathcal{E}[\mathcal{S}_2]$. It means that if the quantity s has the value $s'=1/2$ in S_k' of $\mathcal{S}_1$, then the quantity s has the value $s''=-1/2$ in S_k'' of $\mathcal{S}_2$. One measures the quantity s in all N stochastic systems S_k', $k=1,2,...N$ and obtains that the value $s'=1/2$ appears in stochastic systems $S_{(k_1)}', S_{(k_2)}', ...S_{(k_m)}'$ of $\mathcal{E}[\mathcal{S}_1]$. Then due to correlation the quantity s has the value $s'=-1/2$ in stochastic systems $S_{(k_1)}'', S_{(k_2)}'', ...S_{(k_m)}''$ of $\mathcal{E}[\mathcal{S}_2]$. One can form the statistical ensemble $\mathcal{E}[\mathcal{S}_1']=\mathcal{E}\left[S_{(k_l)}'\right]$. Its state is described by the wave function ψ_2', where $s=1/2$. One can form the statistical ensemble $\mathcal{E}[\mathcal{S}_2']=\mathcal{E}\left[S_{(k_l)}''\right]$ .of the stochastic systems $S_{(k_1)}'', S_{(k_2)}'', ...S_{(k_m)}''$. Its state is described by the wave function ψ_2'', where $s=-1/2$. However, the numbers $(k_1), (k_2), ...(k_m)$ are not known at the point A_2, where the system $\mathcal{S}_2$ is found. In order to construct $\mathcal{E}[\mathcal{S}_2']=\mathcal{E}\left[S_{(k_l)}''\right]$ with $s=-1/2$, one needs to transmit these numbers from A_1 to A_2. This transmission cannot be realized with the speed, which is greater, than the speed of the light.

The united method of description of dynamic systems and stochastic ones is presented in [22]. Here we present only a short scheme of this method application in the example of a free quantum particle.

Statistical ensemble $\mathcal{E}[\mathcal{S}_{\text{cl}}]$ of free nonrelativistic classical particles $\mathcal{S}_{\text{cl}}$ is described by the action

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\text{cl}}]}[\mathbf{x}]=\int\int_{V_\xi}\frac{m}{2}\dot{\mathbf{x}}^2\rho_0(\boldsymbol{\xi})\,dtd\boldsymbol{\xi},\qquad \dot{\mathbf{x}}\equiv\frac{d\mathbf{x}}{dt}\tag{2.2}$$

where $\mathbf{x}=\mathbf{x}(t,\boldsymbol{\xi})$, $\boldsymbol{\xi}=\{\xi_1,\xi_2,\xi_3\}$ are parameters, labelling the particles of the statistical ensemble, and ρ_0 is a weight factor.

If the particles of the ensemble are stochastic, the stochasticity is taken into account by additional dynamical variables in the action. The action for the statistical ensemble $\mathcal{E}[\mathcal{S}_{\text{st}}]$ of stochastic particles $\mathcal{S}_{\text{st}}$ is written in the form

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\text{st}}]}[\mathbf{x},\mathbf{u}]=\int\int_{V_\xi}\left\{\frac{m}{2}\dot{\mathbf{x}}^2+\frac{m}{2}\mathbf{u}^2-\frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}\right\}\rho_0(\boldsymbol{\xi})\,dtd\boldsymbol{\xi},\qquad \dot{\mathbf{x}}\equiv\frac{d\mathbf{x}}{dt}\tag{2.3}$$

The variable $\mathbf{x}=\mathbf{x}(t,\boldsymbol{\xi})$ describes the regular component of the particle motion. The variable $\mathbf{u}=\mathbf{u}(t,\mathbf{x})$ describes the mean value of the stochastic velocity component, $\hbar$ is the quantum constant. The second term in (2.3) describes the kinetic energy of the stochastic velocity component. The third term describes interaction between the stochastic component $\mathbf{u}(t,\mathbf{x})$ and the regular component $\dot{\mathbf{x}}(t,\boldsymbol{\xi})$. The operator

$$\boldsymbol{\nabla}=\left\{\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3}\right\} \tag{2.4}$$

is defined in the space of coordinates $\mathbf{x}$.

Description of a stochastic physical system distinguishes from that of a deterministic physical system only by additional terms and by additional dynamic variables in the Lagrangian function. The additional dynamic variables describe stochasticity of the particle motion.

Dynamic equations for the dynamic system $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ are obtained as a result of variation of the action (2.3) with respect to dynamic variables $\mathbf{x}$ and $\mathbf{u}$.

To obtain the action functional for $\mathcal{S}_{\mathrm{st}}$ from the action (2.3) for $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$, we should omit integration over $\boldsymbol{\xi}$ in (2.3). We obtain

$$\mathcal{A}_{\mathcal{S}_{\mathrm{st}}}[\mathbf{x},\mathbf{u}]=\int\left\{\frac{m}{2}\dot{\mathbf{x}}^2+\frac{m}{2}\mathbf{u}^2-\frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}\right\}dt,\qquad \dot{\mathbf{x}}\equiv\frac{d\mathbf{x}}{dt} \tag{2.5}$$

where $\mathbf{x}=\mathbf{x}(t)$ and $\mathbf{u}=\mathbf{u}(t,\mathbf{x})$ are dependent dynamic variables. The action functional (2.5) is not well defined for $\hbar\neq 0$, because the operator $\boldsymbol{\nabla}$ is defined in some 3-dimensional vicinity of point $\mathbf{x}$, but not at the point $\mathbf{x}$ itself. As far as the action functional (2.5) is not well defined, one cannot obtain dynamic equations for $\mathcal{S}_{\mathrm{st}}$. By definition it means that the particle $\mathcal{S}_{\mathrm{st}}$ is stochastic. Setting $\hbar=0$ in (2.3), we transform the action (2.3) into the action (2.2), because in this case $\mathbf{u}=0$ in virtue of dynamic equations.

The quantum constant $\hbar$ has been introduced in the action (2.3), in order the description by means of the action (2.3) be equivalent to the quantum description by means of the Schrödinger equation. If we substitute the term $-\hbar\boldsymbol{\nabla}\mathbf{u}/2$ by some function $f(\mathbf{u},\boldsymbol{\nabla}\mathbf{u})$, we obtain statistical description of other stochastic system with other form of stochasticity, which does not coincide with the quantum stochasticity. In other words, the form of the last term in (2.3) describes the type of the stochasticity.

To obtain dynamic equations for the statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ of stochastic systems $\mathcal{S}_{\mathrm{st}}$, one needs to vary the action (2.3). Variation of (2.3) with respect to $\mathbf{u}$ gives

$$\begin{aligned}
\delta\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}[\mathbf{x},\mathbf{u}] &= \int\int\limits_{V_\xi}\left\{m\mathbf{u}\delta\mathbf{u}-\frac{\hbar}{2}\boldsymbol{\nabla}\delta\mathbf{u}\right\}\rho_0(\boldsymbol{\xi})\,dtd\boldsymbol{\xi}\\
&= \int\int\limits_{V_\mathbf{x}}\left\{m\mathbf{u}\delta\mathbf{u}-\frac{\hbar}{2}\boldsymbol{\nabla}\delta\mathbf{u}\right\}\rho_0(\boldsymbol{\xi})\,\frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(x^1,x^2,x^3)}dtd\mathbf{x}\\
&= \int\int\limits_{V_\mathbf{x}}\delta\mathbf{u}\left\{m\mathbf{u}\rho+\frac{\hbar}{2}\boldsymbol{\nabla}\rho\right\}dtd\mathbf{x}-\int\oint\frac{\hbar}{2}\rho\delta\mathbf{u}dtd\mathbf{S}
\end{aligned}$$

where

$$\rho=\rho_{0}\left(\boldsymbol{\xi}\right)\frac{\partial\left(\xi_{1},\xi_{2},\xi_{3}\right)}{\partial\left(x^{1},x^{2},x^{3}\right)}=\rho_{0}\left(\boldsymbol{\xi}\right)\left(\frac{\partial\left(x^{1},x^{2},x^{3}\right)}{\partial\left(\xi_{1},\xi_{2},\xi_{3}\right)}\right)^{-1} \tag{2.6}$$

We obtain the following dynamic equation

$$\delta\mathbf{u}:\qquad m\rho\mathbf{u}+\frac{\hbar}{2}\boldsymbol{\nabla}\rho=0 \tag{2.7}$$

where $\rho=\rho\left(t,\mathbf{x}\right)$ is defined by the relation (2.6). Resolving (2.7) with respect to $\mathbf{u}$, we obtain the equation

$$\mathbf{u}=\mathbf{u}\left(t,\mathbf{x}\right)=-\frac{\hbar}{2m}\boldsymbol{\nabla}\ln\rho, \tag{2.8}$$

which reminds the expression for the mean velocity of the Brownian particle with the diffusion coefficient $D=\hbar/2m$.

Variation of the action (2.3) with respect to $\mathbf{x}$ is produced at fixed form of $\mathbf{u}$, but $\mathbf{u}=\mathbf{u}\left(t,\mathbf{x}\right)$, and argument $\mathbf{x}$ of the function $\mathbf{u}$ should be varied. Variation of (2.3) with respect to $\mathbf{x}$ gives

$$\delta\mathcal{A}_{\mathcal{S}_{\mathrm{st}}}\left[\mathbf{x},\mathbf{u}\right]=\int\left\{m\dot{\mathbf{x}}\delta\dot{\mathbf{x}}+\delta\left(\frac{m}{2}\mathbf{u}^{2}-\frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}\right)\right\}\rho_{0}\left(\boldsymbol{\xi}\right)dtd\boldsymbol{\xi}, \tag{2.9}$$

One obtains dynamic equation

$$\delta\mathbf{x}:\qquad -m\frac{d^{2}\mathbf{x}}{dt^{2}}+\boldsymbol{\nabla}\left(\frac{m}{2}\mathbf{u}^{2}-\frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}\right)=0 \tag{2.10}$$

Substituting (2.8) in (2.10) and considering ρ as a function of $t,\mathbf{x}$, one obtains

$$m\frac{d^{2}\mathbf{x}}{dt^{2}}=-\boldsymbol{\nabla}U_{\mathrm{B}} \tag{2.11}$$

where d/dt means the substantial derivative with respect to time t

$$\frac{dF}{dt}\equiv\frac{\partial\left(F,\xi_{1},\xi_{2},\xi_{3}\right)}{\partial\left(t,\xi_{1},\xi_{2},\xi_{3}\right)}$$

$\boldsymbol{\nabla}$ is gradient in the space of coordinates x, and U_{B} is so-called Bohm potential

$$\begin{aligned}U_{\mathrm{B}}\left(t,\mathbf{x}\right)&=-\frac{m}{2}\mathbf{u}^{2}+\frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}=U\left(\rho,\boldsymbol{\nabla}\rho,\boldsymbol{\nabla}^{2}\rho\right)\\&=\frac{\hbar^{2}}{8m}\frac{\left(\boldsymbol{\nabla}\rho\right)^{2}}{\rho^{2}}-\frac{\hbar^{2}}{4m}\frac{\boldsymbol{\nabla}^{2}\rho}{\rho}=-\frac{\hbar^{2}}{2m}\frac{1}{\sqrt{\rho}}\boldsymbol{\nabla}^{2}\sqrt{\rho}\end{aligned} \tag{2.12}$$

where for calculation of U_{B} one uses the relation (2.8)

One obtains

$$m\frac{d^{2}\mathbf{x}}{dt^{2}}=\frac{\hbar^{2}}{2m}\boldsymbol{\nabla}\left(\frac{1}{\sqrt{\rho}}\boldsymbol{\nabla}^{2}\sqrt{\rho}\right) \tag{2.13}$$

However, the relation (2.6) determines the variable ρ as a function of variables $x^{\alpha,\beta} \equiv \partial x^\alpha / \partial \xi_\beta$, and one needs to take into account this circumstance in the dynamic equation (2.13).

In the Euler representation (in terms of independent variables $t, \mathbf{x}$) the equation (2.11) can be written in the form

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\boldsymbol{\nabla})\,\mathbf{v} = -\frac{1}{m}\boldsymbol{\nabla} U_{\mathrm{B}}, \qquad \mathbf{v} = \mathbf{v}\,(t, \mathbf{x}) \tag{2.14}$$

Using the relation (2.6), let us represent the quantity $\rho\mathbf{v}$ in the form

$$\rho\mathbf{v}\,(t, \mathbf{x}) = \rho_0\,(\boldsymbol{\xi})\,\frac{\partial\,(t, \xi_1, \xi_2, \xi_3)}{\partial\,(t, x^1, x^2, x^3)}\frac{\partial\,(\mathbf{x}, \xi_1, \xi_2, \xi_3)}{\partial\,(t, \xi_1, \xi_2, \xi_3)} = \rho_0\,(\boldsymbol{\xi})\,\frac{\partial\,(\mathbf{x}, \xi_1, \xi_2, \xi_3)}{\partial\,(t, x^1, x^2, x^3)} \tag{2.15}$$

Then using identity

$$\frac{\partial}{\partial t}\left(\rho_0\,(\boldsymbol{\xi})\,\frac{\partial\,(\xi_1, \xi_2, \xi_3)}{\partial\,(x^1, x^2, x^3)}\right) + \frac{\partial}{\partial x^\alpha}\left(\rho_0\,(\boldsymbol{\xi})\,\frac{\partial\,(x^\alpha, \xi_1, \xi_2, \xi_3)}{\partial\,(t, x^1, x^2, x^3)}\right) \equiv 0 \tag{2.16}$$

one obtains the continuity equation for variables $\rho = \rho\,(t, \mathbf{x})$ and $\mathbf{v} = \mathbf{v}\,(t, \mathbf{x})$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^\alpha}\,(\rho v^\alpha) = 0 \tag{2.17}$$

Equations (2.14), (2.17) together with (2.12) form dynamic equations for the statistical ensemble of stochastic particles in Euler representation, when independent dynamic variables are $t, \mathbf{x}$.

Any reference to the stochastic velocity distribution or to some other probability distribution is absent. Influence of this distribution on the mean motion of the particles is described by the form of Bohm potential U_{B} (2.12). The situation reminds the case of the gas dynamics, where the action of the Maxwell velocity distribution on the gas motion is described by the internal gas energy. Of course, such a description is not comprehensive, however, it is sufficient for a description of the mean motion of the stochastic particle. As a result we obtain a *purely dynamic description* of the mean motion of a stochastic particle.

The fluid described by dynamic equations (2.14), (2.17) can be described in terms of two-component wave function [8] or [7]. One obtains the following dynamic equation for wave function ψ

$$i\hbar\partial_0\psi + \frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi + \frac{\hbar^2}{8m}\boldsymbol{\nabla}^2 s_\alpha \cdot (s_\alpha - 2\sigma_\alpha)\,\psi - \frac{\hbar^2}{4m}\frac{\boldsymbol{\nabla}\rho}{\rho}\boldsymbol{\nabla} s_\alpha \sigma_\alpha \psi = 0 \tag{2.18}$$

where

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \rho = \psi^*\psi, \qquad s_\alpha = \frac{\psi^*\sigma_\alpha\psi}{\rho}, \qquad \alpha = 1, 2, 3 \tag{2.19}$$

σ_α are 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.20}$$

In the case of non-rotational flow the wave function becomes to be one component, because at $\psi = \psi_1 = a\psi_2$, a =const and s_α =const, $\alpha = 1,2,3$. In this case the equation (2.18) turns to the linear equation (Schrödinger equation)

$$i\hbar\partial_0\psi + \frac{\hbar^2}{2m}\boldsymbol{\nabla}^2\psi = 0 \tag{2.21}$$

One should note a specificity of description in terms of the wave function. Any ideal fluid may be described in terms of the wave function [8]

3 The case of relativistic particles.

The form of stochasticity of nonrelativistic stochastic particle in (2.5) is defined by two last terms. In the relativistic case the action for the statistical ensemble (2.5) is replaced by the action [23]

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}\left[x,\kappa\right] = -\int\int_{V_{\boldsymbol{\xi}}} mcK\sqrt{g_{ik}\dot{x}^i\dot{x}^k}\rho_0\left(\boldsymbol{\xi}\right)d\tau d\boldsymbol{\xi}, \qquad \dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{d\tau} \tag{3.1}$$

$$K = \sqrt{1+\lambda^2\left(g_{kl}\kappa^k\kappa^l + \partial_k\kappa^k\right)}, \qquad \lambda = \frac{\hbar}{mc} \tag{3.2}$$

where $x = \left\{x^k\right\} = \left\{x^k\left(\tau,\boldsymbol{\xi}\right)\right\}$, $k = 0,1,2,3$. The quantity g_{kl} =diag$\{c^2,-1,-1,-1\}$ is the metric tensor. The independent variables $\boldsymbol{\xi} = \{\xi_1,\xi_2,\xi_3\}$ label the particles of the statistical ensemble. The dependent variables $\kappa^k = \kappa^k\left(x\right)$, $k = 0,1,2,3$ form some force field, connected with the mean stochastic component u^l of the particle 4-velocity by the relation $\kappa^l = \frac{m}{\hbar}u^l$, and λ is the Compton wave length of the particle.

In the nonrelativistic approximation one may neglect the temporal component $\kappa^0 = \frac{m}{\hbar}u^0$ with respect to the spatial one $\boldsymbol{\kappa} = \frac{m}{\hbar}\mathbf{u}$. Setting $\tau = t = x^0$ in (3.1), (3.2) we obtain in the nonrelativistic approximation instead of (3.1)

$$\mathcal{A}_{\mathcal{E}[\mathcal{S}_{\mathrm{st}}]}\left[\mathbf{x},\mathbf{u}\right] = \int\int_{V_{\boldsymbol{\xi}}}\left\{-mc^2 + \frac{m}{2}\dot{\mathbf{x}}^2 + \frac{m}{2}\mathbf{u}^2 - \frac{\hbar}{2}\boldsymbol{\nabla}\mathbf{u}\right\}\rho_0\left(\boldsymbol{\xi}\right)dtd\boldsymbol{\xi}, \qquad \dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt} \tag{3.3}$$

The action (3.3) coincides with the action (2.3) except for the first term, which does not contribute to dynamic equations.

Let us add to the action (3.1) the term describing interaction with the electromagnetic field and write it in the form

$$\mathcal{A}\left[x,\kappa\right] = \int\left\{-mcK\sqrt{g_{ik}\dot{x}^i\dot{x}^k} - \frac{e}{c}A_k\dot{x}^k\right\}d^4\xi, \quad d^4\xi = d\xi_0 d\boldsymbol{\xi} \tag{3.4}$$

$$K = \sqrt{1+\lambda^2\left(\kappa_l\kappa^l + \partial_l\kappa^l\right)}, \qquad \lambda = \frac{\hbar}{mc}, \quad \tau = \boldsymbol{\xi}_0 \tag{3.5}$$

Here $x = \{x^i(\xi_0, \boldsymbol{\xi})\}$, $i = 0,1,2,3$ are dependent variables. $\xi = \{\xi_0, \boldsymbol{\xi}\} = \{\xi_k\}$, $k = 0,1,2,3$ are independent variables, and $\dot{x}^i \equiv dx^i/d\xi_0$. The quantities $\kappa^l = \{\kappa^l(x)\}$, $l = 0,1,2,3$ are dependent variables, describing stochastic component of the particle motion, $A_k = \{A_k(x)\}$, $k = 0,1,2,3$ is the potential of electromagnetic field. We shall refer to the dynamic system, described by the action (3.4), (3.5) as $\mathcal{S}_{\mathrm{KG}}$, because irrotational flow of $\mathcal{S}_{\mathrm{KG}}$ is described by the Klein-Gordon equation [24]. We present here this transformation to the Klein-Gordon form. Here and farther a summation is produced over repeated Latin indices $(0-3)$ and over Greek indices $(1-3)$.

Let us consider variables $\xi = \xi(x)$ in (3.4) as dependent variables and variables x as independent variables. Let the Jacobian

$$J = \frac{\partial(\xi_0, \xi_1, \xi_2, \xi_3)}{\partial(x^0, x^1, x^2, x^3)} = \det\left|\left|\xi_{i,k}\right|\right|, \qquad \xi_{i,k} \equiv \partial_k \xi_i, \qquad i,k = 0,1,2,3 \tag{3.6}$$

be considered to be a multilinear function of $\xi_{i,k}$ $J = J(\xi_{i,k})$. Then

$$d^4\xi = Jd^4x, \qquad \dot{x}^i \equiv \frac{dx^i}{d\xi_0} \equiv \frac{\partial(x^i, \xi_1, \xi_2, \xi_3)}{\partial(\xi_0, \xi_1, \xi_2, \xi_3)} = J^{-1}\frac{\partial J}{\partial \xi_{0,i}} \tag{3.7}$$

After transformation to dependent variables ξ the action (3.4) takes the form

$$\mathcal{A}[\xi, \kappa] = \int \left\{-mcK\sqrt{g_{ik}\frac{\partial J}{\partial \xi_{0,i}}\frac{\partial J}{\partial \xi_{0,k}}} - \frac{e}{c}A_k\frac{\partial J}{\partial \xi_{0,k}}\right\} d^4x, \tag{3.8}$$

Let us introduce new variables

$$j^k = \frac{\partial J}{\partial \xi_{0,k}}, \qquad k = 0,1,2,3 \tag{3.9}$$

by means of Lagrange multipliers p_k

$$\mathcal{A}[\xi, \kappa, j, p] = \int \left\{-mcK\sqrt{g_{ik}j^ij^k} - \frac{e}{c}A_kj^k + p_k\left(\frac{\partial J}{\partial \xi_{0,k}} - j^k\right)\right\} d^4x, \tag{3.10}$$

The variable ξ_0 is fictitious. Variation with respect to ξ_i gives

$$\frac{\delta \mathcal{A}}{\delta \xi_i} = -\partial_l\left(p_k\frac{\partial^2 J}{\partial \xi_{0,k}\partial \xi_{i,l}}\right) = 0, \qquad i = 0,1,2,3 \tag{3.11}$$

Using identities

$$\frac{\partial^2 J}{\partial \xi_{0,k}\partial \xi_{i,l}} \equiv J^{-1}\left(\frac{\partial J}{\partial \xi_{0,k}}\frac{\partial J}{\partial \xi_{i,l}} - \frac{\partial J}{\partial \xi_{0,l}}\frac{\partial J}{\partial \xi_{i,k}}\right) \tag{3.12}$$

$$\frac{\partial J}{\partial \xi_{i,l}}\xi_{k,l} \equiv J\delta_k^i, \qquad \partial_l\frac{\partial^2 J}{\partial \xi_{0,k}\partial \xi_{i,l}} \equiv 0 \tag{3.13}$$

one can test by direct substitution that the general solution of linear equations (3.11) has the form

$$p_k = b_0 \left(\partial_k \varphi + g^\alpha (\boldsymbol{\xi}) \partial_k \xi_\alpha\right), \qquad k = 0, 1, 2, 3 \tag{3.14}$$

where $b_0 \neq 0$ is a constant, $g^\alpha (\boldsymbol{\xi})$, $\alpha = 1, 2, 3$ are arbitrary functions of $\boldsymbol{\xi} = \{\xi_1, \xi_2, \xi_3\}$, and φ is the dynamic variable ξ_0, which ceases to be fictitious. Let us substitute (3.14) in (3.10). The term of the form $\partial_k \varphi \partial J / \partial \xi_{0,k}$ is reduced to Jacobian and does not contribute to dynamic equations. The terms of the form $\xi_{\alpha,k} \partial J / \partial \xi_{0,k}$ vanish due to identities (3.13). We obtain

$$\mathcal{A} [\varphi, \boldsymbol{\xi}, \kappa, j] = \int \left\{ -mcK \sqrt{g_{ik} j^i j^k} - j^k \pi_k \right\} d^4 x, \tag{3.15}$$

where quantities π_k are determined by the relations

$$\pi_k = b_0 \left(\partial_k \varphi + g^\alpha (\boldsymbol{\xi}) \partial_k \xi_\alpha\right) + \frac{e}{c} A_k, \qquad k = 0, 1, 2, 3 \tag{3.16}$$

Integration of (3.11) in the form (3.14) is that integration, which admits one to introduce a wave function. Note that coefficients in the system of equations (3.11) for p_k are constructed of minors of the Jacobian (3.6). It is the circumstance that admits one to produce a general integration.

Variation of (3.15) with respect to κ^l gives

$$\frac{\delta \mathcal{A}}{\delta \kappa^l} = -\frac{\lambda^2 mc \sqrt{g_{ik} j^i j^k}}{K} \kappa_l + \partial_l \frac{\lambda^2 mc \sqrt{g_{ik} j^i j^k}}{2K} = 0 \tag{3.17}$$

It can be written in the form

$$\kappa^l = g^{lk} \partial_k \kappa, \quad \kappa = \frac{1}{2} \ln \frac{\lambda^2 mc \sqrt{g_{ik} j^i j^k}}{2K \rho_0} \tag{3.18}$$

where ρ_0 is a constant of integration. It means that the stochastic component of velocity $u^l = \frac{m}{\hbar} \kappa^l$ can be presented in the form

$$u_l = \frac{\hbar}{m} \kappa_l = \frac{\hbar}{m} \partial_l \kappa = \frac{\hbar}{2m} \partial_l \ln \frac{\lambda^2 mc \sqrt{j_s j^s}}{2K \rho_0} = \frac{\hbar}{2m} \partial_l \ln \frac{\lambda^2 mc \sqrt{j_s j^s}}{2\rho_0 \sqrt{1 + \lambda^2 e^{-\kappa} \partial_s \partial^s e^\kappa}} \tag{3.19}$$

Substituting (3.5) in (3.18), we obtain dynamic equation for κ

$$\hbar^2 \left(\partial_l \kappa \cdot \partial^l \kappa + \partial_l \partial^l \kappa\right) = \frac{e^{-4\kappa} j_s j^s}{\rho_0^2} - m^2 c^2 \tag{3.20}$$

It can be transformed to the form

$$\begin{aligned} j_s j^s &= m^2 c^2 \rho_0^2 e^{4\kappa} \left(1 + \lambda^2 e^{-\kappa} \partial_l \partial^l e^\kappa\right) \\ &= m^2 c^2 \rho_0^2 e^{4\kappa} \left(1 - \lambda^2 \partial_l \kappa \partial^l \kappa + \frac{\lambda^2}{2} e^{-2\kappa} \partial_l \partial^l e^{2\kappa}\right) \end{aligned} \tag{3.21}$$

Variation of (3.15) with respect to j^k gives

$$\pi_k = -\frac{mcKj_k}{\sqrt{g_{ls}j^l j^s}} \tag{3.22}$$

or

$$\pi_k g^{kl}\pi_l = m^2c^2K^2 \tag{3.23}$$

It follows from (3.20), (3.22) and (3.21) that

$$j_k = -\frac{\sqrt{g_{ls}j^l j^s}}{mcK}\pi_k = -\rho_0 e^{2\kappa}\pi_k, \tag{3.24}$$

Now we eliminate the variables j^k from the action (3.15), using relation (3.24) and (3.21). We obtain

$$\mathcal{A}[\varphi,\boldsymbol{\xi},\kappa] = \int m^2c^2\rho_0 e^{2\kappa}\left\{-K\sqrt{\left(1-\lambda^2\partial_l\kappa\partial^l\kappa+\frac{\lambda^2}{2}e^{-2\kappa}\partial_l\partial^l e^{2\kappa}\right)}+\pi^k\pi_k\right\}d^4x$$

or

$$\mathcal{A}[\varphi,\boldsymbol{\xi},\kappa] = \int m^2c^2\rho_0 e^{2\kappa}\left\{-\left(1-\lambda^2\partial_l\kappa\partial^l\kappa+\frac{\lambda^2}{2}e^{-2\kappa}\partial_l\partial^l e^{2\kappa}\right)+\pi^k\pi_k\right\}d^4x \tag{3.25}$$

where π_k is determined by the relation (3.16). The bracket in the action (3.25) can be transformed as follows.

$$\begin{aligned} & -m^2c^2e^{2\kappa}\left(1-\lambda^2\partial_l\kappa\partial^l\kappa+\frac{\lambda^2}{2}e^{-2\kappa}\partial_l\partial^l e^{2\kappa}\right) \\ = & -m^2c^2e^{2\kappa}+\hbar^2e^{2\kappa}\partial_l\kappa\partial^l\kappa-\frac{\hbar^2}{2}\partial_l\partial^l e^{2\kappa} \end{aligned}$$

Let us take into account that the last term has the form of divergence. It does not contribute to dynamic equations and can be omitted. Omitting this term, we obtain instead of (3.25)

$$\mathcal{A}[\varphi,\boldsymbol{\xi},\kappa] = \int \rho_0 e^{2\kappa}\left\{-m^2c^2+\hbar^2\partial_l\kappa\partial^l\kappa+\pi^k\pi_k\right\}d^4x, \tag{3.26}$$

Instead of dynamic variables $\varphi,\boldsymbol{\xi},\kappa$ we introduce n-component complex function

$$\psi = \{\psi_\alpha\} = \left\{\sqrt{\rho}e^{i\varphi}w_\alpha(\boldsymbol{\xi})\right\} = \left\{\sqrt{\rho_0}e^{\kappa+i\varphi}w_\alpha(\boldsymbol{\xi})\right\}, \qquad \alpha = 1,2,...n \tag{3.27}$$

Here w_α are functions of only $\boldsymbol{\xi} = \{\xi_1,\xi_2,\xi_3\}$, having the following properties

$$\sum_{\alpha=1}^{\alpha=n} w_\alpha^* w_\alpha = 1, \qquad -\frac{i}{2}\sum_{\alpha=1}^{\alpha=n}\left(w_\alpha^*\frac{\partial w_\alpha}{\partial\xi_\beta}-\frac{\partial w_\alpha^*}{\partial\xi_\beta}w_\alpha\right) = g^\beta(\boldsymbol{\xi}) \tag{3.28}$$

where (*) denotes complex conjugation. The number n of components of the wave function ψ is chosen in such a way, that equations (3.28) have a solution. Then we obtain

$$\psi^*\psi \equiv \sum_{\alpha=1}^{\alpha=n}\psi_\alpha^*\psi_\alpha=\rho=\rho_0 e^{2\kappa},\qquad \partial_l\kappa=\frac{\partial_l\left(\psi^*\psi\right)}{2\psi^*\psi} \tag{3.29}$$

$$\pi_k = -\frac{ib_0\left(\psi^*\partial_k\psi-\partial_k\psi^*\cdot\psi\right)}{2\psi^*\psi}+\frac{e}{c}A_k,\qquad k=0,1,2,3 \tag{3.30}$$

Substituting relations (3.29), (3.30) in (3.26), we obtain the action, written in terms of the wave function ψ

$$\mathcal{A}\left[\psi,\psi^*\right] = \int\left\{\left[\frac{ib_0\left(\psi^*\partial_k\psi-\partial_k\psi^*\cdot\psi\right)}{2\psi^*\psi}-\frac{e}{c}A_k\right]\left[\frac{ib_0\left(\psi^*\partial^k\psi-\partial^k\psi^*\cdot\psi\right)}{2\psi^*\psi}-\frac{e}{c}A^k\right]\right.$$
$$\left.+\hbar^2\frac{\partial_l\left(\psi^*\psi\right)\partial^l\left(\psi^*\psi\right)}{4\left(\psi^*\psi\right)^2}-m^2c^2\right\}\psi^*\psi d^4x \tag{3.31}$$

Let us use the identity

$$\frac{\left(\psi^*\partial_l\psi-\partial_l\psi^*\cdot\psi\right)\left(\psi^*\partial^l\psi-\partial^l\psi^*\cdot\psi\right)}{4\psi^*\psi}+\partial_l\psi^*\partial^l\psi$$
$$\equiv\frac{\partial_l\left(\psi^*\psi\right)\partial^l\left(\psi^*\psi\right)}{4\psi^*\psi}+\frac{g^{ls}}{2}\psi^*\psi\sum_{\alpha,\beta=1}^{\alpha,\beta=n}Q_{\alpha\beta,l}^*Q_{\alpha\beta,s} \tag{3.32}$$

where

$$Q_{\alpha\beta,l}=\frac{1}{\psi^*\psi}\begin{vmatrix}\psi_\alpha & \psi_\beta\\ \partial_l\psi_\alpha & \partial_l\psi_\beta\end{vmatrix},\qquad Q_{\alpha\beta,l}^*=\frac{1}{\psi^*\psi}\begin{vmatrix}\psi_\alpha^* & \psi_\beta^*\\ \partial_l\psi_\alpha^* & \partial_l\psi_\beta^*\end{vmatrix} \tag{3.33}$$

Then we obtain

$$\mathcal{A}\left[\psi,\psi^*\right] = \int\left\{\begin{array}{c}\left(ib_0\partial_k+\frac{e}{c}A_k\right)\psi^*\left(-ib_0\partial^k+\frac{e}{c}A^k\right)\psi\\ +\frac{b_0^2}{2}\sum\limits_{\alpha,\beta=1}^{\alpha,\beta=n}g^{ls}Q_{\alpha\beta,l}Q_{\alpha\beta,s}^*\psi^*\psi\end{array}\right.$$
$$\left.-m^2c^2\psi^*\psi+\left(\hbar^2-b_0^2\right)\frac{\partial_l\left(\psi^*\psi\right)\partial^l\left(\psi^*\psi\right)}{4\psi^*\psi}\right\}d^4x \tag{3.34}$$

Let us consider the case of irrotational flow, when $g^\alpha\left(\boldsymbol{\xi}\right)=0$ and the function ψ has only one component. It follows from (3.33), that $Q_{\alpha\beta,l}=0$, and only the last term in (3.34) is not bilinear with respect to ψ,ψ^*. The constant b_0 is an arbitrary integration constant. One may set $b_0=\hbar$. Then we obtain instead of (3.34)

$$\mathcal{A}\left[\psi,\psi^*\right]=\int\left\{\left(i\hbar\partial_k+\frac{e}{c}A_k\right)\psi^*\left(-i\hbar\partial^k+\frac{e}{c}A^k\right)\psi-m^2c^2\psi^*\psi\right\}d^4x \tag{3.35}$$

Variation of the action (3.35) with respect to ψ^* generates the Klein-Gordon equation

$$\left(-i\hbar\partial_k + \frac{e}{c}A_k\right)\left(-i\hbar\partial^k + \frac{e}{c}A^k\right)\psi - m^2c^2\psi = 0 \tag{3.36}$$

Thus, description in terms of the Klein-Gordon equation is a special case of the stochastic system description by means of the action (3.4), (3.5).

In the case of rotational flow the wave function is two-component, and the dynamic equation has the form (see for details in [24]):

$$\begin{aligned} &\left(-i\hbar\partial_k + \frac{e}{c}A_k\right)\left(-i\hbar\partial^k + \frac{e}{c}A^k\right)\psi - \left(m^2c^2 + \frac{\hbar^2}{4}\left(\partial_l s_\alpha\right)\left(\partial^l s_\alpha\right)\right)\psi \\ = \; & -\hbar^2\frac{\partial_l\left(\rho\partial^l s_\alpha\right)}{2\rho}\left(\sigma_\alpha - s_\alpha\right)\psi \end{aligned} \tag{3.37}$$

where 3-vector $\mathbf{s} = \{s_1, s_2, s_3,\}$ is defined by the relation

$$\rho = \psi^*\psi, \qquad s_\alpha = \frac{\psi^*\sigma_\alpha\psi}{\rho}, \qquad \alpha = 1,2,3 \tag{3.38}$$

$$\psi = \left(\begin{smallmatrix}\psi_1\\ \psi_2\end{smallmatrix}\right), \qquad \psi^* = \left(\psi_1^*, \psi_2^*\right), \tag{3.39}$$

and Pauli matrices $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ have the form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.40}$$

The gradient of the unit 3-vector $\mathbf{s} = \{s_1, s_2, s_3\}$ describes rotational component of the fluid flow. Equation (3.37) turns to the conventional Klein-Gordon equation (3.36), if $\mathbf{s}$ =const. Curl of the vector field π_k, determined by the relation

$$\partial_k\pi_l - \partial_l\pi_k = -4b_0\left[\partial_k\mathbf{n}\times\partial_l\mathbf{n}\right]\mathbf{z} + \frac{e}{c}\left(\partial_k A_l - \partial_l A_k\right), \qquad k,l = 0,1,2,3 \tag{3.41}$$

Here the quantities $\mathbf{n}$ and $\mathbf{z}$ are obtained from the wave function, presented in the form

$$\psi = \sqrt{\rho}e^{i\varphi}\left(\mathbf{n}\boldsymbol{\sigma}\right)\chi, \qquad \psi^* = \sqrt{\rho}e^{-i\varphi}\chi^*\left(\boldsymbol{\sigma}\mathbf{n}\right), \qquad \mathbf{n}^2 = 1, \qquad \chi^*\chi = 1 \tag{3.42}$$

by means of relations

$$\mathbf{s} = 2\mathbf{n}\left(\mathbf{nz}\right) - \mathbf{z}, \qquad \mathbf{n} = \frac{\mathbf{s}+\mathbf{z}}{\sqrt{2\left(1+(\mathbf{sz})\right)}} \tag{3.43}$$

$$\mathbf{z} = \chi^*\boldsymbol{\sigma}\chi, \qquad \mathbf{z}^2 = \chi^*\chi = 1 \tag{3.44}$$

The fundamental difference between the nonrelativistic description (2.8) and the relativistic description (3.19) is as follows. The nonrelativistic equation (2.8) does not contain temporal derivatives, and the field $\mathbf{u}$ is determined uniquely by its source (the particle density ρ). The relativistic equation (3.19) contains temporal derivatives, and the κ-field $u^k = \hbar\kappa^k/m$ can exist without its source. The relativistic κ-field $u^k = \hbar\kappa^k/m$ can escape from its source. Besides, the κ-field changes the effective particle mass, as one can see from the relations (3.1), (3.2). If $\boldsymbol{\kappa}^2$ is large enough, or $\partial_k\kappa^k < 0$ and $\left|\partial_k\kappa^k\right|$ is large enough, the effective particle mass may be imaginary. In this case the mean world line may turn-round in the time direction, and this turn-round may appear to be connected with the pair production, or with the pair annihilation.

In the nonrelativistic case the mean stochastic velocity $\mathbf{u}$ may be eliminated and replaced by its source (the particle density ρ). In the relativistic case the κ-field has in addition its own degrees of freedom, which cannot be eliminated, replacing the κ-field by its source. The κ-field can travel from one space-time region to another one.

The uniform formalism of the particle dynamics (with the statistical ensemble as a basic object of dynamics) admits one to describe such a physical phenomena, which cannot be described in the framework of the conventional dynamic formalism, when the basic object is a single particle. In particular, one can describe the pair production effect, which cannot been described in the framework of the conventional relativistic mechanics, as well as in the framework of the nonrelativistic quantum mechanics.

4 Deterministic models of elementary particles

Stochastic (and quantum) particles $\mathcal{S}_{\mathrm{st}}$ are described by the statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$. Dynamic equations for $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ form a system of partial differential equations (PDE). Is it possible to simplify description of stochastic particle, reducing the system of PDE to a system of ODE, describing a statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{d}}]$ of deterministic particles $\mathcal{S}_{\mathrm{d}}$? It is possible. One needs only to project all derivatives in the system of PDE onto direction of the particle current j^k by means of (1.5). After such a projection the system of PDE turns to the system of ODE. This system of ODE form a dynamic equations for a statistical ensemble $\mathcal{E}[\mathcal{S}_{\mathrm{d}}]$ of deterministic particles $\mathcal{S}_{\mathrm{d}}$. The deterministic particle $\mathcal{S}_{\mathrm{d}}$ is called a *deterministic model of stochastic particle* $\mathcal{S}_{\mathrm{st}}$. Such a procedure is called dynamic disquantization [9]. The dynamic disquantization (D-disquantization) transforms the dynamic system $\mathcal{E}[\mathcal{S}_{\mathrm{st}}]$ to a simpler dynamic system $\mathcal{E}[\mathcal{S}_{\mathrm{d}}]$, where wobbling of the stochastic particle world line is removed. One can obtain dynamic equations for a single $\mathcal{S}_{\mathrm{d}}$ from dynamic equations for $\mathcal{E}[\mathcal{S}_{\mathrm{d}}]$. Introduction of deterministic model is founded on the fact, that in the coordinate system, where the state of the statistical ensemble is uniform, the stochastic component (2.8) does not contribute in the dynamic equations of the statistical ensemble. The dynamic disquantization is a purely dynamic procedure which removes stochastic fluctuations and generates a deterministic model. In general, the dynamic disquantization removes fluctuations of any kind, but not only quantum fluctuations. For the nonrelativistic equations (Schrödinger equation) the

D-disquantization is equivalent to a transition from nonrelativistic quantum particle to a nonrelativistic classical particle. However, for the relativistic quantum particle (Klein-Gordon equation) the D-disquantization leads to a transition to a relativistic classical particle equipped by a κ-field, which is responsible for the pair production.

To obtain deterministic model of a relativistic quantum particle, let us vary the action (3.4), (3.5) taken in the form

$$\mathcal{A}[x,\kappa]=\int\left\{-mcK\sqrt{g_{ik}\dot{x}^i\dot{x}^k}-\frac{e}{c}A_k\dot{x}^k\right\}d^4\xi,\qquad d^4\xi=d\xi_0 d\boldsymbol{\xi},\quad \tau=\boldsymbol{\xi}_0 \tag{4.1}$$

$$K=\sqrt{1+\lambda^2\left(\kappa_l\kappa^l+\partial_l\kappa^l\right)},\qquad \lambda=\frac{\hbar}{mc} \tag{4.2}$$

Here $x=\{x^i(\xi_0,\boldsymbol{\xi})\}$, $i=0,1,2,3$ are dependent variables. $\xi=\{\xi_0,\boldsymbol{\xi}\}=\{\xi_k\}$, $k=0,1,2,3$ are independent variables, and $\dot{x}^i\equiv dx^i/d\xi_0$. The quantities $\kappa^l=\left\{\kappa^l(x)\right\}$, $l=0,1,2,3$ are dependent variables, describing stochastic component of the particle velocity, $A_k=\{A_k(x)\}$, $k=0,1,2,3$ is the potential of electromagnetic field. Variation of (4.1) gives

$$\begin{aligned}\frac{\delta\mathcal{A}[x,\kappa]}{\delta x^k} &= \frac{d}{d\tau}\frac{mcKg_{ik}\dot{x}^i}{\sqrt{\dot{x}_s\dot{x}^s}}-\frac{e}{c}\left(\frac{\partial A_i}{\partial x^k}-\frac{\partial A_k}{\partial x^i}\right)\dot{x}^i\\ &\quad -\frac{\lambda^2 mc\sqrt{\dot{x}_s\dot{x}^s}}{K}\left(\kappa_{l,k}\kappa^l+\frac{1}{2}\partial_k\partial_l\kappa^l\right)=0\end{aligned} \tag{4.3}$$

$$\frac{\delta\mathcal{A}[x,\kappa]}{\delta\kappa^k}=\frac{mc\sqrt{\dot{x}_s\dot{x}^s}J}{K}\kappa_k-\frac{\partial}{\partial x^k}\frac{mc\sqrt{\dot{x}_s\dot{x}^s}J}{2K}=0 \tag{4.4}$$

Here J is the Jacobian (3.6), which appears, because κ_l is a function of x, and one needs to go to integration over x in (4.1), in order to obtain (4.4). One obtains from (4.4)

$$\kappa_k=\partial_k\kappa=\frac{1}{2}\partial_k\ln\frac{mc\sqrt{\dot{x}_s\dot{x}^s}J}{K}=\frac{1}{2}\partial_k\ln\frac{mc\sqrt{j_sj^s}}{K},\qquad j^k=\dot{x}^kJ \tag{4.5}$$

Using (4.2), one can write (4.5) in the form of dynamic equation for the variable κ which can be written in the form

$$e^{3\kappa}\left(e^{\kappa}+\lambda^2\partial_s\partial^s e^{\kappa}\right)=C^2m^2c^2j_sj^s \tag{4.6}$$

where C is the integration constant. The quantity C does not depend on x, but it may depend on coordinates of other particles.

Introducing new variable

$$w=e^{\kappa} \tag{4.7}$$

the equation (4.6) can be written in the form

$$\hbar^2\partial_s\partial^s w+m^2c^2w=\frac{C^2m^2c^2j_sj^s}{w^3} \tag{4.8}$$

$$K=\sqrt{1+\frac{\lambda^2}{w}\partial_s\partial^s w} \tag{4.9}$$

Equation (4.3) is written in the form

$$\frac{d}{d\tau}\frac{mcKg_{ik}\dot{x}^i}{\sqrt{\dot{x}_s\dot{x}^s}}-\frac{e}{c}\left(\frac{\partial A_i}{\partial x^k}-\frac{\partial A_k}{\partial x^i}\right)\dot{x}^i-mc\sqrt{\dot{x}_s\dot{x}^s}\partial_k K \tag{4.10}$$

The relativistic stochastic particle $\mathcal{S}_{\mathrm{st}}$ is described by equations (4.8) - (4.10). Its stochasticity is conditioned by the field w, which depends on the state of the whole statistical ensemble $\mathcal{E}\left[\mathcal{S}_{\mathrm{st}}\right]$ and maybe on other particles via the constant C. Operation of disquantization cannot be applied to the field w, because this field is an external field (at least, partly). Thus, the exact equations (4.8) - (4.10) are dynamic equations for deterministic model simultaneously.

5 Dirac equation in terms of hydrodynamic variables

The Dirac particle is a dynamic system $\mathcal{S}_{\mathrm{D}}$, whose dynamic equation is the Dirac equation

$$i\gamma^k\partial_k\psi+mc\psi=0 \tag{5.1}$$

The Dirac dynamic system $\mathcal{S}_{\mathrm{D}}$ was investigated by many researchers. There is no possibility to list all them, and we mention only some of them. First, this is transformation of the Dirac equation on the base of quantum mechanics [26, 27]. The complicated structure of Dirac particle was discovered by Schrödinger [28], who interpreted it as some complicated quantum motion (zitterbewegung). Investigation of this quantum motion and different models of Dirac particle can be found in [29, 30, 31, 32, 33] and references therein. Our investigation differs in absence of any suppositions on the Dirac particle model and in absence of referring to the quantum principles. We use only dynamic methods and *investigate the Dirac particle* $\mathcal{S}_{\mathrm{D}}$ *simply as a dynamic system.* To obtain the deterministic model of the Dirac particle, one needs to write the Dirac equation in terms of hydrodynamic variables.

The action for a free Dirac particle is written in the form

$$\mathcal{S}_{\mathrm{D}}: \qquad \mathcal{A}_{\mathrm{D}}[\bar{\psi},\psi]=\int(-m\bar{\psi}\psi+\frac{i}{2}\hbar\bar{\psi}\gamma^l\partial_l\psi-\frac{i}{2}\hbar\partial_l\bar{\psi}\gamma^l\psi)d^4x \tag{5.2}$$

Here ψ is four-component complex wave function, ψ^* is the Hermitian conjugate wave function, and $\bar{\psi}=\psi^*\gamma^0$ is conjugate one. γ^i, $i=0,1,2,3$ are 4×4 complex constant matrices, satisfying the relation

$$\gamma^l\gamma^k+\gamma^k\gamma^l=2g^{kl}I, \qquad k,l=0,1,2,3. \tag{5.3}$$

where I is the unit 4×4 matrix, and g^{kl} =diag$(c^{-2},-1,-1,-1)$ is the metric tensor. Considering dynamic system $\mathcal{S}_{\rm D}$, we choose for simplicity such units, where the speed of the light $c=1$.

In our calculations we used the mathematical technique [35, 36], where γ-matrices are represented as hypercomplex numbers. Using designations

$$\gamma_5=\gamma^{0123}\equiv\gamma^0\gamma^1\gamma^2\gamma^3, \tag{5.4}$$

$$\boldsymbol{\sigma}=\{\sigma_1,\sigma_2,\sigma_3,\}=\{-i\gamma^2\gamma^3,-i\gamma^3\gamma^1,-i\gamma^1\gamma^2\} \tag{5.5}$$

we make the change of variables

$$\psi=Ae^{i\varphi+\frac{1}{2}\gamma_5\kappa}\exp\left(-\frac{i}{2}\gamma_5\boldsymbol{\sigma\eta}\right)\exp\left(\frac{i\pi}{2}\boldsymbol{\sigma}\mathbf{n}\right)\Pi \tag{5.6}$$

$$\psi^*=A\Pi\exp\left(-\frac{i\pi}{2}\boldsymbol{\sigma}\mathbf{n}\right)\exp\left(-\frac{i}{2}\gamma_5\boldsymbol{\sigma\eta}\right)e^{-i\varphi-\frac{1}{2}\gamma_5\kappa} \tag{5.7}$$

where (*) means the Hermitian conjugation, and

$$\Pi=\frac{1}{4}(1+\gamma^0)(1+\mathbf{z}\boldsymbol{\sigma}),\qquad \mathbf{z}=\{z^\alpha\}=\text{const},\qquad \alpha=1,2,3;\qquad \mathbf{z}^2=1 \tag{5.8}$$

is a zero divisor. The quantities A, κ, φ, $\boldsymbol{\eta}=\{\eta^\alpha\}$, $\mathbf{n}=\{n^\alpha\}$, $\alpha=1,2,3,$ $\mathbf{n}^2=1$ are eight real parameters, determining the wave function ψ. These parameters may be considered as new dependent variables, describing the state of dynamic system $\mathcal{S}_{\rm D}$. The quantity φ is a scalar, and κ is a pseudoscalar. Six remaining variables A, $\boldsymbol{\eta}=\{\eta^\alpha\}$, $\mathbf{n}=\{n^\alpha\}$, $\alpha=1,2,3,$ $\mathbf{n}^2=1$ can be expressed through the flux 4-vector

$$j^l=\bar{\psi}\gamma^l\psi,\qquad l=0,1,2,3 \tag{5.9}$$

and spin 4-pseudovector

$$S^l=i\bar{\psi}\gamma_5\gamma^l\psi,\qquad l=0,1,2,3 \tag{5.10}$$

Because of two identities

$$S^lS_l\equiv-j^lj_l,\qquad j^lS_l\equiv 0, \tag{5.11}$$

there are only six independent components among eight components of quantities j^l, and S^l.

After change of variables (5.6), (5.7) the γ-matrices disappear from the action and from dynamic equations. One obtains the action (5.2) in terms of hydrodynamic variables j,φ,ξ,κ (see details of calculations in [11, 34])

$$\mathcal{S}_{\rm D}:\qquad \mathcal{A}_D[j,\varphi,\kappa,\boldsymbol{\xi}]=\int\mathcal{L}d^4x,\qquad \mathcal{L}=\mathcal{L}_{\rm cl}+\mathcal{L}_{\rm q1}+\mathcal{L}_{\rm q2} \tag{5.12}$$

$$\mathcal{L}_{\rm cl}=-m\rho-\hbar j^i\partial_i\varphi-\frac{\hbar j^l}{2\left(1+\boldsymbol{\xi}\mathbf{z}\right)}\varepsilon_{\alpha\beta\gamma}\xi^\alpha\partial_l\xi^\beta z^\gamma,\qquad \rho\equiv\sqrt{j^lj_l} \tag{5.13}$$

$$\mathcal{L}_{\mathrm{q1}} = 2m\rho\sin^2(\frac{\kappa}{2}) - \frac{\hbar}{2}S^l\partial_l\kappa, \tag{5.14}$$

$$\mathcal{L}_{\mathrm{q2}} = \frac{\hbar(\rho + j_0)}{2}\varepsilon_{\alpha\beta\gamma}\partial^\alpha\frac{j^\beta}{(j^0 + \rho)}\xi^\gamma - \frac{\hbar}{2(\rho + j_0)}\varepsilon_{\alpha\beta\gamma}\left(\partial^0 j^\beta\right)j^\alpha\xi^\gamma \tag{5.15}$$

Lagrangian is a function of 4-vector j^l, scalar φ, pseudoscalar κ, and unit 3-pseudovector $\boldsymbol{\xi}$, which is connected with the spin 4-pseudovector S^l (5.10) by means of the relations

$$\xi^\alpha = \rho^{-1}\left[S^\alpha - \frac{j^\alpha S^0}{(j^0 + \rho)}\right], \qquad \alpha = 1, 2, 3; \qquad \rho \equiv \sqrt{j^l j_l} \tag{5.16}$$

$$S^0 = \mathbf{j}\boldsymbol{\xi}, \qquad S^\alpha = \rho\xi^\alpha + \frac{(\mathbf{j}\boldsymbol{\xi})j^\alpha}{\rho + j^0}, \qquad \alpha = 1, 2, 3 \tag{5.17}$$

After change of variables the description of $\mathcal{S}_{\mathrm{D}}$ ceases to be relativistically covariant, because the constant matrix 4-vector γ^k is transformed to dynamical variables (see discussion in [37, 39]).

6 Dynamic disquantization of Dirac equation

Let us produce dynamical disquantization [9, 41] of the action (5.12)–(5.15), making the change (1.5). The action (5.12)–(5.15) takes the form

$$\begin{aligned}\mathcal{A}_{\mathrm{Dqu}}[j, \varphi, \kappa, \boldsymbol{\xi}] \;=\; & \int\Big\{-m\rho\cos\kappa - \hbar j^i\left(\partial_i\varphi + \frac{\varepsilon_{\alpha\beta\gamma}\xi^\alpha\partial_i\xi^\beta z^\gamma}{2\left(1 + \boldsymbol{\xi}\mathbf{z}\right)}\right) \\ & + \frac{\hbar j^k}{2(\rho + j_0)\rho}\varepsilon_{\alpha\beta\gamma}\left(\partial_k j^\beta\right)j^\alpha\xi^\gamma\Big\}\,d^4x \end{aligned} \tag{6.1}$$

Note that the second term $-\frac{\hbar}{2}S^l\partial_l\kappa$ in the relation (5.14) is neglected, because 4-pseudovector S^k is orthogonal to 4-vector j^k, and the derivative $S^l\partial_{||l}\kappa = S^l\rho^{-2}j_l j^k\partial_k\kappa/j_s j^s$ vanishes. The action (6.1) is also non-relativistically invariant, because the dynamic disquantization (1.5) is a relativistic procedure.

Although the action (6.1) contains a non-classical variable κ, in fact this variable is a constant. Indeed, a variation with respect to κ leads to the dynamic equation

$$\frac{\delta\mathcal{A}_{Dqu}}{\delta\kappa} = m\rho\sin\kappa = 0, \quad \rho \equiv \sqrt{j_s j^s} \tag{6.2}$$

which has solutions

$$\kappa = n\pi, \qquad n = \text{integer} \tag{6.3}$$

Thus, the effective mass $m_{\mathrm{eff}} = m\cos\kappa$ has two values

$$m_{\mathrm{eff}} = m\cos\kappa = \kappa_0 m = \pm m \tag{6.4}$$

where κ_0 is a dichotomic quantity $\kappa_0 = \pm 1$ introduced instead of $\cos\kappa$. The quantity κ_0 is a parameter of the dynamic system $\mathcal{S}_{\mathrm{Dqu}}$. It is not to be varying. The action

(6.1), turns into the action

$$\mathcal{A}_{\mathrm{Dqu}}[j,\varphi,\boldsymbol{\xi}] = \int \Big\{ -\kappa_0 m\rho - \hbar j^i \left(\partial_i\varphi + \frac{\varepsilon_{\alpha\beta\gamma}\xi^\alpha\partial_i\xi^\beta z^\gamma}{2\,(1+\boldsymbol{\xi}\mathbf{z})}\right) + \frac{\hbar j^k}{2(\rho+j_0)\rho}\varepsilon_{\alpha\beta\gamma}\left(\partial_k j^\beta\right) j^\alpha\xi^\gamma \Big\}\, d^4x \qquad (6.5)$$

Let us introduce Lagrangian coordinates $\tau = \{\tau_0, \boldsymbol{\tau}\} = \{\tau_i(x)\}$, $i = 0,1,2,3$ as functions of coordinates x in such a way that only coordinate τ_0 changes along the direction j^l. The action (6.5) is transformed to the form

$$\mathcal{A}_{\mathrm{Dqu}}[x,\boldsymbol{\xi}] = \int \mathcal{A}_{\mathrm{Dd}}[x,\boldsymbol{\xi}]d\boldsymbol{\tau}, \qquad \mathbf{d}\boldsymbol{\tau} = d\tau_1 d\tau_2 d\tau_3 \qquad (6.6)$$

where

$$\mathcal{S}_{\mathrm{Dd}}: \quad \mathcal{A}_{\mathrm{Dd}}[x,\boldsymbol{\xi}] = \int \left\{ \begin{array}{c} -\kappa_0 m\sqrt{\dot{x}^i\dot{x}_i} + \hbar\frac{(\dot{\boldsymbol{\xi}}\times\boldsymbol{\xi})\mathbf{z}}{2(1+\boldsymbol{\xi}\mathbf{z})} \\ +\hbar\frac{(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\boldsymbol{\xi}}{2\sqrt{\dot{x}^s\dot{x}_s}(\sqrt{\dot{x}^s\dot{x}_s}+\dot{x}^0)} \end{array} \right\} d\tau_0 \qquad (6.7)$$

After dynamic disquantization the Dirac particle is a statistical ensemble of dynamic systems $\mathcal{S}_{\mathrm{Dd}}$, as it follows from (6.6) and (6.7). Any dynamic system $\mathcal{S}_{\mathrm{Dd}}$ has 10 degrees of freedom. Six degrees of freedom describe a progressive motion of a particle and 4 degrees of freedom describe the rotational motion of the particle. It is a deterministic model of the Dirac particle, which contains the quantum constant. The quantum constant appears in classical dynamic equations, because these equations are to contain magnetic moment. But the magnetic moment, (classical quantity!) depends on the quantum constant. The variables $\boldsymbol{\xi}$ describe rotation, which is a deterministic analog of so-called "zitterbewegung". The Dirac particle is not a pointlike particle [34]. Description of internal degrees of freedom in terms of $\boldsymbol{\xi}$ appears to be nonrelativistic [39, 37], although the translational degrees of freedom in terms of x are described relativistically.

It is easy to see that the action (6.7) is invariant with respect to transformation $\tau_0 \to \tilde{\tau}_0 = F(\tau_0)$, where F is an arbitrary monotone function. This transformation admits one to choose the variable $t = x^0$ as a parameter τ_0, or to choose the parameter τ_0 in such a way that $\dot{x}^l\dot{x}_l = \dot{x}_0^2 - \dot{\mathbf{x}}^2 = 1$. In the last case the parameter τ_0 is the proper time along the world line of deterministic Dirac particle. Besides, invariance with respect to transformation $\tau_0 \to \tilde{\tau}_0 = F(\tau_0)$ leads to a connection between the components of the canonical momentum

$$p_k = \frac{\partial L}{\partial\dot{x}^k} - \frac{d}{d\tau_0}\frac{\partial L}{\partial\ddot{x}^k}, \qquad k = 0,1,2,3$$

where L is the Lagrange function for the action (6.7).

We shall not consider here problems connected with relativistic non-invariance of terms, describing internal degrees of freedom, referring to [9], where these problems are discussed. We obtain dynamic equations generated by the action (6.7), solve them and try to interpret the obtained solution.

Variation of the action (6.7) with respect to $\mathbf{x}$ gives the dynamic equation

$$\frac{d}{d\tau_0}\left(-\kappa_0 m\frac{\dot{\mathbf{x}}}{\sqrt{\dot{x}^s\dot{x}_s}}+\frac{\hbar Q}{2}(\boldsymbol{\xi}\times\ddot{\mathbf{x}})-\frac{\hbar}{2}\frac{\partial Q}{\partial\dot{\mathbf{x}}}(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\boldsymbol{\xi}+\frac{\hbar}{2}\frac{d}{d\tau_0}\left(Q(\boldsymbol{\xi}\times\dot{\mathbf{x}})\right)\right)=0 \quad (6.8)$$

where

$$Q=Q\left(\dot{x}\right)=\left(\sqrt{\dot{x}^s\dot{x}_s}(\sqrt{\dot{x}^s\dot{x}_s}+\dot{x}^0)\right)^{-1},\qquad \dot{x}^s\dot{x}_s=\dot{x}_0^2-\dot{\mathbf{x}}^2 \quad (6.9)$$

Varying the action (6.7) with respect to x^0, we obtain

$$\frac{d}{d\tau_0}\left(\kappa_0 m\frac{\dot{x}^0}{\sqrt{\dot{x}^s\dot{x}_s}}-\frac{\hbar}{2}\frac{\partial Q}{\partial\dot{x}^0}(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\boldsymbol{\xi}\right)=0 \quad (6.10)$$

Varying the action (6.7) with respect to $\boldsymbol{\xi}$, one should take into account the side constraint $\boldsymbol{\xi}^2=1$. Setting

$$\xi^\alpha=\frac{\zeta^\alpha}{\sqrt{\boldsymbol{\zeta}^2}},\qquad \alpha=1,2,3 \quad (6.11)$$

where $\boldsymbol{\zeta}$ is an arbitrary 3-pseudovector, one obtains

$$\frac{\delta\mathcal{A}_{\mathrm{dcl}}}{\delta\zeta^\mu}=\frac{\delta\mathcal{A}_{\mathrm{dcl}}}{\delta\xi^\alpha}\frac{\delta\xi^\alpha}{\delta\zeta^\mu}=\frac{\delta\mathcal{A}_{\mathrm{dcl}}}{\delta\xi^\alpha}\frac{\delta^{\alpha\mu}-\xi^\alpha\xi^\mu}{\sqrt{\boldsymbol{\zeta}^2}}=0 \quad (6.12)$$

It means that there are only two independent equations among three dynamic equations (6.12). They are orthogonal to 3-pseudovector $\boldsymbol{\xi}$ and can be obtained from equation $\delta\mathcal{A}_{\mathrm{dcl}}/\delta\xi^\alpha=0$ by means of vector product with $\boldsymbol{\xi}$.

$$-\hbar\frac{\left(\dot{\boldsymbol{\xi}}\times\mathbf{z}\right)\times\boldsymbol{\xi}}{2(1+\mathbf{z}\boldsymbol{\xi})}+\hbar\left(-\frac{d}{d\tau_0}\frac{(\boldsymbol{\xi}\times\mathbf{z})}{2(1+\mathbf{z}\boldsymbol{\xi})}-\frac{(\dot{\boldsymbol{\xi}}\times\boldsymbol{\xi})\mathbf{z}}{2(1+\mathbf{z}\boldsymbol{\xi})^2}\mathbf{z}\right)\times\boldsymbol{\xi}+\hbar\frac{(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\times\boldsymbol{\xi}}{2}Q=0 \quad (6.13)$$

After transformations this equation reduces to the equation (see Appendix)

$$\dot{\boldsymbol{\xi}}=-\boldsymbol{\xi}\times(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})Q, \quad (6.14)$$

which does not contain the vector $\mathbf{z}$. It means that $\mathbf{z}$ determines a fictitious direction in the space-time.

Using invariance of the action (6.7) with respect to transformation of the parameter τ_0, we choose τ_0 in such a way, that

$$\sqrt{\dot{x}^s\dot{x}_s}=\sqrt{\dot{x}_0^2-\dot{\mathbf{x}}^2}=1,\qquad \dot{x}_0=\sqrt{1+\dot{\mathbf{x}}^2} \quad (6.15)$$

Then, using condition (6.15), we obtain from (6.9) for quantities Q, $\partial Q/\partial\dot{x}_0$, $\partial Q/\partial\dot{\mathbf{x}}$

$$Q=\frac{1}{1+\dot{x}_0},\qquad \frac{\partial Q}{\partial\dot{x}_0}=-1,\qquad \frac{\partial Q}{\partial\dot{\mathbf{x}}}=\frac{\dot{\mathbf{x}}\left(2+\dot{x}_0\right)}{\left(1+\dot{x}_0\right)^2} \quad (6.16)$$

Integration of equation (6.10) leads to

$$\kappa_0 m\dot{x}_0 + \frac{\hbar}{2}\left(\dot{\mathbf{x}} \times \ddot{\mathbf{x}}\right) \boldsymbol{\xi} = -p_0 \tag{6.17}$$

where p_0 is an integration constant. This constant p_0 describes the time component of the dynamic system $\mathcal{S}_{\mathrm{Dd}}$ canonical 4-momentum.

Integration of equation (6.8) gives

$$-\kappa_0 m\frac{\dot{\mathbf{x}}}{\sqrt{\dot{x}^s\dot{x}_s}} + \frac{\hbar Q}{2}(\boldsymbol{\xi} \times \ddot{\mathbf{x}}) - \frac{\hbar}{2}\frac{\partial Q}{\partial \dot{\mathbf{x}}}(\dot{\mathbf{x}} \times \ddot{\mathbf{x}})\boldsymbol{\xi} + \frac{\hbar}{2}\frac{d}{d\tau_0}\left(Q(\boldsymbol{\xi} \times \dot{\mathbf{x}})\right) = -\mathbf{p} = \text{const} \tag{6.18}$$

where $\mathbf{p}$ is the 3-momentum of the dynamic system $\mathcal{S}_{\mathrm{Dd}}$ as a whole.

Using the gauge (6.9) and relations (6.16), we rewrite the equation (6.18) in the form

$$-m\dot{\mathbf{x}} + \frac{\hbar}{2}\frac{(\boldsymbol{\xi} \times \ddot{\mathbf{x}})}{1 + \dot{x}_0} - \frac{\hbar}{2}\frac{\dot{\mathbf{x}}\,(2 + \dot{x}_0)}{(1 + \dot{x}_0)^2}(\dot{\mathbf{x}} \times \ddot{\mathbf{x}})\boldsymbol{\xi} + \frac{\hbar}{2}\frac{d}{d\tau_0}\left(\frac{(\boldsymbol{\xi} \times \dot{\mathbf{x}})}{1 + \dot{x}_0}\right) = -\mathbf{p} \tag{6.19}$$

If we set $\hbar = 0$ in (6.19), we obtain conventional connection $\mathbf{p} = m\dot{\mathbf{x}}$ between the velocity $\dot{\mathbf{x}} = d\mathbf{x}/d\tau_0$ and the momentum of a free particle. But the quantum constant $\hbar$ is a coefficient before the highest time derivative, and setting $\hbar = 0$, we suppress some degrees of freedom.

If these additional degrees of freedom are not excited (or suppressed), the classical Dirac particle has six degrees of freedom. We shall see that characteristic energy associated with additional degrees of freedom is of the order of the particle rest energy m. At low energetic processes (calculation of atomic spectra, quantum electrodynamics) one may neglect these degrees of freedom, remaining only numerical characteristics (spin, magnetic momentum) of these degrees of freedom. However, in the case of high energies (ultrarelativistic collisions, structure of elementary particles), one cannot neglect these degrees of freedom. Of course, using the Dirac equation, we take into account these additional degrees of freedom automatically. But it is important also to take into account these additional degrees of freedom in our interpretation of the high energetic processes.

Transformation and solution of equation (6.18) is rather bulky. Many efforts were used to prove that the 3-vectors $\boldsymbol{\xi}$, $\dot{\mathbf{x}}$, and $\ddot{\mathbf{x}}$ are mutually orthogonal and their modules are constant [9] in the coordinate system, where $\mathbf{p} = 0$. We shall not spend time for this proof. Instead, we choose the coordinate system in such a way that $\mathbf{p} = 0$

$$\boldsymbol{\xi} = \{0, 0, \varepsilon_0\}, \qquad \varepsilon_0 = \pm 1 \tag{6.20}$$

and impose constraints

$$\dot{\mathbf{x}}^2 = \text{const}, \qquad (\dot{\mathbf{x}}\boldsymbol{\xi}) = 0, \qquad (\ddot{\mathbf{x}}\boldsymbol{\xi}) = 0, \qquad (\dot{\mathbf{x}} \times \ddot{\mathbf{x}})\,\boldsymbol{\xi} = \text{const} \tag{6.21}$$

We use constraints (6.21) in solution of the system of dynamic equations (6.14), (6.17), (6.19) and show that the constraints (6.21) are compatible with dynamic equations (6.14), (6.17), (6.19).

Taking into account (6.21) and (6.15), we introduce new variables

$$\mathbf{y} = \frac{\dot{\mathbf{x}}}{\sqrt{1+\dot{x}_0}} = \frac{\dot{\mathbf{x}}}{\sqrt{1+\sqrt{1+\dot{\mathbf{x}}^2}}}, \qquad \dot{\mathbf{x}} = \mathbf{y}\sqrt{(\mathbf{y}^2+2)} \tag{6.22}$$

$$\dot{x}_0 = \sqrt{1+\mathbf{y}^2\left(\mathbf{y}^2+2\right)} = \mathbf{y}^2+1 \tag{6.23}$$

Introducing designation

$$\mathbf{y}^2 = \gamma - 1 = \text{const}, \tag{6.24}$$

we obtain

$$\dot{x}_0 = \sqrt{1+\mathbf{y}^2\left(\mathbf{y}^2+2\right)} = \mathbf{y}^2+1 = \gamma = \text{const} \tag{6.25}$$

Then at $\mathbf{p}=0$ the equation (6.19) takes the form

$$-\kappa_0 m\mathbf{y}\left(\gamma+1\right) + \frac{\hbar}{2}(\boldsymbol{\xi}\times\dot{\mathbf{y}}) - \frac{\hbar}{2}\left(\gamma+2\right)\left((\mathbf{y}\times\dot{\mathbf{y}})\boldsymbol{\xi}\right)\mathbf{y} + \frac{\hbar}{2}\frac{d}{d\tau_0}\left((\boldsymbol{\xi}\times\mathbf{y})\right) = 0 \tag{6.26}$$

The equation (6.14) takes the form

$$\dot{\boldsymbol{\xi}} = -(\mathbf{y}\times\dot{\mathbf{y}})\times\boldsymbol{\xi} = 0 \tag{6.27}$$

because of constraints (6.21). In terms of variables $\mathbf{y}$ conditions (6.21) have the form

$$\mathbf{y}^2 = \gamma-1, \qquad (\boldsymbol{\xi}\mathbf{y}) = 0, \qquad (\boldsymbol{\xi}\dot{\mathbf{y}}) = 0, \qquad (\mathbf{y}\dot{\mathbf{y}}) = 0 \tag{6.28}$$

where γ is a constant of integration. In accordance with (6.25) and (6.28) we obtain

$$(\mathbf{y}\times\dot{\mathbf{y}})\boldsymbol{\xi} = \varepsilon_0\omega\left(\gamma-1\right) \tag{6.29}$$

where ω is an indefinite constant (some angular velocity).

Substituting (6.29) in (6.26), we obtain after simplification

$$(\boldsymbol{\xi}\times\dot{\mathbf{y}}) - \left(\frac{1}{2}\left(\gamma+2\right)\left(\gamma-1\right)\varepsilon_0\omega + \frac{\kappa_0 m}{\hbar}\left(\gamma+1\right)\right)\mathbf{y} = 0 \tag{6.30}$$

As far as $\mathbf{y}^2 = \gamma-1$, the equation (6.29) describes rotation of the vector $\mathbf{y}$ with the angular frequency ω. Equation (6.30) describes rotation of the vector $\mathbf{y}$ around the vector $\boldsymbol{\xi}$ with the angular frequency $\frac{1}{2}\left(\gamma+2\right)\left(\gamma-1\right)\varepsilon_0\omega + \frac{\kappa_0 m}{\hbar}\left(\gamma+1\right)$. Equations (6.29) and (6.30) are compatible, if these frequencies coincide. According to (6.28) vectors $\mathbf{y}$ and $\dot{\mathbf{y}}$ are orthogonal to $\boldsymbol{\xi}$. Then in accordance with (6.20) the vectors $\mathbf{y}$ and $\dot{\mathbf{y}}$ can be represented in the form

$$\mathbf{y} = \left\{\sqrt{\gamma-1}\cos\Phi, \sqrt{\gamma-1}\sin\Phi, 0\right\} \tag{6.31}$$

$$\dot{\mathbf{y}} = \left\{-\sqrt{\gamma-1}\omega\sin\Phi, \sqrt{\gamma-1}\omega\cos\Phi, 0\right\}, \qquad \omega = \frac{d\Phi}{d\tau_0} \tag{6.32}$$

By means of (6.31), and (6.32) the equations (6.30) take the form

$$-\varepsilon_0\omega y_1-\left(\frac{1}{2}(\gamma+2)(\gamma-1)\varepsilon_0\omega+\frac{\kappa_0 m}{\hbar}(\gamma+1)\right)y_1 = 0 \tag{6.33}$$

$$-\varepsilon_0\omega y_2-\left(\frac{1}{2}(\gamma+2)(\gamma-1)\varepsilon_0\omega+\frac{\kappa_0 m}{\hbar}(\gamma+1)\right)y_2 = 0 \tag{6.34}$$

Equations (6.33), (6.34) are satisfied, provided

$$\varepsilon_0\omega+\left(\frac{1}{2}(\gamma+2)(\gamma-1)\varepsilon_0\omega+\frac{\kappa_0 m}{\hbar}(\gamma+1)\right)=0 \tag{6.35}$$

Solution of (6.35) has the form

$$\omega=-\frac{2\varepsilon_0\kappa_0 m}{\hbar\gamma} \tag{6.36}$$

According to (6.22) and (6.23) the dynamic equation (6.17) takes the form

$$-p_0=\kappa_0 m\gamma+\frac{\hbar}{2}(\mathbf{y}\times\dot{\mathbf{y}})\boldsymbol{\xi}(\gamma+1) \tag{6.37}$$

Using relations (6.29) and (6.36) we obtain from (6.37)

$$-p_0=\kappa_0 m\left(\gamma-\frac{\gamma^2-1}{\gamma}\right)=\frac{\kappa_0 m}{\gamma},\qquad \kappa_0=\pm 1 \tag{6.38}$$

Then we obtain for the total mass M_{Dd} of the dynamic system $\mathcal{S}_{\mathrm{Dd}}$.

$$M_{\mathrm{Dd}}=\sqrt{p_0^2-\mathbf{p}^2}=|p_0|=\frac{m}{\gamma} \tag{6.39}$$

Note, that writing the relation (6.39), we do not act quite consequently. Writing the relation (6.39), we suppose that the dynamic equations (6.17) and (6.18) are relativistically invariant, and solution of equations (6.17), (6.18) for arbitrary $\mathbf{p}$ can be obtained from the solution for $\mathbf{p}=0$ by means of a corresponding Lorentz transformation. Unfortunately, dynamic equations (6.17), (6.18) are not relativistically invariant, and for arbitrary $\mathbf{p}$ the solution is not a helix, in general, although it is a helix for $\mathbf{p}=0$. World line is a helix approximately in the nonrelativistic case, when $|\mathbf{p}|\ll m$.

Let us transit from independent variable τ_0 to the independent variable $x^0=t$. We have

$$\Omega t=-\varepsilon_0\kappa_0\omega\tau_0,\qquad -\varepsilon_0\kappa_0\omega=\Omega\dot{x}_0=\Omega\gamma=\frac{2m}{\hbar\gamma},\qquad \Omega=\frac{2m}{\hbar\gamma^2} \tag{6.40}$$

Returning from variables $\mathbf{y}$ to variables $\dot{\mathbf{x}}$, we obtain instead of (6.31) and (6.32)

$$\frac{d\mathbf{x}}{dt}=\left\{\frac{\sqrt{\gamma^2-1}}{\gamma}\cos(\Omega t),-\frac{\sqrt{\gamma^2-1}}{\gamma}\sin(\Omega t),0\right\},\qquad \Omega=\frac{2m}{\hbar\gamma^2} \tag{6.41}$$

$$\mathbf{x}=\left\{\frac{\hbar\gamma\sqrt{\gamma^2-1}}{2m}\sin\left(\frac{2m}{\hbar\gamma^2}t\right),\frac{\hbar\gamma\sqrt{\gamma^2-1}}{2m}\cos\left(\frac{2m}{\hbar\gamma^2}t\right),0\right\} \tag{6.42}$$

where $\gamma \geq 1$ is an arbitrary constant.

Thus, in the coordinate system, where the canonical momentum four-vector has the form

$$P_k=\{p_0,\mathbf{p}\}=\left\{-\frac{\kappa_0 m}{\gamma},0,0,0\right\} \tag{6.43}$$

the world line of the deterministic Dirac particle is a helix, which is described by the relation

$$\{t,\mathbf{x}\} = \{t, a_{\mathrm{Dd}}\sin(\Omega t), a_{\mathrm{Dd}}\cos(\omega_{\mathrm{Dd}}t), 0\} \tag{6.44}$$

$$a_{\mathrm{Dd}} = \frac{\hbar\gamma\sqrt{\gamma^2-1}}{2m}, \qquad \omega_{\mathrm{Dd}}=\frac{2m}{\hbar\gamma^2} \tag{6.45}$$

It follows from (6.41) that the classical Dirac particle velocity $\mathbf{v}=d\mathbf{x}/dt$ is expressed as follows

$$\mathbf{v}^2=1-\frac{1}{\gamma^2}, \qquad \gamma=\frac{1}{\sqrt{1-\mathbf{v}^2}} \tag{6.46}$$

In other words, the quantity γ is the Lorentz factor of the classical Dirac particle.

We see that the characteristic frequency, connected with the internal degrees of freedom is $2m/\gamma^2$, and the characteristic energy is of the order $|-m\gamma+m\gamma^{-1}|$.

Parameters γ and ω_{Dd} as functions of the radius a_{Dd} and the Dirac mass m have the form

$$\gamma=\sqrt{\frac{1}{2}\left(1+\sqrt{1+\zeta^2}\right)}, \qquad \omega_{\mathrm{Dd}}=\frac{4m}{\hbar\left(1+\sqrt{1+\zeta^2}\right)} \qquad \zeta=\frac{4ma_{\mathrm{Dd}}}{\hbar} \tag{6.47}$$

7 Discrete space-time geometry

Foundation of quantum mechanics as a statistical description of stochastically moving particles brings up the question: Why do free microparticles move stochastically? It appears that the space-time is discrete, and particles of small mass feel this discreteness. As a result the particles of small mass move stochastically. World function σ_{d} of the discrete space-time geometry $\mathcal{G}_{\mathrm{d}}$ is restricted by the relation (1.1).

In the nonrelativistic physics the particle state is described as a point in the phase space of coordinates and momenta. The particle world line is supposed to be smooth and the particle 4-momentum p_k is described by the relation

$$p_k=g_{kl}\frac{dx^l}{d\tau}=g_{kl}\lim_{d\tau\to 0}\frac{x^l(\tau+d\tau)-x^l(\tau)}{d\tau} \tag{7.1}$$

where $x^l=x^l(\tau)$, $l=0,1,2,3$ is an equation of the world line. In the relativistic case the particle state is described by the world line. In $\mathcal{G}_{\mathrm{d}}$ a world line cannot be smooth, because the limit (7.1) does not exist in $\mathcal{G}_{\mathrm{d}}$. In $\mathcal{G}_{\mathrm{d}}$ a smooth world line is replaced by sequential set of points $...P_0, P_1, P_2, ...$, or by a broken line, whose links are straight segments of the same length

$$\mathcal{C} = \bigcup_s \mathbf{P}_s\mathbf{P}_{s+1}, \quad |\mathbf{P}_s\mathbf{P}_{s+1}| = \mu, \quad s = ...0, 1, .. \tag{7.2}$$

where the length $|\mu| \geq \lambda_0$, and λ_0 is a parameter (elementary length) of the world function σ_{d}, defined by (1.4). The term "world chain" will be used for such a broken world line. The quantity μ is the geometric mass of the particle. It is connected with the usual mass m by the relation

$$m = b\mu \tag{7.3}$$

where b is some universal constant.

For a free particle the adjacent vectors $\mathbf{P}_s\mathbf{P}_{s+1}$ and $\mathbf{P}_{s+1}\mathbf{P}_{s+2}$ are in parallel

$$(\mathbf{P}_s\mathbf{P}_{s+1}.\mathbf{P}_{s+1}\mathbf{P}_{s+2}) = |\mathbf{P}_s\mathbf{P}_{s+1}| \cdot |\mathbf{P}_{s+1}\mathbf{P}_{s+2}|, \quad s = ...0, 1, .. \tag{7.4}$$

where the scalar product $(\mathbf{P}_s\mathbf{P}_{s+1}.\mathbf{P}_{s+1}\mathbf{P}_{s+2})$ is defined by relation

$$(\mathbf{AB}.\mathbf{CD}) = \sigma(A, D) + \sigma(B, C) - \sigma(A, C) - \sigma(B, D) \tag{7.5}$$

In the geometry of Minkowski, when $\lambda_0 \to 0$, the timelike world chain turns to a smooth world line. The wobbling of a spacelike world chain (7.2) does not disappear, because the multivariance of the spacelike vectors equivalence remains at $\lambda_0 \to 0$.

The discrete geometry $\mathcal{G}_{\mathrm{d}}$ is a multivariant geometry, and it is the most essential property of $\mathcal{G}_{\mathrm{d}}$. Furthermore, the discrete geometry is a nonaxiomatizable geometry, which cannot be constructed on the basis of a finite number of axioms. As any generalized geometry, the discrete geometry $\mathcal{G}_{\mathrm{d}}$ is a generalization of the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$.

The proper Euclidean geometry as well as the geometry of Minkowski are continuous geometries. They are described by methods of differential geometry. However, there may exist discrete geometries, where the distance between any two points of the space-time is larger, than some elementary length λ_0. If characteristic scale of the problem is much larger, than the elementary length λ_0, one may set $\lambda_0 = 0$ and consider the space-time geometry as a continuous geometry. However, in microcosm, where characteristic scale is of the order of λ_0, one should consider a discrete space-time geometry, because the real space-time geometry may be discrete, and such a possibility is to be investigated.

At the conventional construction of the Euclidean geometry one uses such concepts as manifold, dimension, coordinate system, linear vector space, which might be used only in continuous (differential) geometries. A discrete geometry is considered as a generalization of the proper Euclidean geometry, because it is the only geometry, whose consistency has been proved. Constructing a discrete geometry as a generalization of the proper Euclidean geometry, one may not use above-mentioned concepts. The only concept, which may be used in the continuous geometry and in the discrete one, is the distance ρ. But the distance ρ is to be introduced as a fundamental quantity. In the Riemannian geometry the distance ρ is introduced as an integral along the geodesic from the infinitesimal distance

$$ds = \sqrt{g_{ik}dx^i dx^k}$$

Such a method of introduction of the distance ρ is inadequate in the discrete geometry, because it uses infinitesimal distance, which does not exist in the discrete geometry. Besides, in the case, when there are several geodesics, connecting two points, one obtains many-valued expressions for the distance or for the world function. Many-valued world function is inadmissible in a geometry.

To construct a discrete geometry, one needs to use the metric approach to geometry. One represents the proper Euclidean geometry in terms of the distance ρ (or in terms of the world function $\sigma = \frac{1}{2}\rho^2$) and uses this representation for generalization of the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ on the case of a discrete geometry $\mathcal{G}_{\mathrm{d}}$. Such a replacement of basic concepts of the Euclidean geometry means a logical reloading of the Euclidean geometry conception. Representation of a geometry in terms of a world function will be referred to as σ-immanent representation. The σ-immanent representation of the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ is always possible.

The distance function ρ_{d} of a discrete geometry $\mathcal{G}_{\mathrm{d}}$ satisfies the condition (1.1) It means that in the geometry $\mathcal{G}_{\mathrm{d}}$ there are no distances, which are shorter, than the elementary length λ_0. The distance $\rho_{\mathrm{d}}(P,Q) = 0$ is admissible. This condition takes place, if $P = Q$.

Note, that the condition (1.1) is a restriction on the values of the distance function, but not on values of its argument (points of Ω), although one considers usually a discrete geometry as a geometry on a lattice. It is true, that the geometries on a lattice are discrete geometries (they satisfy the relation (1.1)), but they form a very special case of the discrete geometries. Such a geometry is essentially a conventional differential geometry, given on a countable set of points, where the distances are the same as in the differential geometry, given on a continual set of points. Besides, such a discrete geometry cannot be uniform and isotropic. A general case of a discrete geometry takes place, when restrictions are imposed on the admissible values of the world function (distance function).

The simplest case of a discrete space-time geometry $\mathcal{G}_{\mathrm{d}}$ is obtained, if $\mathcal{G}_{\mathrm{d}} = \{\sigma_{\mathrm{d}}, \Omega_{\mathrm{M}}\}$ is given on the manifold Ω_{M}, where the geometry of Minkowski $\mathcal{G}_{\mathrm{M}} = \{\sigma_{\mathrm{M}}, \Omega_{\mathrm{M}}\}$ is given. The world function σ_{d} is chosen in the form (1.4), It is easy to verify, that $\rho_{\mathrm{d}} = \sqrt{2\sigma_{\mathrm{d}}}$, defined by (1.4) satisfies the constraint (1.1). Such a discrete geometry is uniform and isotropic as well as the geometry of Minkowski.

8 Metric approach to geometry

There is another circumstance, which prevents from constructing a discrete geometry. The proper Euclidean geometry is an axiomatizable geometry. It means, that all statements of the proper Euclidean geometry can be deduced from a system of several axioms (basic statements of the geometry). Usually one considers the axiomatizability of a geometry as an inherent property of any geometry. One believes that there are no nonaxiomatizable geometries. The reason of such a belief is rather simple. During two thousand years we knew the only geometry - the proper Euclidean geometry, which is axiomatizable. All differential geometries, constructed as a generalization of the proper Euclidean geometry, are also axiomatizable. One knows no other method of a geometry construction other, than the Euclidean method of the geometry deduction from some system of axioms. All differential geometries are constructed by means of this method. Mathematicians believe that any geometry is a logical construction. Such a discipline as the symplectic geometry is used in dynamics, but not for description of the geometric objects properties. Nevertheless it is called a geometry, because its structure reminds the structure of the Euclidean geometry.

In reality any geometry investigates a shape and a mutual disposition of geometrical objects in the space, or in the space-time. This property is an original property of a geometry. However, one used the only Euclidean method of the geometry construction during two thousand years, and as a result the axiomatizability of a geometry is considered now as an inherent property of any geometry, whereas a description of geometrical objects is considered as a secondary property of discipline, called geometry.

In general, there is a metrical approach to geometry, when a geometry is considered as a science, investigating a shape and a mutual disposition of geometrical objects. Such a geometry is known as a metric geometry (metric space), if it uses the triangle axiom. If the triangle axiom is not used, the geometry is called the distant geometry [42, 43]. It is supposed, that the distant geometry $\mathcal{G}_{\mathrm{ds}} = \{\sigma, \Omega\}$ is described completely by the world function $\sigma = \frac{1}{2}\rho^2$

$$\sigma : \quad \Omega \times \Omega \rightarrow \mathbb{R}, \quad \sigma(P, Q) = \sigma(Q, P), \quad \forall P, Q \in \Omega \tag{8.1}$$

where Ω is the point set, where the geometry is given. The world function σ is used instead of the distance function ρ, because in the geometry of Minkowski the distance ρ may be either positive, or pure imaginary, whereas $\sigma = \frac{1}{2}\rho^2$ is always real.

At the metric approach to geometry, a geometry can be constructed on any point set (but not necessarily on a manifold) without a use of coordinates. In the metric space the distance function ρ satisfies additional constraints

$$\rho(P, Q) \geq 0, \quad \forall P, Q \in \Omega, \quad \rho(P, Q) = \sqrt{2\sigma(P, Q)} \tag{8.2}$$

$$\rho(P, Q) + \rho(P, R) \geq \rho(Q, R), \quad \forall P, Q, R \in \Omega \tag{8.3}$$

The condition (8.3) is known as the triangle axiom. This axiom admits one to introduce a straight line in the metric space as a shortest line between two points. In the distant geometry, where the constraint (8.3) is absent, one failed to introduce the straight line in terms of the distance function ρ. Blumental [43] introduced a curve as a continuous mapping $(0,1) \rightarrow \Omega$. The continuous mapping is an operation, which cannot be expressed only in terms of the distance function. As a result a purely metric approach to geometry, when geometry is described completely in terms of the distance function ρ, failed. The reason of this failure lies in the fact, that Blumental believed that the straight line has no thickness, whereas in reality in the distant geometry $\mathcal{G}_{\mathrm{d}}$ the straight line is a hollow tube. In reality the distant geometry is nonaxiomatizable geometry, which cannot be constructed by the Euclidean method.

What is on the bottom of the Euclidean method of the geometry construction? Let us get outside of this method. One cannot perceive the distance directly. One can perceive physical bodies. Geometrical object is an abstraction of space-time properties of a physical body. A physical body, evolving in the space-time, may pass from one space-time region with the space-time geometry $\{\sigma_1, \Omega_1\}$ to another space-time region with the space-time geometry $\{\sigma_2, \Omega_2\}$. We must have a possibility to recognize and to identify the same geometrical object in different space-time geometries. In order, that it should be possible, any geometrical object is to be described in terms of the distance function ρ and only in terms of ρ. Any geometrical object is described by its skeleton and its envelope. We consider a simple examples of geometrical objects. (The general definition of a geometrical object will be given later).

The simplest geometrical object is a sphere $\mathcal{SP}_{P_0 P_1}$, determined by two points P_0, P_1 (skeleton). The point P_0 is a center of the sphere, P_1 is some point on the surface of the sphere. The points $\{P_0, P_1\}$ form the sphere skeleton. The surface of the sphere (its envelope) is a set of points

$$\mathcal{SP}_{P_0 P_1} = \{R | \rho(P_0, R) = \rho(P_0, P_1)\}, \quad \rho = \sqrt{2\sigma} \tag{8.4}$$

The sphere is a hollow geometrical object in the sense, that there are internal points of the sphere, which do not belong to the sphere surface (envelope).

Another simple geometrical object is an ellipsoid $\mathcal{EL}_{F_1 F_2 P}$, determined by three points F_1, F_2, P. The points F_1, F_2 are focuses of the ellipsoid, and the point P is some point on the surface of the ellipsoid

$$\mathcal{EL}_{F_1 F_2 P} = \{R | \rho(F_1, R) + \rho(F_2, R) = \rho(F_1, P) + \rho(F_2, P)\}, \quad \rho = \sqrt{2\sigma} \tag{8.5}$$

If $F_1 \neq P \wedge F_2 \neq P$, the ellipsoid $\mathcal{EL}_{F_1 F_2 P}$ is a hollow geometrical object.

If $F_1 = P \vee F_2 = P$, the ellipsoid degenerates into a straight line segment $\mathcal{T}_{[P_0 P_1]}$

$$\mathcal{T}_{[P_0 P_1]} \equiv \mathcal{EL}_{P_0 P_1 P_1} = \mathcal{EL}_{P_0 P_1 P_0} = \{R | \rho(P_0, R) + \rho(P_1, R) = \rho(P_0, P_1)\} \tag{8.6}$$

The degenerate ellipsoid $\mathcal{EL}_{P_0 P_1 P_1}$ is a straight line segment $\mathcal{T}_{[P_0 P_1]}$ by definition. This name is used, because in the proper Euclidean geometry a degenerate ellipsoid is a straight line segment. In other geometries the geometric object (8.6) may be a hollow geometrical object. It means, that it is not one-dimensional point set, as in the proper Euclidean geometry, but nevertheless we shall refer to it as a straight line segment.

The segment $\mathcal{T}_{[P_0P_1]}$ is determined by two points. All points of $\mathcal{T}_{[P_0P_1]}$ are points of the envelope, which consists of boundary points only. In the proper Euclidean geometry it is not a hollow geometrical object, because it does not contain internal points.

Is the straight line segment $\mathcal{T}_{[P_0P_1]}$ a hollow geometrical object in other distant geometries? It depends on the constraints (8.2),(8.3). If they are satisfied, the segment $\mathcal{T}_{[P_0P_1]}$ is entire (not hollow). If the distance function ρ does not satisfy the triangle axiom (8.3) the segment $\mathcal{T}_{[P_0P_1]}$ may be hollow. In other words, the segment $\mathcal{T}_{[P_0P_1]}$ may be a hollow tube.

Why is the segment entire, if the triangle axiom (8.3) is fulfilled? Let us consider a closed surface $\mathcal{S}$ defined by the relation

$$\mathcal{S}: \quad S_{P_0P_1}(R) = 0, \quad S_{P_0P_1}(R) = \rho(P_0, R) + \rho(P_1, R) - \rho(P_0, P_1) \tag{8.7}$$

Internal points R' (points inside the closed surface $\mathcal{S}$) satisfy the relation $S_{P_0P_1}(R') < 0$. External points R'' satisfy the relation $S_{P_0P_1}(R'') > 0$. If the triangle axiom is fulfilled, it may be written in the form

$$\rho(P_0, R) + \rho(P_1, R) \geq \rho(P_0, P_1), \quad \forall P_1, P_2, R \in \Omega \tag{8.8a}$$

It follows from (8.7) and (8.8a), that $S_{P_0P_1}(R') \geq 0, \quad \forall R' \in \Omega$. It means that the surface $\mathcal{S}$, which coincides with the segment $\mathcal{T}_{[P_0P_1]}$, cannot contain internal points.

Why it is important, whether or not the segment $\mathcal{T}_{[P_0P_1]}$ is hollow? Geometry is reduced to construction of geometrical objects and to investigation of their properties. In the proper Euclidean geometry all geometrical objects are constructed of blocks (point, straight segment). Blocks are to be simple entire (not hollow) geometrical objects. The segment $\mathcal{T}_{[P_0P_1]}$ is determined by two points, and it is entire in the proper Euclidean geometry. It may be used as a constructive block for construction of geometrical objects. For instance, in the proper Euclidean geometry a cube can be filled by straight segments placed in parallel with the cube edge in such a way, that any point of a cube belongs to one and only one segment. Such a situation is impossible, if the blocks are hollow geometrical objects. If the blocks are hollow tubes, one cannot fill the cube by these tubes in such a way, that any point of a cube belongs to one and only one tube. It means, that a cube cannot be constructed of hollow blocks. The same relates to any geometrical object.

The Euclidean method of the geometric object construction is based on the possibility of construction of any geometrical object from blocks. There is a finite number of rules, describing the blocks properties, and there is a finite number of rules for description of the blocks combinations at a construction of a geometrical object. Euclid formulated these rules in the form of axioms of a logical construction. Thus, the axiomatics of the proper Euclidean geometry describes the procedure of a construction of geometrical objects from blocks. If the segment $\mathcal{T}_{[P_0P_1]}$ is entire, the distant geometry is an axiomatizable geometry, because it can be realized as a geometry, where any geometric object can be constructed of blocks, i.e. by means of the Euclidean method.

If blocks are hollow, they cannot be used for construction of geometrical objects. In this case the distant geometry is nonaxiomatizable, because in this case one cannot use the Euclidean method for construction of geometric objects. Formally the segment $\mathcal{T}_{[P_0P_1]}$ is hollow, if the equivalence relation is intransitive (and the geometry is multivariant). If the equivalence relation is transitive, the segment $\mathcal{T}_{[P_0P_1]}$ may be entire.

The constructive block $\mathcal{T}_{[P_0P_1]}$ is a directed object, whose direction is described by the vector $\mathbf{P}_0\mathbf{P}_1 = \overrightarrow{P_0P_1} = \{P_0, P_1\}$, which is an ordered set of two points. The point P_0 is the origin of the vector, the point P_1 is the end of the vector. Any vector $\mathbf{P}_0\mathbf{P}_1$ is described by its module

$$|\mathbf{P}_0\mathbf{P}_1| = \rho\left(P_0, P_1\right) = \sqrt{2\sigma\left(P_0, P_1\right)} \tag{8.9}$$

Vectors are directed quantities, and interrelation of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ is described by an angle φ between them. In the proper Euclidean geometry there is a lot of vectors $\mathbf{Q}_0\mathbf{Q}_1$, which form the angle $\varphi \neq 0$ with the vector $\mathbf{P}_0\mathbf{P}_1$. However, in the proper Euclidean geometry there is only one vector $\mathbf{Q}_0\mathbf{Q}_1$ at the point Q_0 with fixed length $|\mathbf{Q}_0\mathbf{Q}_1|$, which forms with the vector $\mathbf{P}_0\mathbf{P}_1$ the angle $\varphi = 0$. By definition such a vector $\mathbf{Q}_0\mathbf{Q}_1$ is called the vector, which is parallel ($\mathbf{Q}_0\mathbf{Q}_1 \upuparrows \mathbf{P}_0\mathbf{P}_1$) to the vector $\mathbf{P}_0\mathbf{P}_1$.

Instead of the angle φ the mutual direction of two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ may be described by the scalar product $(\mathbf{Q}_0\mathbf{Q}_1.\mathbf{P}_0\mathbf{P}_1)$ of these vectors, defined by the relation

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = |\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1| \cos\varphi \tag{8.10}$$

In the proper Euclidean geometry the definition of the scalar product may be expressed in terms of the world function

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = \sigma\left(P_0, Q_1\right) + \sigma\left(P_1, Q_0\right) - \sigma\left(P_1, Q_1\right) - \sigma\left(P_0, Q_0\right) \tag{8.11}$$

As far as the definition of the scalar product is produced in terms of the world function, this definition may be used for any distant geometry.

Then condition of the vectors parallelism is obtained from (8.10) at $\varphi = 0$. It is written in the form

$$(\mathbf{Q}_0\mathbf{Q}_1 \upuparrows \mathbf{P}_0\mathbf{P}_1): \quad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = |\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1| \tag{8.12}$$

In the proper Euclidean geometry all vectors $\mathbf{P}_0\mathbf{P}_1$,$\mathbf{P}_0\mathbf{P}_1'$, $\mathbf{P}_0\mathbf{P}_1''$, which are parallel to vector $\mathbf{Q}_0\mathbf{Q}_1$, are parallel between themselves. Such a situation is rather special. It is connected with a degenerate character of the proper Euclidean geometry. In the distant geometry vectors $\mathbf{P}_0\mathbf{P}_1$,$\mathbf{P}_0\mathbf{P}_1'$, $\mathbf{P}_0\mathbf{P}_1''$, which are parallel to vector $\mathbf{Q}_0\mathbf{Q}_1$, are not parallel between themselves, in general. This circumstance generates hollowness of straight segments $\mathcal{T}_{[P_0P_1]}$. It depends on properties of the world function σ, which describes a distant geometry completely.

In the proper Euclidean geometry two vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{Q}_0\mathbf{Q}_1$ are equivalent by definition, if they are parallel ($\mathbf{P}_0\mathbf{P}_1 \upuparrows \mathbf{Q}_0\mathbf{Q}_1$) and their lengths are equal $|\mathbf{P}_0\mathbf{P}_1| =$

$|\mathbf{Q}_0\mathbf{Q}_1|$

$$(\mathbf{P}_0\mathbf{P}_1\mathrm{eqv}\mathbf{Q}_0\mathbf{Q}_1): \quad (\mathbf{P}_0\mathbf{P}_1.\mathbf{Q}_0\mathbf{Q}_1) = |\mathbf{P}_0\mathbf{P}_1| \cdot |\mathbf{Q}_0\mathbf{Q}_1| \wedge |\mathbf{P}_0\mathbf{P}_1| = |\mathbf{Q}_0\mathbf{Q}_1| \quad (8.13)$$

This definition of two vectors equivalency (equality) together with the definitions (8.9), (8.11) formulates the equivalence of two vectors in terms of the world function and only in these terms. It does not refer to a dimension, to a coordinate system and other means of description. This definition of two vectors equivalence should be used in any distant geometry.

There are such distant geometries, where the straight segments $\mathcal{T}_{[P_0P_1]}$ are hollow tubes. Then the definition (8.13), (8.12) appears to be intransitive, and the distant geometry appears to be nonaxiomatizable. Some mathematicians object, that the definition (8.13), (8.11) cannot be used as an equivalence relation, because the equivalence relation is transitive by definition. They insist, that one should use another term for the definition (8.13), (8.11), (for instance, general equivalency). The reason of such an objection lies in the fact, that the mathematicians dealt only with axiomatizable geometries, which are logical constructions. Indeed, if one uses a logical construction, one can deduce conclusions, only if the equivalence relation is transitive, and from $a \backsim b$ and $b \backsim c$ it follows, that $a \backsim c$. If the the equivalence relation has not this property, one cannot deduce corollaries of axioms and theorems. Thus, if one insists on the transitivity of the equivalence relation, one insists on impossibility of nonaxiomatizable geometries, in particular, on impossibility of discrete space-time geometries, where the straight segments $\mathcal{T}_{[P_0P_1]}$ are hollow tubes. We believe that imperfection of the description methods cannot be a reason of the discrete geometry discard. Nonaxiomatizability of the discrete geometry $\mathcal{G}_{\mathrm{d}}$ does not mean that $\mathcal{G}_{\mathrm{d}}$ does not exist .

The transitivity of the equivalence relation has been obtained from our experience of work with axiomatizable geometries (Euclidean geometry and its modifications). We have no authority to generalize this property to all space-time geometries. Whether or not the real space-time geometry is discrete, is a question of experimental data, but not a question of mathematical scholasticism. Another problem lies in the fact, that we could construct only axiomatizable geometries, and we could not construct discrete geometries. As a result we constructed only geometries on a lattice, which are not rigorous discrete geometries. How to construct discrete (nonaxiomatizable) geometries, we consider a few later.

9 Description of geometric objects

If the distant geometry includes indefinite metrics (as in the geometry of Minkowski), the condition (8.2) is to be omitted, and description of the geometry is produced in terms of the world function. The geometry described completely by the world function (8.1) will be referred to as a physical geometry.

A geometrical object is a geometrical image of a physical body. Any geometrical object is some subset of points in the space-time. However, geometrical object is not an arbitrary set of points. Geometrical object is to be defined in the physical geometry in such a way, that similar geometrical objects (which are images of similar physical bodies) could be recognized in different space-time geometries.

Definition 1: A geometrical object $g_{\mathcal{P}_n,\sigma}$ of the geometry $\mathcal{G}=\{\sigma,\Omega\}$ is a subset $g_{\mathcal{P}_n,\sigma}\subset\Omega$ of the point set Ω. This geometrical object $g_{\mathcal{P}_n,\sigma}$ is a set of roots $R\in\Omega$ of the function $F_{\mathcal{P}_n,\sigma}$

$$g_{\mathcal{P}_n,\sigma}=\{R|F_{\mathcal{P}_n,\sigma}(R)=0\}, \quad F_{\mathcal{P}_n,\sigma}: \quad \Omega\rightarrow\mathbb{R} \tag{9.1}$$

where $F_{\mathcal{P}_n,\sigma}$ depends on the point R via world functions of arguments $\{\mathcal{P}_n,R\}=\{P_0,P_1,...P_n,R\}$

$$F_{\mathcal{P}_n,\sigma}: \quad F_{\mathcal{P}_n,\sigma}(R)=G_{\mathcal{P}_n,\sigma}(u_1,u_2,...u_s), \qquad s=\frac{1}{2}(n+1)(n+2) \tag{9.2}$$

$$u_l=\sigma(w_i,w_k), \quad i,k=0,1,...n+1, \quad l=1,2,...\frac{1}{2}(n+1)(n+2) \tag{9.3}$$

$$w_k=P_k\in\Omega, \qquad k=0,1,...n, \qquad w_{n+1}=R\in\Omega \tag{9.4}$$

Here $\mathcal{P}_n=\{P_0,P_1,...,P_n\}\subset\Omega$ are $n+1$ points which are parameters, determining the geometrical object $g_{\mathcal{P}_n,\sigma}$

$$g_{\mathcal{P}_n,\sigma}=\{R|F_{\mathcal{P}_n,\sigma}(R)=0\}, \qquad R\in\Omega, \qquad \mathcal{P}_n\in\Omega^{n+1} \tag{9.5}$$

$F_{\mathcal{P}_n,\sigma}(R)=G_{\mathcal{P}_n,\sigma}(u_1,u_2,...u_s)$ is a function of $\frac{1}{2}(n+1)(n+2)$ arguments u_k and of $n+1$ parameters $\mathcal{P}_n$. The set $\mathcal{P}_n=\{P_0,P_1,...P_n\}\in\Omega^{n+1}$ of the geometric object parameters will be referred to as the skeleton of the geometrical object. The subset $g_{\mathcal{P}_n,\sigma}\subset\Omega$ will be referred to as the envelope of the skeleton. The skeleton is an analog of a frame of reference, attached rigidly to a physical body. Tracing the skeleton motion, one can trace the motion of the physical body. When a particle is considered as a geometrical object, its motion in the space-time is described by the skeleton $\mathcal{P}_n$ motion. At such an approach (the rigid body approximation) the shape of the envelope is of no importance.

Remark: An arbitrary subset Ω' of the point set Ω is not a geometrical object, in general. It is supposed, that physical bodies may have only a shape of a geometrical object, because only in this case one can identify identical physical bodies (geometrical objects) in different space-time geometries.

Existence of the same geometrical objects in different space-time regions, having different geometries, brings up the question on equivalence of geometrical objects in different space-time geometries. Such a question did not brought up before, because one does not consider such a situation, when a physical body moves from one space-time region to another space-time region, having another space-time geometry. In general, mathematical technique of the conventional space-time geometry (differential geometry) is not applicable for simultaneous consideration of several different geometries of different space-time regions.

We can perceive the space-time geometry only via motion of physical bodies in the space-time, or via construction of geometrical objects corresponding to these physical bodies. As it follows from the *definition 1* of the geometrical object, the function $G_{\mathcal{P}_n,\sigma}$ as a function of its arguments u_k, $k = 1, 2, ...n(n+1)/2$ (of world functions of different points) is the same in all physical geometries. It means, that a geometrical object $\mathcal{O}_1$ in the geometry $\mathcal{G}_1 = \{\sigma_1, \Omega_1\}$ is obtained from the same geometrical object $\mathcal{O}_2$ in the geometry $\mathcal{G}_2 = \{\sigma_2, \Omega_2\}$ by means of the replacement $\sigma_2 \rightarrow \sigma_1$ in the definition of this geometrical object.

Definition 2: Geometrical object $g_{P'_n,\sigma'}$ ($\mathcal{P}'_n = \{P'_0, P'_1, ..P'_n\}$) in the geometry $\mathcal{G}' = \{\sigma', \Omega'\}$ and the geometrical object $g_{P_n,\sigma}$ ($\mathcal{P}_n = \{P_0, P_1, ..P_n\}$) in the geometry $\mathcal{G} = \{\sigma, \Omega\}$ are similar geometrical objects, if

$$\sigma'(P'_i, P'_k) = \sigma(P_i, P_k), \qquad i, k = 0, 1, ..n \tag{9.6}$$

and the functions $G'_{\mathcal{P}'_n,\sigma'}$ for $g_{P'_n,\sigma'}$ and $G_{\mathcal{P}_n,\sigma}$ for $g_{P_n,\sigma}$ in the formula (9.2) are the same functions of arguments $u_1, u_2, ...u_s$

$$G'_{\mathcal{P}'_n,\sigma'}(u_1, u_2, ...u_s) = G_{\mathcal{P}_n,\sigma}(u_1, u_2, ...u_s) \tag{9.7}$$

In this case

$$u_l \equiv \sigma(P_i, P_k) = u'_l \equiv \sigma'(P'_i, P'_k), \quad i, k = 0, 1, ...n, \quad l = 1, 2, ..n(n+1)/2 \tag{9.8}$$

The functions $F'_{\mathcal{P}'_n,\sigma'}$ for $g_{P'_n,\sigma'}$ and $F_{\mathcal{P}_n,\sigma}$ for $g_{P_n,\sigma}$ in the formula (9.2) have the same roots, if the relation (9.7) is fulfilled. As a result one-to-one connection between the geometrical objects $g_{P'_n,\sigma'}$ and $g_{P_n,\sigma}$ arises.

As far as the physical geometry is determined by its geometrical objects construction, a physical geometry $\mathcal{G} = \{\sigma, \Omega\}$ can be obtained from some known standard geometry $\mathcal{G}_{\mathrm{st}} = \{\sigma_{\mathrm{st}}, \Omega\}$ by means of a deformation of the standard geometry $\mathcal{G}_{\mathrm{st}}$. Deformation of the standard geometry $\mathcal{G}_{\mathrm{st}}$ is realized by the replacement $\sigma_{\mathrm{st}} \rightarrow \sigma$ in all definitions of the geometrical objects in the standard geometry. The proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ is an axiomatizable geometry. It has been constructed by means of the Euclidean method as a logical construction. Simultaneously the proper Euclidean geometry is a physical geometry. It may be used as a standard geometry $\mathcal{G}_{\mathrm{st}}$. Construction of a physical geometry as a deformation of the proper Euclidean geometry will be referred to as the deformation principle [44, 45]. The most physical geometries are nonaxiomatizable geometries. They can be constructed only by means of the deformation principle.

10 General geometric relations

Describing a physical geometry in terms of the world function, one should distinguish between general geometric relations and specific geometric relations. The general geometric relations are definitions of the proper Euclidean geometry, which are written in terms and only in terms of the world function. The general geometric relations are valid for any physical geometry.

The first general geometric definition is the definition of the scalar product of two vectors (8.11). Definition of the two vector equivalence (8.13) is also a general geometric relation.

Linear dependence of n vectors $\mathbf{P}_0\mathbf{P}_1, \mathbf{P}_0\mathbf{P}_2, ...\mathbf{P}_0\mathbf{P}_n$ is defined by the relation,

$$F_n(\mathcal{P}_n) = 0, \quad F_n(\mathcal{P}_n) \equiv \det ||(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_k)||, \qquad i,k = 1,2,...n \tag{10.1}$$

where $\mathcal{P}_n = \{P_0, P_1, ...P_n\}$ and $F_n(\mathcal{P}_n)$ is the Gram's determinant. Vanishing of the Gram's determinant $F_n(\mathcal{P}_n)$ is the necessary and sufficient condition of the linear dependence of n vectors. Condition of linear dependence is considered usually as properties of the linear vector space $\mathcal{L}_n$, because it is defined via operations in $\mathcal{L}_n$. It seems rather meaningless to use it, if the linear vector space cannot be introduced. Nevertheless, the relation (10.1) written as a general geometric relation describes some general geometric properties of vectors, which are transformed to the property of linear dependence in the proper Euclidean geometry . In particular, the metric dimension of the proper Euclidean geometry is defined in terms of the world function by means of the relations of the type (10.1) as a maximal number of linear independent vectors, which is possible in the Euclidean space. This circumstance seems to be rather unexpected, because in conventional presentation of the Euclidean geometry the geometry dimension is postulated in the beginning of the presentation.

As we have seen, a definition of geometrical objects in the form of general geometric relations (i.e. in terms of the world function) is necessary to recognize the same physical body (and corresponding geometrical object) in different space-time geometries.

The general geometric relations are parametrized by the form of the world function σ. Changing the form of the world function σ, one obtains the general geometric relations at a new value of the parameter σ (new form of the world function).

11 Specific properties of the n-dimensional Euclidean space

Along of general geometric properties, describing mainly definitions of the linear vector space, there are special geometric relations, describing properties of the world function. For instance, there are relations, which are necessary and sufficient conditions of the fact, that the world function $\sigma = \sigma_{\mathrm{E}}$ is the world function of n-dimensional Euclidean space. They have the form [46]:

I. Definition of the dimension:

$$\exists \mathcal{P}^n \equiv \{P_0, P_1, ...P_n\} \subset \Omega, \quad F_n(\mathcal{P}^n) \neq 0, \quad F_k\left(\Omega^{k+1}\right) = 0, \qquad k > n \tag{11.1}$$

where $F_n(\mathcal{P}^n)$ is the n-th order Gram's determinant (10.1) Vectors $\mathbf{P}_0\mathbf{P}_i, \quad i = 1,2,...n$ are basic vectors of the rectilinear coordinate system K_n with the origin at the point P_0. The metric tensors $g_{ik}(\mathcal{P}^n)$, $g^{ik}(\mathcal{P}^n)$, $i,k = 1,2,...n$ in K_n are defined by the relations

$$\sum_{k=1}^{k=n} g^{ik}\left(\mathcal{P}^n\right) g_{lk}\left(\mathcal{P}^n\right) = \delta_l^i, \quad g_{il}\left(\mathcal{P}^n\right) = \left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}_l\right), \quad i,l = 1,2,...n \tag{11.2}$$

$$F_n\left(\mathcal{P}^n\right) = \det\left|\left|g_{ik}\left(\mathcal{P}^n\right)\right|\right| \neq 0, \qquad i,k = 1,2,...n \tag{11.3}$$

II. Linear structure of the Euclidean space:

$$\sigma\left(P,Q\right) = \frac{1}{2}\sum_{i,k=1}^{i,k=n} g^{ik}\left(\mathcal{P}^n\right)\left(x_i\left(P\right) - x_i\left(Q\right)\right)\left(x_k\left(P\right) - x_k\left(Q\right)\right), \quad \forall P,Q \in \Omega \tag{11.4}$$

where coordinates $x_i\left(P\right), x_i\left(Q\right), i = 1,2,...n$ of the points P and Q are covariant coordinates of the vectors $\mathbf{P}_0\mathbf{P}$, $\mathbf{P}_0\mathbf{Q}$ respectively in the coordinate system K. The covariant coordinates are defined by the relation

$$x_i\left(P\right) = \left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}\right), \qquad i = 1,2,...n \tag{11.5}$$

III: The metric tensor matrix $g_{lk}\left(\mathcal{P}^n\right)$ has only positive eigenvalues g_k

$$g_k > 0, \qquad k = 1,2,...,n \tag{11.6}$$

IV. The continuity condition: the system of equations

$$\left(\mathbf{P}_0\mathbf{P}_i.\mathbf{P}_0\mathbf{P}\right) = y_i \in \mathbb{R}, \qquad i = 1,2,...n \tag{11.7}$$

considered to be equations for determination of the point P as a function of coordinates $y = \{y_i\}$, $i = 1,2,...n$ has always one and only one solution.

Conditions I – IV contain a reference to the dimension n of the Euclidean space, which is defined by the relations (11.1). All relations I – IV are written in terms of the world function. They are constraints on the form of the world function of the proper Euclidean geometry. Constraints (11.1), determining the dimension via the form of the world function, look rather unexpected. They contain a lot of constraints imposed on the world function of the proper Euclidean geometry, and they are necessary. At the conventional approach to geometry one uses a very simple supposition: "Let the dimension of the Euclidean space be n." Conventionally one uses this very short postulate instead of numerous constraints (11.1), used in the σ-representation (description in terms of the world function) of a geometry.

In the vector representation of the proper Euclidean geometry, which is based on a use of the linear vector space, the dimension is considered as a primordial property of the linear vector space and as a primordial property of the Euclidean geometry. Situation, when the geometry dimension is different at different points of the space Ω, or when the dimension is indefinite, is not considered. In the vector representation of the Euclidean geometry one does not distinguish between the general geometric relations and the specific relations of the geometry.

Instead of constraints (11.1) – (11.7) one may use an explicit form of the world function

$$\sigma_{\mathrm{E}}(x,x') = \frac{1}{2}\sum_{k=1}^{k=n}\left(x^k - x'^k\right)^2 \tag{11.8}$$

where $x^k, x'^k \in \mathbb{R}$, $k = 1,2,...n$ are Cartesian coordinates of points P and P' respectively. The relation (11.8) satisfies all constraints (11.1) – (11.7). It uses concepts of dimension and of coordinates as primordial concepts of geometry. Using the world function only in such an explicit form, one cannot imagine a generalized geometry without such concepts as a dimension and a coordinate system, although these concepts are only means of a geometry description.

In general, after the logical reloading to σ-representation the proper Euclidean geometry looks rather unexpected. Some concepts look very simple in the vector representation. The same concepts look complicated in the σ-representation and vice versa. As a result the proper Euclidean geometry in the σ-representation is perceived hardly.

In the vector representation one has several fundamental concepts and quantities: dimension, coordinate system, linear dependence, whereas in the σ-representation there is only one fundamental quantity: world function. The dimension, the coordinate system and the linear dependence are derivative quantities and concepts. Agreement between these quantities is achieved in any physical geometry, because they are defined as some attributes of the world function.

12 Skeleton conception of particle dynamics

An elementary particle is a physical body. In the discrete space-time geometry a position of a physical body is described by its skeleton $\mathcal{P}_n = \{P_0, P_1, ..P_n\}$. Of course, such a description of a physical body position may be used in any space-time geometry. The skeleton is an analog of the frame of reference attached rigidly to the particle (physical body). Tracing the skeleton motion, one traces the physical body motion. Direction of the skeleton displacement is described by the leading vector $\mathbf{P}_0\mathbf{P}_1$.

The skeleton motion is described by a world chain $\mathcal{C}$ of connected skeletons

$$\mathcal{C} = \bigcup_{s=-\infty}^{s=+\infty} \mathcal{P}_n^{(s)} \tag{12.1}$$

Skeletons $\mathcal{P}_n^{(s)}$ of the world chain are connected in the sense, that the point P_1 of a skeleton is a point P_0 of the adjacent skeleton. It means

$$P_1^{(s)} = P_0^{(s+1)}, \qquad s = ...0, 1, ... \tag{12.2}$$

The vector $\mathbf{P}_0^{(s)}\mathbf{P}_1^{(s)} = \mathbf{P}_0^{(s)}\mathbf{P}_0^{(s+1)}$ is the leading vector, which determines the direction of the world chain. The case (7.2), when the skeleton $\mathcal{P}_1 = \{P_s, P_{s+1}\}$ of a pointlike particle is described by two points is a special case of (12.1).

If the particle motion is free, the adjacent skeletons are equivalent

$$\mathcal{P}_n^{(s)}\text{eqv}\mathcal{P}_n^{(s+1)}: \qquad \mathbf{P}_i^{(s)}\mathbf{P}_k^{(s)}\text{eqv}\mathbf{P}_i^{(s+1)}\mathbf{P}_k^{(s+1)}, \qquad i,k=0,1,...n, \qquad s=..0,1,.. \tag{12.3}$$

If the particle is described by the skeleton $\mathcal{P}_n^{(s)}$, the world chain (12.1) has $n(n+1)/2$ invariant quantities

$$\mu_{ik}=\left|\mathbf{P}_i^{(s)}\mathbf{P}_k^{(s)}\right|^2=2\sigma\left(P_i^{(s)},P_k^s\right), \qquad i,k=0,1,...n, \qquad s=...0,1,... \tag{12.4}$$

which are constant along the whole world chain.

Equations (12.3) form a system of $n(n+1)$ difference equations for evolution of nD coordinates of n skeleton points $\{P_1,..P_n\}$, where D is the dimension of the space-time. The number of dynamical variables, which are liable for determination of the world chain, distinguishes, in general, from the number of dynamic equations. It is the main difference between the skeleton conception of particle dynamics and the conventional conception of particle dynamics, where the number of dynamic variables coincides with the number of dynamic equations.

In the case of pointlike particle, when $n=1$, $D=4$, the number of equations $n_{\mathrm{e}}=2$, whereas the number of variables $n_{\mathrm{v}}=4$. The number of equations is less, than the number of dynamic variables. In the discrete space-time geometry (1.4) the position of the adjacent skeleton is not determined uniquely. As a result the world chain wobbles. In the nonrelativistic approximation a statistical description of the stochastic world chains leads to the Schrödinger equations [10], if the elementary length λ_0 has the form $\lambda_0^2=\hbar/bc$, where $\hbar$ is the quantum constant, c is the speed of the light and b is a universal constant, connecting the particle mass m with the length (geometric mass) μ of the world chain link

$$m=b\mu$$

Dynamic equations (12.3) are difference equations. At the large scale, when one may go to the limit $\lambda_0=0$, the dynamic equations (12.3) turn to the differential dynamic equations. In the case of pointlike particle ($n=1$) and of the Kaluza-Klein five-dimensional space-time geometry these equation describe the motion of a charged particle in the given electromagnetic field. One can see in this example, that the space-time geometry "assimilates" the electromagnetic field. It means that one may consider only a free particle motion, keeping in mind, that the space-time geometry can "assimilate" all force fields.

Dynamic equations (12.3) realize the skeleton conception of particle dynamics in the microcosm. The skeleton conception of dynamics distinguishes from the conventional conception of particle dynamics in the relation, that the number of dynamic equations may differ from the number of dynamic variables, whose solution is to be determined. In the conventional conception of particle dynamics the number of dynamic equations (first order) coincides always with the number of dynamic variables, which are to be determined. As a result the motion of a particle (or of an averaged particle) appears to be deterministic. In the case of quantum particles, whose motion is stochastic (indeterministic), the dynamic equations are written for a statistical ensemble of indeterministic particles (or for the statistically averaged particle).

In the conventional conception of dynamics one can obtain dynamic equation for the statistically averaged particle (i.e. statistical ensemble normalized to one particle), but there are no dynamic equations for a single stochastic particle. In the skeleton conception of the particle dynamics there are dynamic equations for a single particle. These equations are many-valued (multivariant), but they do exist. In the conventional conception of the particle dynamics one can derive dynamic equations for the statistically averaged particle, which are a kind of equations for a fluid (continuous medium). But one cannot obtain dynamic equations for a single indeterministic particle [7].

The skeleton conception of the particle dynamics realizes a more detailed description of elementary particle. One may hope to obtain some information on the elementary particle structure.

We have now only two examples of the skeleton conception application. Considering compactification in the 5-dimensional discrete space-time geometry of Kaluza-Klein, and imposing condition of uniqueness of the world function, one obtains that the value of the electric charge of a stable elementary particle is restricted by the elementary charge [47]. This result has been known from experiments, but it could not be explained theoretically, because in the continuous space-time geometry nobody considers the world function as a fundamental quantity, and one does not demand its uniqueness.

Another example concerns structure of Dirac particles (fermions). Consideration in framework of skeleton conception [48] shows, that a world chain of a fermion is a (spacelike or timelike) helix with timelike axis. The averaged world chain of a free fermion is a timelike straight line. The helical motion of a skeleton generates an angular moment (spin) and magnetic moment. Such a result looks rather reasonable. In the conventional conception of the particle dynamics the spin and magnetic moment of a fermion are postulated without a reference to its structure. Thus, deterministic model of the Dirac particle gives a more detailed information on arrangement of the Dirac particle. In the classical model the spin and the magnetic moment are axiomatic quantities containing quantum constant. The classical model gives no information on arrangement of spin and magnetic moment.

To obtain the helical world chain in the skeleton conception, one consider space-time geometry $\mathcal{G}_{\mathrm{g}}$ described by the quasi-discrete world function

$$\sigma_{\mathrm{g}}=\sigma_{\mathrm{M}}+d\left(\sigma_{\mathrm{M}}\right), \qquad d\left(\sigma_{\mathrm{M}}\right)=\lambda_0^2 f\left(\frac{\sigma_{\mathrm{M}}}{\sigma_0}\right), \qquad f\left(x\right)=\begin{cases} 1 & \text{if} \quad x\geq 1 \\ x^3 & \text{if} \quad -1<x<1 \\ -1 & \text{if} \quad x\leq -1 \end{cases} \tag{12.5}$$

where $d\left(\sigma_{\mathrm{M}}\right)$ is a distortion describing deflection of the world function σ_{g} from the world function σ_{M} of the geometry of Minkowski $\mathcal{G}_{\mathrm{M}}$. The quantity λ_0 is the elementary length and σ_0 is some constant. The geometry $\mathcal{G}_{\mathrm{g}}$ does not pretend to be a real space-time geometry. The geometry $\mathcal{G}_{\rm g}$ is a granulated geometry, i.e. $\mathcal{G}_{\rm g}$ is the space-time geometry, which is discrete only partly. It is considered as a possible space-time geometry, where the particle world chain may be a helix. The particle skeleton consists of three points $\mathcal{P}_2 = \{P_0, P_1, P_2\}$. The leading vector $\mathbf{P}_0\mathbf{P}_1$ may be timelike or spacelike. The vector $\mathbf{P}_0\mathbf{P}_2$ is timelike. It is directed along the helix axis. The vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_0\mathbf{P}_2$ satisfy the restrictions

$$\left||\mathbf{P}_0\mathbf{P}_1|^2\right| < \sigma_0, \quad |\mathbf{P}_0\mathbf{P}_2|^2 < \sigma_0 \tag{12.6}$$

Under these restrictions the world chain wobbles, but the wobbling amplitude can be restricted, even if the vector $\mathbf{P}_0\mathbf{P}_1$ is spacelike. Details of this investigation may be found in [48].

13 Tachyons

Tachyons are particles with spacelike world chain. They were not detected experimentally and they are not envisaged by the Standard model of elementary particles. Impossibility of the tachyon existence is conditioned by a usage of linear vector space operations in the case, when they are not adequate. Operations of the linear vector space are applied to spacelike vectors of the space-time geometry of Minkowski. It is a mistake, because the equivalence relation is intransitive and multivariant for spacelike vectors in $\mathcal{G}_{\rm M}$. For instance, all vectors $\{r, r\cos\phi, r\sin\phi, z\}$, where r and ϕ are arbitrary number, are equivalent to the vector $\{0,0,0,z\}$, but they are not equivalent between themselves. It is a reason, why the tachyon world chain wobbles with infinite amplitude. Such a tachyon cannot been detected because of this wobbling. Impossibility of a single tachyon detection does not mean that tachyons do not exist. A single tachyon cannot be detected, but the tachyon gas may be detected by its gravitational field. Properties of the the tachyon gas are such, that the tachyon gas is the best candidate for the dark matter [49, 50].

According to (7.2) the world chain for two-point skeleton $\mathcal{P}_1 = \{P_0, P_1\}$ have the form

$$\mathcal{C} = \bigcup_s \mathbf{P}_s\mathbf{P}_{s+1}, \quad |\mathbf{P}_s\mathbf{P}_{s+1}| = \mu = \text{const}, \quad s = ...0,1,2,... \tag{13.1}$$

The adjacent vectors $\mathbf{P}_s\mathbf{P}_{s+1}$ and $\mathbf{P}_{s+1}\mathbf{P}_{s+2}$ are equivalent ($\mathbf{P}_s\mathbf{P}_{s+1}\text{eqv}\mathbf{P}_{s+1}\mathbf{P}_{s+2}$) for a free particle. The equivalence conditions (7.4), (7.5) can be written in the form

$$\begin{aligned} \sigma(P_s, P_{s+2}) &= 4\sigma(P_s, P_{s+1}), \quad \sigma(P_s, P_{s+1}) = \sigma(P_{s+1}, P_{s+2}) \\ s &= 0, \pm 1, \pm 2, ... \end{aligned} \tag{13.2}$$

If there exist the limit $\mu \to 0$, the world chain (13.1) turns into a smooth world line. Keeping in mind that world function $\sigma(P_s, P_{s+1}) = \frac{1}{2}\rho^2(P_s, P_{s+1})$, where ρ is the distance between the points P_s and P_{s+1}, one can see, that in the proper Euclidian geometry $\mathcal{G}_{\rm E}$ the relation (13.2) describes the rule of the straight line construction by means of only compasses.

In the case of tachyon $\sigma(P_s, P_{s+1}) < 0$ and μ is imaginary $\mu^2 = -|\mu^2|$. We consider three adjacent points P_0, P_1, P_2 of the world chain

$$P_0 = \{x_0, \mathbf{x}\}, \quad P_1 = \{x_0 + p_0, \mathbf{x} + \mathbf{p}\}, \quad P_2 = \{x_0 + 2p_0 + \alpha_0, \mathbf{x} + 2\mathbf{p} + \boldsymbol{\alpha}\} \tag{13.3}$$

The 4-vector $\alpha = \{\alpha_0, \boldsymbol{\alpha}\}$ is a discrete analog of the acceleration vector. We write equations (13.2) for the points (13.3). The quantities $x = \{x_0, \mathbf{x}\}$ and $\{x_0 + p_0, \mathbf{x} + \mathbf{p}\}$ are supposed to be given, and the four components of the 4-vector $\alpha = \{\alpha_0, \boldsymbol{\alpha}\}$ are to be determined from two equations (13.2) (acceleration is determined from the dynamic equations).

One considers the space-time geometry with the world function

$$\sigma(x, x') = \frac{1}{2}\left(\left(c^2 - 2V(\mathbf{y})\right)(x_0 - x_0')^2 - (\mathbf{x} - \mathbf{x}')^2\right), \quad \mathbf{y} = \frac{\mathbf{x} + \mathbf{x}'}{2} \tag{13.4}$$

where $\{x^0, \mathbf{x}\} = \{x^0, x^1, x^2, x^1\}$ are coordinates in some inertial coordinate system, $V = V(x)$ is the gravitational potential ($V \ll c^2$).

One obtains the following nonunique solution [49, 50]

$$\alpha_{\parallel} = \frac{r\sqrt{c^2 - 2V}}{\sqrt{(v^2 - c^2 + 2V)}}, \quad v = \frac{p}{p_0} \tag{13.5}$$

$$\alpha_{\perp 1} = r\cos\phi, \quad \alpha_{\perp 2} = r\sin\phi \quad v = \frac{p}{p_0} = \frac{p\sqrt{(c^2 - 2V)}}{\sqrt{\mathbf{p}^2 - |\mu|^2}} \tag{13.6}$$

$$\alpha_0 = \frac{\boldsymbol{\alpha}\mathbf{p}}{p_0(c^2 - 2V)} = \frac{p}{p_0}\left(\frac{r}{\sqrt{(v^2 - c^2 + 2V)(c^2 - 2V)}}\right) \tag{13.7}$$

where r, ϕ are arbitrary real numbers $r \geq 0$. The length $|\boldsymbol{\alpha}|$ of multivariant 3-vector $\boldsymbol{\alpha}$ is of the order r and components of $\boldsymbol{\alpha}$ are defined by the relations

$$\boldsymbol{\alpha}_{\parallel} = \mathbf{p}\frac{(\boldsymbol{\alpha}\mathbf{p})}{\mathbf{p}^2}, \quad \boldsymbol{\alpha}_{\perp} = \boldsymbol{\alpha} - \boldsymbol{\alpha}_{\parallel}, \quad \alpha_{\parallel}^2 = \frac{(\boldsymbol{\alpha}\mathbf{p})^2}{\mathbf{p}^2}, \quad \alpha_{\parallel} = \frac{\boldsymbol{\alpha}\mathbf{p}}{p}, \quad p = |\mathbf{p}| \tag{13.8}$$

Here $\boldsymbol{\alpha}_{\parallel}$ is the component of 3-vector $\boldsymbol{\alpha}$ which is in parallel with the 3-vector $\mathbf{p}$, whereas $\boldsymbol{\alpha}_{\perp}$ is the component of 3-vector $\boldsymbol{\alpha}$, which is perpendicular to the 3-vector $\mathbf{p}$. As far as the quantity r may be infinite, the wobbling of the tachyon world chain may have infinite amplitude.

Averaging over r and ϕ, one obtains macroscopic parameters of the tachyon gas (the mean components of the tachyon gas velocity) [50].

$$\langle u_{\parallel}\rangle = \left\langle \frac{p\sqrt{c^2 - 2V}}{r} \right\rangle = 0, \quad \langle \mathbf{u}_{\perp}\rangle = 0 \tag{13.9}$$

$$\langle \mathbf{u}_{\perp}^2\rangle = \left\langle \left|\frac{\boldsymbol{\alpha}_{\perp}}{\alpha_0}\right|^2 \right\rangle = \left\langle \frac{r^2}{r^2}\left(c^2 - 2V\right)\right\rangle = c^2 - 2V \tag{13.10}$$

$$\left\langle u_{\|}^{2}\right\rangle =\left\langle u_{\|}\right\rangle ^{2}=0,\quad \left\langle \mathbf{u}^{2}\right\rangle =\left\langle u_{\|}^{2}\right\rangle +\left\langle \mathbf{u}_{\perp}^{2}\right\rangle =c^{2}-2V \tag{13.11}$$

One can see from (13.9) - (13.11) that results for $\left\langle u_{\|}\right\rangle$, $\left\langle \mathbf{u}_{\perp}\right\rangle$, $\left\langle u_{\|}^{2}\right\rangle$, $\left\langle \mathbf{u}_{\perp}^{2}\right\rangle$ do not depend on the geometric mass μ of tachyon.

The energy-momentum tensor does not depend on μ also [50]

$$T^{00}=\rho,\quad T^{\alpha 0}=T^{0\alpha}=\rho\left\langle u^{\alpha}\right\rangle \tag{13.12}$$

$$T^{\alpha\beta}=\rho\left\langle u^{\alpha}\right\rangle \left\langle u^{\beta}\right\rangle +P^{\alpha\beta},\quad \alpha,\beta=1,2,3 \tag{13.13}$$

$$P^{\alpha\beta}=\frac{1}{2}\rho\left(\delta^{\alpha\beta}-\frac{\left\langle u^{\alpha}\right\rangle \left\langle u^{\beta}\right\rangle }{\left\langle \mathbf{u}\right\rangle ^{2}}\right)\left(c^{2}-2V-\left\langle \mathbf{u}\right\rangle ^{2}\right) \tag{13.14}$$

In other words, macroscopic parameters of the tachyon gas are the same, as for usual gas of very high pressure. One may work with the tachyon gas as with usual gas, whose molecules cannot be detected. One can detect only the gravitational field of the tachyon gas.

14 Tachyon world chain with two-point skeleton

We investigate now, whether a world chain with a spacelike leading vector may form a helix with timelike axis. If it is possible, then we try to investigate, under which world function such a situation is possible. We consider the world function σ_{g} of the form

$$\sigma_{\mathrm{g}}=\sigma_{\mathrm{M}}+\frac{\lambda_{0}^{2}}{2}f\left(\frac{\sigma_{\mathrm{M}}}{\sigma_{0}}\right),\qquad f\left(x\right)=\left\{\begin{array}{ccc}\mathrm{sgn}\left(x\right) & \text{if} & \left|x\right|>1\\ Cx+\varepsilon g\left(x\right) & \text{if} & \left|x\right|\leq 1\end{array}\right., \tag{14.1}$$

$$\sigma_{0}=\text{const}>0,\quad g\left(x\right)=-g\left(-x\right),\quad 0\leq\varepsilon\ll 1 \tag{14.2}$$

where C is a constant, which is determined from the relation

$$C+\varepsilon g\left(1\right)=1$$

Such a choice of the space-time geometry does not pretend to a real space-time. This is only a model, which is easy for investigation. The function $f\left(\frac{\sigma_{\mathrm{M}}}{\sigma_{0}}\right)$ should be determined from the condition that the world chain with spacelike leading vectors $\mathbf{P}_{0}^{(s)}\mathbf{P}_{1}^{(s)}$ forms a helix with timelike axis. The shape of the chain is determined by leading vectors.

To estimate the form of σ_{g} as a function of σ_{M} at $\sigma_{\mathrm{M}}<\sigma_{0}$, it is useful to consider the world chain, consisting only of spacelike leading vectors $\mathbf{P}_{0}\mathbf{P}_{1}$, $\mathbf{P}_{1}\mathbf{P}_{2}$, $\mathbf{P}_{2}\mathbf{P}_{3}$,...Other vectors of the skeleton will be considered later, when one needs to reduce the chain wobbling. The chain describes the free particle motion, and its links satisfy the equations (12.3). We suppose that the chain is a helix with timelike axis in the space-time. Let the points $...P_{0},P_{1},...$ have the coordinates

$$P_{k}=\left\{kl_{0},R\cos\left(k\varphi\right),R\sin\left(k\varphi\right),0\right\},\qquad k=...0,1,2,... \tag{14.3}$$

All points (14.3) lie on a helix with timelike axis. The quantities R, l_0, φ are parameters of the chain.

We investigate, if it is possible such a space-time geometry (14.1), that the world chain, consisting of connected vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_1\mathbf{P}_2$, $\mathbf{P}_2\mathbf{P}_3$,... form a helix with the radius R. The parameters l_0, $l_1 = 2R\sin\frac{\varphi}{2}$ are small in the sense, that

$$|l_0|\,, |l_1| < \sqrt{2\sigma_0}, \quad l_1 = 2R\sin\frac{\varphi}{2} \tag{14.4}$$

To obtain connection between parameters l_0, l_1, φ, it is sufficient to solve equations, connecting adjacent leading vectors $\mathbf{P}_0\mathbf{P}_1$, $\mathbf{P}_1\mathbf{P}_2$. The dynamic equations have the form

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_1\mathbf{P}_2)_{\mathrm{g}} = |\mathbf{P}_0\mathbf{P}_1|_{\mathrm{g}}^2 \tag{14.5}$$

$$|\mathbf{P}_0\mathbf{P}_1|_{\mathrm{g}}^2 = |\mathbf{P}_1\mathbf{P}_2|_{\mathrm{g}}^2 \tag{14.6}$$

Here index "g" means that the quantities are calculated in the space-time geometry $\mathcal{G}_{\mathrm{g}}$, whose world function σ_{g} is chosen in the form (14.1) where g is some function $g(x) = -g(-x)$, $x \in (-1,1)$ and $\varepsilon \ll 1$.

We are to verify that two adjacent vectors $\mathbf{P}_0\mathbf{P}_1$ and $\mathbf{P}_1\mathbf{P}_2$ satisfy the relations (14.5), (14.6), if

$$P_0 = \{0,0,0,0\}\,, \quad P_1 = \{l_0, l_1, 0, 0\}\,, \quad P_2 = \{2l_0, l_1\cos\varphi, l_1\sin\varphi, 0\} \tag{14.7}$$

and $l_0^2 < l_1^2$. If parameter $l_1 = 2R\sin\frac{\varphi}{2}$, the points (14.7) correspond to three points of the helix (14.3). It is sufficient to verify, that the points (14.7) satisfy equations (14.5), (14.6), because in this case all other pairs of adjacent points (14.3) will satisfy equations of the form (14.5), (14.6).

It is important to keep in mind that the vectors

$$\mathbf{P}_0\mathbf{P}_1 = \{l_0, l_1, 0, 0\}\,, \quad \mathbf{P}_1\mathbf{P}_2 = \{l_0, l_1(\cos\varphi - 1), l_1\sin\varphi, 0\} \tag{14.8}$$

are not unique solution of the equations (14.5), (14.6). There is a lot of other solutions, which lead to unpredictable wobbling of the world chain (14.3). Amplitude of this wobbling is infinite. The world chain of a pointlike particle, described by two-point skeleton $\mathcal{P}_2 = \{P_0, P_1\}$ with spacelike vector $\mathbf{P}_0\mathbf{P}_1$, is unobservable, because it is impossible to trace such a world chain. One cannot trace the world chain, because the spatial distance between points P_s and P_{s+1} may be infinite in any coordinate system. It means that the statement of the relativity theory on impossibility of the tachyons existence is strongly overstated. Tachyons may exist, but they are unobservable.

Considering equations (14.5), (14.6), we write them in the Minkowski space-time, setting

$$\sigma_{\mathrm{g}}(P_0,P_1) = \sigma_{\mathrm{M}}(P_0,P_1) + d(P_0,P_1)\,, \quad d(P_0,P_1) \equiv \frac{\lambda_0^2}{2} f\left(\frac{\sigma_{\mathrm{M}}(P_0,P_1)}{\sigma_0}\right) \tag{14.9}$$

Then equations (14.5), (14.6) take the form

$$(\mathbf{P}_0\mathbf{P}_1.\mathbf{P}_1\mathbf{P}_2)_{\mathrm{M}}+w\left(P_0,P_1,P_1,P_2\right)=\left|\mathbf{P}_0\mathbf{P}_1\right|_{\mathrm{M}}^2+2d\left(P_0,P_1\right) \tag{14.10}$$

$$\left|\mathbf{P}_0\mathbf{P}_1\right|_{\mathrm{M}}^2=\left|\mathbf{P}_1\mathbf{P}_2\right|_{\mathrm{M}}^2 \tag{14.11}$$

where

$$w\left(P_0,P_1,P_3,P_4\right)=d\left(P_0,P_4\right)+d\left(P_1,P_3\right)-d\left(P_0,P_3\right)-d\left(P_1,P_4\right) \tag{14.12}$$

Dynamic equations (14.10), (14.11) may be treated as a description of the particle motion in the space-time geometry of Minkowski under influence of force fields w and d. In other words, we pass from description in $\mathcal{G}_{\mathrm{g}}$ to description in the Minkowski space-time geometry $\mathcal{G}_{\mathrm{M}}$, introducing additional force fields, generated by the geometry $\mathcal{G}_{\mathrm{g}}$. Such a passage admits one to use conventional mathematical technique of the Minkowski geometry.

Further we shall use the scalar product only in the space-time of Minkowski. Index "M" will be omitted for brevity. We present points (14.7) in the form

$$P_0=\{0,0,0,0\}, \qquad P_1=l, \qquad P_2=l+q+\alpha \tag{14.13}$$

$$\mathbf{P}_0\mathbf{P}_1=l, \quad \mathbf{P}_1\mathbf{P}_2=q+\alpha, \quad \mathbf{P}_0\mathbf{P}_2=l+q+\alpha \tag{14.14}$$

Here

$$l=\{l_0,l_1,0,0\}, \quad q=\{l_0,l_1\cos\varphi,l_1\sin\varphi,0\} \tag{14.15}$$

$$\alpha=\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}=\{\alpha_0,\boldsymbol{\alpha}\} \tag{14.16}$$

Vector α describes wobbling of the point P_2 near the "helical" position of the point $P_2=l+q$.

To determine the form of the world function , we set $\alpha=0$ in (14.13), (14.14). For $\left|\mathbf{P}_0\mathbf{P}_1\right|^2$, $\left|\mathbf{P}_1\mathbf{P}_2\right|^2$, $\left|\mathbf{P}_0\mathbf{P}_2\right|^2$ and w in (14.10) one obtains dynamic equations

$$\left|\mathbf{P}_0\mathbf{P}_1\right|_{\mathrm{M}}^2=\left|\mathbf{P}_1\mathbf{P}_2\right|_{\mathrm{M}}^2=2\sigma_{\mathrm{M}}\left(P_0,P_1\right)=l_0^2-l_1^2\equiv l^2, \quad l_0^2<l_1^2 \tag{14.17}$$

$$\left|\mathbf{P}_0\mathbf{P}_2\right|_{\mathrm{M}}^2=4l^2+4l_1^2\sin^2\frac{\varphi}{2}, \quad l^2<0, \quad l_0^2,l_1^2<\sigma_0 \tag{14.18}$$

$$w\left(P_0,P_1,P_1,P_2\right)=\frac{\lambda_0^2}{2}\left(f\left(\frac{2l_1^2\sin^2\frac{\varphi}{2}+2\left(l_0^2-l_1^2\right)}{\sigma_0}\right)-2f\left(\frac{l_0^2-l_1^2}{2\sigma_0}\right)\right) \tag{14.19}$$

Setting

$$l^2=l_0^2-l_1^2=-2\nu\sigma_0, \qquad \nu>0 \tag{14.20}$$

$$a=\frac{2l_1^2}{\sigma_0}\sin^2\frac{\varphi}{2}, \qquad \varkappa=\frac{\sigma_0}{\lambda_0^2} \tag{14.21}$$

dynamic equation (14.5) may be written in the form

$$a\varkappa+f\left(a-4\nu\right)=-4f\left(\nu\right) \tag{14.22}$$

Here the function f is an antisymmetric function, defined by the relation (14.1). Dynamic equation (14.6) transforms to the identity.

After a use of (14.1) equation (14.22) turns into

$$a(\varkappa+1)-\varepsilon g(1)-\varepsilon g(4\nu-a)+4\varepsilon g(\nu)=0 \tag{14.23}$$

$$a=\frac{\varepsilon(g(4\nu)-4g(\nu))}{\varkappa+1-\varepsilon g(1)-\varepsilon g'(4\nu)}=\frac{\varepsilon(g(4\nu)-4g(\nu))}{\varkappa+1}+O\left(\varepsilon^2\right) \tag{14.24}$$

It follows from (14.24), that a may be a small quantity, if $\varepsilon \ll 1$. According to (14.21) a must be positive. It is possible, if

$$g(4\nu)>4g(\nu), \quad \nu>0, \quad 0<\varepsilon \ll 1 \tag{14.25}$$

According to (14.4) and (14.21) one obtains

$$R=\frac{l_1}{2\sin\frac{\varphi}{2}}=\frac{l_1^2}{\sqrt{2a\sigma_0}}=\frac{l_1}{\sqrt{\varepsilon}}\frac{\frac{l_1}{\sqrt{2\sigma_0}}\sqrt{1+\frac{\sigma_0}{\lambda_0^2}}}{\sqrt{(g(4\nu)-4g(\nu))}} \tag{14.26}$$

It means that the radius R of helix may be macroscopic, if ε is small enough.

The result obtained

$$\mathbf{P}_1\mathbf{P}_2=q, \quad \mathbf{P}_0\mathbf{P}_2=l+q \tag{14.27}$$

corresponds to position of the point P_2 on the helix (14.3). However, there are another solutions of equations (14.5), (14.6), where the point P_2 is described by relations (14.13) and vectors (14.14)

$$\mathbf{P}_1\mathbf{P}_2=q+\alpha, \quad \mathbf{P}_0\mathbf{P}_2=l+q+\alpha \tag{14.28}$$

Here vector α describes wobbling of the point P_2. It satisfies the dynamic equations

$$l^2=(q+\alpha)^2 \tag{14.29}$$

$$(l.q+\alpha)+w(P_0,P_1,P_1,P_2)=l^2+2d\left(\frac{l^2}{2}\right) \tag{14.30}$$

which are reduced to the form

$$\alpha^2+2(q.\alpha)=0 \tag{14.31}$$

$$2l_1^2\sin^2\frac{\varphi}{2}+(l.\alpha)+\frac{\lambda_0^2}{2}f\left(\frac{2l^2+2l_1^2\sin^2\frac{\varphi}{2}+(l.\alpha)}{\sigma_0}\right)-2\lambda_0^2 f\left(\frac{l^2}{2\sigma_0}\right)=0 \tag{14.32}$$

Supposing that $(l.\alpha)=l_0\alpha_0-\mathbf{l}\boldsymbol{\alpha}$ is a small quantity, one expands (14.32) over $(l.\alpha)$. As far as the zeroth term of expansion coincides with (14.22), The first term of expansion of (14.32) has the form

$$(l.\alpha)+\varepsilon\frac{\lambda_0^2}{2}g'\left(\frac{2l^2+2l_1^2\sin^2\frac{\varphi}{2}}{\sigma_0}\right)\frac{(l.\alpha)}{\sigma_0}=0 \tag{14.33}$$

or

$$(l.\alpha) = l_0\alpha_0 - l_1\alpha_1 = 0, \quad \alpha_0 = \frac{l_1\alpha_1}{l_0} \tag{14.34}$$

Substituting α_0 from (14.34) in (14.31), one obtains

$$2(l_1 - l_1\cos\varphi)\alpha_1 - 2l_1\sin\varphi\alpha_2 + \left(\frac{l_1\alpha_1}{l_0}\right)^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2 = 0 \tag{14.35}$$

Taking into account that φ is small and setting for simplicity $\varphi = 0$, one obtains for spatial components of vector α

$$\left(\left(\frac{l_1}{l_0}\right)^2 - 1\right)\alpha_1^2 - \alpha_2^2 - \alpha_3^2 = 0 \tag{14.36}$$

As far as $l_1^2 > l_0^2$, the first term in (14.36) is positive, components of 3-vector $\boldsymbol{\alpha}$ may be infinitely large. It means that the wobbling amplitude is infinite. Thus, the helical world chain (14.3) with the two-point spacelike skeleton $\mathcal{P}_1^{(s)} = \left\{P_0^{(s)}, P_1^{(s)}\right\}$ is unstable with respect to the wobbling.

15 Helical world chain with three-point skeleton

Reduction of the wobbling of the world chain, consisting of spacelike vectors, can be achieved, if we consider the world chain with more complicated links, whose skeleton consists of three points $\{P_k, P_{k+1}, Q_{k+1}\}$, $k = ...1.2, ...$ Let $\mathbf{P}_k\mathbf{P}_{k+1}$ be a spacelike vector, whereas the vector $\mathbf{P}_k\mathbf{Q}_{k+1}$ be a timelike vector in $\mathcal{G}_\mathrm{M}$. To investigate the effect of stabilization, it is sufficient to consider the points P_0, P_1, P_2, Q_1, Q_2, having coordinates

$$\begin{aligned} P_0 &= \{0\}, \qquad P_1 = \{l\}, \qquad P_2 = \{l+q+\alpha\}, \\ Q_1 &= \{s\}, \qquad Q_2 = \{s+q+\beta\}, \end{aligned} \tag{15.1}$$

Corresponding vectors have the form

$$\begin{aligned} \mathbf{P}_0\mathbf{P}_1 &= l, \qquad \mathbf{P}_1\mathbf{P}_2 = q+\alpha, \qquad \mathbf{P}_0\mathbf{P}_2 = l+q+\alpha, && (15.2)\\ \mathbf{P}_0\mathbf{Q}_1 &= s, \quad \mathbf{P}_1\mathbf{Q}_2 = s+q-l+\beta, \quad \mathbf{P}_0\mathbf{Q}_2 = s+q+\beta, && (15.3)\\ \mathbf{P}_1\mathbf{Q}_1 &= s-l, \qquad \mathbf{P}_2\mathbf{Q}_2 = s-l+\gamma, \qquad \mathbf{Q}_1\mathbf{Q}_2 = q+\beta, && (15.4)\\ \mathbf{Q}_1\mathbf{P}_2 &= l+q-s+\alpha, \quad \gamma = \beta-\alpha && (15.5) \end{aligned}$$

Here l, q, s are 4-vectors of the Minkowski space-time

$$l = \{l_0, l_1, 0, 0\} \qquad q = \{l_0, l_1\cos\varphi, l_1\sin\varphi, 0\}, \qquad s = \{s_0, s_1, s_2, 0\} \tag{15.6}$$

Vectors α, β, $\gamma = \beta - \alpha$ are vectors describing wobbling, connected with points P_2 and Q_2. On needs to write six dynamic equations corresponding to equalities

$\mathbf{P}_0\mathbf{P}_1$eqv$\mathbf{P}_1\mathbf{P}_2$, $\mathbf{P}_0\mathbf{Q}_1$eqv$\mathbf{P}_1\mathbf{Q}_2$, and $\mathbf{P}_1\mathbf{Q}_1$eqv$\mathbf{P}_2\mathbf{Q}_2$. Two equations, corresponding to $\mathbf{P}_0\mathbf{P}_1$eqv$\mathbf{P}_1\mathbf{P}_2$, have been written and investigated (equations (14.5), (14.6))

In the case $\mathbf{P}_0\mathbf{Q}_1$eqv$\mathbf{P}_1\mathbf{Q}_2$ one obtains

$$s^2 = (s + q - l + \beta)^2 \tag{15.7}$$

$$s^2 + (\beta.s) + w(P_0, Q_1, P_1, Q_2) = s^2 + 2d\left(\frac{s^2}{2}\right) \tag{15.8}$$

where according to (14.12) and (15.2) - (15.5)

$$w(P_0, Q_1, P_1, Q_2) = d(P_0, Q_2) + d(Q_1, P_1) - d(P_0, P_1) - d(Q_1, Q_2)$$

$$= \frac{\lambda_0^2}{2}\left(f\left(\frac{(s+q+\beta)^2}{2\sigma_0}\right) + f\left(\frac{(s-l)^2}{2\sigma_0}\right) - f\left(\frac{l^2}{2\sigma_0}\right) - f\left(\frac{(q+\beta)^2}{2\sigma_0}\right)\right) \tag{15.9}$$

We define s in such a way, that

$$2(s.q - l) = -(q - l)^2 = 4l_1^2 \sin^2\frac{\varphi}{2} \tag{15.10}$$

Then

$$s = \{s_0, l_1(1 - \cos\varphi), l_1 \sin\varphi, 0\} \tag{15.11}$$

Equations (15.7), (15.8) are transformed to the form

$$2(\beta.s + q - l) + \beta^2 = 0 \tag{15.12}$$

$$(s.\beta) + 2l_1^2 \sin^2\frac{\varphi}{2} + \frac{\lambda_0^2}{2}\left(\begin{array}{c} f\left(\frac{(s+q+\beta)^2}{2\sigma_0}\right) + f\left(\frac{(s-l)^2}{2\sigma_0}\right) \\ -f\left(\frac{l^2}{2\sigma_0}\right) - f\left(\frac{(q+\beta)^2}{2\sigma_0}\right) + 2f\left(\frac{s^2}{2\sigma_0}\right)\end{array}\right)$$

$$= \quad -2d\left(\frac{s^2}{2}\right) \tag{15.13}$$

The necessary condition of the fact, that equation (15.13) has the solution $\beta = 0$, has the form

$$2l_1^2 \sin^2\frac{\varphi}{2} + \frac{\lambda_0^2}{2}\left(\begin{array}{c} f\left(\frac{(s+q)^2}{2\sigma_0}\right) + f\left(\frac{(s-l)^2}{2\sigma_0}\right) \\ -2f\left(\frac{l^2}{2\sigma_0}\right) - 4f\left(\frac{s^2}{2\sigma_0}\right)\end{array}\right) = 0 \tag{15.14}$$

Substituting f from (14.1) in (15.14), one obtains

$$\frac{4l_1^2}{\lambda_0^2}\sin^2\frac{\varphi}{2} + \varepsilon g\left(\frac{(s_0 + l_0)^2 - l_1^2(1 + 4\sin^2\varphi)}{2\sigma_0}\right) + \varepsilon g\left(\frac{(s_0 - l_0)^2 - l_1^2}{2\sigma_0}\right)$$

$$-2\varepsilon g\left(\frac{l_0^2 - l_1^2}{2\sigma_0}\right) - 4\varepsilon g\left(\frac{s_0^2}{2\sigma_0}\right) = -\frac{(1 - \varepsilon g(1))}{\sigma_0}s_0^2 + \mathcal{O}\left(\varepsilon^2\right) \tag{15.15}$$

This equation together with (14.22) determine parameters of the helical world chain: l_0, l_1, s_0, R, where R is defined by equation (14.21), (14.26) $R = l_1 \left(2\sin\frac{\varphi}{2}\right)^{-1}$). These parameters depend on the form of function g.

In the case $\mathbf{P}_1\mathbf{Q}_1\text{eqv}\mathbf{P}_2\mathbf{Q}_2$ we obtain

$$(s-l)^2 = (s-l+\gamma)^2 \tag{15.16}$$

$$(s-l.s-l+\gamma) + w(P_1, Q_1, P_2, Q_2) = (s-l)^2 + 2d\left(\frac{(s-l)^2}{2}\right) \tag{15.17}$$

where according to (14.12) and (15.2) - (15.5)

$$\begin{aligned} & w(P_1, Q_1, P_2, Q_2) \\ = \ & d(\sigma_{\mathrm{M}}(P_1, Q_2)) + d(\sigma_{\mathrm{M}}(Q_1, P_2)) - d(\sigma_{\mathrm{M}}(P_1, P_2)) - d(\sigma_{\mathrm{M}}(Q_1, Q_2)) \end{aligned}$$

$$= f\left(\frac{(s+q-l+\beta)^2}{2\sigma_0}\right) + f\left(\frac{(l+q-s+\alpha)^2}{2\sigma_0}\right) - f\left(\frac{l^2}{2\sigma_0}\right) - f\left(\frac{(q+\beta)^2}{2\sigma_0}\right)$$

Equations (15.16) and (15.17) take the form

$$\gamma^2 + 2((s-l).\gamma) = 0, \quad \gamma = \beta - \alpha \tag{15.18}$$

$$\begin{aligned} & ((s-l).\gamma) + \frac{\lambda_0^2}{2} f\left(\frac{(s+q-l+\beta)^2}{2\sigma_0}\right) + \frac{\lambda_0^2}{2} f\left(\frac{(l+q-s+\alpha)^2}{2\sigma_0}\right) \\ & -\frac{\lambda_0^2}{2} f\left(\frac{l^2}{2\sigma_0}\right) - \frac{\lambda_0^2}{2} f\left(\frac{(q+\beta)^2}{2\sigma_0}\right) - \lambda_0^2 f\left(\frac{(s-l)^2}{2\sigma_0}\right) = 0 \end{aligned} \tag{15.19}$$

In the case $\alpha = \beta = \gamma = 0$ equation (15.19) turns to the equation

$$\varepsilon\left(g\left(\frac{(s+q-l)^2}{2\sigma_0}\right) + g\left(\frac{(l+q-s)^2}{2\sigma_0}\right) - 2g\left(\frac{l^2}{2\sigma_0}\right) - 2g\left(\frac{(s-l)^2}{2\sigma_0}\right)\right) = 0$$

or

$$\begin{aligned} & \varepsilon g\left(\frac{s_0^2 - 4l_1^2\sin^2\varphi}{2\sigma_0}\right) + \varepsilon g\left(\frac{(2l_0 - s_0)^2 - 4l_1^2\cos^2\varphi}{2\sigma_0}\right) \\ = \ & 2\varepsilon g\left(\frac{l_0^2 - l_1^2}{2\sigma_0}\right) + 2g\left(\frac{(s-l)^2}{2\sigma_0}\right) \end{aligned} \tag{15.20a}$$

One supposes that the function g has such a form, that system of three equations (14.22), (15.15), (15.20a), considered as system of equations for variables l_0, l_1, s_0, R ($l_0^2, l_1^2, s_0^2 < \sigma_0$) has a solution. Thus, one supposes that parameters l_0, l_1, s_0, R are not arbitrary. They satisfy equations (14.22), (15.15), (15.20a). There may be other solutions with $\alpha, \beta \neq 0$

Let us return to equations (15.18), (15.19)

$$\gamma^2 + 2\left(\left(s-l\right).\gamma\right) = 0, \quad \gamma = \beta - \alpha \tag{15.21}$$

Expanding them over γ, and taking into account (15.20a), one obtains

$$\begin{aligned}
&\left(\left(s-l\right).\gamma\right) + \varepsilon\frac{\lambda_0^2}{2}g'\left(\frac{s^2}{2\sigma_0}\right)\left(2\left(s+q-l.\beta\right)+\beta^2\right)\\
&+\varepsilon\frac{\lambda_0^2}{2}g'\left(\frac{\left(l+q-s\right)^2}{2\sigma_0}\right)\left(\alpha^2+2\left(l+q-s+\alpha\right)\right)\\
&-\varepsilon\frac{\lambda_0^2}{2}f\left(\frac{q^2}{2\sigma_0}\right)\left(\beta^2+2\left(q.\beta\right)\right) = 0
\end{aligned} \tag{15.22}$$

Taking into account (15.12) and (14.31) one obtains from (15.22)

$$\begin{aligned}
&\left(\left(s-l\right).\gamma\right) + \varepsilon\lambda_0^2 g'\left(\frac{\left(s_0-2l_0\right)^2-4l_1^2\cos^2\varphi}{2\sigma_0}\right)\left(\left(l-s.\alpha\right)\right)\\
&-\varepsilon\lambda_0^2 g'\left(\frac{\left(l_0\right)^2-l_1^2}{2\sigma_0}\right)\left(l-s.\beta\right) = 0
\end{aligned} \tag{15.23}$$

Or

$$\left(\left(s-l\right).\gamma\right) = \mathcal{O}\left(\varepsilon\right) \tag{15.24}$$

Supposing that β is small and expanding (15.13) over β, one obtains

$$\left(s.\beta\right) + 2l_1^2\sin^2\frac{\varphi}{2} + \varepsilon\frac{\lambda_0^2}{2}\left(\begin{array}{c} g'\left(\frac{(s+q)^2}{2\sigma_0}\right)\left(2\left(\beta.l\right)\right) \\ -g'\left(\frac{q^2}{2\sigma_0}\right)\left(\beta^2+2\left(\beta.l-s\right)\right) \end{array}\right) = 0$$

As far as $\sin^2\frac{\varphi}{2} = \mathcal{O}\left(\varepsilon\right)$, one obtains

$$\left(s.\beta\right) = \mathcal{O}\left(\varepsilon\right) \tag{15.25}$$

It follows from (15.25) and (15.11)

$$\beta_0 = \frac{\boldsymbol{\beta}\mathbf{s}}{s_0} + \mathcal{O}\left(\varepsilon\right) = \frac{l_1\left(\beta_1\left(1-\cos\varphi\right)+\beta_2\sin\varphi\right)}{s_0} + \mathcal{O}\left(\varepsilon\right) = \mathcal{O}\left(\sqrt{\varepsilon}\right) \tag{15.26}$$

Substituting (15.26) in (15.12) and taking into account that

$$s+q-l = \left\{s_0, 0, 2l_1\sin\varphi, 0\right\} \tag{15.27}$$

one obtains

$$\left(\frac{\boldsymbol{\beta}\mathbf{s}}{s_0}\right)^2 - \boldsymbol{\beta}^2 - 4l_1\sin\varphi\beta_2 = \mathcal{O}\left(\varepsilon\right), \quad \boldsymbol{\beta}^2 = -4l_1\sin\varphi\beta_2 + \mathcal{O}\left(\varepsilon\right) = \mathcal{O}\left(\sqrt{\varepsilon}\right) \tag{15.28}$$

$$\beta_1^2 + \left(1 - \frac{4l_1^2 \sin^2 \varphi}{s_0^2}\right) \beta_2^2 + 4\beta_2 l_1 \sin \varphi + \beta_3^2 = \mathcal{O}\left(\sqrt{\varepsilon}\right) \qquad (15.29)$$

Then

$$\beta_1, \beta_3 = \mathcal{O}\left(\sqrt{\varepsilon}\right), \quad \beta_2 = \mathcal{O}\left(1\right), \quad \text{if } s_0^2 > 4l_1^2 \sin^2 \varphi \qquad (15.30)$$

Let us consider equations for γ (15.24), (15.21). It follows from (15.24) and (15.11)

$$\gamma_0 = \frac{\boldsymbol{\gamma}\left(\mathbf{s} - \mathbf{l}\right)}{s_0 - l_0} = \frac{-l_1 \cos \varphi \gamma_1 + l_1 \sin \varphi \gamma_2}{s_0 - l_0} = \frac{-l_1 \gamma_1}{s_0 - l_0} + \mathcal{O}\left(\sqrt{\varepsilon}\right) \qquad (15.31)$$

Substituting (15.31) in (15.21), one obtains

$$\left(\frac{l_1}{s_0 - l_0}\right)^2 \gamma_1^2 - \gamma_1^2 - \gamma_2^2 - \gamma_3^2 = \mathcal{O}\left(\sqrt{\varepsilon}\right) \qquad (15.32)$$

It follows from (15.32) that

$$\gamma_1, \gamma_2, \gamma_3 = \mathcal{O}\left(\varepsilon^{1/4}\right), \quad \text{if } \ l_1^2 < \left(s_0 - l_0\right)^2 \qquad (15.33)$$

Restriction (14.36) on $\boldsymbol{\alpha}$ is valid, but $\boldsymbol{\alpha} = \boldsymbol{\gamma} + \boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ are restricted by the conditions (15.30) and (15.33), then $\boldsymbol{\alpha}$ is restricted by the condition

$$\alpha_1, \alpha_3 = \mathcal{O}\left(\sqrt{\varepsilon}\right), \quad \alpha_2 = \mathcal{O}\left(1\right), \text{ if } \ l_1^2 < \left(s_0 - l_0\right)^2, \quad \varepsilon \ll 1 \qquad (15.34)$$

Thus, in the case of three-point skeleton wobbling of the helical world chain is restricted, provided s_0 component of the timelike vector $\mathbf{P}_s\mathbf{Q}_{s+1}$ is and large enough. According to (14.26) the radius of the helix is of the order $\varepsilon^{-1/2}$, whereas the amplitude of wobbling is of the order 1. It means, that in the case, when $\varepsilon \ll 1$ and $l_1^2 < \left(s_0 - l_0\right)^2$, the wobbling of the world chain violate slightly the shape of helix.

This property of the tachyon with three point skeleton reminds discrete states of atomic electrons. The discreteness of the atomic electron states is conditioned by the electromagnetic emanation of the atom. It radiates until the charge density of the electron envelope changes in time. As soon as the electric charge density ceases to change, the atom ceases to radiate and the electron state becomes stable. In the case of the tachyon helix there is wobbling of the tachyon world chain. At some values of parameters l, q, s the world chain wobbling reduces, and a quasi-stable tachyon world chain arises.

16 Conclusion

In our way to tachyon model of neutrino we followed to physical principles (not to arbitrary hypotheses), correcting mistakes and defects. At first we substituted nonrelativistic concept of the particle state by the relativistic one. As a result we succeeded to construct a united formalism for description of deterministic and stochastic relativistic particles. It appeared that nonrelativistic quantum mechanics is a relativistic conception in the sense that stochastic component of the quantum particle motion is relativistic. One should use relativistic description of the nonrelativistic quantum particle motion. This stochastic component vanishes after averaging. The mean regular component remains. It is nonrelativistic. As a result the nonrelativistic quantum theory looks as a nonrelativistic conception, although in reality it can be understood only from the viewpoint of relativistic statistical description.

The united formalism of dynamics admits one to interpret the quantum mechanics as a dynamics of relativistic stochastic particles. Then the idea of uniting of special relativity with principles of quantum theory appears to be unnecessary. The question on reasons of the free elementary particles stochasticity appears instead. Usually the stochastic behavior of quantum particle is explained by the quantum principles, i.e. axiomatically. Now, when the quantum principles are not used, one should find reasons of the elementary particle stochasticity. A reason of the free elementary particle stochasticity appears a discreteness of the space-time geometry. Exactly the reason of stochasticity is a multivariance of the discrete space-time geometry. Elementary length λ_0 of the space-time geometry is connected with the quantum constant $\hbar$. As a result the quantum constant appears to be a parameter of the discrete space-time geometry. This fact explains the overall character of the quantum constant (it is explained by properties of the space-time).

Using description of quantum mechanics, any quantum particle is labelled by a classical particle which is a simplified (classical) model of the quantum particle. For instance, from classical viewpoint a free electron has spin s and magnetic moment μ, which depend on $\hbar$. World line of the free electron is a straight line. Classical model for appearance of s and μ is absent. The quantities s and μ are simply quantum numbers ascribed to electron. The quantities s and μ are obtained from the concept of the quantum electron as a result of the limit $\hbar \to 0$, but nevertheless s and μ depend on $\hbar$.

Using statistical description of a free electron as a stochastic particle, one can label the stochastic particle by a deterministic ("classical") particle. However, in this case the world line of the deterministic particle is a helix. Spin s and magnetic moment μ are explained by the helical character of the world line. In this case the deterministic model admits one to construct a more detailed arrangement of the electron. In this example we see, that the statistical approach admits one to determine more detailed arrangement of the elementary particle. This becomes more clear, if we compare situation of the elementary particles arrangement with the situation at the investigation of the atoms arrangement.

In the investigation of the atoms properties there are two different approaches: (1) structural approach, (2) empirical approach. At the structural approach one investigates the atoms arrangement, its components (nucleus and electron envelope), dynamics of these components and their interaction. At the structural approach one uses quantum mechanics and atomic physics. At the empirical approach one investigates properties of different chemical elements, classification of chemical elements over their properties, reactions between chemical elements. The empirical approach is used in chemistry.

If one knows arrangement of atoms, one can calculate in principle properties of chemical elements. But these calculations are very complicated, and one does not use them in practice. One prefer to use periodical system of chemical elements in order to classify and to investigate chemical reactions. The periodical system was obtained empirically. It is more simple technically, although in principle the chemical reaction can be calculated, if arrangement of atoms is known. However, one cannot to investigate arrangement of the atoms, basing on empirical data, obtained at empirical approach (periodical system of chemical elements). In this sense the structural approach is more fundamental, than the empirical approach.

In the contemporary investigations of elementary particles one uses only empirical approach. One cannot hope to investigate the arrangement of elementary particles, using only empirical approach, which ascribes quantum numbers to elementary particles instead of investigation of their structure. Founded on quantum theory, the empirical approach cannot explain, from where these quantum numbers appear. Formalism of quantum theory does not admit one to obtain such an explanation. Disruption between the structural approach and the empirical approach is more in the elementary particle theory, than in the atomic theory.

Note that the structural approach uses a new formalism of the particle dynamics and a new formalism of the space-time geometry. A new operation such as the dynamical disquantization is used in the structural approach.

17 Appendix. Transformation of equation for variable $\boldsymbol{\xi}$

Multiplying equation (6.13) by $2(1+\mathbf{z}\boldsymbol{\xi})/\hbar$ and keeping in mind that $\boldsymbol{\xi}^2 = 1$ and $\mathbf{z}^2 = 1$, we obtain

$$\begin{aligned} & -\left(\dot{\boldsymbol{\xi}}\times\mathbf{z}\right)\times\boldsymbol{\xi}+\left(-(\dot{\boldsymbol{\xi}}\times\mathbf{z})+\frac{(\boldsymbol{\xi}\times\mathbf{z})\mathbf{z}\dot{\boldsymbol{\xi}}}{(1+\mathbf{z}\boldsymbol{\xi})}-\frac{(\dot{\boldsymbol{\xi}}\times\boldsymbol{\xi})\mathbf{z}}{(1+\mathbf{z}\boldsymbol{\xi})}\mathbf{z}\right)\times\boldsymbol{\xi} \\ = & -(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\times\boldsymbol{\xi}Q(1+\mathbf{z}\boldsymbol{\xi}) \end{aligned} \tag{17.1}$$

$$\begin{aligned} & -\left(\dot{\boldsymbol{\xi}}\times\mathbf{z}\right)\times\boldsymbol{\xi}-(\dot{\boldsymbol{\xi}}\times\mathbf{z})\times\boldsymbol{\xi}+\left(\frac{(\boldsymbol{\xi}\times\mathbf{z})\left(\mathbf{z}\dot{\boldsymbol{\xi}}\right)}{(1+\mathbf{z}\boldsymbol{\xi})}-\mathbf{z}\frac{\left((\boldsymbol{\xi}\times\mathbf{z})\dot{\boldsymbol{\xi}}\right)}{(1+\mathbf{z}\boldsymbol{\xi})}\right)\times\boldsymbol{\xi} \\ = & -(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\times\boldsymbol{\xi}Q(1+\mathbf{z}\boldsymbol{\xi}) \end{aligned} \tag{17.2}$$

The term in brackets is written as a double vector product

$$-2\left(\dot{\boldsymbol{\xi}}\times\mathbf{z}\right)\times\boldsymbol{\xi}+\frac{\dot{\boldsymbol{\xi}}\times((\boldsymbol{\xi}\times\mathbf{z})\times\mathbf{z})}{(1+\mathbf{z}\boldsymbol{\xi})}\times\boldsymbol{\xi}+(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\times\boldsymbol{\xi}Q(1+\mathbf{z}\boldsymbol{\xi})=0 \tag{17.3}$$

$$-2\left(\dot{\boldsymbol{\xi}}\times\mathbf{z}\right)\times\boldsymbol{\xi}-\frac{\dot{\boldsymbol{\xi}}\times(\boldsymbol{\xi}-\mathbf{z}(\boldsymbol{\xi}\mathbf{z}))}{(1+\mathbf{z}\boldsymbol{\xi})^2}\times\boldsymbol{\xi}+(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\times\boldsymbol{\xi}Q(1+\mathbf{z}\boldsymbol{\xi})=0 \tag{17.4}$$

Transforming the double vector products in the first and second terms, one obtains

$$\dot{\boldsymbol{\xi}}\left(\boldsymbol{\xi}\left(2\mathbf{z}-\frac{(\mathbf{z}\boldsymbol{\xi})\,\mathbf{z}-\boldsymbol{\xi}}{(1+\mathbf{z}\ \boldsymbol{\xi})}\right)\right)+(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\times\boldsymbol{\xi}Q(1+\mathbf{z}\boldsymbol{\xi})=0 \tag{17.5}$$

$$\dot{\boldsymbol{\xi}}\left((2\mathbf{z}\boldsymbol{\xi}-(\mathbf{z}\boldsymbol{\xi}-1))\right)+(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\times\boldsymbol{\xi}Q(1+\mathbf{z}\boldsymbol{\xi})=0 \tag{17.6}$$

$$\dot{\boldsymbol{\xi}}=-\left(\boldsymbol{\xi}\times(\dot{\mathbf{x}}\times\ddot{\mathbf{x}})\right)Q \tag{17.7}$$

References

[1] A. Sommerfeld, Simplified deduction of the field and the forces of an electron moving in any given way". *Knkl. Acad. Wetensch* **7**, *345–367, (1904).*

[2] O.-M. P. Bilaniuk, V. K. Deshpande, E. C. G. Sudarshan, "'Meta' Relativity". *American Journal of Physics* **30***(10), 718, (1962).*

[3] Ya.P. Terletsky, Positive, negative and imaginary rest masses. *J. de Physique at le Radium* **23,** *iss 11, 910-920 (1963).*

[4] G. Feinberg, "Possibility of Faster-Than-Light Particles". *Physical Review* **159 (5)**: *1089–1105. (1967)*

[5] Yu. A.Rylov, Discrete space-time geometry and skeleton conception of particle dynamics. *Int. J. Theor. Phys.* **51**, *iss. 6 1847-1865, (2012)*, see also *e-print 1110.3399v1*

[6] L.Smolin, The *trouble with physics: the rise of string theory, the fall of science, and what comes next.* Houghton Mifflin, Boston, 2006.

[7] Yu. A.Rylov, Uniform formalism for description of dynamic, quantum and stochastic systems. *http://arXiv.org/abs/physics/0603237v6*

[8] Yu.A. Rylov, Spin and wave function as attributes of ideal fluid. (*Journ. Math. Phys.* **40**, *pp. 256 - 278, (1999).*

[9] Yu.A.Rylov, Dynamic disquantization of Dirac equation *e-print /quant-ph/0104060.*

[10] Yu.A.Rylov, "Non-Riemannian model of the space-time responsible for quantum effects. *Journ. Math. Phys.* **32(8)**, *2092-2098, (1991)*

[11] Yu.A.Rylov, Dirac equation in terms of hydrodynamic variables. *Advances in Applied Clifford Algebras*, **5**, *pp 1-40, (1995)*). See also *e-print /1101.5868.*

[12] Yu. A.Rylov, Discrete space-time geometry and skeleton conception of particle dynamics. *Int. J. Theor. Phys.* **51**, iss. 6 (2012), pp/ 1847-1865, (2012), see also *e-print* /1110.3399v1.

[13] J.E. Moyal, Quantum mechanics as a statistical theory. *Proc.Phil. Soc.* **45**, 99 (1949).

[14] I.Fenyes, Foundation and interpretation of quantum mechanics from viewpoint of probability theory. *Zs. f. Physics* **132**, 81, (1952)

[15] E. Madelung, Quantentheorie in hydrodynamischer Form, Z. Physik, **40**, 322-326, (1926).

[16] D. Bohm, On interpretation of quantum mechanics on the basis of the "hidden" variable conception. *Phys.Rev.* **85,** 166, 180, (1952).

[17] P. Holland, *The Quantum Theory of Motion*, (Cambridge University Press, Cambridge, 1993) and references therein.

[18] Yu.A.Rylov, Quantum Mechanics as a theory of relativistic Brownian motion. *Ann. Phys. (Leipzig).* **27**, *1-11, (1971)*

[19] Yu.A.Rylov, Quantum mechanics as relativistic statistics.I: The two-particle case. *Int. J. Theor. Phys.* **8**, *65-83,(1973).*

[20] Yu.A.Rylov, Quantum mechanics as relativistic statistics.II: The case of two interacting particles. *Int. J. Theor. Phys.* 8, *123-139, (1973*).

[21] Yu. A. Rylov, Hydrodynamical interpretation of quantum mechanics: the momentum distribution. *e-print, /physics/0402068.*

[22] Yu. A.Rylov, Incompatibility of the Copenhagen interpretation with quantum formalism and its reasons, *e-print /physics/0604111. Concepts of Physics* **5**, *iss.2, 323-328, (2008).*

[23] Yu.A.Rylov, Classical description of pair production, *e-print /abs/physics/0301020)*

[24] Yu.A.Rylov, Quantum mechanics as a dynamic construction. *Found. Phys.* **28**, No.2, 245-271, (1998).

[25] Yu. A.Rylov. Model conception of quantum phenomena: logical structure and investigation methods. *e-print /physics/0310050v2.*

[26] P. A. M. Dirac, *Principles of Quantum Mechanics,* 4th ed. Oxford, 1958.

[27] L. L. Foldy, and S. A. Wouthuysen, *Phys. Rev.,* **78**, 29, (1950).

[28] E. Schrödinger, *Sitzungsber. Preuss. Wiss. Phys. Math. Kl.* **24**, 418, (1930).

[29] A. O. Barut, N. Zanghi, *Phys. Rev. Lett.* **52**, 2009, (1984).

[30] A. O. Barut, A. J. Bracken, *Phys. Rev.* **D23**, 2454, (1981).

[31] J. C. Aron, *Found. Phys.* **11**, 863, (1981).

[32] D. Hestenes, *Found. Phys.* **20**, 45, (1990).

[33] W. A. Rodriges Jr., J. Vaz Jr. *Phys. Lett.* **B318**, 623, (1993)

[34] Yu. A.Rylov, Is the Dirac particle composite? *e-print* /physics/0410045.

[35] F. Sauter, *Zs. Phys.* **63,** 803, (1930), **64**, 295, (1930).

[36] A. Sommerfeld, *Atombau and Spektrallinien.* bd.2, Braunschweig, 1951.

[37] Yu. A. Rylov (2004), Is the Dirac particle completely relativistic? *e-print* /physics/0412032.

[38] J. L. Anderson, *Principles of relativity physics.* Academic Press, New-York, 1967, pp 75-88.

[39] Yu. A. Rylov, Anderson's absolute objects and constant timelike vector hidden in Dirac matrices. *e-print quant-ph/0112091*

[40] Yu.A.Rylov, Ptolemyness of conventional program for microcosm investigations and alternative research program. *e-print /quant-ph/0011044. (*Russan version: *Fizicheskaya Mysl Rossii*, iss.1, pp. 1-23, (2001)).

[41] Yu. A. Rylov, Formalized procedure of transition to classical limit in application to the Dirac equation. *e-print physics/0507183*

[42] K. Menger, Untersuchen über allgemeine Metrik, *Mathematische Annalen,* **100**, 75-113, (1928).

[43] L.M. Blumenthal, *Theory and Applications of Distance Geometry*, Oxford, Clarendon Press, 1953

[44] Yu. A.Rylov, Deformation principle and further geometrization of physics.*e-print* /0704.3003

[45] Yu. A.Rylov, Non-Euclidean method of the generalized geometry construction and its application to space-time geometry. in *Pure and Applied Differential geometry* pp.238-246. eds. Franki Dillen and Ignace Van de Woestyne. Shaker Verlag, Aachen, 2007. see also *e-print* /Math.GM/0702552

[46] Yu.A.Rylov, Geometry without topology as a new conception of geometry. *Int. Jour. Mat. & Mat. Sci.* **30**, *iss. 12, 733-760, (2002)*, see also /math.MG/0103002.

[47] Yu. A. Rylov, Discriminating properties of compactification in discrete uniform isotropic space-time. *e-print 0809.2516v2*

[48] Yu. A. Rylov, Geometrical dynamics: spin as a result of rotation with superluminal speed. *e-print 0801.1913.*

[49] Rylov Yu. A., Tachyon gas as a candidate for dark matter. *Vestnik RUDN, mathematics, informatics, physics (2013) iss 2 pp.159-173.*

[50] Yu. A. Rylov, Dynamic equations for tachyon gas. *submitted to Int. J. Theor. Phys.* (2012)